Utility theory front to back – inferring utility from agents' choices

A. M. G. Cox*
Dept. of Mathematical Sciences
University of Bath
Bath BA2 7AY

David Hobson†
Department of Statistics
University of Warwick
Coventry CV4 7AL

Jan Obłój‡
Mathematical Institute *and*
Oxford–Man Institute of Quantitative Finance
University of Oxford
Oxford OX1 3LB

January 20, 2011

Abstract

We pursue an inverse approach to utility theory and consumption & investment problems. Instead of specifying an agent's utility function and deriving her actions, we assume we observe her actions (i.e. her consumption and investment strategies) and ask if it is possible to derive a utility function for which the observed behaviour is optimal. We work in continuous time both in a deterministic and stochastic setting. In a deterministic setup, we find that there are infinitely many utility functions generating a given consumption pattern. In the stochastic setting of the Black-Scholes complete market it turns out that the consumption and investment strategies have to satisfy a consistency condition (PDE) if they come from a classical utility maximisation problem. We further show that agent's important characteristics such as attitude towards risk (e.g. DARA) can be directly deduced from her consumption/investment choices.

1 Introduction

The study of investment and consumption problems in finance has a long history, and there is large literature relating to these problems. In general, however, the set-up and solution of the problems take the following form: specify a utility function which describes the investor's 'desire' for wealth/consumption, and then solve a stochastic optimisation problem to find the optimal investment and consumption behaviour. Unfortunately, although we can postulate a simple

---

[*] e-mail: `A.M.G.Cox@bath.ac.uk`; web: `www.maths.bath.ac.uk/~mapamgc/`

[†] e-mail: `D.Hobson@warwick.ac.uk`; web: `www.warwick.ac.uk/go/dhobson/`

[‡] e-mail: `obloj@maths.ox.ac.uk`; web: `www.maths.ox.ac.uk/~obloj/`

parametric form for the utility function, and hope to deduce simple forms for the optimal consumption/investment strategies, it is difficult to justify any claim that such a utility function accurately represents the preferences of the agent. Moreover, attempts to elicit utility functions directly are notoriously difficult, and prone to paradoxes and inconsistencies.

In this work we approach consumption/investment problems from a different, and possibly more natural, perspective. Rather than supposing that we have previously divined an investor's utility function, we suppose that we know their future consumption and investment patterns, and ask whether we can compute a corresponding utility function from the given behaviour. We believe that there are a number of reasons why this is a natural question to ask:

- consumption and investment strategies are 'observables' in that they can actually be measured from investors' actions, are therefore they are a more natural concept around which to build a model than the intangible utility function;
- the framework will allow us to see how natural behaviour patterns in the consumption/investment setting relate to properties of the underlying utility function;
- the analysis mirrors the robust approach to pricing and hedging (cf. Cox and Obłój (2010); Hobson (2010)) where one takes the vanilla option prices as observables and attempts to infer information about the prices of exotics and the dynamics of the price process of the underlying.

Our general question regarding how much information about an agent's preferences and optimality criteria we can recover from her behaviour and choices falls under the heading of *revealed preferences* in Economics. It dates back (at least) to Samuelson (1948) and is sufficiently central and important that it deserves an entry in the New Palgrave Dictionary of Economics (Richter (2008)). Other related work in the economics literature include Green and Srivastava (1985), who consider when a given consumption may be optimal for a utility maximising investor in a one-period, finite state model, and Mas-Colell (1977), where the observed quantities are the consumers demand function, and the aim is to recover the consumers preferences. In the financial literature a similar "reversed" point of view was adopted by Dybvig and Rogers (1997) who considered the recovery of an agent's utility function from a single realisation of her consumption path, working under the (strong) assumption of time homogeneity of agent's utility function.

One of our main results in the continuous time Black-Scholes market setting is that consumption/investment strategies are compatible with a utility maximisation framework if and only if they satisfy a certain PDE. We call this Black's PDE as it was derived by Black (1968) who first considered the inverse problem. His analysis was then extended and made rigorous in a remarkable paper of He and Huang (1994)[1]. We comment on the relation of our paper to these works in Remark 3.8 below. We note also that the main result linking an agent's action via a PDE is similar in spirit to results in Wang (1993). However

---

[1] We thank Thaleia Zariphopoulou who indicated these two valuable references to us when this article neared completion. The manuscript Black (1968) was published in a modified form as Black (1988).

in Wang (1993) a full equilibrium model for a representative agent is considered and we have a partial equilibrium for a price taking agent.

The paper is organised as follows. In the first part of the paper, Section 2, we work in a deterministic setup. After the problem setup and a heuristic derivation of the solution, the main theorem is given in Section 2.3. In Section 2.4 we analyse what can be deduced from agents' consumption about their risk attitudes and present explicit examples. Finally, Section 2.5 comments on our assumptions and the resulting restrictions. In the second part of the paper, Section 3, we work in the stochastic setting of a Black-Scholes market. Sections 3.1 and 3.2 give a heuristic derivation of the main result using the primal and the dual approach respectively. The main theorem and its proof are then given in Section 3.3. Section 3.4 discusses the implication for reading off the risk attitudes of agents from their actions and Section 3.5 focuses on the case of time-homogenous strategies. Section 4 presents possible extensions and future challenges.

Notation. We make the following notational assumptions: throughout, an optimal wealth path will be denoted by $c(t,w)$, where t is the current time, and w the investors wealth at time t. Similarly, an optimal investment strategy (in terms of the cash amount invested in a risky asset) will be $\pi(t,w)$. A general consumption and investment process will be (C_t, Π_t). All stochastic processes will be denoted by capital letters. Partial derivatives will be written $c_w(t,w)$ and $c_t(t,w)$. There should be no confusion over subscripts t since applied to a (upper case) process it refers to a time parameter, and applied to a (lower case) function it is a derivative.

2 Deterministic setting

2.1 Problem set-up

We begin by considering the case where there is no stochastic investment opportunity, so that we only observe the investor's consumption over an infinite horizon. More specifically, suppose we know that the investor who has wealth w at time t will consume an amount $c(t,w)\mathrm{d}t$ in the time interval $t, t+\mathrm{d}t$, where $c(t,w) \geq 0$, and suppose that we work in a situation with no interest on savings (or equivalently, all amounts are discounted back to their time-zero values). Then an investor with initial wealth $x \geq 0$ will have time-t wealth $w(t,x)$ described by:

$$\begin{aligned} \frac{\partial w}{\partial t}(t,x) &= -c(t, w(t,x)) \\ w(0,x) &= x. \end{aligned} \tag{1}$$

Further, we impose that the budget constraint $w(t,x) \geq 0$ holds for all x and t, or, in terms of $c(t,w)$, that:

$$\int_0^\infty c(t, w(t,x))\mathrm{d}t \leq x.$$

Our main concern is then the following. Suppose $c(t,w)$ is as above, and suppose we are told $c(t, w(t, x))$ is optimal for the problem:

$$v(x) = \sup_{\substack{C_t \geq 0, \\ \int_0^\infty C_t \,\mathrm{d}t \leq x}} \int_0^\infty u(t, C_t)\,\mathrm{d}t, \tag{2}$$

where the supremum is taken over processes $(C_t)_{t\geq 0}$. What can we infer about the function u?

2.2 Heuristics

By introducing a Lagrangian term into (2) we get:

$$\begin{aligned} v(x) &= \inf_{\lambda \geq 0} \sup_{C_t \geq 0} \left[\int_0^\infty u(t, C_t)\,\mathrm{d}t - \lambda \left(\int_0^\infty C_t \,\mathrm{d}t - x \right) \right] \\ &= \inf_{\lambda \geq 0} \sup_{C_t \geq 0} \left[\int_0^\infty (u(t, C_t) - \lambda(x) C_t)\,\mathrm{d}t + x\lambda(x) \right]. \end{aligned}$$

In the second line we write $\lambda = \lambda(x)$ to emphasise that λ will depend on the initial wealth.

Hence, for the optimal C_t, we get (supposing that u is suitably differentiable):

$$u_c(t, c(t, w(t, x))) = \lambda(x), \tag{3}$$

where the optimality of λ implies $v_x(x) = \lambda(x)$. Moreover, if we differentiate (3) again, we get:

$$u_{cc}(t, c(t, w(t, x))) = -\frac{D(x)}{\frac{\partial}{\partial x}\left[c(t, w(t, x))\right]}, \tag{4}$$

where $D(x) = -\lambda_x(x) = -v_{xx}(x)$. To then find $u_{cc}(t, c)$, we need to assume that we can recover x as a function of c and t. This will be the case if we assume that $c(t, w)$ is increasing as a function of w. It seems to be a fairly natural assumption to make in terms of investor behaviour, although note that the assumption does then imply $|w(t, x_0) - w(t, x_1)|$ is decreasing in t — that is, the 'wealth paths' corresponding to different initial wealths are 'getting closer together' as time increases. Moreover, one could imagine paths corresponding to two different initial wealths merge at some later point. To rule out such behaviour we will also impose that $\frac{\partial}{\partial x} c(t, w(t, x)) > 0$. Note as well that if this is combined with the assumption that $D(x) > 0$ (or equivalently that the value function is concave in x) we will have u concave — or a decreasing marginal utility of additional consumption. Since these all seem fairly plausible economic assumptions, we will work from this point on under these assumptions.

Finally, observe that u will necessarily be undetermined at least up to addition of a function of the form $A(t) + Bc$ and we would not expect to fully recover u from (4).

Example 2.1. CRRA: Suppose the optimal consumption strategy is: $c(t, w) = \kappa w$, for some $\kappa > 0$ — so the investor consumes a constant proportion of her wealth, and that she always consumes all of her wealth. Then it follows that

$w(t,x) = x\mathrm{e}^{-\kappa t}$ and $c(t, w(t,x)) = \kappa x \mathrm{e}^{-\kappa t}$, and we can invert this to see that if she is consuming c at time t, then her initial wealth was $\frac{c}{\kappa}\mathrm{e}^{\kappa t}$. Hence we get:

$$u_{cc}(s,c) = -\frac{D\left(\frac{c}{\kappa}\mathrm{e}^{\kappa t}\right)}{\kappa \mathrm{e}^{-\kappa t}}.$$

Motivated by our knowledge of the form of the solution in the CRRA case, suppose we assume further that $D(x) = -v_{xx}(x) = \gamma x^{-\gamma-1}$ for some positive γ. Then

$$u_{cc}(t,c) \quad = \quad \frac{-\gamma\left(\frac{c}{\kappa}\mathrm{e}^{\kappa t}\right)^{-\gamma-1}}{\kappa \mathrm{e}^{-\kappa t}} = -\gamma \kappa^{\gamma} \mathrm{e}^{-\gamma \kappa t} c^{-\gamma-1}$$

Integrating this expression in c, we get:

$$\begin{aligned} u_c(t,c) &= c^{-\gamma}\kappa^{\gamma}\mathrm{e}^{-\gamma\kappa t} + \beta(t) = \left(\kappa x \mathrm{e}^{-\kappa t}\right)^{-\gamma}\kappa^{\gamma}\mathrm{e}^{-\gamma\kappa t} + \beta(t) \\ &= x^{-\gamma} + \beta(t), \end{aligned}$$

but by (3), we know this expression must be independent of t, i.e. $\beta(t) \equiv \beta$, and integrating once more, we get (provided $\gamma \neq 1$):

$$u(t,c) = \frac{c^{-\gamma+1}\kappa^{\gamma}\mathrm{e}^{-\gamma\kappa t}}{1-\gamma} + A(t) + \beta c$$

where A is an unknown function of time, and β is a constant. Note that these will not affect the choice of the optimal strategies derived from the utility function (assuming that $\int_0^\infty A(t)\,\mathrm{d}t$ is finite).

We remark that in the above example, we could have chosen any sensible function $D(\cdot)$, and we would end up with the corresponding value function at time 0 given by $v_{xx}(x) = -D(x)$, with exactly the same optimal consumption paths. This suggests we can interpret the paths $(t, c(t, w(t,x)))$ as the contours where the gradient of u is constant, while the function D encodes our relative valuation of the different paths. Knowledge of consumption paths does not reveal the relative valuations of the different paths since there is no natural way of comparing the path with initial wealth x and the path with initial wealth y, simply from the specification of the optimal paths. Specifying the function $D(\cdot)$, however, does give an indication as to the relative valuation of the different paths, and in order to recover $u(t,c)$, we would expect to need to specify this function. We come back to this issue below in Section 2.4 and Example 2.8. Parallels in the setting where a risky asset is traded and an agent also has to specify also her investment strategy are drawn in Remark 3.6 in Section 3.

2.3 Main results

Before we transform the above remarks into a theorem, we also note that there may be a 'maximal' solution to (1), given by $\bar{w}(t) = \sup_{x\geq 0} w(t,x)$ which may be finite for $t > 0$. In such a case, there is a 'maximal' wealth path which comes down from infinity in finite time, and since we assume we only see behaviour from time zero, we will not observe any behaviour at higher wealths, and therefore at higher consumptions than $\bar{c}(t) = \sup_{x\geq 0} c(t, w(t,x))$. Thus we will not be able to infer features of u for levels of consumption above $\bar{c}(t)$. Some thought confirms that $\bar{c}(t)$ may be finite even if $\bar{w}(t)$ is equal to infinity for all t. Mathematically, we will represent this fact by assuming the function $u(t,c)$ is constant above $\bar{c}(t)$, but note that there may be other possible choices of u which produce the same optimal choice of c and w.

Theorem 2.2. *Suppose we are given functions* $\{c(t,w) : w \in \mathbb{R}_+, t \geq 0\}$ *such that* $c(t,0) \equiv 0$*,* $c(t,w)$ *is continuous and strictly increasing in* w *and suppose that if* $w(t,x)$ *is the solution to:*

$$\begin{aligned} \frac{\partial w}{\partial t}(t,x) &= -c(t,w(t,x)) \\ w(0,x) &= x, \end{aligned} \tag{5}$$

then

$$\int_0^\infty c(t,w(t,x))\mathrm{d}t = x,$$

and the function $\frac{\partial}{\partial x}c(t,w(t,x))$ *exists and is strictly positive. Then there exists a function* $u(t,c)$ *such that* $u_c(t,c) \geq 0$ *and* $u_{cc}(t,c) \leq 0$*, for which the problem:*

$$v(x) = \sup_{\substack{C_t \geq 0: \\ \int_0^\infty C_t \,\mathrm{d}t \leq x}} \int_0^\infty u(t,C_t)\,\mathrm{d}t \tag{6}$$

is solved by the choice of consumption: $C_t = c(t,w(t,x))$ *for each* $x \geq 0$*.*

Remark 2.3. In fact, as we shall see, there is a family of solutions u for which the choice $C_t = c(t,w(t,x))$ is optimal. It should also be clear from the proof that Theorem 2.2 could be modified into an if and only if statement, albeit more technical and complicated than the current version.

