From minimal embeddings to minimal diffusions

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Abstract

We show that there is a one-to-one correspondence between diffusions and the solutions of the Skorokhod Embedding Problem due to Bertoin and Le-Jan. In particular, the minimal embedding corresponds to a 'minimal local martingale diffusion', which is a notion we introduce in this article. Minimality is closely related to the martingale property. A diffusion is minimal if it it minimises the expected local time at every point among all diffusions with a given distribution at an exponential time. Our approach makes explicit the connection between the boundary behaviour, the martingale property and the local-time characteristics of time-homogeneous diffusions.

1 Introduction

In this article, we show that there is a one-to-one correspondence between a (generalised) diffusion in natural scale and a classical solution to the Skorokhod Embedding Problem due to Bertoin and Le-Jan [4]. The correspondence is a consequence of the following two related facts: i.) Bertoin Le-Jan (BLJ) Skorokhod embeddings can be represented as time changes evaluated at an independent exponential time and ii.) diffusions can be uniquely characterised in terms of their law at an independent exponential time.

In [7], Cox et. al. identify a connection between martingale diffusions and the minimal Skorokhod embedding which is used to prove existence of a (martingale) diffusion with a given law at an exponentially distributed time. Subsequently it was shown in [13] that a diffusion's speed measure can be represented in terms of its exponential time law. In this article we complete the characterisation of diffusions by characterising the relationship between diffusions and Bertoin Le-Jan (BLJ) solutions to the Skorokhod Embedding Problem for Brownian motion. We illustrate, for instance, the connection between the Wronskian of a diffusion and the parameter in the BLJ embedding which 'sets' the expected local time of the process up until the stopping time.

An important notion in the literature on the Skorokhod Embedding Problem is the concept of a minimal stopping time, first introduced by Monroe, [16]. Loosely speaking, a stopping time is minimal if there is no almost surely smaller stopping time for the process which results in the same distribution. An important contribution of this paper is that we show that this notion extends naturally to the diffusion context. We also provide a novel characterisation of minimal stopping times in terms of the local times, which has a natural corresponding interpretation in the diffusion setting.

The correspondence between the classical BLJ solution to the Skorokhod Embedding Problem and time-homogeneous diffusions provides insight into the local time properties of local martingale diffusions modulo a fixed exponential-time law. In particular minimal BLJ embeddings correspond naturally to the notion of a 'minimal local martingale diffusion', which we introduce in this article. Relating quantities well known in diffusion theory (the speed measure and the Wronskian) to quantities in the BLJ embedding leads to improved intuition and understanding of diffusions and the embeddings.

2 Preliminaries

2.1 Generalised Diffusions

Let m be a non-negative, non-zero Borel measure on an interval $I \subseteq \mathbb{R}$, with left endpoint a and right endpoint b (either or both of which may be infinite). Let $x_0 \in (a, b)$ and let $B = (B_t)_{t \geq 0}$ be a Brownian motion started at $B_0 = x_0$ supported on a filtration $\mathbb{F}^B = (\mathcal{F}_u^B)_{u \geq 0}$ with local time process $\{L_u^x; u \geq 0, x \in \mathbb{R}\}$. Define Γ to be the continuous, increasing, additive functional

$$\Gamma_u = \int_{\mathbb{R}} L_u^x m(dx),$$

and define its right-continuous inverse by

$$A_t = \inf\{u \geq 0 : \Gamma_u > t\}.$$

If $X_t = B_{A_t}$ then $X = (X_t)_{t \geq 0}$ is a one-dimensional generalised diffusion in natural scale with $X_0 = x_0$ and speed measure m. Moreover, $X_t \in I$ almost surely for all $t \geq 0$. In contrast to diffusions, which are continuous by definition, generalised diffusions may be discontinuous if the speed measure places no mass on an interval. For instance, if the speed measure is purely atomic, then the process is a birth-death process in the sense of Feller [10]. See also Kotani and Watanabe [15]. In the sequel, we will use diffusion to denote the class of generalised diffusions, rather than continuous diffusions.

Let $H_x = \inf\{u : X_u = x\}$. Then for $\lambda > 0$ (see e.g. [20]),

$$\mathbb{E}_x[e^{-\lambda H_y}] = \begin{cases} \frac{\varphi_\lambda(x)}{\varphi_\lambda(y)} & x \leq y \\ \frac{\phi_\lambda(x)}{\phi_\lambda(y)} & x \geq y, \end{cases} \tag{2.1}$$

where φ_λ and ϕ_λ are respectively a strictly increasing and a strictly decreasing solution to the differential equation

$$\frac{1}{2} \frac{d^2}{dm dx} f = \lambda f. \tag{2.2}$$

The two solutions are linearly independent with Wronskian $W_\lambda = \varphi'_\lambda \phi_\lambda - \phi'_\lambda \varphi_\lambda$, which is a positive constant.

The solutions to (2.2) are called the λ-eigenfunctions of the diffusion. We will scale the λ-eigenfunctions so that $\varphi_\lambda(x_0) = \phi_\lambda(x_0) = 1$.

2.2 The Skorokhod Embedding Problem

We recall some important notions relating to the Skorokhod Embedding Problem (SEP). The SEP can be stated as follows: given a Brownian motion $(B_t)_{t\geq 0}$ (or, more generally, some stochastic process) and a measure μ on $\mathbb{R}$, a solution to the SEP is a stopping time τ such that $B_\tau \sim \mu$. We refer to Obłój [17] for a comprehensive survey of the history of the Skorokhod Embedding Problem.

