

# Moduli of sheaves from moduli of Kronecker modules

Luis Álvarez-Cónsul

*Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM  
Serrano 113 bis, 28006 Madrid, Spain*

Alastair King

*Mathematical Sciences, University of Bath  
Bath BA2 7AY, UK*

*To Peter Newstead on his 65th birthday*

This article is an expository survey of our paper [AK], which provides a new way to think about the construction of moduli spaces of coherent sheaves on projective schemes and the closely related construction of theta functions on such moduli spaces.

More precisely, for any projective scheme  $X$ , over an algebraically closed field of arbitrary characteristic, we are interested in the moduli spaces (schemes)  $\mathcal{M}_X^{ss}(P)$  of semistable coherent sheaves of  $\mathcal{O}_X$ -modules, with a fixed Hilbert polynomial  $P$  with respect to a very ample invertible sheaf  $\mathcal{O}(1)$ .

Such moduli spaces were first constructed for vector bundles on smooth projective curves by Mumford [Mu] and Seshadri [S1], and this was the context where the key ideas were first developed, namely the notions of stability, semistability and S-equivalence. Thus Mumford showed that there was a quasi-projective variety parametrising isomorphism classes of stable bundles, while Seshadri showed that this is a dense open set in a projective variety parametrising S-equivalence classes of semistable bundles.

For modern account of moduli spaces of sheaves and their construction in higher dimensions, see [HL]. Recall that every semistable sheaf has a Jordan-Hölder filtration, or *S-filtration*, with stable factors and two semistable sheaves are *S-equivalent* if their associated graded sheaves are isomorphic, i.e. their S-filtrations have isomorphic stable factors (counted with multiplicity). The importance of this notion lies in the fact that, since any semistable sheaf can degenerate to its associated graded sheaf, S-equivalent sheaves must correspond to the same point in a moduli space (see e.g. [HL, Lemma 4.1.2]).

One of the key properties of  $\mathcal{M}_X^{ss}(P)$  is then that its (closed) points correspond precisely to S-equivalence classes of semistable sheaves. Indeed, one way to think of the construction of  $\mathcal{M}_X^{ss}(P)$  is the specification of a scheme structure on the underlying set of S-equivalence classes with an appropriate universal property with respect to families of semistable sheaves, nowadays called ‘corepresenting the moduli functor’ (see Section 4 for more details and [Ne, §1.2] or [HL, §4.1] for a full discussion).

Since Mumford and Seshadri’s original work, and subsequent generalisations to higher dimensions and arbitrary characteristic by Gieseker [Gi], Maruyama [Ma], Simpson [Si] and Langer [La], the basic method for the construction of  $\mathcal{M}_X^{ss}(P)$  has proceeded in two steps. First, ‘rigidify’ by identifying isomorphism classes of sheaves with orbits in a certain Quot-scheme for a certain action of a reductive group. Second, ‘linearise’ by finding a projective embedding of the Quot-scheme to obtain a problem in Geometric Invariant Theory (GIT), as developed for precisely such a purposes by Mumford [MF] (see also Newstead’s article in this volume). It is the second step where the essential difficulties and variations of approach occur.

Once one has an intrinsic definition of semistability for sheaves, the basic problem is to find a linearisation where semistable sheaves correspond to GIT semistable orbits in (a suitably chosen subscheme of) the Quot-scheme, and furthermore S-equivalence of sheaves corresponds to closure equivalence of orbits.

The widely-accepted intrinsic notion of semistability, that generalises Mumford’s for curves, was formulated by Gieseker and refined by Simpson. However, there have been many projective embeddings used to try to capture this notion geometrically. One of the most natural, first used by Simpson in this context, is into the Grassmannian originally used by Grothendieck [Gr] to construct the Quot-scheme.

The fundamental change of view point introduced in [AK] may be encapsulated by saying that, in Simpson’s version of the construction, it is possible to linearise before rigidifying. To be more precise, now to ‘linearise’ means to embed the category of sheaves (subject to some regularity condition) in a simpler and more ‘linear’ category; in this case, in the category of Kronecker modules for the vector space  $H = H^0(\mathcal{O}(n))$ , for suitably large  $n$ . In Sections 1 and 2, we will explain in detail how this is done.

Such a Kronecker module is a linear map  $\alpha: V_0 \otimes H \rightarrow V_1$ , for finite dimensional vector spaces  $V_i$ , or equivalently a representation of the

quiver

$$\bullet \xrightarrow{H} \bullet$$

One may also say that  $V = V_0 \oplus V_1$  is a module for the path algebra  $A$  (see Section 2 for details) and we shall use the language of  $A$ -modules and Kronecker modules interchangeably. When we ‘rigidify’ the problem of classifying Kronecker modules up to isomorphism, by fixing the vector spaces  $V_i$ , we obtain the *linear* space  $R = \text{Hom}(V_0 \otimes H, V_1)$  with a *linear* action of the reductive group  $G = \text{GL}(V_0) \times \text{GL}(V_1)$ .

A notable property of the category of Kronecker modules is that there is a unique intrinsic notion of semistability, which corresponds to the unique GIT problem associated to the action of  $G$  on  $R$ , which thus constructs the moduli space  $\mathcal{M}_A^{ss}$  of semistable  $A$ -modules (see Section 4 for details). This construction should be considered to be straightforward and transparent, including the correspondence between S-equivalence of  $A$ -modules and closure equivalence of orbits in the GIT problem. Indeed it is a simple case of a more general, but equally transparent, theory of moduli of representations of quivers [Ki].

