Exercise Sheet 1— Brownian motion and Quadratic Variation

1.1: Show that if $(W_t)_{t\geq 0}$ is a standard Brownian motion, then $\mathbb{C}ov(W_s, W_t) = \min\{s,t\}$.

1.2: Let $(W_t)_{t\geq 0}$ be a standard Brownian motion. For a sufficiently differentiable, bounded function f, we write

$$P_t f(x) = \mathbb{E}\left[f(W_{t+s})|W_s = x\right]$$

and note that this is well defined (independent of s) by the Markov property. By using a Taylor expansion of f, show that

$$\lim_{h\searrow 0} \frac{P_h f(x) - f(x)}{h} = \frac{1}{2} f''(x).$$

1.3: Fix $x, y \in \mathbb{R}$. Let $(X_t)_{t\in[0,1]}$ be the $\mathbb{R}$-valued process with finite dimensional distributions given by:

$$\mathbb{E}\left[f(X_{t_1}, X_{t_2}, \ldots, X_{t_n})\right] = \int f(x_1, \ldots, x_n) \frac{p(t_1, x, x_1)\prod_{i=2}^n p(t_i - t_{i-1}, x_{i-1}, x_i) p(1 - t_n, x_n, y)}{p(1, x, y)} \,\mathrm{d}x_1 \cdots \mathrm{d}x_n$$

for $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < 1$. Here $p(t, x, y) = \frac{\mathrm{e}^{-(x-y)^2/2t}}{\sqrt{2\pi t}}$.

(a) Show that there is a modification of this process which is continuous.

(b) Show that this is a Markov process, i.e.

$$\mathbb{E}\left[f(X_{t_n})|X_{t_1}, \ldots, X_{t_{n-1}}\right] = \mathbb{E}\left[f(X_{t_n})|X_{t_{n-1}}\right].$$

(c) Let f be a sufficiently smooth, bounded function. Find an expression involving f', f'' for

$$\lim_{h\searrow 0} \frac{\mathbb{E}\left[f(X_{t+h})|X_t\right] - f(X_t)}{h}$$

for $t \in [0, 1)$.

(d) Let W_t be a standard Brownian motion with $W_0 = 0$. Show that the process X'_t defined by:

$$X'_t := x + W_t + t\left((y - x) - W_1\right)$$

is the same process as the continuous modification of X_t.

1.4: Find (by trial and error) functions $\sigma : (0, 1) \to \mathbb{R}$ and $\mu : (0, 1) \to \mathbb{R}$ such that the numerical solutions X_t to equation (1) given in the lectures satisfy:

(a) for $X_0 \in (0, 1)$, $X_t \in (0, 1)$ for all $t \geq 0$;

(b) as $t \to \infty$, $X_t \to 0$ or $X_t \to 1$.

1.5: Let W_t be a standard Brownian motion with $W_0 = 0$.

(a) Using *iii)* of Theorem 2.2 show that

$$\mathbb{P}\left(\frac{|W_t|}{t} > \alpha C, \text{ some } t \in \left[0, \frac{1}{\alpha^2}\right]\right)$$

is independent of α.

(b) Hence show that

$$\mathbb{P}\left(\frac{|W_t|}{t} > \alpha, \text{ some } t \in \left[0, \frac{1}{\alpha^4}\right]\right) \to 1$$

as $\alpha \to \infty$, and deduce that almost surely Brownian motion is not differentiable at 0.

(c) Hence deduce that, for fixed $t > 0$, Brownian motion is almost surely not differentiable at t.

1.6: **[Hard]** Our aim in this question is to show that the Brownian path on $[0,1]$ is almost surely α-Hölder continuous for $\alpha < \frac{1}{2}$, that is:

$$\sup\left\{\frac{|W_t - W_s|}{|t-s|^\alpha} : t, s \in [0,1], t \neq s\right\} < \infty.$$

To do this, we first prove: suppose $(X_t)_{t\in[0,1]}$ is a continuous process and there exist strictly positive constants γ, c, ε such that

$$\mathbb{E}\left[|X_t - X_s|^\gamma\right] \leq c|t-s|^{1+\varepsilon}$$

then we aim to show:

$$\mathbb{E}\left[\left(\sup_{s\neq t}\left(\frac{|X_t - X_s|}{|t-s|^\alpha}\right)\right)^\gamma\right] < \infty,$$

for all $\alpha \in [0, \varepsilon/\gamma)$.

(a) For $m \in \mathbb{N}$, set $D_m := \{i2^{-m}; i = 0, 1, 2, \ldots, 2^m - 1\}$, and let Δ_m be the pairs (s,t) such that $s, t \in D_m$ and $|s-t| = 2^{-m}$. Write $K_i := \sup_{(s,t)\in\Delta_i} |X_s - X_t|$, and show that

$$\mathbb{E}\left[K_i^\gamma\right] \leq \tilde{c}2^{-i\varepsilon}$$

for some constant $\tilde{c}$.

(b) Write $D = \cup_m D_m$, and show that if $s, t \in D$ and $|s-t| \leq 2^{-m}$ then

$$|X_s - X_t| \leq 2\sum_{i=m}^\infty K_i.$$

(c) Define

$$M_\alpha := \sup\left\{\frac{|X_t - X_s|}{|t-s|^\alpha} : s, t \in D, s \neq t\right\},$$

and show that $M_\alpha \leq 2^{\alpha+1}\sum_{i=0}^\infty 2^{i\alpha}K_i$, and hence that

$$\mathbb{E}\left[(M_\alpha)^\gamma\right] < \infty.$$

Deduce (hint: use Fatou's Lemma) that X_t is α-Hölder continuous, and hence show that Brownian motion W_t is α-Hölder continuous, for any $\alpha < 1/2$.