Many solutions to the problem are known, and it is common to require some additional assumption on the process: for example that the stopped process $(B_{t\wedge\tau})_{t\geq 0}$ is uniformly integrable. In the case where $B_0 = x_0$, this requires some additional regularity on μ — specifically that μ is integrable, and $x_0 = \bar{x}_\mu = \int y\,\mu(dy)$. We recall a more general notion due to Monroe [16]:

Definition 2.1. A stopping time τ is *minimal* if, whenever σ is another stopping time with $B_\sigma \sim B_\tau$ then $\sigma \leq \tau$ $\mathbb{P}$-a.s. implies $\sigma = \tau$ $\mathbb{P}$-a.s..

That is, a stopping time τ is minimal if there is no strictly smaller stopping time which embeds the same distribution. It was additionally shown by Monroe that if the necessary condition described above was true (μ integrable with mean x_0) then minimality of the embedding τ is equivalent to uniform integrability of the stopped process. In the case where the means do not agree, minimality of stopping times was investigated in Cox and Hobson [8] and Cox [6]. Most natural solutions to the Skorokhod Embedding problem can be shown to be minimal.

To motivate some of our later results, we give the following alternative characterisation of minimality for stopping times of Brownian motion. We note that this condition was first introduced by Bertoin and Le Jan [4], as a property of the BLJ embedding. We write L_t^a for the local time at the level a. When $a = 0$, we will often simply write L_t.

Lemma 2.2. *Let μ be an integrable measure, and suppose τ embeds μ in a Brownian motion $(B_t)_{t\geq 0}$ with $B_0 = x_0$. Let $a \in \mathbb{R}$ be fixed. Then τ is minimal if and only if τ minimises $\mathbb{E}[L_\sigma^a]$ over all stopping times σ embedding μ.*

Proof. Without loss of generality, we may assume $a = 0$. Let τ be an embedding of μ, and let τ_N be a localising sequence of stopping times. Since $L_t - |B_t|$ is a local martingale, it follows that

$$\mathbb{E}[L_{\tau_N}] = -|x_0| + \mathbb{E}[|B_{\tau_N}|]$$

and hence

$$\lim_{N\to\infty} \mathbb{E}[L_{\tau_N}] = -|x_0| + \lim_{N\to\infty} \mathbb{E}[|B_{\tau_N}|].$$

Since $|x| = x + 2x_-$ (where $x_- = \max\{0, -x\}$) we can write

$$\lim_{N\to\infty} \mathbb{E}[|B_{\tau_N}|] = \lim_{N\to\infty} \mathbb{E}[B_{\tau_N}] + 2\lim_{N\to\infty} \mathbb{E}[(B_{\tau_N})_-].$$

Since τ_N is a localising sequence, $\lim_{N\to\infty} \mathbb{E}[B_{\tau_N}] = x_0$, while, by Fatou's Lemma, $\lim_{N\to\infty} \mathbb{E}[(B_{\tau_N})_-] \geq \mathbb{E}[(B_\tau)_-]$, and the final term depends only on μ. So $\mathbb{E}[L_\tau] \geq x_0 - |x_0| + 2\mathbb{E}[(B_\tau)_-]$.

This is true for any embedding τ, however in the case where τ is minimal, we observe that $\{(B_{\tau_N})_-\}_{N\in\mathbb{N}}$ is a UI family, by Theorem 5 of [8], and so we have the equality: $\lim_{N\to\infty} \mathbb{E}[(B_{\tau_N})_-] = \mathbb{E}[(B_\tau)_-]$, and hence $\mathbb{E}[L_\tau] = x_0 - |x_0| + 2\mathbb{E}[(B_\tau)_-]$.

For the converse, again using Theorem 3 of [8], we observe that it is sufficient to show that $\{(B_{t\wedge\tau})_-\}_{t\geq 0}$ is a UI family.

Note that for any integrable measure μ a minimal embedding exists, and therefore any embedding τ which minimises $\mathbb{E}[L_\tau]$ over the class of embeddings must have $\lim_{t\to\infty}\mathbb{E}[(B_{\tau\wedge t})_-] = \mathbb{E}[(B_\tau)_-]$. However, suppose for a contradiction that $\{(B_{t\wedge\tau})_-\}_{t\geq 0}$ is not a UI family. Since $(B_{t\wedge\tau})_- \to (B_\tau)_-$ in probability as $t\to\infty$, this implies we cannot have convergence in $\mathcal{L}^1$ (otherwise the sequence would be UI). It follows that

$$\mathbb{E}\left[|(B_{t\wedge\tau})_- - (B_\tau)_-|\right] \to \varepsilon > 0,$$

as $t\to\infty$. But

$$\mathbb{E}\left[|(B_{t\wedge\tau})_- - (B_\tau)_-|\right] \leq \mathbb{E}\left[\left|((B_{t\wedge\tau})_- - (B_\tau)_-)\,\mathbb{1}_{\{\tau\leq t\}}\right|\right] + \mathbb{E}[(B_\tau)_-\,\mathbb{1}_{\{\tau>t\}}] + \mathbb{E}[(B_t)_-\,\mathbb{1}_{\{\tau>t\}}].$$

It follows that $\lim_{t\to\infty}\mathbb{E}[(B_t)_-\,\mathbb{1}_{\{\tau>t\}}] \geq \varepsilon$. But $\mathbb{E}[(B_{t\wedge\tau})_-\,\mathbb{1}_{\{\tau\leq t\}}] \to \mathbb{E}[(B_\tau)_-]$, and hence $\lim_{t\to\infty}\mathbb{E}[(B_{\tau\wedge t})_-] > \mathbb{E}[(B_\tau)_-]$.