Note that, in the case  $\dim V_0 = \dim V_1 = 1$ , a (non-zero) Kronecker module is in effect a point in the projective space  $\mathbb{P}(H^*)$  in which  $X$  itself is embedded by the linear system  $H$ . Indeed, the categorical embedding of sheaves in Kronecker modules, when restricted to point sheaves on  $X$  gives precisely this embedding.

From the new view point, the basic problem for this ‘categorical linearisation’ is to show that semistability of sheaves corresponds to semistability of  $A$ -modules and that S-filtrations and S-equivalence are preserved. As we will see in Section 3, this helps to clarify the procedure and identify the delicate parts of the problem. We can then, in Section 4, use the construction and properties of  $\mathcal{M}_A^{ss}$  to deduce the corresponding construction and properties of  $\mathcal{M}_X^{ss}$ .

One corollary is that we obtain an ‘embedding’ of moduli spaces

$$\varphi: \mathcal{M}_X^{ss} \rightarrow \mathcal{M}_A^{ss}.$$

More precisely, this is a scheme-theoretic embedding in characteristic zero and at stable points in characteristic  $p$ . Well-understood subtleties with quotients mean that we only know that it is a set-theoretic embedding at strictly semistable points in characteristic  $p$ .

Up to this technical detail, we can then import known definitions and properties of determinantal semi-invariants of quivers, i.e. the natural

homogeneous coordinates on  $\mathcal{M}_A^{ss}$ , to define and study the corresponding coordinates, or ‘theta functions’, on  $\mathcal{M}_X^{ss}$  (see Section 5 for details). In this way, we are able to strengthen, and generalise to arbitrary  $X$ , results of Faltings [Fa] for smooth curves.

### 1 Simpson’s construction revisited

To explain precisely how our shift in view point occurs, we recall in more detail Simpson’s version of the construction of moduli of sheaves.

For the first (rigidification) step, one chooses an integer  $n_0 \gg 0$  such that any semistable sheaf  $E$  with Hilbert polynomial  $P$  is  $n_0$ -regular in the sense of Castelnuovo-Mumford (see e.g. [HL, §1.7] for details), which in particular guarantees that the natural evaluation map

$$\epsilon_E: H^0(E(n_0)) \otimes \mathcal{O}(-n_0) \rightarrow E \tag{1.1}$$

is surjective and  $\dim H^0(E(n_0)) = P(n_0)$ . Thus, after the choice of an isomorphism  $H^0(E(n_0)) \cong V_0$ , where  $V_0$  is some fixed  $P(n_0)$ -dimensional vector space, we may identify  $E$  with a point in the Quot-scheme parametrising quotients of  $V_0 \otimes \mathcal{O}(-n_0)$  with Hilbert polynomial  $P$ . Changing the choice of isomorphism is given by the natural action of the reductive group  $\mathrm{SL}(V_0)$  on the Quot-scheme.

For the second (linearisation) step, one chooses  $n_1 \gg n_0$  so that applying the functor  $H^0(- \otimes \mathcal{O}(n_1))$  to (1.1) yields a surjective map

$$\alpha_E: H^0(E(n_0)) \otimes H \rightarrow H^0(E(n_1)) \tag{1.2}$$

where  $H = H^0(\mathcal{O}(n_1 - n_0))$  and  $\dim H^0(E(n_1)) = P(n_1)$ . More precisely, this construction is applied after choosing the isomorphism

$$H^0(E(n_0)) \cong V_0$$

and thus  $\alpha_E$  determines a point in the Grassmannian of  $P(n_1)$ -dimensional quotients of  $V_0 \otimes H$ . Note that the kernel  $\beta_E: U \hookrightarrow V_0 \otimes H$  of  $\alpha_E$  determines  $E$ , and indeed the corresponding point in the Quot-scheme, as the cokernel of the corresponding map

$$U \otimes \mathcal{O}(-n_1) \xrightarrow{\beta_E} V_0 \otimes \mathcal{O}(-n_0) \rightarrow E \rightarrow 0. \tag{1.3}$$

This is how to see that the map from the Quot-scheme to the Grassmannian is an embedding.

Now we may observe that, by not choosing the rigidifying isomorphism  $H^0(E(n_0)) \cong V_0$ , one may interpret Simpson’s method as a sin-

gle functorial procedure whereby the sheaf  $E$  determines the Kronecker module  $\alpha_E$ , from which the sheaf  $E$  may in turn be recovered.

From this point of view, the importance of regularity is also clear. Applying Serre's construction to a Veronese embedding of  $X$ , we know that a sheaf  $E$  is determined by the graded module

$$V_{\bullet}(E) = \bigoplus_{k \geq 0} H^0(E(n_0 + nk))$$

for the algebra  $\text{Sym}^{\bullet}(H)$ , for any  $n_0$  and  $n = n_1 - n_0 > 0$ . The regularity of  $E$  and of  $\mathcal{O}_X$  determine how large  $n_0$  and  $n$  must be to guarantee that the generators and relations of  $V_{\bullet}$  are all in degree  $k = 0$ , so that  $V_{\bullet}$  (and hence  $E$ ) is determined by the Kronecker module  $V_0 \otimes H \rightarrow V_1$ .

## 2 The functorial point of view

We may describe the above procedure more formally as follows. Let  $T = \mathcal{O}(n_0) \oplus \mathcal{O}(n_1)$  and let  $A \subset \text{End}_X(T)$  be the algebra spanned by the two projections  $e_0, e_1$  and  $H = \text{Hom}_X(\mathcal{O}(n_0), \mathcal{O}(n_1))$ . Indeed, in most cases, when  $H^0(\mathcal{O})$  consists just of scalars, we actually have  $A = \text{End}_X(T)$ . An  $A$ -module  $V$  is equivalent to a Kronecker module  $\alpha: V_0 \otimes H \rightarrow V_1$ , where  $V_i = e_i V$  and conversely  $V = V_0 \oplus V_1$ . The *dimension vector* of  $V$  is  $v = (\dim V_0, \dim V_1)$ .