Proof. Define $\bar{c}(t) = \sup_{x \geq 0} c(t,w(t,x))$, then for $0 \leq c < \bar{c}(t)$, we can find a unique x such that $c = c(t,w(t,x))$. Write this as $y(t,c)$, and note therefore that $y(t,c(t,w(t,x))) = x$, and $y(t,\bar{c}(t)) = \infty$. Also, by the assumption that $\frac{\partial}{\partial x}c(t,w(t,x))$ exists and is strictly positive, $y(t,c)$ is a differentiable function of c with derivative

$$y_c(t,c) = \frac{1}{\frac{\partial}{\partial x}c(t,w(t,x))\big|_{x=y(t,c)}}.$$

Let $D(x)$ be a positive function satisfying

$$\int_x^\infty D(y)\,\mathrm{d}y < \infty, \quad \text{for every } x > 0. \tag{7}$$

Then we can define a function u by:

$$u_c(t,c) = \begin{cases} \int_c^{\bar{c}(t)} \frac{D(y(t,\kappa))}{\frac{\partial}{\partial x}c(t,w(t,x))\big|_{x=y(t,\kappa)}}\,\mathrm{d}\kappa & : c \leq \bar{c}(t) \\ 0 & : c > \bar{c}(t) \end{cases}, \tag{8}$$

where (7) ensures that the integral is finite for $c > 0$. Indeed, using the substitution $\xi = y(t,\kappa)$, we get:

$$\begin{aligned} u_c(t,c) &= \int_c^{\bar{c}(t)} \frac{D(y(t,\kappa))}{\frac{\partial}{\partial x} c(t,w(t,x))\big|_{x=y(t,\kappa)}} \,\mathrm{d}\kappa \\ &= \int_{y(t,c)}^{y(t,\bar{c}(t))=\infty} \frac{D(\xi)}{\frac{\partial}{\partial x} c(t,w(t,x))\big|_{x=\xi}} \left.\frac{\partial}{\partial x} c(t,w(t,x))\right|_{x=\xi} \mathrm{d}\xi \\ &= \int_{y(t,c)}^{\infty} D(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{9}$$

Then $u_c(t,c) \geq 0$ and $u_{cc}(t,c) \leq 0$ so that $u(t,c)$ is concave in c. Also, writing $c = c(t,w(t,x))$ we find

$$u_c(t, c(t,w(t,x))) = \int_x^{\infty} D(\xi)\,\mathrm{d}\xi. \tag{10}$$

Now we consider a general consumption path C_t satisfying $\int_0^\infty C_t \mathrm{d}t \leq x$. Then, using the concavity of $u(t,\cdot)$, and (10) we conclude:

$$\begin{aligned} \int_0^\infty \left[u\left(t,C_t\right) - u(t,c(t,w(t,x)))\right] \mathrm{d}t &\leq \int_0^\infty u_c(t,c(t,w(t,x)))(C_t - c(t,w(t,x))\,\mathrm{d}t \\ &= \left(\int_x^\infty D(\xi)\,\mathrm{d}\xi\right) \int_0^\infty (C_t - c(t,w(t,x))\,\mathrm{d}t \\ &\leq 0 \end{aligned}$$

where the budget constraint gives the final step. Hence the given $c(t,w(t,x))$ is the optimal path as required. □

2.4 Inferring risk aversion from optimal consumption

So far, we have discussed the derivation of a utility function from an initial choice of consumption behaviour. Can we extend this, and say something about some other classical methods of describing investor behaviour? For example, a natural question in this direction would be: given a set of consumption paths can we determine whether the investor has decreasing absolute/relative risk aversion?

As already observed in Section 2.2 above, it turns out that specifying the consumption paths alone is not sufficient. We present examples below of two utility functions, one with decreasing absolute risk aversion and one with increasing absolute risk aversion, which yield the same optimal consumptions paths. In essence, consumption alone does not tell us how the investor compares different wealths. This is specified by the additional function D. We can think of $D(x)$ (or more accurately $\int_x^\infty D(y)\,\mathrm{d}y$) as determining the relative weightings of different initial wealths: when $D(x)$ is large, the additional utility of an agent from a small increase in initial wealth above x is large, when $D(x)$ is small, the additional utility is also small. In what follows, we say that an agent with consumption paths $c(t,w)$ has *relative weighting of initial wealths* $D(x)$ if $D(x)$ is differentiable, satisfies (7), and the agent's utility is specified via (10).

We start with a simple observation about the role of the function D.

Note 2.4. The *Inada condition* — that is, that for all t, $u_c(t,c)$ takes all values in $[0,\infty)$, is equivalent to $\int_x^\infty D(y)\mathrm{d}y \uparrow \infty$ as $x \downarrow 0$.

We now analyse in detail the risk aversion of the investor. We concentrate on absolute risk aversion, but we should point out that similar results can be derived for relative risk aversion.

Definition 2.5. For a utility function u, the absolute risk aversion is given by

$$\rho(t,c) = -\frac{u_{cc}(t,c)}{u_c(t,c)}.$$

We say that an investor is DARA (decreasing absolute risk aversion) if $\rho_c(t,c) \leq 0$ for all $t, c \geq 0$. Similarly, we say an investor is CARA (constant absolute risk aversion) or IARA (increasing absolute risk aversion) if respectively $\rho_c(t,c) = 0$ or $\rho_c(t,c) \leq 0$, for all $t, c \geq 0$.

Recall that u is recovered only up to an affine function. We should note that our normalisation $u_c(t,\infty) = 0$, or more precisely $\lim_{x\to\infty} u_c(t, c(t, w(t,x))) = 0$, which is implicit in the equation (3) and explicit in (10), and follows from the use of $\bar{c}(t)$ as a reference point in (8), has a consequence on the value of the function $\rho(t,c)$. A different reference point might change the absolute risk aversion.

Proposition 2.6. *Suppose an investor has consumption paths $c(t,w)$ and relative weighting of initial wealths $D(x)$. Then the sign of $\rho_c(t,c)$ is the same as the sign of:*

$$\frac{D_x(x)}{D(x)} + \frac{D(x)}{\int_x^\infty D(y)\,\mathrm{d}y} - \frac{\frac{\partial^2}{\partial x^2} c(t,w(t,x))}{\frac{\partial}{\partial x} c(t,w(t,x))} \equiv \frac{\partial}{\partial x} \ln\left(\frac{D(x)}{\frac{\partial}{\partial x} c(t,w(t,x)) \int_x^\infty D(y)\,\mathrm{d}y}\right),$$

evaluated at $x = y(t,c)$.

Corollary 2.7. *An investor is DARA if and only if:*

$$\frac{D_x(x)}{D(x)} + \frac{D(x)}{\int_x^\infty D(y)\,\mathrm{d}y} \leq \inf_{t\geq 0} \frac{\frac{\partial^2}{\partial x^2} c(t,w(t,x))}{\frac{\partial}{\partial x} c(t,w(t,x))}, \quad x > 0. \tag{11}$$

An investor is CARA if and only if

$$\frac{D_x(x)}{D(x)} + \frac{D(x)}{\int_x^\infty D(y)\,\mathrm{d}y} = \frac{\frac{\partial^2}{\partial x^2} c(t,w(t,x))}{\frac{\partial}{\partial x} c(t,w(t,x))}$$

so that in particular, the right hand side of the equation is independent of t. Finally an investor is IARA if and only if:

$$\frac{D_x(x)}{D(x)} + \frac{D(x)}{\int_x^\infty D(y)\,\mathrm{d}y} \geq \sup_{t\geq 0} \frac{\frac{\partial^2}{\partial x^2} c(t,w(t,x))}{\frac{\partial}{\partial x} c(t,w(t,x))}, \quad x > 0. \tag{12}$$

Proof of Proposition 2.6. It follows from (10) that:

$$\rho(t, c(t, w(t,x))) = \frac{D(x)}{\frac{\partial}{\partial x} c(t,w(t,x)) \int_x^\infty D(y)\,\mathrm{d}y}.$$

Since $c(t,w(t,x))$ is increasing in x, $\rho(t,c)$ is increasing in c if and only if the right-hand-side of the above expression is increasing in x, if and only if the logarithm of the right-hand-side is increasing in x. $\square$

Example 2.8. Consider the consumption function of Example 2.1, so that $c(t,w) = \kappa w$ and $c(t,w(t,x)) = \kappa x \mathrm{e}^{-\kappa t}$. Then:

$$\frac{\frac{\partial^2}{\partial x^2} c(t,w(t,x))}{\frac{\partial}{\partial x} c(t,w(t,x))} = 0.$$

If we consider a function $D(x) = \gamma x^{-\gamma-1}$ with $\gamma > 0$, then

$$\frac{D_x(x)}{D(x)} + \frac{D(x)}{\int_x^\infty D(y)\,\mathrm{d}y} = -\frac{1}{x} < 0,$$

so the corresponding investor is DARA. On the other hand, for the choice $D(x) = x\mathrm{e}^{-x^2/2}$,

$$\frac{D_x(x)}{D(x)} + \frac{D(x)}{\int_x^\infty D(y)\,\mathrm{d}y} = \frac{1}{x} > 0,$$

and the investor is IARA. The case $D(x) = \mathrm{e}^{-\zeta x}$ gives a CARA investor. Note that in the last two cases we necessarily have $u(t,\infty) < \infty$, whereas in the first case the finiteness of $u(t,\infty)$ depends on the sign of $(\gamma - 1)$.

Example 2.9. The purpose of this example is to show that explicit answers may still be available beyond the CRRA case in which consumption is proportional to wealth. Again we find that knowledge of the consumption path alone is not sufficient to determine the attitude to risk.

Suppose we have a concave, increasing function $G(z)$ of class $\mathcal{C}^3$ and such that $G(0) = 0, G_z(0) = 1$. Let $w(t,x) = \frac{1}{t}G(xt)$. Then it follows that $w(0,x) = x$ and:

$$\begin{aligned} c(t,w(t,x)) &= -\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{t}G(xt)\right] \\ &= \frac{1}{t^2}\left[G(xt) - xtG_z(xt)\right] = \frac{w}{t} - \frac{1}{t^2}G^{-1}(tw)G_z(G^{-1}(tw)) \end{aligned}$$

which is positive by concavity. In particular, we get:

$$\frac{\frac{\partial^2}{\partial x^2} c(t,w(t,x))}{\frac{\partial}{\partial x} c(t,w(t,x))} = \frac{1}{x} + t\frac{G_{zzz}(xt)}{G_{zz}(xt)}$$

One simple example of such a function is $G(z) = \ln(1+z)$, in which case we get $c(t,w) = \frac{1}{t^2}(tw + e^{-wt} - 1)$ and:

$$\frac{\frac{\partial^2}{\partial x^2} c(t,w(t,x))}{\frac{\partial}{\partial x} c(t,w(t,x))} = \frac{1}{x} - \frac{2t}{1+xt}.$$

This expression is decreasing in t, so we can conclude that

$$\inf_{t \geq 0} \frac{\frac{\partial^2}{\partial x^2} c(t,w(t,x))}{\frac{\partial}{\partial x} c(t,w(t,x))} = \lim_{t\to\infty}\left[\frac{1}{x} - \frac{2t}{1+xt}\right] = -\frac{1}{x}$$

and we see that the corresponding investor is DARA if we take $D(x) = \gamma x^{-\gamma-1}$ for $\gamma > 0$. On the other hand,

$$\sup_{t \geq 0} \frac{\frac{\partial^2}{\partial x^2} c(t,w(t,x))}{\frac{\partial}{\partial x} c(t,w(t,x))} = \lim_{t\to 0}\left[\frac{1}{x} - \frac{2t}{1+xt}\right] = \frac{1}{x}.$$

so the choice $D(x) = x\mathrm{e}^{-x^2}$, gives an IARA investor.

Another example arises by taking $G(z) = 1 - \mathrm{e}^{-z}$. In this case, we have $c(t, w) = \frac{w}{t} + (1 - wt)t^{-2} \ln(1 - wt)$, and

$$\frac{\frac{\partial^2}{\partial x^2} c(t, w(t, x))}{\frac{\partial}{\partial x} c(t, w(t, x))} = \frac{1}{x} - t.$$

As before, taking e.g. $D(x) = x\mathrm{e}^{-x^2}$, gives an IARA investor. However there is no choice of $D(x)$ for which the investor will be DARA. Note that in this example, since G is bounded by 1, the investor's wealth will be below $\frac{1}{t}$ at time t, no matter how large their initial wealth.

2.5 Admissible utility functions

It is natural to ask if we can recover all utility functions u (up to addition of a function $A(t) + Bc$) from the above setup? The answer is no.

Consider for example functions of the form: $u(t, c) = U(c)$ for some increasing concave function $U(\cdot)$. Such functions correspond to optimal paths which are constant, but of course, these have infinite total consumption. Agents with finite initial wealth will try to spread the total consumption as evenly as possible across the whole time horizon, but there will be no sensible 'optimal' consumption. There may also be cases when optimal consumptions exist but are not covered by our framework. For example, one may construct utility functions for which optimal consumption paths are zero for a while and then leave zero to follow a positive path.

Our aim in Section 2 was to consider the extent to which knowledge of optimal consumption paths can be used to determine the utility in the deterministic case. To obtain a complete and coherent description we worked under plausible, but not necessary, assumptions e.g. that consumption levels are strictly increasing in current wealth. The key discovery is that in the deterministic case there is no way to compare utilities across different optimal consumption paths. We shall see that this situation is rather special, and that the picture is different in the stochastic case.

3 Stochastic Setting

We now turn to a more sophisticated version of the above problem, by considering what happens when we add the possibility of investment in a stochastic asset. Specifically, we suppose there is a risky asset P_t, where P_t is a Black-Scholes asset so that it has dynamics:

$$\frac{dP_t}{P_t} = \sigma(dB_t + \theta\, \mathrm{d}t) + r\, \mathrm{d}t.$$

Here σ is the asset volatility, $\theta > 0$ is the Sharpe ratio, and r is the interest rate, which are all assumed to be constant, and B_t a standard Brownian motion. The investor now has to choose a rate of consumption C_t and also an amount, Π_t, which is to be invested in the risky asset. Then her wealth at time t, W_t, is the solution to:

$$dW_t = rW_t \mathrm{d}t - C_t \mathrm{d}t + \Pi_t \sigma(\mathrm{d}B_t + \theta \mathrm{d}t), \tag{13}$$

subject to $W_0 = x$. Where we wish to highlight the dependence on initial wealth x or strategy (C, Π) we may write this as $W_t^{x,C,\Pi}$.