□

2.3 Summary of the main result

Given a Brownian motion $B = (B_t)_{t\geq 0}$, Bertoin and Le-Jan [4], show how to construct a family of stopping times $T(\mu) = \{\tau_\kappa; \kappa \geq \kappa_0\}$ such that $B_{\tau_\kappa} \sim \mu$. Moreover, they show τ_{κ_0} is optimal in that it minimises the expected local time of Brownian motion at the starting point over all embeddings of μ in Brownian Motion. It follows from Lemma 2.2 that this is equivalent to minimality of the stopping time.

One of the three proofs given in [7] for the existence of a martingale diffusion with a given distribution at an exponential time establishes a connection between the minimal BLJ embedding and martingale diffusions. Here we are interested not only in the optimal BLJ stopping time τ_{κ_0}, but in the whole family $T(\mu)$ of stopping times. In particular we show how each embedding corresponds to a diffusion with law μ at an exponentially distributed time. Moreover, we show that in general, each minimal BLJ embedding corresponds to a minimal local martingale diffusion. The correspondence is summarized as follows.

Correspondence between diffusions and BLJ embeddings: Given a target law μ with mean $\bar{x}_\mu$, each BLJ embedding τ_κ of μ in Brownian motion corresponds to a local-martingale diffusion X in the sense that $\tau_\kappa = \inf\{u \geq 0 : \lambda\kappa\Gamma_u > L_u\}$, where Γ_u is the quadratic variation process of the time-homogenous diffusion X_t. The diffusion corresponding to τ_κ has Wronskian $W_\lambda = \frac{2}{\kappa}$. Moreover, if a is finite and b is infinite and inaccessible then $X_{t\wedge H_a}$ is a martingale if and only if $W_\lambda = \frac{2}{\kappa_0}$ and $X_0 \leq \bar{x}_\mu$.

3 Diffusions with a given law at an exponential time

Let us begin by recalling the construction of a diffusion's speed measure in terms of its exponential time law in [13].

Given an integrable probability measure μ on I, let $U^\mu(x) = \int_I |x-y|\mu(dy)$, $C^\mu(x) = \int_I (y-x)^+\mu(dy)$ and $P^\mu(x) = \int_I (x-y)^+\mu(dy)$. Let T be an exponentially distributed random variable, independent of B with mean $1/\lambda$. The following theorem summarises the main results in [13].

Theorem 3.1. *Let $X = (X_t)_{t\geq 0}$ be a diffusion in natural scale with Wronskian W_λ. Then $X_T \sim \mu$ if and only if the speed measure of X satisfies*

$$m(dx) = \begin{cases} \frac{1}{2\lambda}\frac{\mu(dx)}{P^\mu(x)-P^\mu(x_0)+1/W_\lambda}, & a < x \leq x_0 \\ \frac{1}{2\lambda}\frac{\mu(dx)}{C^\mu(x)-C^\mu(x_0)+1/W_\lambda}, & x_0 \leq x < b. \end{cases} \tag{3.1}$$

Since m is positive, $1/W_\lambda \geq \max\{C^\mu(x_0), P^\mu(x_0)\}$. Note that $C^\mu(x_0) \geq (\leq)\ P^\mu(x_0)$ if $x_0 \leq\ (\geq)\ \bar{x}_\mu$.

The decomposition of the speed measure in Theorem 3.1 is essentially related to the λ-potential of a diffusion. Recall that for a diffusion X, the λ-potential (also known as the resolvent density) of X is defined as $u_\lambda(x,y) = \mathbb{E}_x\left[\int_0^\infty e^{-\lambda t} dL^y_{A_t}(t)\right]$. This has the natural interpretation that $\lambda u_\lambda(x_0,y) = \mathbb{E}_{x_0}[L^y_{A_T}]$ is the expected local time of X at y up until the exponentially distributed time T.

Corollary 3.2. *The λ-potential of a diffusion X with $X_T \sim \mu$ satisfies*

$$u_\lambda(x_0,y) = \begin{cases} 2(P^\mu(y) - P^\mu(x_0)) + 2/W_\lambda, & a < y \leq x_0 \\ 2(C^\mu(y) - C^\mu(x_0)) + 2/W_\lambda, & x_0 \leq y < b. \end{cases}$$

Moreover, $\varphi(x) = u_\lambda(x_0,x)$ for $x \leq x_0$ and $\phi(x) = u_\lambda(x_0,x)$ for $x \geq x_0$.

Proof. It follows from the calculations in [13] (p. 4) that

$$\mathbb{E}_y[e^{-\lambda H_{x_0}}] = \begin{cases} W_\lambda(P^\mu(y) - P^\mu(x_0)) + 1, & a < y \leq x_0 \\ W_\lambda(C^\mu(y) - C^\mu(x_0)) + 1, & x_0 \leq y < b. \end{cases}$$

Since $u_\lambda(x_0,y) = \frac{2}{W_\lambda}\mathbb{E}_y[e^{-\lambda H_{x_0}}]$ (cf. Theorem 50.7, V.50 in Rogers and Williams [19] for classical diffusions and Itô and McKean [12] for generalised diffusions), the result follows. □