Now, the assignment of  $\alpha_E$  to  $E$  is achieved by the functor

$$\Phi: \text{Coh}(X) \rightarrow \text{Mod}(A): E \mapsto H^0(T \otimes E) = \text{Hom}_X(T^{\vee}, E). \quad (2.1)$$

Moreover, the recovery of  $E$  from  $\alpha_E$  is achieved by the adjoint functor

$$\Phi^{\vee}: \text{Mod}(A) \rightarrow \text{Coh}(X): V \mapsto T^{\vee} \otimes_A V.$$

Explicitly,  $T^{\vee} \otimes_A V$  may be described in terms of the Kronecker module  $\alpha: V_0 \otimes H \rightarrow V_1$  as the pushout of the natural diagram

$$\begin{array}{ccc} V_0 \otimes H \otimes \mathcal{O}(-n_1) & \xrightarrow{1 \otimes \mu} & V_0 \otimes \mathcal{O}(-n_0) \\ \alpha \otimes 1 \downarrow & & \\ V_1 \otimes \mathcal{O}(-n_1) & & \end{array}$$

When  $\alpha$  is surjective, this is equivalent to the procedure as in (1.3). The fact that this procedure works, when it does, may then be formulated as follows [AK, Thm 3.4].

**Theorem 2.1** *Suppose that  $\mathcal{O}_X$  is  $n$ -regular and that  $T = \mathcal{O}(n_0) \oplus \mathcal{O}(n_1)$  for  $n_1 - n_0 \geq n$ . Then the functor  $\Phi$  of (2.1) is fully faithful on the full subcategory of  $n_0$ -regular sheaves. In other words, if  $E$  is  $n_0$ -regular, then the natural map  $\varepsilon_E: \Phi^\vee \Phi(E) \rightarrow E$  is an isomorphism.*

Note that the natural ‘counit’ map  $\varepsilon_E$  in the theorem is the evaluation map

$$\varepsilon_E: T^\vee \otimes_A \text{Hom}_X(T^\vee, E) \rightarrow E.$$

We may paraphrase Theorem 2.1 by saying that  $\Phi$  gives a functorial embedding of  $n_0$ -regular sheaves into  $A$ -modules. In fact, regularity is also crucial to extending this embedding to families of sheaves.

Let  $\mathcal{M}_X^{\text{reg}}(n_0; P)$  be the moduli functor of  $n_0$ -regular sheaves on  $X$  with Hilbert polynomial  $P$ , that is, the (contravariant) functor that assigns to any scheme  $S$  the set of isomorphism classes of flat families over  $S$  of such sheaves. Similarly, let  $\mathcal{M}_A(v)$  be the moduli functor of  $A$ -modules with dimension vector  $v$ . Then regularity also implies that  $\Phi$  preserves flat families [AK, Prop. 4.1] and so induces an embedding of moduli functors, i.e. a natural transformation

$$[\Phi]: \mathcal{M}_X^{\text{reg}}(n_0; P) \rightarrow \mathcal{M}_A(P(n_0), P(n_1)) \tag{2.2}$$

such that  $[\Phi]_S$  is injective for all  $S$ . Note in particular that this means that we only need to look among  $A$ -modules of dimension vector

$$(P(n_0), P(n_1))$$

for the images of all  $n_0$ -regular sheaves of Hilbert polynomial  $P$ .

The general machinery of adjunction provides an explicit condition that determines when an  $A$ -module  $V$  is in the image of this functorial embedding. The adjunction also has a natural ‘unit’ map

$$\eta_V: V \rightarrow \Phi \Phi^\vee(V) = \text{Hom}_X(T^\vee, T^\vee \otimes_A V) \tag{2.3}$$

and  $V \cong \Phi(E)$ , for some sheaf  $E$  for which  $\varepsilon_E$  is an isomorphism, if and only if  $\eta_V$  is an isomorphism, in which case  $E = \Phi^\vee(V)$  is the appropriate sheaf and whether  $E$  is  $n_0$ -regular may be considered a property of  $V$ .

Now, the set of (isomorphism classes of) Kronecker modules with dimension vector  $v = (v_0, v_1)$  is in natural bijection with the set of orbits in the *representation space*

$$R = R_A(v) = \text{Hom}(V_0 \otimes H, V_1) \tag{2.4}$$

for the action of the symmetry group  $G = \mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$  by conjugation, where  $V_i$  is some fixed vector space of dimension  $v_i$ . Note that  $R$  carries a tautological  $G$ -equivariant family  $\mathbb{V}$  of  $A$ -modules, which is ‘equivariantly locally universal’ in the sense that the induced natural transformation from the quotient functor  $\underline{R}/G \rightarrow \mathcal{M}_A$  is a local isomorphism, i.e. an isomorphism after sheafification [AK, Prop. 4.4]. Here  $\underline{Z}$  denotes the functor of points of a scheme  $Z$ , i.e.  $\mathrm{Hom}(-, Z)$ .

Now, using  $\Phi^\vee$  and its associated flattening stratification (see [AK, Prop. 4.2] for details), we can determine a locally closed  $G$ -invariant subscheme  $Q \subset R$  over which  $\mathbb{E} = \Phi^\vee(\mathbb{V})$  is a flat family of  $n_0$ -regular sheaves with Hilbert polynomial  $P$ . This family is also equivariantly locally universal in same sense as above [AK, Thm 4.5]. The functorial embedding (2.2) of moduli functors can naturally be enhanced to an embedding of moduli stacks, which is thus modelled on the embedding of quotient stacks  $[Q/G] \rightarrow [R/G]$ .