The investor will specify an optimal pair of investment and consumption strategies, Π_t and C_t which attain the supremum in:

$$\sup_{C_t, \Pi_t : W_t^{C,\Pi} \geq 0} \mathbb{E}\left[\int_0^\infty u(t, C_t)\,\mathrm{d}t\right] \tag{14}$$

subject to a budget constraint $W_t \geq 0$ for all $t \geq 0$. Here u is an unknown function which we aim to find.

As usual, the above generalises to an optimal control problem, which has value function:

$$v(t, w) = \sup_{C_t, \Pi_t : W_t^{C,\Pi} \geq 0} \left[\int_t^\infty u(s, C_s)\,\mathrm{d}s \big| W_t = w\right].$$

Standard theory tells us that for a general pair (C_t, Π_t) the process $M_t = \int_0^t u(s, C_s)\,ds + v(t, W_t)$ must be a supermartingale, and under the optimal strategy will be a martingale. Applying Itô to M_t, we see that the relevant drift $(\mathrm{d}t)$ terms are:

$$u(t, C_t) + v_t(t, W_t) + v_w(t, W_t)\left[rW_t - C_t + \Pi_t \sigma\theta\right] + \frac{1}{2} v_{ww}(t, W_t)\sigma^2 \Pi_t^2. \tag{15}$$

We assume that the optimal strategy takes the form $(C_t = c(t, W_t), \Pi_t = \pi(t, W_t))$. Then, by analysing this equation, and considering possible solution terms $v(t, w)$, we prove in Theorem 3.4 that there is a function $u(t, c)$ for which the pair $(c(t, w), \pi(t, w))$ is optimal if and only if these functions satisfy:

$$\frac{c(t, w)}{\pi(t, w)} - \frac{rw}{\pi(t, w)} + \frac{\sigma^2}{2}\pi_w(t, w) + \int_\cdot^w \frac{\pi_t(t, \tilde{w})}{\pi(t, \tilde{w})^2}\,\mathrm{d}\tilde{w} = \beta(t) \tag{16}$$

for some function $\beta(t)$ — in particular, the left hand side is independent of w. This consistency relationship between π and c for them to be the solution of an optimal consumption/investment problem of the type (14) was first derived by Black (1968) (published later in a modified form as Black (1988)), and then re-derived in a more rigorous fashion by He and Huang (1994), see Remark 3.8 below. Before stating and proving our main result, Theorem 3.4, we give heuristic derivations of (16) using both primal and dual approach to (14).

3.1 Heuristics: the primal approach

To motivate the condition in (16), it turns out to be instructive to look at a more general problem: we introduce a function $\Psi(t, w)$ and then consider

$$v(t, w) = \sup_{\Pi_s, C_s} \mathbb{E}\left[\int_t^\infty \left(u(s, C_s) + \Psi(s, W_s)\right)\,\mathrm{d}s \big| W_t = w\right]. \tag{17}$$

As before our starting point is an assumption that the optimal strategy takes the form $(C_t = c(t, W_t), \Pi_t = \pi(t, W_t))$. Then, by deriving an expression for Ψ in terms of the functions $\pi(t, w), c(t, w)$, we will be able to recover the condition (16) in the special case where Ψ is a function of time alone.

In the same way that we derived (15), we can get the martingale condition corresponding to (17) which is

$$\sup_{\Pi,C}\Big[u(t,C)+v_t(t,W_t)\,+\,v_w(t,W_t)\left[rW_t-C+\Pi\sigma\theta\right] \\ +\frac{1}{2}v_{ww}(t,W_t)\sigma^2\Pi^2+\Psi(t,W_t)\Big]=0.$$

Rearranging this expression, we get:

$$\sup_{C}\left[u(t,C)-v_w(t,W_t)C\right]+\sup_{\Pi}\left[\Pi\sigma\theta v_w(t,W_t)+\frac{1}{2}\sigma^2\Pi^2 v_{ww}(t,W_t)\right] \\ +v_t(t,W_t)+v_w(t,W_t)rW_t+\Psi(t,W_t)=0. \tag{18}$$

In particular, the optimal choice of Π, namely $\pi(t,w)$, should satisfy:

$$\pi(t,w)=-\frac{\theta v_w(t,w)}{\sigma v_{ww}(t,w)}, \tag{19}$$

which in turn suggests we can write:

$$v_w(t,w)=\exp\left\{A(t)-\int_{\cdot}^{w}\frac{\theta}{\sigma\pi(t,\tilde{w})}\,\mathrm{d}\tilde{w}\right\}, \tag{20}$$

where $A(t)$ is some function of t to be specified.

Suppose that u is concave and differentiable in c and introduce the convex dual, $\tilde{u}(t,\xi)=\sup_c\left(u(t,c)-\xi c\right)$. Note that we have $\tilde{u}_\xi(t,\xi)=-c^*$, where c^* is the choice of c which attains the supremum.

Substituting the optimal actions $c(t,w)$ and $\pi(t,w)$ into (18), we get:

$$0=\tilde{u}(t,v_w(t,w))+\frac{1}{2}\pi(t,w)\sigma\theta v_w(t,w)+v_t(t,w)+v_w(t,w)rw+\Psi(t,w), \tag{21}$$

and differentiating (21) with respect to w, we obtain:

$$0=-v_{ww}(t,w)c(t,w)+\frac{1}{2}\pi(t,w)\sigma\theta v_{ww}(t,w)+\frac{1}{2}\sigma\theta v_w(t,w)\pi_w(t,w)+v_{tw}(t,w) \\ +v_{ww}(t,w)rw+v_w(t,w)r+\Psi_w(t,w). \tag{22}$$

If we now differentiate (20) in the time variable, we see that we must have:

$$v_{tw}(t,w)=\left[\frac{\mathrm{d}A}{\mathrm{d}t}(t)+\int_{\cdot}^{w}\frac{\theta}{\sigma\pi(t,\tilde{w})^2}\pi_t(t,\tilde{w})\,\mathrm{d}\tilde{w}\right]v_w(t,w), \tag{23}$$

so that (22) becomes:

$$0=\left[\frac{1}{2}\pi(t,w)\sigma\theta+rw-c(t,w)\right]v_{ww}(t,w)+\Psi_w(t,w) \\ +\left[\frac{1}{2}\sigma\theta\pi_w(t,w)+r+\frac{\mathrm{d}A}{\mathrm{d}t}(t)+\int_{\cdot}^{w}\frac{\theta}{\sigma\pi(t,\tilde{w})^2}\pi_t(t,\tilde{w})\,\mathrm{d}\tilde{w}\right]v_w(t,w).$$

Finally, dividing through by $v_w(t,w)$ and using (19), we have:

$$-\frac{\Psi_w(t,w)}{v_w(t,w)}=\left[\frac{1}{2}\pi(t,w)\sigma\theta+rw-c(t,w)\right]\frac{v_{ww}(t,w)}{v_w(t,w)}+\frac{1}{2}\sigma\theta\pi_w(t,w) \\ +r+\frac{\mathrm{d}A}{\mathrm{d}t}(t)+\int_{\cdot}^{w}\frac{\theta}{\sigma\pi(t,\tilde{w})^2}\pi_t(t,\tilde{w})\,\mathrm{d}\tilde{w},$$

and therefore:

$$\begin{aligned}\Psi_w(t,w)\exp\left\{-A(t)+\int_{\cdot}^{w}\frac{\theta}{\sigma\pi(t,\tilde{w})}\,\mathrm{d}\tilde{w}\right\}\\ =\frac{1}{2}\theta^2+\frac{rw\theta}{\sigma\pi(t,w)}-\frac{\theta c(t,w)}{\pi(t,w)\sigma}-\frac{1}{2}\sigma\theta\pi_w(t,w)-r-\frac{\mathrm{d}A}{\mathrm{d}t}(t)\\ -\int_{\cdot}^{w}\frac{\theta}{\sigma\pi(t,\tilde{w})^2}\pi_t(t,\tilde{w})\,\mathrm{d}\tilde{w}.\end{aligned}\tag{24}$$

Since we have not yet fixed the constant of integration $A(t)$, we are free to choose this. Because our main interest is in the case where $\Psi(t,w)$ is independent of w, it follows that we are interested in cases where we can make the expression on the right-hand side of (24) disappear, which will occur whenever the expression

$$\frac{\theta^2}{2}+\frac{r\theta w}{\sigma\pi(t,w)}-\frac{\theta c(t,w)}{\sigma\pi(t,w)}-\frac{\sigma\theta}{2}\pi_w(t,w)-r-\int_{1}^{w}\frac{\theta}{\sigma\pi(t,\tilde{w})^2}\pi_t(t,\tilde{w})\,\mathrm{d}\tilde{w}$$

is independent of w. Differentiating once more in w, and rearranging, we see that this is equivalent to π, c satisfying:

$$\pi_t(t,w)=-\frac{\sigma^2}{2}\pi(t,w)^2\pi_{ww}(t,w)+(c(t,w)-rw)\pi_w(t,w)-\pi(t,w)c_w(t,w)+r\pi(t,w),\tag{25}$$

This is Black's equation (Black (1968)[Equation (9)], see also He and Huang (1994)[Equation (26)]). Equivalently, defining $R(t,w):=\frac{c(t,w)}{\pi(t,w)}$, we have that R solves:

$$R_w(t,w)=\frac{r}{\pi(t,w)}-\frac{1}{\pi(t,w)^2}\left(\pi_t(t,w)+rw\pi_w(t,w)\right)-\frac{\sigma^2}{2}\pi_{ww}(t,w).\tag{26}$$

3.2 Heuristics: the dual approach

We now give a second derivation of Black's equation using a dual approach to the consumption/investment problem. We will use this approach to give our main theorem below.

For the problem (14) we can rewrite the budget constraint as

$$\mathbb{E}\left[\int_0^\infty C_t Z_t \mathrm{d}t\right]=x,$$

where $(Z_t)_{t\geq 0}$ is the state-price density process and is given by

$$Z_t=\exp\left(-rt-\theta B_t-\frac{\theta^2}{2}t\right).$$

With this formulation the problem becomes to find

$$\sup_{C_t}\mathbb{E}\left[\int_0^\infty u(t,C_t)\mathrm{d}t-\lambda\left(\int_0^\infty C_t Z_t\mathrm{d}t-x\right)\right]$$

for an appropriate Lagrange multiplier $\lambda=\lambda(x)$. This expression is bounded by $\lambda x+\mathbb{E}[\int_0^\infty \tilde{u}(t,\lambda Z_t)\mathrm{d}t]$ and for optimality we must have that $u_c(t,C_t)=\lambda Z_t$

so that writing $I(t,\cdot)$ for the inverse in space of $u_c(t,\cdot)$ we must have that the optimal consumption takes the form $C_t = I(t, \lambda Z_t)$.

Now assume that the optimal strategy is a given function $c = c(t, w)$ of time and wealth so that $C_t = c(t, W_t)$. It follows that we must have $W_t = f(t, \lambda Z_t)$ for some $f = f(t, z)$ which depends on the (now unknown u) through $f = c^{-1} \circ I$.

Then, by Itô's formula,

$$\begin{aligned} \mathrm{d}W_t &= \lambda f_z(t, \lambda Z_t)\mathrm{d}Z_t + f_t(t, \lambda Z_t)\mathrm{d}t + (1/2) f_{zz}(t, \lambda Z_t)\lambda^2 \mathrm{d}\langle Z\rangle_t \\ &= -\theta\lambda Z_t f_z \mathrm{d}B_t + \{f_t + (1/2)\theta^2\lambda^2 Z^2 f_{zz} - r\lambda Z_t f_z\}\mathrm{d}t. \end{aligned}$$

Comparing this with the wealth dynamics (13) we must have

$$\sigma\pi(t, f(t,z)) = -\theta z f_z(t,z), \tag{27}$$

$$r f(t,z) - c(t, f(t,z)) + \theta\sigma\pi(t, f(t,z)) = f_t(t,z) + (1/2)\theta^2 z^2 f_{zz}(t,z) - r z f_z(t,z). \tag{28}$$

Then $\sigma\pi_w f_z = -\theta f_z - \theta z f_{zz}$ and $\sigma\pi_t + \sigma\pi_w f_t = -\theta z f_{tz}$ so that

$$\theta^2 z^2 f_{zz} = (\theta + \sigma\pi_w)\sigma\pi, \tag{29}$$

$$f_{tz}/f_z = \pi_t/\pi + \pi_w f_t/\pi. \tag{30}$$

Putting (29) into (28) gives

$$rf - c + \sigma\theta\pi = \frac{\sigma}{2}\pi(\theta + \sigma\pi_w) + f_t + \frac{r\sigma}{\theta}\pi. \tag{31}$$

Differentiating wrt z, dividing by $f_z = -\sigma\pi/(\theta z)$, using (30) and (31) to eliminate f_{tz} and f_t and multiplying by π we finally get

$$\begin{aligned} (r - c_w(t,w))\pi(t,w) - \frac{\sigma^2}{2}(\pi(t,w))^2\pi_{ww}(t,w) - \pi_t(t,w) \\ - r\pi_w(t,w)w + \pi_w(t,w)c(t,w) = 0, \end{aligned} \tag{32}$$

which is equivalent to Black's pde (25). Dividing by π^2 and integrating we arrive at Black's pde in integrated form (16)

$$\int_1^w \frac{\pi_t(t,\xi)}{(\pi(t,\xi))^2}\mathrm{d}\xi + \frac{\sigma^2}{2}\pi_w(t,w) + \frac{c(t,w)}{\pi(t,w)} - r\frac{w}{\pi(t,w)} = \beta(t) \tag{33}$$

for some function $\beta(t)$, independent of w.

Remark 3.1. Our motivation so far has been the following: we have supposed that both the consumption and investment functions have been stated for all times and wealths, and we have derived the consistency condition (33) as a necessary condition that these functions must satisfy. However, the above calculations also suggest an alternative way of viewing the setup. Suppose instead our agent specifies her consumption (at all times and wealths), and her initial investment strategies at all wealths (i.e. $\{\pi(0,w)\}_{w\geq 0}$). Then, under the assumption that the agent is a utility maximiser, we can solve the parabolic PDE (32) to deduce $\pi(t,w)$ at times $t \geq 0$. Note that the utility function itself is bypassed in the sense that we do not need to specify it to deduce $\pi(t,w)$. This was one of the motivating observations for Black (1968).