Example 3.3. *Let $\mu = \frac{1}{3}\delta_0 + \frac{1}{3}\delta_{1/2} + \frac{1}{3}\delta_1$. Then $P^\mu(x) = \frac{1}{3}x$ for $0 \leq x \leq \frac{1}{2}$ and $C^\mu(x) = \frac{1}{3} - \frac{1}{3}x$ for $1/2 \leq x \leq 1$. Let $u_\lambda(1/2,x)$ be the λ-resolvent of a diffusion started at $1/2$ such that $X_T \sim \mu$. Then*

$$u_\lambda(1/2,x) = \begin{cases} 2(x/3 - 1/6) + 2/W_\lambda, & 0 \leq x \leq 1/2 \\ 2(1/6 - x/3) + 2/W_\lambda, & 1/2 \leq x \leq 1. \end{cases}$$

where $W_\lambda \leq 6$. The speed measure of the consistent diffusion charges only the points 0, $1/2$ and 1 and is given by

$$2\lambda m(\{x\}) = \begin{cases} \frac{3}{6/W_\lambda - 1}, & x = 0, \\ \frac{W_\lambda}{2}, & x = 1/2, \\ \frac{3}{6/W_\lambda - 1}, & x = 1. \end{cases}$$

Consistent diffusions are birth-death processes in the sense of Feller [10]. $\frac{d\Gamma_t}{dt} > 0$ whenever $B_t \in \{0,1,2\}$ and the process is 'sticky' there. Γ is constant whenever $B_t \notin \{0,1,2\}$, so that A skips over these time intervals and $X_t = B_{A_t}$ spends no time away from these points. Note that if $W_\lambda = 6$ then $m(\{0\}) = m(\{1\}) = \infty$ so that $\Gamma_t = \infty$ for t greater than the first hitting time of 0 or 1 implying that the endpoints are absorbing. If $W_\lambda < 6$, the endpoints are reflecting but sticky.

Let $H_x = \inf\{u \geq 0 : X_u = x\}$ and let $y \in (a, b)$.

Lemma 3.4. *$\int_{a+}(|x|+1)m(dx) = \infty$ (resp. $\int^{b-}(|x|+1)m(dx) = \infty$) if and only if $1/W_\lambda = P^\mu(x_0)$ (resp. $1/W_\lambda = C^\mu(x_0)$).*

Proof. We prove the second statement, the first follows similarly.

Clearly, if $1/W_\lambda > C^\mu(x_0)$, then $\int^{b-} \frac{(1+|x|)\mu(dx)}{C^\mu(x)-C^\mu(x_0)+1/W_\lambda} < \infty$. Conversely, suppose that $1/W_\lambda = C^\mu(x_0)$. Suppose first that $b = \infty$. Then $\lim_{x\uparrow\infty} \phi_\lambda(x) = \lim_{x\uparrow\infty} u_\lambda(x_0, x) = 0$, since $C^\mu(x) \downarrow 0$ as $x \uparrow \infty$. It follows by Theorem 51.2 in [19] (which holds in the generalised diffusion case) that $\int^\infty xm(dx) = \infty$. Now suppose instead that $b < \infty$. Observe that $C^\mu(x)$ is a convex function on I, so it has left and right derivatives, while its second derivative can be interpreted as a measure; moreover, $C^\mu(b) = 0 = (C^\mu)'(b+)$ and $(C^\mu)''(dx) = \mu(dx)$.

From the convexity and other properties of C^μ, we note that $\frac{(x-b)(C^\mu)'(x-)}{C^\mu(x)} \to 1$ as $x \nearrow b$, and also $\frac{C^\mu(x)}{x-b} \geq (C^\mu)'(x-)$ for $x < b$, so $\frac{1}{x-b} \geq \frac{(C^\mu)'(x-)}{C^\mu(x)}$ and $\frac{(C^\mu)'(x-)}{C^\mu(x)} \searrow -\infty$ as $x \nearrow b$. The claim will follow provided we can show $\lim_{v\nearrow b} \int_u^v \frac{(C^\mu)''(dx)}{C^\mu(x)} = \infty$ for $u < b$. Using Fubini/integration by parts, and the fact that $(C^\mu)'(x)$ is increasing and negative, we see that:

$$\begin{aligned}
\int_u^{v-} \frac{(C^\mu)''(dx)}{C^\mu(x)} &= \frac{(C^\mu)'(v-)}{C^\mu(v)} - \frac{(C^\mu)'(u-)}{C^\mu(u)} + \int_u^v \left(\frac{(C^\mu)'(x-)}{C^\mu(x)}\right)^2 dx \\
&\geq \frac{(C^\mu)'(v-)}{C^\mu(v)} - \frac{(C^\mu)'(u-)}{C^\mu(u)} + (C^\mu)'(v-)\int_u^v \frac{(C^\mu)'(x-)}{C^\mu(x)^2}\, dx \\
&\geq \frac{(C^\mu)'(v-)}{C^\mu(v)} - \frac{(C^\mu)'(u-)}{C^\mu(u)} + (C^\mu)'(v-)\left(\frac{1}{C^\mu(u)} - \frac{1}{C^\mu(v)}\right) \\
&\geq \frac{(C^\mu)'(v-) - (C^\mu)'(u-)}{C^\mu(u)}.
\end{aligned}$$

Letting first $v \to b$, the right hand side is equal (in the limit) to $-\frac{(C^\mu)'(u-)}{C^\mu(u)}$, but we observed above that this is unbounded as $u \nearrow b$, and since the whole expression is increasing in u, it must be infinite, as required. $\square$

As Lemma 3.4 demonstrates, the behaviour of X at the boundaries is determined by the value of W_λ. When I is unbounded, the boundary behaviour determines whether or not X is a martingale diffusion. Suppose that a is finite, $b = \infty$. Then Kotani [14] (see also Delbaen and Shirakawa ([9])) show that $X_{t\wedge H_a \wedge H_b}$ is a martingale if and only if $\int^{\infty-} xm(dx) = \infty$ (i.e. ∞ is not an entrance boundary). By Lemma 3.4 this is equivalent to the conditions $x_0 \leq \bar{x}_\mu$ and $1/W_\lambda = C^\mu(x_0)$ being satisfied. An analogous observation holds when b is finite and a is infinite.