It is the space  $Q$  with  $G$ -action that plays the role in our story of the Quot-scheme with  $\mathrm{SL}(V_0)$ -action, or, more strictly speaking, of the open set of  $n_0$ -regular sheaves in the Quot-scheme. Then, the embedding  $Q \subset R$  plays the role of the embedding of the Quot scheme in the Grassmannian.

### 3 Semistability

We now turn to the essential goal of the ‘categorical linearisation’ of sheaves by Kronecker modules, namely of demonstrating the relationship between the semistability of a sheaf  $E$  and the semistability of the corresponding  $A$ -module  $\Phi(E)$ . It is this which will enable us to use the moduli spaces  $\mathcal{M}_A^{ss}$  to construct the moduli spaces  $\mathcal{M}_X^{ss}$ .

Recall the usual (Gieseker-Simpson) definition of semistability for sheaves. Note that this notion depends just on the Hilbert polynomial  $P_E$  of a sheaf  $E$  and of its subsheaves. The ‘multiplicity’  $r_E$  of  $E$  is the leading coefficient of  $P_E$  and the dimension of the support of  $E$  is the degree of  $P_E$ . Then  $E$  is ‘pure’ if it has no proper subsheaves with lower dimensional support.

**Definition 3.1** *A sheaf  $E$  is semistable if  $E$  is pure and, for all  $E' \subset E$ ,*

$$\frac{P_{E'}(n)}{r_{E'}} \leq \frac{P_E(n)}{r_E} \quad \text{for } n \gg 0.$$

For our purposes, this definition has a crucial reformulation, which

gives a cleaner dependence on the Hilbert polynomial and for which purity is an automatic consequence (see also [Ru] for another formulation).

**Lemma 3.2** *A sheaf  $E$  is semistable if and only if, for all  $E' \subset E$ ,*

$$\frac{P_{E'}(n_0)}{P_{E'}(n_1)} \leq \frac{P_E(n_0)}{P_E(n_1)} \quad \text{for } n_1 \gg n_0 \gg 0.$$

This formulation is manifestly related to the (essentially unique) notion of semistability for Kronecker modules.

**Definition 3.3** *An  $A$ -module  $V$  is semistable if, for all  $V' \subset V$ ,*

$$\frac{\dim V'_0}{\dim V'_1} \leq \frac{\dim V_0}{\dim V_1}.$$

Thus  $E$  is semistable if and only if for all  $E' \subset E$ ,  $\Phi(E')$  does not destabilise  $\Phi(E)$  for  $n_1 \gg n_0 \gg 0$ . Note that this is still some way from saying the  $E$  is semistable if and only if  $\Phi(E)$  is semistable, but this would seem to be the result that one could hope for. In fact, we do not prove this ideal result, and it is quite possible that it is not true, demonstrating the more subtle role of purity in this problem. What we do show is the following [AK, Thm 5.10a],

**Theorem 3.4** *Given  $P$ , for  $n_1 \gg n_0 \gg 0$ , suppose that  $E$  is  $n_0$ -regular and pure with Hilbert polynomial  $P$ . Then  $E$  is semistable if and only if  $\Phi(E)$  is semistable.*

Note that  $n_0$  is, in particular, chosen large enough that all semistable sheaves with Hilbert polynomial  $P$  are  $n_0$ -regular, and, of course, all semistable sheaves are pure.

For the proof, we need to show that  $n_0, n_1$  can be chosen so that  $\Phi$  and  $\Phi^\vee$  provide a one-one correspondence between the critical (i.e. most destabilising) subsheaves of  $E$  and the critical submodules of  $\Phi(E)$  and furthermore that the numbers  $n_0, n_1$  in Lemma 3.2 can be chosen uniformly in  $P$ , i.e. independently of  $E$  and  $E'$ . It is this that makes the proof very delicate and in particular seems to require purity as an explicit condition.

This result shows that the Kronecker module  $\Phi(E)$  is semistable whenever the sheaf  $E$  is semistable and thus the embedding of moduli functors (2.2) restricts to an embedding

$$[\Phi]: \mathcal{M}_X^{ss}(P) \rightarrow \mathcal{M}_A^{ss}(P(n_0), P(n_1)), \tag{3.1}$$

where  $\mathcal{M}_X^{ss}(P)$  and  $\mathcal{M}_A^{ss}(v)$  are respectively the moduli functors of semistable sheaves with Hilbert polynomial  $P$  and semistable modules with dimension vector  $v$ .

The other key result [AK, Thm 5.10b-d, Cor. 5.11] is that  $\Phi$  and  $\Phi^\vee$  provide mutually inverse identifications between the S-filtrations of a semistable sheaf  $E$  (and the associated graded sheaf of stable factors) and the S-filtrations of  $\Phi(E)$  (and the associated graded module).

Thus one may see that there is a well-defined and injective map from S-equivalence classes of semistable sheaves with Hilbert polynomial  $P$  to S-equivalence classes of semistable  $A$ -modules with dimension vector  $v = (P(n_0), P(n_1))$ . In other words, we have at least a set-theoretic embedding of moduli spaces  $\mathcal{M}_X^{ss}(P) \rightarrow \mathcal{M}_A^{ss}(v)$ .