3.3 Main results

Given a pair of processes $(C,\Pi) \equiv (C_s,\Pi_s)_{s\geq 0}$ define the associated wealth process $(W_s^{x,C,\Pi})_{s\geq 0}$ for initial wealth x by

$$W_s^{x,C,\Pi} = x + \int_0^s \Pi_u \sigma(\mathrm{d}B_u + \theta \mathrm{d}u) + \int_0^s (rW_u^{x,C,\Pi} - C_u)\mathrm{d}u. \tag{34}$$

We say that C,Π is admissible if $W_s^{x,C,\Pi} \geq 0$ for all s and we write $\mathcal{A} = \mathcal{A}(x)$ for the space of admissible strategies. Note that if C,Π is admissible then, writing W for $W^{x,C,\Pi}$,

$$\mathrm{d}(Z_sW_s) = Z_s(\sigma\Pi_s - \theta W_s)\mathrm{d}B_s - Z_sC_s\mathrm{d}s. \tag{35}$$

Hence $(Z_sW_s)_{s\geq 0}$ is a non-negative supermartingale so that if $W_s = 0$ then, for $t \geq s$, $\mathbb{E}[W_tZ_t|\mathcal{F}_s] \leq W_sZ_s = 0$ and hence $W_t = 0$ almost surely. Thus zero is absorbing for any admissible strategy.

We suppose we are given functions $c = c(t,w)$ and $\pi = \pi(t,w)$ and we aim to find, where possible, u such that $C_t = c(t, W_t^{x,C,\Pi}), \Pi_t = \pi(t, W_t^{x,C,\Pi})$ is optimal for (14). We start by defining the class of utility functions we consider and imposing some minimal regularity assumptions on our inputs.

Definition 3.2. We say that a function $u : [0,\infty)^2 \to \mathbb{R}$ is a *regular utility function* if $u(t,\cdot)$ is continuously differentiable, satisfies the Inada condition: $u_c(t,0) = \infty$ and $u_c(t,\infty) = 0$, and is strictly concave and increasing on $(0,\overline{c}(t))$, where $\overline{c}(t) = \inf\{c : u_c(t,c) = 0\}$.

If a utility function u is given we denote the set of admissible strategies for which the reward in (14) is well defined by $\mathcal{A}^u(x) = \{(C,\Pi) \in \mathcal{A}(x) : \mathbb{E}\int_0^\infty u(t,C_t)^+\mathrm{d}t < \infty \text{ or } \mathbb{E}\int_0^\infty u(t,C_t)^-\mathrm{d}t < \infty\}$.

Assumption 3.3 (Regularity Conditions)**.** Assume that $c(t,0) = 0 = \pi(t,0)$ and $c(t,w), \pi(t,w)$ are strictly positive for $w > 0$. Assume also that $c = c(t,w)$ is jointly continuous and strictly increasing in w, and that $\pi = \pi(t,w)$ is continuously differentiable in both arguments. Finally assume that the SDE

$$\mathrm{d}W_t^x = \pi(t,W_t^x)\sigma(\mathrm{d}B_t + \theta\mathrm{d}t) + (rW_t^x - c(t,W_t^x))\mathrm{d}t, \qquad W_0^x = x. \tag{36}$$

has a strong pathwise unique solution, which we denote $W^x = W^{x,c,\pi}$ depending on the context.

Under the above regularity conditions define $Y(t,c)$ to be the inverse to $c(t,w)$ so that $Y(t,c(t,w)) = w$. We put $Y(t,c) = \infty$ for all $c \geq c(t,\infty)$. Suppose further that c,π satisfy (33) and let $A(t) = -\frac{\theta}{\sigma}\int_0^t \beta(s)\mathrm{d}s + (\frac{\theta^2}{2} - r)t$, and define $F(t,w)$ by

$$F(t,w) = e^{A(t)} \exp\left\{-\frac{\theta}{\sigma}\int_1^w \frac{\mathrm{d}\xi}{\pi(t,\xi)}\right\}. \tag{37}$$

For each t, $F(t,w)$ is $\mathcal{C}^{1,2}$ and decreasing in w, so we can define its inverse $f = F^{-1}$ such that $f(t,F(t,w)) = w$. Finally set $H(t,c) = \int_1^c F(t,Y(t,b))\mathrm{d}b$. Note that we have

$$\begin{aligned} f_z(t,z)F_w(t,f(t,z)) &= 1; && (38)\\ f_t(t,z) + f_z(t,z)F_t(t,f(t,z)) &= 0; && (39)\\ f_z(t,z)F_{ww}(t,f(t,z)) + f_{zz}(t,z)F_w(t,f(t,z))^2 &= 0, && (40) \end{aligned}$$

and that f is $\mathcal{C}^{1,2}$.

Theorem 3.4. *Suppose an agent's actions* c, π *satisfy Assumption 3.3. For any* $x > 0$*, the following two are equivalent:*

(i) $c(t, W_t^x)$ *and* $\pi(t, W_t^x)$ *achieve a finite maximum in the problem*

$$\max_{C,\Pi \in \mathcal{A}^u(x)} \mathbb{E}[\int_0^\infty u(t, C_t)\mathrm{d}t], \tag{41}$$

for a regular utility function u*, as in Definition 3.2, for which*

$$\exists\lambda > 0 \text{ such that } x = \mathbb{E}\int_0^\infty Z_t I(t, \lambda Z_t)\mathrm{d}t, \tag{42}$$

where $u_c(t, I(t,z)) = z$*, and such that* $Y(t, I(t,z))$ *is* $\mathcal{C}^{1,2}$ *on* $(0,\infty)^2$*.*

(ii) $c(t,w), \pi(t,w)$ *satisfy* (33)*, for all* $t > 0$ $F(t,0) = \infty$ *and* $F(t,\infty) = 0$*,*

$$\mathbb{E}\left[\int_0^\infty Z_t c(t, W_t^x)\mathrm{d}t\right] = x, \tag{43}$$

and for some $0 < x_0 \leq x$, $\mathbb{E}[|H(t, c(t, W_t^{x_0}))|] < \infty$ *for almost all* $t \geq 0$ *and* $\int_0^\infty \mathbb{E}[H(t, c(t, W_t^{x_0})) - h(t)]^+ \mathrm{d}t < \infty$*, where* $h(t) = \mathbb{E}[H(t, c(t, W_t^{x_0}))]$*.*

Moreover, we then have $u_c(t,c) = H_c(t,c)$*,* $\mathcal{A}^u(x) = \mathcal{A}(x)$ *and in (i) one may take* $u(t,c) = H(t,c) - h(t)$*.*

Remark 3.5. In the initial hypotheses of the theorem and in (*ii*) it is equivalent to use $f(t, F(0,x)Z_t)$ in place of W_t^x throughout. This condition may be easier to check.

Remark 3.6. It is interesting to observe the analogy with the deterministic setup considered in Section 2. There, given an agent's consumption, we recovered their utility function as $u_c(t,c) = \tilde{F}(t, y(t,c))$, where $y(t,c)$ was the inverse of consumption and $\tilde{F}(x) = \int_x^\infty D(s)\mathrm{d}s$ was an arbitrary absolutely continuous decreasing non-negative function. In Theorem 3.4 above, we recover the utility function in the same form $u_c(t,c) = F(t, Y(t,c))$ but now F is uniquely specified in terms of the agent's investment strategy coupled with the discounting term $A(t)$ read off from Black's equation (33).

Remark 3.7. There are close parallels between different conditions in (*i*) and (*ii*):

- The fundamental point of the theorem is the equivalence between (33) and optimality of c, π for the problem (41). If (33) fails, c, π may still be optimal but for a more general problem of the type (17).
- The conditions on F correspond to the Inada condition on u. The final condition, $Y(t, I(t,z))$ is $\mathcal{C}^{1,2}$, corresponds to $f(t,z) \in \mathcal{C}^{1,2}$ which is granted by Assumption 3.3. Note that we allow $u(t,\cdot)$ to be constant on some $[\bar{c}(t), \infty)$ and this corresponds to $c(t,\infty) < \infty$.

- Equations (42) and (43) are essentially the same and encode the budget constraint. We show below that if $\mathbb{E}[\int_0^\infty Z_t c(t, W_t^x)\mathrm{d}t] > x$ then (C, Π) is not admissible. Conversely, if $\mathbb{E}[\int_0^\infty Z_t c(t, W_t^x)\mathrm{d}t] = x - \delta$ for $\delta > 0$ then c, π is typically not optimal. Indeed, if $\tilde{c}(t, w) = c(t, w) + \delta e^{-s}/Z_s$ then (by Theorem 9.4 of Karatzas and Shreve (1998)) there exists a process $\tilde{\Pi}$ such that $\tilde{c}, \tilde{\Pi}$ is admissible, and achieves a strictly higher expected utility of consumption over time in (41), with a possible exception when the utility is constant on some interval $(\bar{c}(t), \infty)$.

In general the assumptions of Theorem 3.4 may be non-trivial to verify. However we provide a wide class of examples where they hold, see Lemma 3.12 below.

Remark 3.8. Theorem 3.4 above is similar to combined Theorems 1 & 3 of He and Huang (1994) and we should comment in more detail on our contribution. In fact, He and Huang (1994) considered a more general setup than we do. They allowed the stock price P_t to be a generic diffusion (local volatility) process and c and π to depend on the state (i.e. P_t) as well. However, they imposed more regularity assumptions on agents actions. Compared with our assumptions, He and Huang (1994) took $\pi \in \mathcal{C}^{1,2}$ and $\pi > 0$, instead of $\pi \in \mathcal{C}^{1,1}$, and $c \in \mathcal{C}^{1,1}$ instead of jointly continuous. They imposed linear growth restrictions on π and assumed $W_t^x > 0$ a.s. for all $t \geq 0$. The former corresponds to our assumptions $F(t, \infty) = 0$ and the latter is equivalent to $F(t, 0) = \infty$. A significant difference between the two set-ups is that He and Huang (1994) considered maximisation of utility from consumption and terminal wealth over a finite time horizon $[0, T]$. They comment that their analysis extends to infinite setup, as in (41), with the additional condition $\mathbb{E}[Z_t W_t] \to 0$ as $t \to \infty$ which, under all their assumptions, implies our budget constraint (43). However this does not seem to be so immediate due to the restrictions on c in He and Huang (1994). They assume c has linear growth and further that $\mathbb{E}[\int_0^T c(t, W_t)^2 \mathrm{d}t] < \infty$. Simply replacing T by ∞ in the integral would rule out any examples considered in their paper. Finally, the well-posedness and finiteness of the expected utility is not commented.

We believe that our approach has at least two advantages compared with He and Huang (1994).

Firstly, we make less restrictive assumptions. This allows us to obtain a general class of actions for which we can verify the assumptions, see Lemma 3.12 below, which includes interesting examples. In contrast He and Huang (1994) are unable to verify their assumptions for most of the examples in the paper.

Secondly, we present a unified *if and only if* statement with a straightforward proof which should be appealing to a contemporary reader. Naturally, He and Huang (1994) remains an impressive work which, due to its more general scope of asset dynamics, is able to address questions we can not not consider in our framework, see also Section 4.

Proof of Theorem 3.4. We first show that $(ii) \Rightarrow (i)$.
We take $u(t, c) = H(t, c) - h(t)$. Observe that by the assumption on F, f is well defined and $\mathcal{C}^{1,2}$ on $(0, \infty)^2$. Let $\lambda = \lambda(x) = F(0, x)$ and set $W_t = f(t, \lambda(x) Z_t)$. By construction $W_0 = f(0, \lambda(x)) = x = W_0^{x,\pi,c}$. To show that $W_t = W_t^{x,\pi,c}$, by Assumption 3.3, it suffices to show that the two processes satisfy the same SDE for $t > 0$.

Note that by construction we have $u_c(s, c(s, W_s)) = F(s, W_s) = \lambda(x)Z_s$. By Itô

$$\begin{aligned}
\mathrm{d}W_t =& \lambda(x)f_z(t, \lambda(x)Z_t)\mathrm{d}Z_t + f_t(t, \lambda(x)Z_t)\mathrm{d}t + \frac{\lambda(x)^2}{2} f_{zz}(t, \lambda Z_t) d\langle Z\rangle_t \\
=& -\theta\lambda(x)Z_t f_z(t, \lambda(x)Z_t)\mathrm{d}B_t \\
& + \left(f_t(t, \lambda(x)Z_t) - r\lambda(x)Z_t f_z(t, \lambda(x)Z_t) + \frac{\theta^2\lambda(x)^2}{2} Z_t^2 f_{zz}(t, \lambda(x)Z_t) \right) \mathrm{d}t \ .
\end{aligned}$$

Then $W_t = W_t^x$ provided that for $w = f(t, z)$,

$$-\theta z f_z(t, z) = \sigma\pi(t, w)$$

and

$$\theta\sigma\pi(t, w) + rw - c(t, w) = f_t(t, z) - rzf_z(t, z) + \frac{1}{2}\theta^2 z^2 f_{zz}(t, z).$$

For the first of these, using $z = F(t, w)$ and the definition of F and (38), we have

$$-\theta z f_z(t, z) = -\theta \frac{F(t, w)}{F_w(t, w)} = \sigma\pi(t, w).$$

For the second, using also (39) and (40),

$$\begin{aligned}
& f_t(t, z) - rzf_z(t, z) + \frac{1}{2}\theta^2 z^2 f_{zz}(t, z) \\
&= -f_z(t, z) \left(F_t(t, w) + rF(t, w) + \frac{\theta^2}{2} \frac{F(t, w)^2 F_{ww}(t, w)}{F_w(t, w)^2} \right) \\
&= -\frac{F(t, w)}{F_w(t, w)} \left(A'(t) + \frac{\theta}{\sigma} \int_1^w \frac{\pi_t(t, \xi)}{(\pi(t, \xi))^2} d\xi + r + \frac{1}{2} \left[\theta^2 + \sigma\theta\pi_w(t, w)\right] \right) \\
&= \frac{\sigma\pi(t, w)}{\theta} \left(\theta^2 + \frac{\theta rw}{\sigma\pi(t, w)} - \frac{\theta c(t, w)}{\sigma\pi(t, w)} \right) \\
&= rw + \theta\sigma\pi(t, w) - c(t, w).
\end{aligned}$$

We thus conclude that $W_t^x = f(t, \lambda(x)Z_t)$ a.s. with $\lambda(x) = F(0, x)$.
For the rest of the proof, with slight abuse of notational conventions, let us write $c_t^x := c(t, W_t^x)$. It follows that $c_t^x = c(t, f(t, F(0, x)Z_t))$ and in particular $c_s^x \le c_s^y$ for $0 < x < y$. Further, since $u_c(t, c(t, f(t, z))) = F(t, f(t, z)) = z$, we have $u_c(t, c_t^x) = \lambda(x)Z_t$, so that $c_t^x = I(t, \lambda(x)Z_t)$ and

$$\tilde{u}(t, \lambda(x)Z_t) = u(t, c_t^x) - \lambda(x)Z_t c_t^x, \tag{44}$$

where $\tilde{u}$ is the convex dual of u. It follows that (42) is simply (43).