Theorem 3.1 and Lemma 3.4 provide a natural way of determining boundary properties by inspection of the decomposition of the speed measure in terms of $\mathbb{P}(X_T \in dx)$. Furthermore, the decomposition gives us a canonical way of constructing strict local martingales with a given law at a random time. For instance if b is infinite, we can generate strict local martingale diffusions by choosing a measure μ and setting $W_\lambda < 1/C^\mu(x_0)$.

Example 3.5. *Let $m(dx) = \frac{dx}{x^4}$ and $I = (0, \infty)$. Suppose $X_0 = 1$ and $\lambda = 2$. Then $\phi(x) = \frac{x \sinh(\frac{1}{x})}{\sinh(1)}$ and $\varphi(x) = xe^{1-1/x}$ are respectively the strictly decreasing and strictly increasing eigenfunctions of the inverse Bessel process of dimension three, X, which solves the equation*

$$\frac{1}{2} \frac{d^2}{dmdx} f = 2f.$$

We calculate (cf. Equations (3.2) and (3.3) in [13]),

$$\mu(dx) = \mathbb{P}(X_T \in dx) = \begin{cases} \frac{\sinh(1)}{2x^3} e^{-1/x}\, dx & 0 < x \leq 1, \\ \frac{e^{-1}}{2x^3} \sinh(1)\, dx & 1 \leq x. \end{cases} \tag{3.2}$$

We find $\bar{x}_\mu = 1 - 1/e$. Further, we calculate $P^\mu(x) = \frac{1}{2} \sinh(1) x e^{-1/x}$ for $x \leq 1$ and $C^\mu(x) = e^{-1}(x \sinh(1/x) - 1)$ for $x \geq 1$. Thus

$$m(dx) = dx/x^4 = \begin{cases} \frac{\mu(dx)}{P^\mu(x)} & 0 < x \leq 1, \\ \frac{\mu(dx)}{C^\mu(x) - (2C^\mu(1) + 2P^\mu(1))} & 1 \leq x. \end{cases} \tag{3.3}$$

It follows that X is a strict local-martingale diffusion.

In the example above, the strict local martingale property of the process follows from the fact that C^μ is shifted up in the decomposition of the speed measure (3.1). In general, we expect reflection at the left boundary $X = a$, when we have had to shift the function P^μ up, and we expect the process stopped at $t = H_a \wedge H_b$ to be a strict local martingale when we shift C^μ up.

Example 3.6. *Let us reconsider the diffusion in Example 3.5. Let us construct a diffusion Y with the same law as the inverse Bessel process of dimension three at an exponential time, such that $Y_{t \wedge H_0}$ is a martingale (and therefore, such that $Y_0 = \bar{x}_\mu$). As we are shifting the starting point, we must adjust the speed measure between $\bar{x}_\mu$ and the starting point of the diffusion in Example 3.5. By Lemma 3.4 we know that the speed measure of the martingale diffusion must satisfy*

$$m(dx) = \begin{cases} \frac{\mu(dx)}{P^\mu(x)} & 0 < x \leq \bar{x}_\mu, \\ \frac{\mu(dx)}{C^\mu(x)} & \bar{x}_\mu \leq x, \end{cases}$$

Recall that $\bar{x}_\mu = 1 - 1/e$. We calculate

$$m(dx) = \begin{cases} \frac{1}{x^4}\, dx & 0 < x \leq \bar{x}_\mu, \\ \frac{(e - e^{-1}) e^{-1/x}\, dx}{x^3((e - e^{-1}) x e^{-1/x} + 2(\bar{x}_\mu - x))} & \bar{x}_\mu < x \leq 1, \\ \frac{\sinh(1/x)\, dx}{x^3(x \sinh(1/x) - 1)} & 1 \leq x. \end{cases}$$

4 The BLJ embedding for Brownian motion

The BLJ construction is remarkably general and can be used to embed distributions in general Hunt processes. Our interest, however, lies in the specific case of embedding a law μ in Brownian Motion. In this setting, the rich structure of the embedding translates into a one-to-one correspondence to the family of diffusions with a given law at an exponential time.

Suppose that $x_0 \in [a,b]$, $\mu(\{x_0\}) = 0$ and define

$$V_\mu^{x_0}(x) = \int \mathbb{E}_y[L^x_{H_{x_0}}]\mu(dy). \tag{4.1}$$

We will assume from now on that μ has a finite first moment and that $\mu(\{x_0\}) = 0$. Then $\kappa_0 = \sup\{V_\mu(x); x \in [a,b]\} < \infty$ (see, for instance, p. 547 in [4]). By the Corollary on p. 540 in [4] it follows that for each $\kappa \geq \kappa_0$ the stopping time

$$\tau_\kappa = \inf\left\{t > 0 : \kappa \int \frac{L^x_t \mu(dx)}{\kappa - V_\mu^{x_0}(x)} > L^{x_0}_t\right\} \tag{4.2}$$

embeds μ in B^{x_0}, i.e. $B^{x_0}_{\tau_\kappa} \sim \mu$. Moreover, we have $\mathbb{E}_{x_0}[L^{x_0}_{\tau_\kappa}] = \kappa$. Finally, the stopping time τ_{κ_0} is optimal in the following sense; if σ is another embedding of μ in B_{x_0}, then for every $x \in I$, $\mathbb{E}_{x_0}[L^x_\sigma] \geq \mathbb{E}_{x_0}[L^x_{\tau_{\kappa_0}}]$. From Lemma 2.2, it follows that the stopping time τ_{κ_0} is minimal in the sense of Definition 2.1.