#### 4 Moduli spaces

We now consider more carefully how it is that  $\mathcal{M}_X^{ss}$  and  $\mathcal{M}_A^{ss}$  are defined as moduli spaces, that is, as schemes that ‘corepresent’ the corresponding moduli functors. This means that there is a natural transformation  $\mathcal{M}^{ss} \rightarrow \underline{\mathcal{M}}^{ss}$  through which any other natural transformation  $\mathcal{M}^{ss} \rightarrow \underline{\mathcal{Z}}$  uniquely factorises. Note that such a universal property uniquely characterises  $\mathcal{M}^{ss}$  as a scheme. In particular, we will show how this property for  $\mathcal{M}_X^{ss}$  follows from the same property for  $\mathcal{M}_A^{ss}$ , thereby justifying the claim that  $\mathcal{M}_X^{ss}$  is ‘constructed’ using the functor  $\Phi$ .

Recall that the set of isomorphism classes of  $A$ -modules of dimension vector  $v$  is in natural one-one correspondence with the set of  $G$ -orbits in the representation space  $R = R_A(v)$ , as defined in (2.4). Consider the character  $\chi$  on  $G$  given by

$$\chi(g_0, g_1) = (\det g_0)^{-k_0} (\det g_1)^{k_1}, \quad (4.1)$$

where  $k_1/k_0 = v_0/v_1$  with  $k_0, k_1$  coprime. This determines the graded ring

$$\mathrm{SI}_\chi^\bullet = \bigoplus_{d \geq 0} \mathrm{SI}_\chi^d$$

of associated semi-invariants, i.e. a polynomial  $f$  on  $R$  is in  $\mathrm{SI}_\chi^d$  if and only if  $f(gx) = \chi(g)^d f(x)$  for all  $g \in G$  and  $x \in R$ . Then the following hold [Ki]:

- (i) a point  $x \in R$  is  $\chi$ -semistable in the sense of GIT, i.e.  $f(x) \neq 0$  for some  $f \in \mathrm{SI}_\chi^d$  with  $d > 0$ , if and only if the corresponding  $A$ -module is semistable, in the sense of Definition 3.3,

- (ii) two points  $x, y$  in the open subset  $R^{ss}$  of semistable points are closure equivalent, i.e.  $\overline{Gx}$  and  $\overline{Gy}$  intersect in  $R^{ss}$ , if and only if the corresponding  $A$ -modules are S-equivalent.

The general machinery of GIT implies that the projective variety

$$M_A = \text{Proj SI}_\chi^\bullet$$

is a ‘good’ quotient of  $R^{ss}$  by  $G$ , meaning in particular that  $R^{ss} \rightarrow M_A$  is a categorical quotient, whose fibres are closure equivalence classes.

Thus  $M_A$  is a scheme whose points correspond to S-equivalence classes of semistable  $A$ -modules and  $M_A$  corepresents the quotient functor

$$\underline{R^{ss}}/G$$

(this is the definition of categorical quotient). But it follows from the observations in Section 2 that this quotient functor is locally isomorphic to the moduli functor  $\mathcal{M}_A^{ss}(v)$  of semistable  $A$ -modules. In other words,  $M_A$  is the moduli space  $\mathcal{M}_A^{ss}(v)$ .

Now, also recall that there is a locally closed  $G$ -invariant subscheme  $Q \subset R$ , which parametrises  $n_0$ -regular sheaves  $E$  with Hilbert polynomial  $P$ . Supposing that  $n_0$  is large enough that all semistable  $E$  with Hilbert polynomial  $P$  are  $n_0$ -regular, it follows from Theorem 3.4 that the open subset  $Q^{[ss]} \subset Q$ , parametrising semistable sheaves, is a locally closed subscheme of  $R^{ss}$ .

Since the moduli functor of semistable sheaves is locally isomorphic to the quotient functor  $\underline{Q^{[ss]}}/G$ , the problem of ‘construction’ of the moduli space  $\mathcal{M}_\chi^{ss}(P)$  amounts to firstly showing that  $Q^{[ss]}$  has a good quotient by  $G$  and secondly showing that the closure equivalence classes in  $Q^{[ss]}$  are in one-one correspondence with the S-equivalence classes of sheaves.

For the first, the fact that  $Q^{[ss]}$  has a good quotient follows from the fact that  $R^{ss}$  does, *provided* we also know that, for any  $G$ -orbit  $O$  in  $Q^{[ss]}$ , if  $O'$  is the closed orbit in the closure of  $O$  in  $R^{ss}$ , then  $O' \subset Q^{[ss]}$  (see [AK, Lemma 6.2]). This follows because we know that, if  $O$  corresponds to  $\Phi(E)$  for a semistable sheaf  $E$ , with associated graded sheaf  $E'$ , then  $O'$  corresponds to the associated graded module of  $\Phi(E)$ , which is equal to  $\Phi(E')$ , and thus it is indeed in  $Q^{[ss]}$ .

The second follows for almost the same reason. Two semistable sheaves  $E'$  and  $E''$  are S-equivalent if and only if  $\Phi(E')$  and  $\Phi(E'')$  are S-equivalent, i.e. the corresponding orbits have the same closed orbit in

their closure within  $R^{ss}$  or equally within  $Q^{[ss]}$  and thus correspond to the same point in the good quotient of  $Q^{[ss]}$ .

Thus we have constructed the moduli space  $\mathcal{M}_X^{ss}(P)$  and it remains to show that it is a projective variety. Let  $Z \subset R$  be the closure of  $Q^{[ss]}$ . Then the projectivity would follow immediately if we could show *a priori* that the inclusion  $Q^{[ss]} \subset Z^{ss}$  is an equality, where  $Z^{ss} = Z \cap R^{ss}$ . However, knowing only this inclusion we can only deduce that  $\mathcal{M}_X^{ss}(P)$  is quasi-projective, being a dense open subset of the GIT quotient of  $Z$ .