By the assumption $\mathbb{E}[|u(t, c(t, W_t^{x_0}))|] \le \mathbb{E}[|H(t, c(t, W_t^{x_0}))|] + |h(t)| < \infty$ and

$$\mathbb{E}[u(t, c(t, W_t^{x_0}))] = \mathbb{E}[H(t, c(t, W_t^{x_0}))] - h(t) = 0.$$

Using the hypothesis $\mathbb{E}[\int_0^\infty [u(t, c(t, W_t^{x_0}))]^+ \mathrm{d}t] < \infty$ we obtain

$$\mathbb{E}[\int_0^\infty u(t, c(t, W_t^{x_0}))\mathrm{d}t] = 0.$$

For $x > x_0$ we write

$$u(t, c_t^x) \leq \tilde{u}(t, \lambda(x_0)Z_t) + \lambda(x_0)Z_t c_t^x = u(t, c_t^{x_0}) + \lambda(x_0)(c_t^x - c_t^{x_0})Z_t$$

and hence

$$\mathbb{E}\int_0^\infty u(t, c_t^x)^+ \mathrm{d}t \leq \mathbb{E}\int_0^\infty u(t, c_t^{x_0})^+ \mathrm{d}t + \lambda(x_0)\mathbb{E}\int_0^\infty Z_t c_t^x \mathrm{d}t < \infty.$$

Hence $\mathbb{E}[\int_0^\infty u(t, c_t^x)\mathrm{d}t]$ is well defined and non-negative.

Consider now arbitrary $C, \Pi \in \mathcal{A}(x)$. From (35) we have

$$0 \leq W_t Z_t = x + \int_0^t Z_s(\sigma\Pi_s - \theta W_s)\mathrm{d}B_s - \int_0^t Z_s C_s \mathrm{d}s.$$

It follows that

$$0 \leq \int_0^t Z_s C_s \mathrm{d}s \leq x + \int_0^t Z_s(\sigma\Pi_s - \theta W_s)\mathrm{d}B_s.$$

In particular $\int_0^t Z_s(\Pi_s - \theta W_s)\mathrm{d}B_s \geq -x$ is bounded below and hence is a supermartingale. We also conclude that for each t, $\mathbb{E}[\int_0^t Z_s C_s \mathrm{d}s] \leq x$ and hence

$$\mathbb{E}\left[\int_0^\infty Z_s C_s \mathrm{d}s\right] \leq x. \tag{45}$$

It follows that, with $\lambda = \lambda(x)$,

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty u(s, C_s)^+ \mathrm{d}s\right] &\leq x\lambda + \mathbb{E}\left[\int_0^\infty (u(s, C_s) - \lambda Z_s C_s)^+ \mathrm{d}s\right] \\
&\leq x\lambda + \mathbb{E}\left[\int_0^\infty \tilde{u}(s, \lambda Z_s)^+ \mathrm{d}s\right] \\
&= x\lambda + \mathbb{E}\left[\int_0^\infty (u(s, c_s^x) - \lambda Z_s c_s^x)^+ \mathrm{d}s\right] \\
&\leq x\lambda + \mathbb{E}\left[\int_0^\infty u(s, c_s^x)^+ \mathrm{d}s\right] < \infty,
\end{aligned}$$

where we used (44). In consequence, $\mathcal{A}(x) = \mathcal{A}^u(x)$. Once we know the expectations exist we proceed with a standard argument:

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty u(s, C_s)\mathrm{d}s\right] &\leq \lambda x + \mathbb{E}\left[\int_0^\infty u(s, C_s) - \lambda Z_s C_s \mathrm{d}s\right] \\
&\leq \lambda x + \mathbb{E}\left[\int_0^\infty \tilde{u}(s, \lambda Z_s)\mathrm{d}s\right] .
\end{aligned} \tag{46}$$

Further, from (43) and (44) it is immediate that there is equality throughout (46) for $C_s = c_s^x$ and $\Pi_s = \pi(s, W_s^x)$ which shows that these are optimal. Finally, given that $c(t, 0) = 0$ and $c(t, \cdot)$ is strictly increasing, we see that $u_c(t, 0) = F(t, Y(t, 0)) = F(t, 0) = \infty$ and $u_c(t, \infty) = F(t, Y(t, \infty)) = F(t, \infty) = 0$. Further, $u(t, \cdot)$ is $\mathcal{C}^1$ and, with $u_c(t, I(t, z)) = z$, we have $Y(t, I(t, z)) = f(t, z)$ which is $\mathcal{C}^{1,2}$ on $(0, \infty)^2$, as required.

We come now to the other implication: $(i) \Rightarrow (ii)$.
Take λ as in (42) and let $C_s := I(s, \lambda Z_s)$. As observed earlier (cf. Theorem 9.4 in Karatzas and Shreve (1998)), there exists (Π_s) such that $(C_s, \Pi_s) \in \mathcal{A}(x)$. We have

$$u(s, c_s^x)^- \geq (\lambda Z_s(c_s^x - C_s) + u(s, C_s))^- \geq u(s, C_s)^- - \lambda Z_s(c_s^x - C_s)^+,$$

and hence

$$\mathbb{E}\left[\int_0^\infty u(s, C_s)^- \mathrm{d}s\right] \leq \mathbb{E}\left[\int_0^\infty u(s, c_s^x)^- \mathrm{d}s\right] + \lambda \mathbb{E}\left[\int_0^\infty Z_s(c_s^x + C_s)\mathrm{d}s\right] < \infty,$$

where we used (45) and the fact that c_s^x induces a finite maximum in (41). This shows that $(C_s, \Pi_s) \in \mathcal{A}^u(x)$. Proceeding as in (46), we obtain

$$\mathbb{E}\left[\int_0^\infty u(s, c_s^x)\mathrm{d}s\right] \leq \lambda x + \mathbb{E}\left[\int_0^\infty \tilde{u}(s, \lambda Z_s)\mathrm{d}s\right] = \mathbb{E}\left[\int_0^\infty u(s, C_s)\mathrm{d}s\right].$$

It follows we have to have equality in the above equation which is true if and only if $c_s^x = C_s$ almost surely. In consequence, (43) is simply (42). Further, the wealth process generated by (c, π) is given as $W_t = W_t^{x,c,\pi} = g(t, \lambda Z_t)$ for $g = Y \circ I$. Note that $I(t, 0) = \overline{c}(t) = c(t, \infty) \leq \infty$.
By assumption, g is $\mathcal{C}^{1,2}$ on $(0, \infty)^2$ and $g(t, z) > 0$ for $z > 0$. Using Itô's formula and equating dW_t with the wealth dynamics in (36) we obtain (27)–(28), which hold for all $t, z > 0$. Note that $\pi(t, g(t, z)) > 0$ and hence also $g_z(t, z) > 0$. We then proceed as in (29)–(32), with g instead of f, to conclude that (33) holds. This means F in (37) is well defined and we may consider $\tilde{W}_t = f(t, F(0, x)Z_t)$, with $f = F^{-1}$ as above. Proceeding as in the first part of the proof it follows that $\tilde{W}_t = W_t^x = g(t, \lambda Z_t)$ for $0 \leq t < \tau$, where

$$\tau = \inf\{s : Z_s \notin (F(s, \infty)/F(0, x), F(s, 0)/F(0, x))\} = \inf\{s : \tilde{W}_s \in \{0, \infty\}\}.$$

However since we know that W_t^x does not hit zero and does not explode (because $W_t^x = g(t, \lambda Z_t)$), it follows that $\tau = \infty$; i.e. $F(t, 0) = \infty$ and $F(t, \infty) = 0$. Finally, from $c_t^x = I(t, \lambda Z_t)$ and $W_t^x = \tilde{W}_t$, it also follows that $u_c(t, c) = H_c(t, c)$ so that $u(t, c) = H(t, c) - \zeta(t)$, for some function ζ. As c_t^x achieves a finite maximum in (41) it follows that $\mathbb{E}[|H(t, c(t, W_t^x))|] < \infty$ for a.e. $t \geq 0$ and Fubini's theorem yields $\int_0^\infty (h(t) - \zeta(t))\mathrm{d}t$ is well defined and finite when we take $h(t) = \mathbb{E}[H(t, c(t, W_t^x))]$. In consequence,

$$\begin{aligned}\mathbb{E}\int_0^\infty [H(t, c(t, W_t^x)) - h(t)]^+ \mathrm{d}t < \mathbb{E}\int_0^\infty &[H(t, c(t, W_t^x)) - \zeta(t)]^+ \mathrm{d}t \\ &+ \int_0^\infty (\zeta(t) - h(t))^+ \mathrm{d}t < \infty.\end{aligned} \tag{47}$$

Hence *(ii)* holds when we take $x = x_0$. It follows from the first part of the proof that we may take $u(t, c) = H(t, c) - h(t)$. □

Example 3.9. Suppose $c(t, w) = \kappa w$ and $\pi(t, w) = \phi w$ for $\kappa, \phi > 0$ with $\phi \neq \theta/\sigma$. Then Black's equation (33) is satisfied, $Y(t, b) = b/\kappa$, $\beta(t) \equiv \beta = (\kappa - r)/\phi + \sigma^2\phi/2$ and $A(t) = \xi t$ where $\xi = -\theta\beta/\sigma + \theta^2/2 - r$.

Let $\gamma = \theta/\phi\sigma$. We have $F(t,w) = e^{\xi t}w^{-\gamma}$ and in particular $F(t,0) = \infty$, $F(t,\infty) = 0$. It follows that $\lambda(x) = x^{-\gamma}$ and $f(t,z) = z^{-1/\gamma}e^{(\xi/\gamma)t}$, which is $\mathcal{C}^{1,2}$ differentiable.

Further, $W_t^x = f(t,\lambda(x)Z_t) = xe^{\phi\sigma B_t+(\sigma\phi\theta+r-\kappa-\sigma^2\phi^2/2)t}$ and a direct computation yields

$$\mathrm{e}^{\xi t}\mathbb{E}[(W_t^x)^{1-\gamma}] = x^{1-\gamma}\mathrm{e}^{\xi t/\gamma}\mathbb{E}[Z_t^{1-1/\gamma}] = x^{1-\gamma}\mathrm{e}^{-\kappa t}. \tag{48}$$

It follows that

$$\mathbb{E}\left[\int_0^\infty Z_t c(t,W_t^x)\mathrm{d}t\right] = \kappa x\mathbb{E}\left[\int_0^\infty (Z_t)^{1-1/\gamma}\mathrm{e}^{\xi t/\gamma}\mathrm{d}t\right] = \kappa x\int_0^\infty \mathrm{e}^{-\kappa t}\mathrm{d}t = x.$$

Also $H(t,c) = \frac{1}{1-\gamma}\mathrm{e}^{\xi t}\kappa^\gamma[c^{1-\gamma}-1]$ so that, taking $x_0 = 1$,

$$h(t) = \mathbb{E}[H(t,c(t,W_t^{x_0}))] = \frac{1}{1-\gamma}\left(\kappa\mathrm{e}^{-\kappa t} - \mathrm{e}^{\xi t}\kappa^\gamma\right),$$

and

$$\begin{aligned} u(t,c) &= \frac{1}{1-\gamma}\left(\kappa^\gamma\mathrm{e}^{\xi t}c^{1-\gamma} - \kappa\mathrm{e}^{-\kappa t}\right) \\ &= \frac{\kappa^\gamma\mathrm{e}^{\xi t}}{1-\gamma}\left(c^{1-\gamma} - (\kappa\mathrm{e}^{-\zeta t})^{1-\gamma}\right), \end{aligned} \tag{49}$$

where $\zeta = (\xi+\kappa)/(1-\gamma) = \kappa - r - \frac{\theta^2}{2\gamma}$. Then, using (48), $\mathbb{E}[u(t,c(t,W_t^{x_0}))^+] < D\mathrm{e}^{-\kappa t}$ for some constant D, and hence $\int_0^\infty \mathbb{E}[H(t,C_t)-h(t)]^+\mathrm{d}t < \infty$ for the optimal policy.

In the above we could take any $x_0 > 0$. So, in conclusion, for any initial capital $x > 0$, π, c solve the optimal consumption problem for admissible strategies for u as given in (49). We note that the choice of consumption and investment which are linear in wealth and time-homogeneous necessarily implies an exponential discounting of utility from a given wealth.

3.4 Risk aversion

We now return to similar questions as those which arose in Theorem 2.6, namely, we investigate what we can say about an investor's risk profile from her actions. Recall the absolute risk aversion $\rho(t,c)$ given in Definition 2.5.

Proposition 3.10. *An investor with investment and consumption strategies $\pi(t,w)$, $c(t,w)$ satisfying the assumptions of (ii) of Theorem 3.4 has decreasing absolute risk aversion (DARA) if and only if*

$$\frac{\pi_w(t,w)}{\pi(t,w)} \geq -\frac{c_{ww}(t,w)}{c_w(t,w)}. \tag{50}$$

In particular, a sufficient condition for an investor to be DARA is convexity (in wealth) of her consumption and investment which is increasing in wealth.

Proof. From the final statement of Theorem 3.4, the investor's utility function u satisfies $u_c(t,c) = H_c(t,c) = F(t,Y(t,c))$ and from (37) it follows that

$$\rho(t,c) = \frac{\theta}{\sigma\pi(t,Y(t,c))}Y_c(t,c), \quad c \leq \bar{c}(t) = c(t,\infty).$$

The absolute risk aversion is decreasing for $c \leq \bar{c}(t)$, i.e. $\rho_c(t,c) \leq 0$ iff:

$$\begin{aligned}
& 0 \geq \frac{\partial}{\partial c}\left[\frac{Y_c(t,c)}{\pi(t,Y(t,c))}\right] \\
\iff \qquad & 0 \geq \frac{Y_{cc}(t,c)}{\pi(t,Y(t,c))} - \frac{\left(Y_c(t,c)\right)^2 \pi_w(t,Y(t,c))}{\pi(t,Y(t,c))^2} \\
\iff \qquad & 0 \geq -\frac{c_{ww}(t,w)}{\pi(t,w)c_w(t,w)^3} - \frac{\pi_w(t,w)}{c_w(t,w)^2\pi(t,w)^2}\bigg|_{w=Y(t,c)} \\
\iff \qquad & 0 \leq \pi_w(t,w) + \pi(t,w)\frac{c_{ww}(t,w)}{c_w(t,w)}\bigg|_{w=Y(t,c)} .
\end{aligned}$$

The above has to hold for all $c \leq \bar{c}(t)$ or equivalently for all $w = Y(t,c) \geq 0$ and all $t > 0$. Transforming the last inequality we arrive at the statement of Proposition. ☐

In a similar manner we derive a condition equivalent to relative risk aversion. We omit the proof for the sake of brevity.