5 Embeddings and diffusions

Our purpose now is to show that each BLJ embedding τ_κ of μ corresponds to a local-martingale diffusion X such that $X_T \sim \mu$, for some independent exponentially distributed random variable T. The rate of the exponential time plays no significant role in the correspondence, being merely a scaling factor.

We will establish the connection between BLJ embeddings and diffusions by relating the λ-potential of diffusions, the potential U^μ of μ and $V^{x_0}_\mu$. A number of solutions to the Skorokhod Embedding Problem, most notably the solutions of Chacon-Walsh [5], Azéma-Yor ([3], [2]) and Perkins [18] can be derived directly from quantities related to the potential U^μ.

Lemma 5.1.

$$V_\mu^{x_0}(x) = \begin{cases} 2(P^\mu(x_0) - P^\mu(x)) & x \leq x_0 \\ 2(C^\mu(x_0) - C^\mu(x)) & x > x_0. \end{cases} \tag{5.1}$$

Proof. Observe that $\mathbb{E}_y[L^x_{H_{x_0}}] = |y - x_0| + |x_0 - x| - |y - x|$ is simply the potential kernel for B^{x_0} killed at x_0. Integrating we find

$$V_\mu^{x_0}(x) = \int (|y - x_0| + |x_0 - x| - |y - x|)\mu(dy) = U^\mu(x_0) - U^\mu(x) + |x_0 - x|.$$

The result follows by re-arrangement. □

Recalling the definition of κ_0 in the BLJ embedding, we now observe that $\kappa_0 = \max\{2C^\mu(x_0), 2P^\mu(x_0)\}$, which is the smallest constant c such that $c - V^{x_0}_\mu(x) \geq 0$ for all x. The situation is illustrated in Figure 1 below.

The following Lemma follows from the Corollary in [4], page 540.

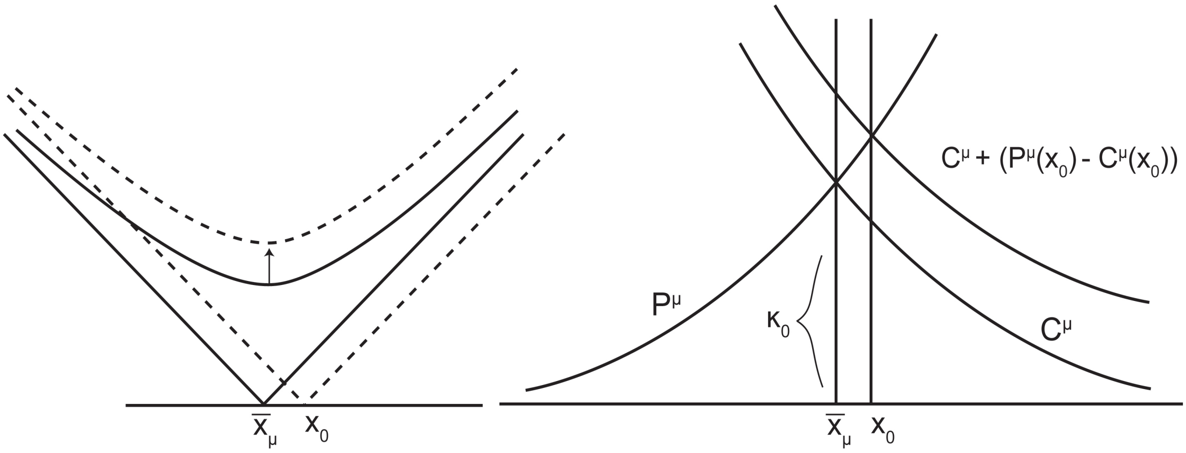

Figure 1: If $x_0 = \bar{x}_\mu$, then $\kappa_0 - V_\mu^{x_0}(x) = U^\mu(x) - |x - \bar{x}_\mu|$. If $x_0 \neq \bar{x}_\mu$, then the potential must be shifted upwards to lie above $|x_0 - x|$ everywhere. Thus, if $x_0 < \bar{x}_\mu$, $\kappa_0 - V_\mu^{x_0}(x) = U^\mu(x) + (2C^\mu(x_0) - U^\mu(x_0)) - |x_0 - x|$, while if $x_0 > \bar{x}_\mu$, $\kappa_0 - V_\mu^{x_0}(x) = U^\mu(x) + (2P^\mu(x_0) - U^\mu(x_0)) - |x_0 - x|$ (see left picture). Note that if $x \geq x_0 > \bar{x}_\mu$, $\frac{1}{2}(\kappa_0 - V_\mu^{x_0}(x)) = P^\mu(x_0) - C^\mu(x_0) + C^\mu(x)$ (see right figure).