On the other hand, we can show that  $\mathcal{M}_X^{ss}(P)$  is proper using Langton’s method [AK, Prop. 6.5] (cf. [Ma, §5]), a well-known application of the valuative criterion for properness. Hence we can deduce that  $\mathcal{M}_X^{ss}(P)$  is projective and thus *a posteriori* that  $Q^{[ss]} = Z^{ss}$  and therefore that  $Q^{[ss]}$  is closed in  $R^{ss}$ .

In conclusion, we have therefore reproved the existence of a projective moduli space of semistable sheaves [AK, Thm 6.4, Prop. 6.6].

**Theorem 4.1** *There is a projective scheme  $\mathcal{M}_X^{ss}(P)$  which is the moduli space of semistable sheaves on  $X$  with Hilbert polynomial  $P$ , i.e. it corepresents the moduli functor  $\mathcal{M}_X^{ss}(P)$ . The closed points of  $\mathcal{M}_X^{ss}$  are in one-one correspondence with the  $S$ -equivalence classes of semistable sheaves.*

We have also obtained an explicit map  $\varphi: \mathcal{M}_X^{ss} \rightarrow \mathcal{M}_A^{ss}$  induced by the inclusion  $Q^{[ss]} \subset R^{ss}$  (see [AK, Prop. 6.3] for details), which fits into the commuting diagram of natural transformations,

$$\begin{array}{ccc}
 \mathcal{M}_X^{ss} & \xrightarrow{[\Phi]} & \mathcal{M}_A^{ss} \\
 \psi_X \downarrow & & \downarrow \psi_A \\
 \underline{\mathcal{M}_X^{ss}} & \xrightarrow{\varphi} & \underline{\mathcal{M}_A^{ss}}
 \end{array} \tag{4.2}$$

where  $\psi_X$  and  $\psi_A$  are the corepresenting transformations. Note that the corepresenting property of  $\psi_X$  means that such a map  $\varphi$  must exist and be uniquely determined by  $[\Phi]$ .

Because we know that a semistable sheaf  $E$  is stable if and only if the semistable Kronecker module  $\Phi(E)$  is stable [AK, Theorem 5.10b], we see that there is an open subscheme  $\mathcal{M}_X^s(P) \subset \mathcal{M}_X^{ss}(P)$  corepresenting the moduli functor  $\mathcal{M}_X^s(P) \subset \mathcal{M}_X^{ss}(P)$  of stable sheaves. Indeed, we have  $\mathcal{M}_X^s = \varphi^{-1}\mathcal{M}_A^s$ , where  $\mathcal{M}_A^s \subset \mathcal{M}_A^{ss}$  is the corresponding (open) moduli space of stable  $A$ -modules.

**Remark 4.2** As already observed at the end of the previous section, the map  $\varphi$  is set-theoretically injective, but, since we now know that it is induced by the closed embedding  $Q^{[ss]} \subset R^{ss}$  we also know that the image of  $\varphi$  is a closed subset of  $\mathcal{M}_A^{ss}$ . Indeed, in characteristic zero, this further shows that  $\varphi$  is a scheme-theoretic embedding. However, the characteristic zero assumption is crucial to this deduction, and we cannot obtain the same conclusion in characteristic  $p > 0$ , although we can still prove (see [AK, Prop. 6.7] for details) that the restriction to  $\mathcal{M}_X^s$  is a scheme-theoretic embedding.

One point we would like to emphasize is that the spaces  $\mathcal{M}_A^{ss}(v)$  are a family of well-behaved (essentially linear) projective varieties, which naturally generalise projective spaces as potential targets for embedding moduli spaces. Because they are constructed as GIT quotients of linear spaces by classical reductive group actions, the spaces  $\mathcal{M}_A^{ss}(v)$  have well-controlled singularities (only at strictly semistable points) and well-understood homogeneous coordinates.

They are also a good test case for developing a theory of ‘non-commutative moduli spaces’ which should in particular carry appropriately universal families of  $A$ -modules. Our functorial construction should then adapt naturally to construct non-commutative moduli of sheaves.

### 5 Theta functions

We finish by explaining how the natural homogeneous coordinates on  $\mathcal{M}_A^{ss}$  are obtained from Schofield’s general theory of determinantal semi-invariants of quivers and how the adjunction between  $\Phi$  and  $\Phi^\vee$  enables us to restrict them to  $\mathcal{M}_X^{ss}$  to obtain natural coordinates that we will call ‘theta functions’.

Let  $P_i = Ae_i$ , for  $i = 0, 1$ , be the two indecomposable projective  $A$ -modules and note that, for any  $A$ -module  $V$  of dimension vector  $v$ , we have  $\text{Hom}_A(P_i, V) = e_i V = V_i$ . Suppose we are given any map

$$\gamma: P_1^{k_1 d} \rightarrow P_0^{k_0 d}$$

where  $k_1/k_0 = v_0/v_1$ , with  $k_0, k_1$  coprime. Then the linear map

$$\text{Hom}_A(\gamma, V): V_0^{k_0 d} \rightarrow V_1^{k_1 d}$$

is between vector spaces of the same dimension and hence there is naturally defined an element

$$\theta_\gamma(V) = \det \text{Hom}_A(\gamma, V)$$

in the line  $\lambda(V)^d$ , where

$$\lambda(V) = (\det V_0)^{-k_0} \otimes (\det V_1)^{k_1}.$$

If  $V$  is a family of  $A$ -modules over a scheme  $S$ , then naturality means that  $\lambda(V)$  is a line bundle over  $S$  and  $\theta_\gamma(V)$  is a global section of  $\lambda(V)^d$ . In particular, if we consider  $V_0$  and  $V_1$  to be fixed, giving rise to the  $G$ -equivariant family  $\mathbb{V}$  on the representation space  $R$ , and if we trivialise  $\lambda(\mathbb{V})$ , then naturality also means that  $\theta_\gamma(\mathbb{V})$  is a ‘determinantal’ semi-invariant in  $\mathrm{SI}_\chi^d$  (cf. (4.1) and after).