Proposition 3.11. *An investor with investment and consumption strategies $\pi(t,w), c(t,w)$ satisfying assumptions of Theorem 3.4 has decreasing relative risk aversion (DRRA) if and only if*

$$(c_w(t,w))^2 - c(t,w)c_{ww}(t,w) \leq \frac{c_w(t,w)c(t,w)\pi_w(t,w)}{\pi(t,w)}$$

or equivalently iff

$$\frac{\partial}{\partial w}\left[\log\frac{c(t,w)}{c_w(t,w)}\right] \leq \frac{\partial}{\partial w}\log\pi(t,w).$$

3.5 Time-homogeneous investment and consumption

We specialise now to the important special case of $\pi(t,w)$, $c(t,w)$ which are independent of time. Suppose first that $\pi(t,w) = \pi(w)$ is independent of time. Equation (26) then simplifies to

$$R_w(t,w) = \frac{r}{\pi(w)} - \frac{rw}{\pi(w)^2}\pi_w(w) - \frac{\sigma^2}{2}\pi_{ww}(w) = \frac{\partial}{\partial w}\left(\frac{rw}{\pi(w)} - \frac{\sigma^2}{2}\pi_w(w)\right)$$

which yields

$$R(t,w) = \frac{rw}{\pi(w)} - \frac{\sigma^2}{2}\pi_w(w) + \beta(t),$$

where $\beta(t)$ is taken such that $R(t,w) > 0$. In consequence

$$c(t,w) = R(t,w)\pi(w) = rw - \frac{\sigma^2}{2}\pi(w)\pi_w(w) + \beta(t)\pi(w) . \tag{51}$$

In particular, if the agent invests a constant proportion of wealth in the risky asset, i.e. $\pi(w) = \phi w$, $\phi > 0$, then

$$c(t,w) = \left(r - \frac{\sigma^2}{2}\phi^2 + \beta(t)\phi\right)w = R(t,1)\phi w$$

is also linear in wealth. It is straightforward to see that for a reasonable $\beta(t)$ (e.g. continuous and bounded) agent's choices c and π verify the assumptions of Theorem 3.4. The case $\beta(t) \equiv \beta$, a constant, was worked out explicitly in Example 3.9 above.

We want to study in more detail the implications of representation (51) on the possible behaviour of admissible investment/consumption strategies. Assume that $c = c(w)$ is also time-homogenous, or equivalently that $\beta(t) \equiv \beta$ is a constant. Recall that we require $\pi(0) = c(0) = 0$ and both $\pi(w), c(w)$ are non-negative and $c(w)$ is increasing. Consider an investment strategy given by $\pi(w) = \phi w^{\alpha}$ with $\alpha > 0$. Then (51) gives

$$c(w) = rw - \frac{\sigma^2\phi^2\alpha}{2} w^{2\alpha-1} + \beta\phi w^{\alpha}. \tag{52}$$

The condition $c(0) = 0$ restricts us to $\alpha > 1/2$. Considering $\alpha \in (0.5, 1)$ we see that the middle term in (52) dominates for small w so that $c_w(0+) = -\infty$ and $c(w)$ is negative for small values of w. On the other hand, if $\alpha > 1$ then the middle term dominates for large values of w and $c(w)$ becomes negative then. We conclude that the only admissible value is the one studied above: $\alpha = 1$. This indicates that an admissible investment strategy has to have linear behaviour near zero and infinity. For such actions we are able to verify the assumptions in Theorem 3.4.

Lemma 3.12. *Suppose $c(t,0) = 0 = \pi(t,0)$, $\pi(t,w) = \pi(w)$ is time homogeneous, $c(t,w)$ is continuous and c, π are continuously differentiable in w. Further, c and π satisfy* (33) *and there exist strictly positive constants $\tilde{\delta}_1, \tilde{\delta}_2, \kappa_1, \kappa_2$ with*

$$\begin{aligned} &\pi_w(w) \xrightarrow[w\to\infty]{} \tilde{\delta}_1, \quad \pi_w(w) \xrightarrow[w\to 0]{} \tilde{\delta}_2, \\ &\tilde{\delta}_1 \wedge \tilde{\delta}_2 \le \pi_w(w) \le \tilde{\delta}_1 \vee \tilde{\delta}_2, \ \text{and } \kappa_1 \le c_w(t,w) \le \kappa_2,\ t \ge 0, w \ge 0. \end{aligned} \tag{53}$$

Finally, assume that $\tilde{\delta}_1 \wedge \tilde{\delta}_2 \le \theta/\sigma \le \tilde{\delta}_1 \vee \tilde{\delta}_2$, or $\int_0^\infty \exp(A(t))\mathrm{d}t < \infty$.

Then c, π satisfy Assumption 3.3 and, for any $x > 0$, $\mathbb{E}[|H(t, c(t, W_t^x))|] < \infty$ and $\int_0^\infty \mathbb{E}[H(t, c(t, W_t^x)) - h(t)]^+\mathrm{d}t < \infty$. Further, $F(t,0) = \infty$, $F(t,\infty) = 0$, for all $t \ge 0$, and (43) *holds for any $x > 0$. In consequence, (i) and (ii) in Theorem 3.4 hold true.*

Proof. Let $\delta_1 := \tilde{\delta}_1 \wedge \tilde{\delta}_2$ and $\delta_2 := \tilde{\delta}_1 \vee \tilde{\delta}_2$. It is immediate that $\int_0^1 d\xi/\pi(\xi) = \int_1^\infty d\xi/\pi(\xi) = \infty$ and hence $F(t,0) = \infty$ and $F(t,\infty) = 0$. Fix $x > 0$ for the rest of the proof. Equation (53) implies global Lipschitz behaviour of c, π which guarantees the existence of a strong unique solution to (36). Let $\mathbb{Q}$ be the risk neutral measure under which $B_t^\theta := B_t + \theta t$ is a Brownian motion and put $\tilde{W}_t := \mathrm{e}^{-rt}W_t$. We then have

$$\tilde{W}_t = x + \sigma M_t - \int_0^t \mathrm{e}^{-rs} c(s, W_s)\mathrm{d}s, \tag{54}$$

where $M_t := \int_0^t \mathrm{e}^{-rs}\pi(W_s)\mathrm{d}B_s^\theta$ is a $\mathbb{Q}$-local martingale. In particular, $\tilde{W}_t$ is a non-negative super-martingale under $\mathbb{Q}$ and hence converges, a.s. as $t \to \infty$. It follows that M_t also converges a.s. which is equivalent to $\langle M\rangle_t$ converging (cf. (Revuz and Yor, 2001, Proposition V.1.8)). However

$$\int_0^t \delta_1^2 \tilde{W}_s^2 \mathrm{d}s \ \le\ \langle M\rangle_t = \int_0^t \mathrm{e}^{-2rs}\pi(W_s)^2\mathrm{d}s \ \le\ \int_0^t \delta_2^2 \tilde{W}_s^2 \mathrm{d}s$$

and it follows that $\tilde{W}_t \to 0$ a.s. Finally, from classical estimates (e.g. (Friedman, 1975, Theorem 5.2.3)), we have that $\mathbb{E}^{\mathbb{Q}}[(\tilde{W}_t)^m] < \infty$ for all $m \geq 1$. It follows that $\mathbb{E}^{\mathbb{Q}}[\langle M \rangle_t] < \infty$ and hence M_t is a $\mathbb{Q}$-martingale with $\mathbb{E}^{\mathbb{Q}} M_t = 0$. In particular

$$\mathbb{E} \int_0^\infty Z_s c(s, W_s) \mathrm{d}s = \lim_{t \to \infty} \mathbb{E}^{\mathbb{Q}} \int_0^t \mathrm{e}^{-rs} c(s, W_s) \mathrm{d}s = x - \lim_{t \to \infty} \mathbb{E}^{\mathbb{Q}}[\tilde{W}_t].$$

To show (43), it remains to argue that $\lim_{t \to \infty} \mathbb{E}^{\mathbb{Q}}[\tilde{W}_t] = 0$. It follows from the above representation that $\mathbb{E}^{\mathbb{Q}}[\tilde{W}_t]$ is decreasing in t. By (54), and $\mathbb{E}^{\mathbb{Q}} M_t = 0$, we have:

$$\mathbb{E}^{\mathbb{Q}} \tilde{W}_t = x - \mathbb{E}^{\mathbb{Q}} \int_0^t \mathrm{e}^{-rs} c(s, \mathrm{e}^{rs} \tilde{W}_s) \mathrm{d}s.$$

Using the fact that $c(s, w) \geq \kappa_1 w$, and applying Fubini's theorem, we get:

$$\mathbb{E}^{\mathbb{Q}} \tilde{W}_t \leq x - \kappa_1 \int_0^t \mathbb{E}^{\mathbb{Q}} \tilde{W}_s \, \mathrm{d}s.$$

The desired conclusion follows immediately. It remains to show the integrability properties of H. From (53) we get instantly that

$$\begin{aligned} w^{-\frac{\theta}{\sigma\delta_2}} &\leq \mathrm{e}^{-A(t)} F(t, w) \leq w^{-\frac{\theta}{\sigma\delta_1}}, \quad 0 \leq w < 1 \\ w^{-\frac{\theta}{\sigma\delta_1}} &\leq \mathrm{e}^{-A(t)} F(t, w) \leq w^{-\frac{\theta}{\sigma\delta_2}}, \quad w \geq 1 \end{aligned} \tag{55}$$

from which it follows that

$$\begin{aligned} \mathrm{e}^{A(t)\frac{\sigma\delta_1}{\theta}} z^{-\frac{\sigma\delta_1}{\theta}} &\leq f(t, z) \leq \mathrm{e}^{A(t)\frac{\sigma\delta_2}{\theta}} z^{-\frac{\sigma\delta_2}{\theta}}, \quad 0 \leq z < \mathrm{e}^{A(t)} \\ \mathrm{e}^{A(t)\frac{\sigma\delta_2}{\theta}} z^{-\frac{\sigma\delta_2}{\theta}} &\leq f(t, z) \leq \mathrm{e}^{A(t)\frac{\sigma\delta_1}{\theta}} z^{-\frac{\sigma\delta_1}{\theta}}, \quad z \geq \mathrm{e}^{A(t)} \ . \end{aligned} \tag{56}$$

The integrability properties we need to establish are invariant under a shift of H by a constant so we are free to redefine H as

$$H(t, z) := \int_{c(t,1)}^z F(t, Y(t, b)) \mathrm{d}b, \text{ so that } H(t, c(t, w)) = \int_1^w F(t, s) c_w(t, s) \mathrm{d}s.$$

In consequence

$$H(t, c(t, w))^+ = \int_1^{w \vee 1} F(t, s) c_w(t, s) \mathrm{d}s \leq \kappa_2 \mathrm{e}^{A(t)} \int_1^{w \vee 1} s^{-\frac{\theta}{\sigma\delta_2}} \mathrm{d}s \tag{57}$$

and similarly

$$H(t, c(t, w))^- = \int_{w \wedge 1}^1 F(t, s) c_w(t, s) \mathrm{d}s \leq \kappa_2 \mathrm{e}^{A(t)} \int_{w \wedge 1}^1 s^{-\frac{\theta}{\sigma\delta_1}} \mathrm{d}s \tag{58}$$

Suppose first that $\sigma\delta_1 < \theta < \sigma\delta_2$. Then $H(t, c(t, w))^+ \leq \mathrm{e}^{A(t)} \frac{\sigma\delta_2\kappa_2}{\sigma\delta_2 - \theta} w^{1 - \frac{\theta}{\sigma\delta_2}}$ and $H(t, c(t, w))^- \leq \mathrm{e}^{A(t)} \frac{\sigma\delta_1\kappa_2}{\theta - \sigma\delta_1} w^{1 - \frac{\theta}{\sigma\delta_1}}$.

Using $W_t = f(t, \lambda(x) Z_t)$ and the estimates in (56), we have

$$\begin{aligned} \mathbb{E} H(t, c(t, W_t))^+ &\leq \mathrm{e}^{A(t)} \frac{\sigma\delta_2\kappa_2}{\sigma\delta_2 - \theta} \mathbb{E}\left[f(t, \lambda(x) Z_t)^{1 - \frac{\theta}{\sigma\delta_2}} \mathbf{1}_{\lambda(x) Z_t \leq \mathrm{e}^{A(t)}} \right] \\ &\leq \frac{\sigma\delta_2\kappa_2}{\sigma\delta_2 - \theta} \mathrm{e}^{A(t)\frac{\sigma\delta_2}{\theta}} \mathbb{E}[(\lambda(x) Z_t)^{1 - \frac{\sigma\delta_2}{\theta}}] \end{aligned} \tag{59}$$

and it follows that $\mathbb{E}H(t, c(t, W_t))^+ < \infty$. Similarly, using (58) and reasoning as in (59) we obtain

$$\begin{aligned} \mathbb{E}H(t, c(t, W_t))^- &\leq \mathrm{e}^{A(t)} \frac{\sigma\delta_1\kappa_2}{\theta - \sigma\delta_2} \mathbb{E}\left[f(t, \lambda(x)Z_t)^{1-\frac{\theta}{\sigma\delta_1}} \mathbf{1}_{\lambda(x)Z_t \geq \mathrm{e}^{A(t)}} \right] \\ &\leq \frac{\sigma\delta_1\kappa_2}{\theta - \sigma\delta_2} \mathrm{e}^{A(t)\frac{\sigma\delta_1}{\theta}} \mathbb{E}[(\lambda(x)Z_t)^{1-\frac{\sigma\delta_1}{\theta}}] \end{aligned} \tag{60}$$

and it follows that $\mathbb{E}H(t, c(t, W_t))^- < \infty$, giving also $\mathbb{E}|H(t, c(t, W_t))| < \infty$.