Lemma 5.2.

$$\mathbb{E}_{x_0}[L^{x_0}_{\tau_{\kappa_0}}] = \begin{cases} 2C^\mu(x_0) & x_0 \leq \bar{x}_\mu \\ 2P^\mu(x_0) & x_0 \geq \bar{x}_\mu. \end{cases} \tag{5.2}$$

Proof. By the Corollary in [4], $\kappa_0 = \mathbb{E}_{x_0}[L^{x_0}_{\tau_{\kappa_0}}]$. The result now follows from the fact that $\kappa_0 = \max\{2C^\mu(x_0), 2P^\mu(x_0)\}$. □

In fact, the Corollary in [4] implies the more general formula $\mathbb{E}_{x_0}[L^{x}_{\tau_{\kappa_0}}] = \kappa_0 - V_\mu^{x_0}(x)$. Pictorially, the distance between the shifted potential and potential kernel is the expected local time at x of Brownian motion started at x_0 until the BLJ stopping time.

So far we have focused on the minimal embedding τ_{κ_0} for μ. However, the picture for the suboptimal embeddings τ_κ, $\kappa > \kappa_0$ follows immediately from the fact that these embeddings correspond to shifting the potential picture $\kappa_0 - V_\mu^{x_0}(x)$ further upwards by $\kappa - \kappa_0$. Thus we have for instance $\mathbb{E}_{x_0}[L^x_{\tau_\kappa}] = \mathbb{E}_{x_0}[L^x_{\tau_{\kappa_0}}] + (\kappa - \kappa_0)$.

Let us summarise the correspondence between diffusions and BLJ embeddings. Fix $x_0 \in \mathrm{supp}(\mu)$ and let $\kappa \geq \max\{2C^\mu(x_0), 2P^\mu(x_0)\}$. Let τ_κ be a BLJ embedding of μ in a Brownian motion B^{x_0}, such that $\mathbb{E}_{x_0}[L^{x_0}_{\tau_\kappa}] = \kappa$. Let L^x_u be the local time of B^{x_0} at x up until time u.

Proposition 5.3. *Fix $\lambda > 0$ and define a Borel measure m via $m(dx) = \frac{1}{\lambda}\frac{\mu(dx)}{\kappa - V_\mu^{x_0}(x)}$. Let X be the diffusion with speed measure m, defined via $X_t = B^{x_0}_{A_t}$, where A is the right-continuous inverse of the functional $\Gamma_u = \int_{\mathbb{R}} L^x_u m(dx)$. Then $\tau_\kappa = \inf\{u \geq 0 : \lambda\kappa\Gamma_u > L^{x_0}_u\}$. X has Wronskian $W_\lambda = \frac{2}{\kappa}$, λ-potential $u_\lambda(x_0, y) = \kappa - V_\mu^{x_0}(y)$ and $X_T \sim \mu$.*

Proof. Since μ has a finite first moment, $\kappa_0 < \infty$ and Hypothesis 1 in [4] is satisfied. Thus by the Corollary in [4], $\tau_\kappa = \inf\left\{u > 0 : \kappa \int \frac{L^x_u \mu(dx)}{\kappa - V_\mu^{x_0}(x)} > L^{x_0}_u\right\} = \inf\{u \geq 0 : \lambda\kappa\Gamma_u > L^{x_0}_u\}$ embeds μ in B^{x_0}. It follows from Lemma 5.1 and Corollary 3.2 that $u_\lambda(x_0, x) = \kappa - V_\mu^{x_0}(x)$ and that $\mathbb{E}_{x_0}[L^{x_0}_{A_T}] = \frac{2}{W_\lambda} = \kappa$. By Theorem 3.1, $X_T \sim \mu$. □

Remark 5.4. The assumption $\mu(\{x_0\}) = 0$ made in Section 4 is necessary in the construction of the BLJ embedding in [4], but not in the construction of diffusions with a given exponential time law in [13]. To make the correspondence independent of this assumption on the target law, a modified version of the BLJ embedding with external randomisation can be constructed.

6 Minimal diffusions

The purpose of this final section is to define a notion of minimality for diffusions which will be analogous to the notion of minimality for the BLJ Skorokhod embeddings.

Suppose we have fixed a starting point x_0 and a law μ and we are faced with a (Skorokhod embedding) problem of finding a stopping time τ, such that $B_\tau \sim \mu$, where B is a standard Brownian motion started at x_0. Then (and especially when we are interested in questions of optimality) it is most natural to search for stopping times τ which are minimal.

In the family of BLJ embeddings, the embedding τ_{κ_0} is minimal and the notion of minimal embedding carries over into a notion of minimality for diffusions in natural scale.

Definition 6.1. We say that a time homogeneous diffusion X in natural scale and started at x_0 is a λ-minimal diffusion if, whenever Y is another time-homogeneous diffusion in natural scale, with $X_T \sim Y_T$ where T an independent exponential random variable with mean $\frac{1}{\lambda}$, then $\mathbb{E}_{x_0}[L_T^{x_0}(X)] \leq \mathbb{E}_{x_0}[L_T^{x_0}(Y)]$.

We say that X is a minimal diffusion in natural scale if X is λ-minimal for all $\lambda > 0$.

By Proposition 5.3, the λ-minimal local-martingale diffusion corresponds to the minimal BLJ embedding τ_{κ_0}. As an analogue of the notion of minimality for the BLJ solution to the Skorokhod Embedding Problem, minimality is a natural probabilistic property. A minimal diffusion has the lowest value of $\mathbb{E}_{x_0}[L_T^y]$ for all $y > 0$ among the diffusions with a given law μ at time T started at x_0 (this follows from Theorem 3.1). In terms of diffusion dynamics, the minimal diffusion moves around the state space slower than all other diffusions with the same exponential time law.