Furthermore, one may show (see [AK, Prop. 7.5] for details) that  $\lambda(\mathbb{V})$  satisfies Kempf’s descent criterion over  $R^{ss}$  and so descends to a line bundle  $\lambda(v)$  on  $\mathcal{M}_A^{ss}(v)$ . Since  $\theta_\gamma(\mathbb{V})$  is an invariant section, it descends to a global section  $\theta_\gamma(v)$  of  $\lambda(v)^d$ . Because  $\mathcal{M}_A^{ss}(v)$  is constructed by GIT as  $\mathrm{Proj} \mathrm{SI}_\chi^\bullet$ , we see that  $\lambda(v)$  is ample and the global sections of  $\lambda(v)^d$  may be naturally identified with  $\mathrm{SI}_\chi^d$ .

The main result about semi-invariants of quivers [DW, SV] is that the functions  $\theta_\gamma$  span  $\mathrm{SI}_\chi^d$ . Thus choosing  $d$  large enough that  $\lambda(v)^d$  is very ample, we see that it is possible to find finitely many

$$\gamma_0, \dots, \gamma_N : P_1^{k_1 d} \rightarrow P_0^{k_0 d}$$

such that the map

$$\Theta_\gamma : \mathcal{M}_A^{ss}(v) \rightarrow \mathbb{P}^N : [V] \mapsto (\theta_{\gamma_0}(V) : \dots : \theta_{\gamma_N}(V)) \tag{5.1}$$

is a scheme-theoretic closed embedding.

We will now see how the ‘embedding’  $\varphi : \mathcal{M}_X^{ss}(P) \rightarrow \mathcal{M}_A^{ss}(v)$  enables us to deduce similar results for  $\mathcal{M}_X^{ss}(P)$ . More precisely, suppose that  $V = \Phi(E)$  for some sheaf (or family of sheaves)  $E$  which is  $n_0$ -regular with Hilbert polynomial  $P$ . Then the adjunction

$$\mathrm{Hom}_A(\gamma, \Phi(E)) = \mathrm{Hom}_X(\Phi^\vee(\gamma), E),$$

enables us to write  $\theta_\gamma(V)$  entirely in terms of  $E$ . Indeed, if

$$\delta = \Phi^\vee(\gamma) : \mathcal{O}(-n_1)^{k_1 d} \rightarrow \mathcal{O}(-n_0)^{k_0 d},$$

with  $k_0$  and  $k_1$  coprime and such that  $k_1/k_0 = P(n_0)/P(n_1)$ , then

$$\mathrm{Hom}_X(\delta, E) : H^0(E(n_0))^{k_0 d} \rightarrow H^0(E(n_1))^{k_1 d}$$

is a linear map between vector spaces of the same dimension and we can define

$$\theta_\delta(E) = \det \mathrm{Hom}_X(\delta, E),$$

as a natural element of the line  $\lambda(E)^d$ , where

$$\lambda(E) = (\det H^0(E(n_0)))^{-k_0} \otimes (\det H^0(E(n_1)))^{k_1}.$$

If  $\mathbb{E} = \Phi^\vee(\mathbb{V})$  is the tautological family of semistable sheaves over  $Q^{[ss]}$ , then  $\lambda(\mathbb{E})$  is the restriction of  $\lambda(\mathbb{V})$  and thus it descends to an ample line bundle  $\lambda(P) = \varphi^*\lambda(v)$  on  $\mathcal{M}_X^{ss}(P)$ . Furthermore, the invariant sections  $\theta_\delta(\mathbb{E})$  descend to global sections  $\theta_\delta(P) = \varphi^*\theta_\gamma(v)$  of  $\lambda(P)^d$ , which may be called ‘determinantal’ theta functions. Note that, even when  $X$  is a smooth curve, this determinant line bundle  $\lambda(P)$  is already an uncontrollably large power (depending on  $n_0, n_1$ ) of the fundamental determinant line bundle on  $\mathcal{M}_X^{ss}(P)$ .

A first consequence of the spanning property of determinantal semi-invariants of Kronecker modules, is that a module  $V$  is semistable if and only if there is a  $\gamma$  such that  $\theta_\gamma(V) \neq 0$ , that is,  $\text{Hom}(\gamma, V)$  is an isomorphism. Thus, using Theorem 3.4, we deduce the following determinantal characterisation of semistable sheaves [AK, Thm 7.2].

**Theorem 5.1** *Given  $P$ , for  $n_1 \gg n_0 \gg 0$ , let  $E$  be  $n_0$ -regular and pure with Hilbert polynomial  $P$ . Then  $E$  is semistable if and only there is a map  $\delta: \mathcal{O}(-n_1)^{m_1} \rightarrow \mathcal{O}(-n_0)^{m_0}$  with  $\text{Hom}_X(\delta, E)$  invertible, i.e.  $\theta_\delta(E) \neq 0$ .*

Note that the invertibility of  $\text{Hom}_X(\delta, E)$  and the regularity of  $E$  automatically imply that  $m_1/m_0 = P(n_0)/P(n_1)$  and so we do not need to impose that condition explicitly here.