To estimate the expectation of the integral in time we need a more careful analysis. From Black's equation (33), given that π is time-homogeneous, we know that

$$\beta(t) = \frac{\sigma^2}{2}\pi_w(w) - r\frac{w}{\pi(w)} + \frac{c(t, w)}{\pi(w)}, \tag{61}$$

is a function of t only. Now, depending on whether $\delta_2 = \tilde{\delta_2}$ or $\delta_2 = \tilde{\delta_1}$, we let $w \to 0$ or $w \to \infty$ on the RHS. We detail the case $\delta_2 = \tilde{\delta_2}$. The first term in (61) then converges to $\sigma^2\delta_2/2$, the second term converges by l'Hôpital's rule to r/δ_2 and hence also the third term converges to some $\kappa_3(t)/\delta_2$, where $\kappa_3(t) \geq \kappa_1 > 0$. We conclude that $\beta(t) = \sigma^2\delta_2/2 + \kappa_3(t)/\delta_2 - r/\delta_2$ and

$$\begin{aligned} A(t) &= -\frac{\theta}{\sigma}\int_0^t \beta(s)ds + \left(\frac{\theta^2}{2} - r\right)t \\ &\leq \left(-\frac{\theta}{\sigma}(\sigma^2\delta_2/2 + \kappa_1/\delta_2 - r/\delta_2) + \frac{\theta^2}{2} - r\right)t \\ &= \left(-\frac{\theta\kappa_1}{\sigma\delta_2} - \left(1 - \frac{\theta}{\sigma\delta_2}\right)\left(\frac{\theta\sigma\delta_2}{2} + r\right)\right)t \end{aligned} \tag{62}$$

Using this last estimate in (59) we recover the situation in (48). Since by assumption $\theta < \sigma\delta_2$, and using the representation $Z_t = e^{-rt-\theta B_t - \theta^2 t/2}$ in (59) we have $\mathbb{E}H(t, c(t, W_t))^+ \leq C\mathrm{e}^{-\kappa_1 t}$ for a constant C, and hence $\mathbb{E}\int_0^\infty H(t, c(t, W_t))^+ \mathrm{d}t < \infty$. To estimate $\mathbb{E}\int_0^\infty H(t, c(t, W_t))^- \mathrm{d}t$ we use similar arguments but consider the asymptotics of (61) when $w \to \infty$ (respectively $w \to 0$ if $\delta_1 = \tilde{\delta_2}$). Using the assumption that $\theta > \sigma\delta_1$, and proceeding analogously to the above, we obtain $\mathbb{E}H(t, c(t, W_t))^- \leq C\mathrm{e}^{-\kappa_1 t}$ for a (possibly different) constant C, so that $\mathbb{E}\int_0^\infty H(t, c(t, W_t))^- \mathrm{d}t < \infty$. It follows that $\mathbb{E}\int_0^\infty |H(t, c(t, W_t))| \mathrm{d}t < \infty$ and $\mathbb{E}\int_0^\infty [H(t, c(t, W_t)) - h(t)]\mathrm{d}t$ is well defined (and equal to zero).

Now suppose that $\theta = \sigma\delta_i$ for some i. We detail the case $\theta = \sigma\delta_2$, the other being similar, but relies on a new estimate of $H(t, c(t, W_t))^-$. Then, for $z < e^{A(t)}$, $f(t, z) \leq e^{A(t)} z^{-1}$ and (57) yields $H(t, c(t, W_t))^+ = \kappa_2 e^{A(t)}(\ln f(t, \lambda(x)Z_t))^+$. But, from (62), and using $\theta = \sigma\delta_2$ we have $A(t) \leq -\kappa_1 t$. It follows that $\int_0^\infty \mathbb{E}[H(t, c(t, W_t))^+]\mathrm{d}t < \int_0^\infty (C_1 + C_2 t)e^{-\kappa_1 t}\mathrm{d}t < \infty$. Note that if $\delta_1 < \delta_2$ then $\theta > \sigma\delta_1$ and $\int_0^\infty \mathbb{E}[H(t, c(t, W_t))^-]\mathrm{d}t < \infty$ exactly as before. Otherwise, if $\delta_1 = \delta_2$, a modification of the above argument works for $H(t, c(t, W_t))^-$.

Finally suppose that $\theta > \sigma\delta_2$ but $\int_0^\infty e^{A(t)}\mathrm{d}t < \infty$. (Again, the case with $\theta < \sigma\delta_1$ but $\int_0^\infty e^{A(t)}\mathrm{d}t < \infty$ is similar.) Then (57) yields $H(t, c(t, W_t))^+ < Ce^{A(t)}$ which is integrable by hypothesis, and $\int_0^\infty \mathbb{E}[H(t, c(t, W_t))^-]\mathrm{d}t < \infty$ as before.

$\square$

Remark 3.13. Recall that inequality in (62) also holds with δ_1 in place of δ_2. It follows that sufficient conditions for $\int_0^\infty e^{A(t)}\mathrm{d}t < \infty$ are either

$$\kappa_1 > (1 - \sigma\tilde{\delta}_2/\theta)(r + \theta\sigma\tilde{\delta}_2/2), \text{ or } \kappa_1 > (1 - \sigma\tilde{\delta}_1/\theta)(r + \theta\sigma\tilde{\delta}_1/2).$$

This is automatic if $\theta < \sigma\left(\tilde{\delta}_1 \vee \tilde{\delta}_2\right)$, when $r > 0$. Note that this final conclusion would allow us to simplify the statement of Lemma 3.12 by ignoring the case where $\tilde{\delta}_1 \wedge \tilde{\delta}_2 \leq \theta/\sigma \leq \tilde{\delta}_1 \vee \tilde{\delta}_2$, however it does not appear that this would simplify the proof, and since this condition is generally simpler to check, we leave the statement as given.

Remark 3.14. From the proof it is clear that we do not need to assume time-homogeneity of π. Instead we take $\pi(t,w) \in \mathcal{C}^{1,1}$ and assume (53) and (33). Then it follows that

$$\kappa_3(t) := \tilde{\delta}_2 \lim_{w\to 0} \frac{c(t,w)}{\pi(t,w)} + \int_1^w \frac{\pi_t(t,\xi)}{\pi(t,\xi)^2}\mathrm{d}\xi,\ \kappa_4(t) := \tilde{\delta}_1 \lim_{w\to\infty} \frac{c(t,w)}{\pi(t,w)} + \int_1^w \frac{\pi_t(t,\xi)}{\pi(t,\xi)^2}\mathrm{d}\xi$$

are well defined. It is then enough to assume that either $\int_0^\infty \exp(A(t))\mathrm{d}t < \infty$ or $\tilde{\delta}_1 \wedge \tilde{\delta}_2 < \theta/\sigma < \tilde{\delta}_1 \vee \tilde{\delta}_2$ and further $\int_0^\infty \exp(-\int_0^t \kappa_3(u)du)\mathrm{d}t < \infty$ and likewise for $\kappa_4(t)$. In the time-homogeneous case above, the integral term disappears and (53) implies that $\kappa_3(t), \kappa_4(t) \geq \kappa_1 > 0$ which is a stronger condition.

Lemma 3.12 is particularly useful as it allows us to construct a wealth of examples of non-linear consumption and investment pairs with prescribed desired behaviour. We explore now a method to obtain convex/concave investment and consumptions pairs and then present a simple parametric family of examples.

Assume that $\pi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a thrice differentiable, strictly increasing, concave function such that $\pi(0) = 0$ and

$$0 < \pi_w(\infty-) \leq \pi_w(w) < \pi_w(0+) < \infty, \quad w \in (0,\infty). \tag{63}$$

Further, let $\chi(w) = (\pi\pi_w)_w(w) = \pi_w(w)^2 + \pi(w)\pi_{ww}(w)$ and assume that there exists $\epsilon > 0$ such that

$$-\epsilon^{-1} < \frac{\sigma^2}{2}\chi(w) < r - \epsilon, \quad w \in (0,\infty). \tag{64}$$

The optimal consumption is given by (51) and we assume that it is time homogenous with $\beta(t) \equiv \beta \geq 0$. We then have $c(0) = 0$ and

$$\epsilon \leq \epsilon + \beta\pi_w(\infty-) < c_w(w) = r + \beta\pi_w(w) - \frac{\sigma^2}{2}\chi(w) < r + \beta\pi_w(0+) + \epsilon^{-1}.$$

In particular, (53) holds. Provided $r > 0$ and the Sharpe ratio satisfies $\theta < \sigma\pi_w(0+)$, by Remark 3.13, we can apply Lemma 3.12 and conclude that (i) and (ii) in Theorem 3.4 hold true.

By hypothesis π is concave: the concavity/convexity of c will depend on the sign of $c_{ww} = \beta\pi_{ww} - \sigma^2\chi_w(w)/2$. Noting that $\pi_{ww} \leq 0$ we have that if

$$C\pi_{ww}(w) < \chi_w(w) < 0, \quad w \in (0,\infty), \tag{65}$$

for some positive constant C then the choice $\beta = 0$ gives that c is strictly convex, whereas the choice $\beta = \sigma^2 C/2$ gives that c is strictly concave.

Similarly, we can produce examples with π convex. Suppose that π is a thrice differentiable, strictly increasing, convex function such that $\pi(0) = 0$ and $0 < \pi_w(0+) \leq \pi_w(w) \leq \pi_w(\infty-) < \infty$. Then, if we assume again that (64) holds, we have c such that $c(0) = 0$ and

$$\epsilon \leq \epsilon + \beta\pi_w(0+) < c_w(w) < r + \beta\pi_w(\infty-) + \epsilon^{-1}.$$

In conclusion, (53) holds and if $\theta < \sigma\pi_w(\infty-)$ the assumptions of Lemma 3.12 are satisfied and (i) and (ii) in Theorem 3.4 follow. If, instead of (65), we have

$$C\pi_{ww}(w) > \chi_w(w) > 0, \quad w \in (0, \infty), \tag{66}$$

then the choice of $\beta = 0$ gives a concave consumption while $\beta = \sigma^2 C/2$ generates a convex c.

Example 3.15. As an example, take in the above $\pi(w) = (\phi w + \psi((1+w)^p - 1))$ with $\phi, \psi > 0$ and $0 < p < 1$. Then π is concave, $\pi_w(\infty-) = \phi$ and $\pi_w(0+) = \phi + p\psi$. Furthermore,

$$\chi_w(w) = \pi(w)\pi_{www}(w) + 3\pi_w(w)\pi_{ww}(w) = -\psi(1+w)^{p-3}p(1-p)\Lambda(w)$$

where

$$\Lambda(w) = (1+p)\phi w + (2-p)\psi + 3\phi + 2(1+w)^p\psi(2p-1).$$

Suppose that the parameters are such that $\Lambda(w)$ is positive, a simple sufficient condition for which is that $p \geq 1/2$. Since $\psi > 0$ it follows that χ is decreasing and

$$\phi^2 = \chi(\infty) \leq \chi(w) \leq \chi(0) = (\phi + p\psi)^2$$

so that (64) follows provided $(\phi + p\psi)^2 < 2r/\sigma^2$, a condition we now impose.

We already have $\chi_w < 0$ so for (65) it suffices to look at the sign of

$$C\pi_{ww} - \chi_w = \psi p(1-p)(1+w)^{p-3}[-C(1+w) + \Lambda(w)]. \tag{67}$$

Since Λ is bounded above by an affine function of w it follows easily that this expression can be made negative on $(0, \infty)$ by choosing C sufficiently large. Choose the parameters such that $\Lambda(w) > 0$, $(\phi + p\psi)^2 < 2r/\sigma^2$ and $\theta < \sigma(\phi + p\psi)$. Taking $\beta = 0$ we obtain an example for which π is concave and c is convex, and conversely, taking β sufficiently large, we obtain an example with both π and c concave.

To obtain examples with convex π consider now $\phi > 0$, $0 < p < 1$ but $\psi < 0$. Assume also that $\phi + p\psi > 0$ and $\phi^2 < 2r/\sigma^2$. Then π is increasing, convex with $\phi + p\psi = \pi_w(0+) < \pi_w(w) < \pi_w(\infty-) = \phi$. Suppose again that the parameters are such that $\Lambda(w)$ is positive. In fact, given $\Lambda(0) = 3(\phi + p\psi) > 0$, $p \leq 1/2$ is a simple sufficient condition. It follows that $\chi(w)$ is increasing with

$$(\phi + p\psi)^2 = \chi(0) \leq \chi(w) \leq \chi(\infty) = \phi^2$$

so that (64) holds under the condition $\phi^2 < 2r/\sigma^2$. Further, (67) can be made uniformly positive for large C and hence (66) holds. Lemma 3.12 applies with e.g. $\theta < \sigma\phi$. We conclude that if we take $\beta = 0$ we obtain an example with convex investment and a concave consumption. Conversely, if we take β sufficiently large we get an example with both π and c convex. A numerical example

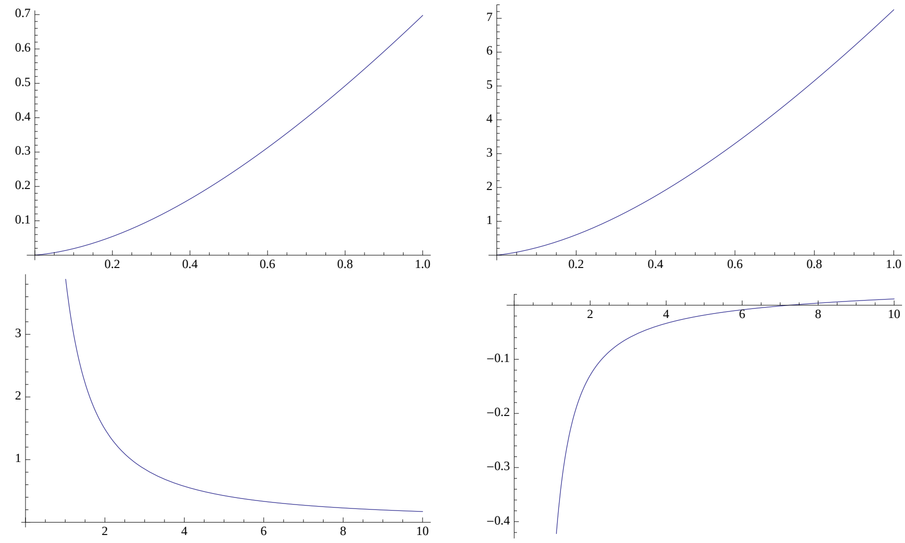

Figure 1: Graphs for Example 3.15 when $\pi_{ww} > 0$. *Top panes*: The investment strategy $\pi(w)$ (left) and the corresponding optimal consumption $c(w)$ (right) for parameters: $r = 0.3$, $\theta = 0.5$, $\sigma = 0.25$, $\beta = 10$, $p = 1/30$, $\phi = 2.1$ and $\psi = -60$. *Bottom panes*: The absolute risk aversion $\rho(1, c)$ (left) inferred from these actions and a compatible utility function $u(1, c)$ (right).

is given in Figure 1 which also presents graphs for the case of π and c convex. Note that convexity of c implies DARA by Proposition 3.10.