Minimality is a necessary condition for $X_{t\wedge H_a \wedge H_b}$ to be a martingale. For instance, if a is finite and b is infinite, then $X_{t\wedge H_a \wedge H_b}$ is a martingale if and only if X is minimal and $x_0 \leq \bar{x}_\mu$, i.e. b is not an entrance boundary. When I is a finite interval, minimality is a more natural property than the martingale property; every (stopped) diffusion on a finite interval is a martingale diffusion, but there is only one minimal diffusion for every exponential time law.

Note also that these definitions extend in an obvious way to diffusions which are natural scale: a diffusion which is not in natural scale is minimal if and only if it is minimal when it is mapped into natural scale. (See Example 6.3.)

We collect these observations in the following result:

Theorem 6.2. *Let X be a time-homogenous diffusion in natural scale. Then the following are equivalent:*

1. *X is λ-minimal;*

2. *if $X_T \sim \mu$ and W_λ is the Wronskian of X, then:*

$$1/W_\lambda = \max\{C^\mu(x_0), P^\mu(x_0)\};$$

3. *X has at most one entrance boundary;*

4. *X is a minimal diffusion.*

Proof. The equivalence of the first two statements follows from Corollary 3.2. Next, if X is minimal then $\mathbb{E}_a[e^{-\lambda H_{x_0}}] = 0$ or $\mathbb{E}_b[e^{-\lambda H_{x_0}}] = 0$, and hence X has at most one entrance boundary. Finally, the properties of the boundary points are independent of the choice of λ, so if X has at most one entrance boundary, then X is minimal. □

Example 6.3. *A natural class of non-minimal diffusions is the following class of Jacobi diffusions in natural scale. On the domain $[0, b]$ let*

$$dX_t = (\alpha - \beta X_t)dt + \sigma\sqrt{X_t(b - X_t)}, \quad X_0 \in (0, b),$$

where $\frac{2\beta}{\sigma^2} - \frac{2\alpha}{\sigma^2 b} - 1 > 0$ and $\frac{2\alpha}{\sigma^2 b} - 1 > 0$. The eigenfunctions φ_λ and ϕ_λ can be calculated explicitly as hypergeometric functions, see Albanese and Kuznetsov [1]. It is shown in [1] that $\lim_{x\downarrow 0} \varphi_\lambda(x) > 0$ and $\lim_{x\uparrow 0} \phi_\lambda(x) > 0$. Let s be the scale function of X and consider $Y = s(X)$ with eigenfunctions $\overline{\varphi}_\lambda$ and $\overline{\phi}_\lambda$. Then $\lim_{x\downarrow s(0)} \overline{\varphi}_\lambda(x) = \lim_{x\downarrow 0} \varphi_\lambda(s^{-1}(x)) > 0$. Similarly $\lim_{x\uparrow s(b)} \overline{\phi}(x) > 0$. Hence Y is non-minimal.

Example 6.4. *Let $I = (0, 1)$ and let $X = (X_t)_{t\geq 0}$ be a diffusion with $X_0 = 1/2$ and speed measure $m(dx) = \frac{dx}{x^2(1-x^2)}$ (sometimes known as the Kimura martingale cf. [11]). The increasing and decreasing eigenfunctions are $\varphi(x) = \frac{2x^2}{1-x}$ and $\phi(x) = \frac{2(1-x)^2}{x}$ respectively. Note that both boundaries are natural, so this diffusion is minimal.*

Non-minimal diffusions with the same exponential time law have increasing/decreasing eigenfunctions $\varphi_\delta(x) = \varphi(x) + \delta$ and $\phi_\delta(x) = \phi(x) + \delta$, for some $\delta > 0$, and with reflection at the endpoints. Accordingly, the consistent speed measures indexed by δ are given by $m_\delta(dx) = \frac{dx}{\sigma_\delta^2(x)}$, where

$$\sigma_\delta^2(x) = \begin{cases} \left(\frac{\delta}{2}(1-x) + x^2\right)(1-x)^2 & x < \frac{1}{2} \\ \left(\frac{\delta}{2}x + (1-x)^2\right)x^2 & x \geq \frac{1}{2}. \end{cases}$$

Example 6.5. *Consider Brownian motion on $[0, 2]$ with instantaneous reflection at the endpoints and initial value $B_0 = x_0 \in [1, 2)$. Let $\lambda = 1/2$, then the 1/2-eigenfunctions solve $\frac{d^2 f}{dx^2} = f$, so the increasing eigenfunction (up to a constant) is $\varphi(x) = \frac{\cosh(x)}{\cosh(x_0)}$ and $\phi(x) = \frac{\cosh(2-x)}{\cosh(2-x_0)}$. Note that this diffusion is non-minimal. To construct the minimal diffusion with the same marginal law at an exponential time, let $\eta = \cosh(x_0)^{-1}$ and set $\varphi_\eta(x) = \varphi(x) - \eta = \frac{\cosh(x)-1}{\cosh x_0}$, and similarly $\phi_\eta(x) = \phi(x) - \eta = \frac{\cosh(2-x)}{\cosh(2-x_0)} - \frac{1}{\cosh(x_0)}$. The diffusion co-efficient of the minimal diffusion is given by $\sigma^2(x) = \frac{\varphi_\eta(x)}{\varphi_\eta''(x)} = 1 - \frac{1}{\cosh(x)}$ for $x \in [0, x_0]$ and by $\sigma^2(x) = \frac{\phi_\eta(x)}{\phi_\eta''(x)} = 1 - \frac{\cosh(2-x_0)}{\cosh(2-x)\cosh(x_0)}$ for $x \in [x_0, 1]$.*

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