If we now combine Theorem 3.4 with the full force of the spanning property, i.e. the projective embedding (5.1), we obtain the following embedding theorem [AK, Thm 7.10], modulo the technical detail of Remark 4.2.

**Theorem 5.2** *There exist  $m_0$  and  $m_1$ , satisfying  $m_1/m_0 = P(n_0)/P(n_1)$ , and finitely many*

$$\delta_0, \dots, \delta_N: \mathcal{O}(-n_1)^{m_1} \rightarrow \mathcal{O}(-n_0)^{m_0},$$

such that the map

$$\Theta_\delta: \mathcal{M}_X^{ss}(P) \rightarrow \mathbb{P}^N: [E] \mapsto (\theta_{\delta_0}(E) : \dots : \theta_{\delta_N}(E))$$

is a set-theoretic closed embedding. This embedding is scheme-theoretic in characteristic zero, while in characteristic  $p > 0$  it is scheme-theoretic on the stable locus.

These two theorems may be compared to two results of a similar flavour proved by Faltings [Fa] (see also [S2]) for vector bundles on a smooth projective curve  $C$ . Firstly, Faltings showed that a vector bundle  $E$  is semistable if and only if there exists a non-zero bundle  $F$  such that

$$\mathrm{Hom}_C(F, E) = 0 = \mathrm{Ext}_C^1(F, E). \quad (5.2)$$

This condition easily implies that  $E$  is semistable [S2, Lemma 8.3], so the main point is to show that, for any semistable  $E$ , such an  $F$  exists.

Note that the condition (5.2) is equivalent to the condition that  $\theta_F(E) \neq 0$ , for a suitably defined theta function  $\theta_F$  (see [Fa, §1], [S2, §2] or [AK, §7.4] for details). Such theta functions have a broader ‘domain of definition’ than our theta functions  $\theta_\delta$ , in the sense that  $\theta_F(E)$  is a well-defined section of a line bundle on the base of *any* family of sheaves  $E$  with  $\chi(F, E) = 0$ , whereas the definition of  $\theta_\delta(E)$  requires that the family also be  $n_0$ -regular. On the other hand, note that the construction of  $\theta_F$  requires crucially that  $X$ , or at least the support of  $F$ , is a smooth curve, whereas  $\theta_\delta$  requires no such restriction.

Secondly, Faltings showed that it is possible to find finitely many bundles  $F_0, \dots, F_N$  which can be used to give a morphism

$$\Theta_F: \mathcal{M}_C^{ss}(P) \rightarrow \mathbb{P}^N: [E] \mapsto (\theta_{F_0}(E) : \dots : \theta_{F_N}(E))$$

which is the normalisation of its image. Esteves [Es] improved this to show that one could arrange that  $\Theta_F$  is injective on points and, in characteristic zero, a scheme-theoretic embedding on the stable locus.

To see more closely the relation to Theorems 5.1 and 5.2, note that, for any  $X$ , because  $\mathcal{O}(n_1 - n_0)$  is very ample and  $m_1 > m_0$ , the generic map  $\delta: \mathcal{O}(-n_1)^{m_1} \rightarrow \mathcal{O}(-n_0)^{m_0}$  is injective as a map of sheaves and thus provides an acyclic resolution of its cokernel  $F$ , with respect to the functor  $\mathrm{Hom}_X(-, E)$  for any  $n_0$ -regular sheaf  $E$ .

Note further that the condition that  $\mathrm{Hom}_X(\delta, E)$  is invertible is an open condition on  $\delta$ , for a fixed  $n_0$ -regular  $E$ . Hence, if this condition holds for some  $\delta$ , then it also holds for some injective  $\delta$ , in which case it is equivalent to the condition, for  $F = \mathrm{coker} \delta$ , that  $\mathrm{Ext}_X^i(F, E) = 0$  for  $i = 0, 1$  and hence indeed for all  $i \geq 0$ , because the others vanish automatically by the nature of  $F$  and the regularity of  $E$ .

Thus, over a smooth curve, Theorem 5.1 reproves the first result of Faltings. Furthermore, we may also suppose that all  $\delta_i$  in Theorem 5.2 are injective and then, over any family of semistable (and hence  $n_0$ -regular) sheaves on a smooth curve,  $\theta_{\delta_i} = \theta_{F_i}$ , for  $F_i = \mathrm{coker} \delta_i$ . Thus, Theorem 5.2 further strengthens Faltings second result.

In fact, we can also use our methods to show that theta functions  $\theta_\delta$  span sufficiently high powers of the line bundle  $\lambda(P)$  on  $\mathcal{M}_X^{ss}$ . Hence, in the curve case, theta functions  $\theta_F$  span sufficiently high powers of the fundamental line bundle, although this result is now superceded by the proof of the strange duality conjecture by Marian and Oprea [MO], which shows that theta functions  $\theta_F$  span all powers of the fundamental line bundle.

Thus, by restricting our attention to regular sheaves, we have been able to define theta functions  $\theta_\delta$ , which satisfy the natural generalisations of Faltings results in higher dimensions, where there are theoretical obstructions to defining  $\theta_F(E)$  for general sheaves  $E$  and  $F$ . Note that the line bundles  $\mathcal{O}(-n_i)$  for  $n_i \geq n$  are effectively projective objects with respect to  $n$ -regular sheaves and thus it is in keeping with Schofield's philosophy that maps between them should provide the natural source of 'homogeneous functions of moduli of sheaves'.

**Acknowledgements.** Partially supported by the Ministerio de Educación y Ciencia under Grant MTM2007-67623. LAC is also supported by the "Programa Ramón y Cajal".

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