Our general approach easily allows us to obtain admissible sets of parameters with additional convexity and concavity properties. Note, however, that even when the arguments for the convexity/concavity fail, it is still straightforward to write down expressions for c_w, and to see that c_w is bounded above. Provided parameters are chosen such that $c_w(w) > \kappa_1 > 0$, and θ is sufficiently small, the hypotheses of Lemma 3.12 are satisfied and (i) and (ii) of Theorem 3.4 hold. A numerical example which satisfies these conditions but for which $\Lambda(w)$ goes negative is given in Figure 2. It also features a risk aversion which is first decreasing, then increasing and then decreasing again.

Example 3.16. We present another example where consumption has a simple convex expresion in wealth. Consider

$$\pi(w) = \frac{2}{\sigma}\sqrt{\frac{r-\kappa}{2}w^2 + \alpha w + \frac{\alpha}{a}(\mathrm{e}^{-aw} - 1)}, \tag{68}$$

where we take $r > \kappa > 0$, $\alpha, a > 0$ with $\kappa > a\alpha$. Clearly $\pi(w)$ is an increasing function of w and

$$\delta_2 = \pi_w(0+) = \frac{\sqrt{2}}{\sigma}\sqrt{r - \kappa + \alpha a}, \quad \delta_1 = \pi_w(\infty-) = \frac{\sqrt{2}}{\sigma}\sqrt{r-\kappa}.$$

From (51), with $\beta(t) \equiv 0$, we recover the optimal consumption as

$$c(w) = \kappa w + \alpha\left(\mathrm{e}^{-aw} - 1\right) \tag{69}$$

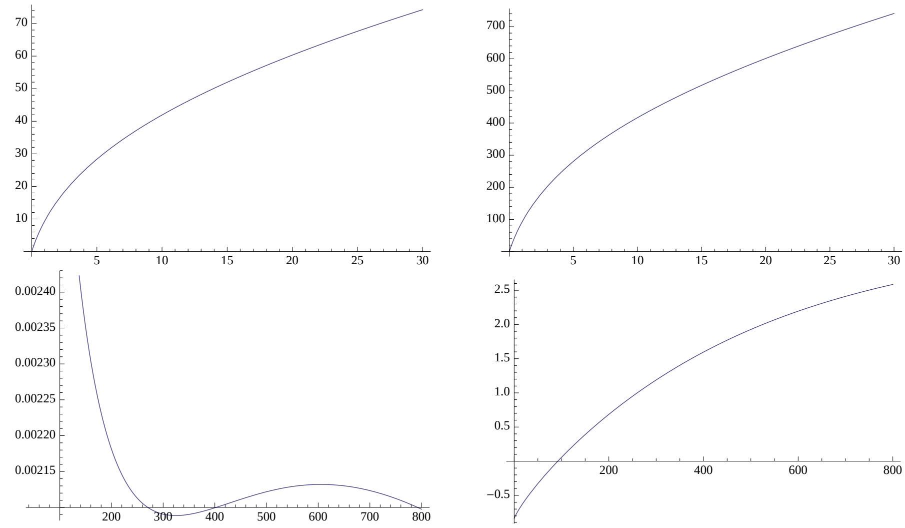

Figure 2: Graphs for Example 3.15 when $\pi_{ww} < 0$. *Top panes*: The investment strategy $\pi(w)$ (left) and the corresponding optimal consumption $c(w)$ (right) for parameters: $r = 0.05$, $\theta = 0.5$, $\sigma = 0.25$, $\beta = 10$, $p = 1/5$, $\phi = 0.5$ and $\psi = 60$. *Bottom panes*: The absolute risk aversion $\rho(1, c)$ (left) inferred from these actions and a compatible utility function $u(1, c)$ (right).

which is an increasing convex function with $c_{ww}(w) = a^2\alpha \mathrm{e}^{-aw}$. Sufficient conditions for π, c to satisfy the assumptions of Lemma 3.12 are then concavity of π (from which we deduce $\delta_1 \leq \pi_w(w) \leq \delta_2$) and $\theta < \sqrt{2r}$ (which implies $\int_0^\infty \exp(A(t))\,\mathrm{d}t < \infty$). We will verify the concavity condition numerically for the cases of interest presented in Figure 3. Note that in this example c is convex and π is increasing so Proposition 3.10 implies that the agent employing these actions necessarily has a decreasing absolute risk aversion.

Example 3.17. Finally, we compute explicitly an example in which c is bounded. Consider

$$\pi(w) = \frac{2\sqrt{\bar{c}}}{\sigma}\sqrt{\frac{r}{2\bar{c}}w^2 - w - \frac{1}{\alpha}\mathrm{e}^{-\alpha w} + \frac{1}{\alpha}}, \tag{70}$$

where we take $r > \alpha\bar{c}$ with $\alpha, \bar{c} > 0$. A simple computation shows that $\frac{r}{2\bar{c}}w^2 - w - \frac{1}{\alpha}(\mathrm{e}^{-\alpha w} - 1)$ is a non-negative and strictly increasing function of w for $w \geq 0$. Further we have $\pi(w)\pi_w(w)\sigma^2/2 = rw - \bar{c}(1 - \mathrm{e}^{-\alpha w})$ and there exist two constants $\delta_1, \delta_2 > 0$ such that $\delta_1 < \pi_w(w) < \delta_2$, $w \geq 0$. From (51), with $\beta(t) \equiv 0$, we recover the optimal consumption as

$$c(w) = \bar{c}\left(1 - \mathrm{e}^{-\alpha w}\right), \tag{71}$$

which increases, as $w \to \infty$, to the upper bound $\bar{c}$, see Figure 4. While it is clear that c, π satisfy Assumption 3.3 it is does not seem easy to verify that assumptions in (ii) of Theorem 3.4 hold. In this example the proposed recovered utility function is constant on $[\bar{c}, \infty)$ so consumption $C_t \equiv \bar{c}$, if admissible, has to be optimal. For $x \geq \bar{c}/r$, taking $\Pi_t \equiv 0$ the wealth dynamics become

$$\mathrm{d}W_t^x = (rW_t^x - \bar{c})\mathrm{d}t, \quad W_0^x = x,$$

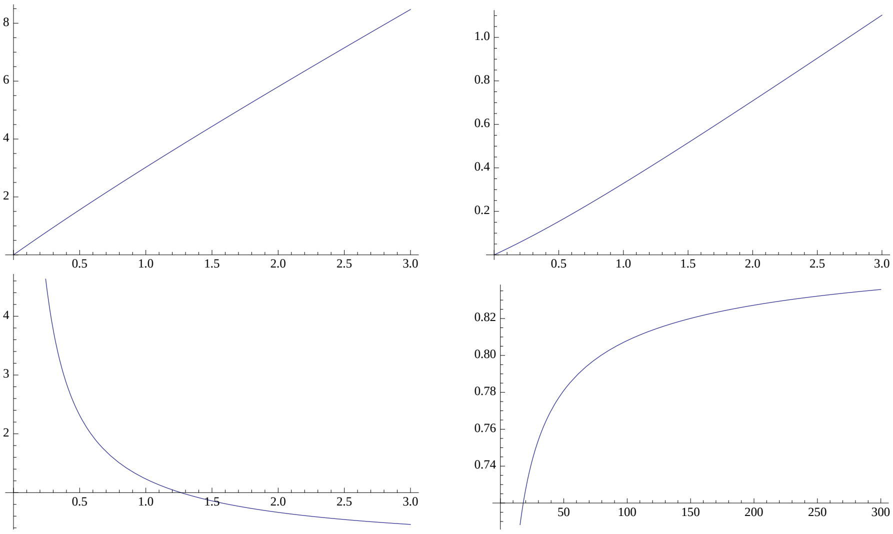

Figure 3: Graphs for Example 3.16. *Top panes*: The investment strategy $\pi(w)$ (left) in (68) and the corresponding optimal consumption $c(w)$ (right) in (69) for parameters: $\kappa = 0.4, \sigma = 0.25, r = 0.6, \alpha = 0.1, a = 1.25, \theta = 0.95$. *Bottom panes*: The absolute risk aversion $\rho(1, c)$ (left) inferred from these actions and a compatible utility function $u(1, c)$ (right).

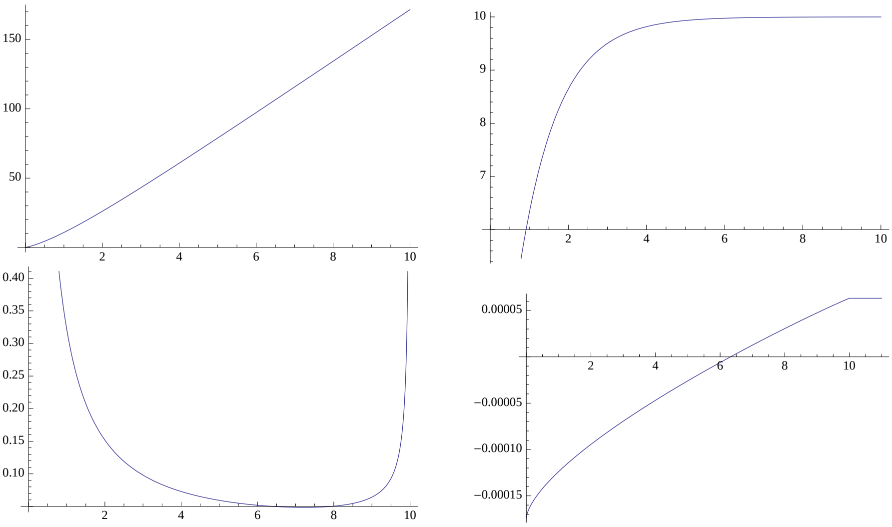

Figure 4: Graphs for Example 3.17. *Top panes*: The investment strategy $\pi(w)$ (left) in (70) and the corresponding optimal consumption $c(w)$ (right) in (71) for parameters: $r = 11$, $\theta = 0.5$, $\sigma = 0.25$, $\bar{c} = 10$, $\alpha = 1$. *Bottom panes*: The absolute risk aversion $\rho(1, c)$ (left) inferred from these actions and a compatible utility function $u(1, c)$ (right).

so that $W_t^x = x\exp(rt) - \bar{c}t$ is an increasing process and $(\Pi_t, C_t) = (0, \bar{c})$ is admissible and optimal. In consequence, Theorem 3.4 may only hold for $x < \bar{c}/r$. Even then this example is not covered by Lemma 3.12 and verification of (43) appears challenging.

4 Extensions and further research

The work presented in this paper may be seen as the first step which motivates exploration of a set of wider related questions. Our underpinning principle is to start with those actions which may be observed in an investor's behaviour, and then attempt to determine whether their actions are consistent with utility maximisation. In this paper, we have considered two cases: a deterministic setup and a stochastic complete market setup. In the deterministic case, our fundamental conclusion is that observing investor's actions for any given wealth is not enough to fully specify their utility. Risk aversion remains unspecified. In the stochastic case, we suppose we observe both consumption and investment. Then the assumption that the investor is maximising utility implies that the consistency constraint (33) holds. These two studies would have natural, and interesting, analogues in other markets such as one period models (where the investor can choose to consume now or in the subsequent period), or, at the other extreme in terms of complexity, continuous-time models which are incomplete (e.g. a stochastic factor model). The questions parallel to those which we have answered here would include:

- is specifying an investor's consumption and investment strategies sufficient to determine their utility function (up to constants)?
- If it is not, is the system over-specified or underspecified?

In the case where the system is over-specified, we might expect a consistency condition such as Black's equation, (33), and it is interesting to ask whether there is a more general optimisation problem such as (17) which may correspond to the general choice of consumption and investment.

A further achievement of the paper is that we have been able to go beyond simply recovering the utility function, and have also been able to provide characterisations of certain aspects of the agent's behaviour (absolute and relative risk aversion) in terms of the given data. In more complex situations, where it may not be possible to fully recover an agent's utility function, it may still be possible to deduce some of these related properties from the given data. We hope to pursue some of these questions in subsequent work.

References

Black, F., 1968. Investment and consumption through time, Financial Note No. 6B. Arthur D. Little, Inc.

Black, F., 1988. Individual investment and consumption under uncertainty. In: Luskin, D. L. (Ed.), Portfolio Insurance: A guide to dynamic hedging. Wiley, pp. 207–225.

Cox, A. M. G., Obłój, J., 2010. Robust pricing and hedging of double no-touch options. Finance Stoch.To appear.

Dybvig, P., Rogers, L., 1997. Recovery of preferences from observed wealth in a single realization. The Review of Financial Studies 10 (1), 151–174.

Friedman, A., 1975. Stochastic differential equations and applications. Vol. 1. Academic Press [Harcourt Brace Jovanovich Publishers], New York, probability and Mathematical Statistics, Vol. 28.

Green, R. C., Srivastava, S., 1985. Risk aversion and arbitrage. The Journal of Finance 40 (1), 257–268.

He, H., Huang, C.-f., 1994. Consumption-portfolio policies: An inverse optimal problem. Journal of Economic Theory 62 (2), 257 – 293.

Hobson, D., 2010. The Skorokhod Embedding Problem and Model-Independent Bounds for Option Prices. In: Carmona, R., Çinlar, E., Ekeland, I., Jouini, E., Scheinkman, J., Touzi, N. (Eds.), Paris-Princeton Lectures on Mathematical Finance 2010. Vol. 2003 of Lecture Notes in Math. Springer, pp. 267–318.

Karatzas, I., Shreve, S. E., 1998. Methods of Mathematical Finance. Vol. 39 of Applications of Mathematics (New York). Springer-Verlag, New York.

Mas-Colell, A., 1977. The recoverability of consumers' preferences from market demand behavior. Econometrica 45 (6), 1409–1430.

Revuz, D., Yor, M., 2001. Continuous martingales and Brownian motion, revised Third Edition. Vol. 293 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin.

Richter, M., 2008. Revealed prefence theory. In: Durlauf, S., Blume, L. (Eds.), The New Palgrave Dictionary of Economics. Palgrave MacMillan.

Samuelson, P., 1948. Consumption theory in terms of revealed preference. Economica 15 (60), 243–253.

Wang, S., 1993. The integrability problem of asset prices. Journal of Economic Theory 59, 199–213.