Model-Independent Arbitrage Bounds on American Put Options

submitted by

Christoph Hoeggerl

for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

December 2014

Für meine Eltern

SUMMARY

The standard approach to pricing financial derivatives is to determine the discounted, risk-neutral expected payoff under a model. This model-based approach leaves us prone to model risk, as no model can fully capture the complex behaviour of asset prices in the real world.

Alternatively, we could use the prices of some liquidly traded options to deduce no-arbitrage conditions on the contingent claim in question. Since the reference prices are taken from the market, we are not required to postulate a model and thus the conditions found have to hold under any model.

In this thesis we are interested in the pricing of American put options using the latter approach. To this end, we will assume that European options on the same underlying and with the same maturity are liquidly traded in the market. We can then use the market information incorporated into these prices to derive a set of no-arbitrage conditions that are valid under any model. Furthermore, we will show that in a market trading only finitely many American and co-terminal European options it is always possible to decide whether or not the prices are consistent with a model. If they are not there has to exist arbitrage in the market.

ACKNOWLEDGEMENTS

First and foremost, I want to thank my supervisor Alexander M. G. Cox for his guidance and support throughout the years. I am very grateful for his interest in my research and for always finding time in his increasingly busy schedule.

I would also like to express my gratitude to the members of the ProbL@b for providing such a friendly and stimulating working environment. Both the weekly seminars and the reading groups have allowed me to gain a more in-depth understanding of various topics in probability.

Further I would like to thank all the people that have made these last few years so enjoyable. In particular, I want to thank Marion, Deepak, Vaios, Curdin and Melina for the odd beer and for having always been there when I needed them. My office mates Yi and Matteo for the great chats as well as Kuntalee, Maren and Horatio for the exciting games of squash and badminton.

A very special thank you to Fani for her love and devotion and for cheering me up when I needed it the most.

Finally, my parents and my sister who have always supported and encouraged me — thank you

Contents

List of Figures . . . iii
List of Algorithms . . . iii

1 Introduction **2**
1.1 Robust pricing of derivatives . . . 3
1.2 Outline . . . 5
1.3 Preliminaries . . . 6
1.3.1 The potential picture . . . 7
1.3.2 The Chacon-Walsh solution to the Skorokhod embedding problem 7
1.3.3 Connecting the potential of a measure to option prices . . . 9
1.3.4 Arbitrage in the model-free setting . . . 11

2 Model-independent no-arbitrage conditions on American put options **12**
2.1 Introduction . . . 12
2.2 Necessary conditions for the American put price function A . . . 15
2.3 Sufficiency of the conditions on the American put price function A . . . 21
2.3.1 Algorithm . . . 24
2.3.2 Existence and calculation of the critical time . . . 24
2.3.3 The splitting procedure . . . 31
2.3.4 Convergence of the Algorithm . . . 37
2.4 Conclusion . . . 41
2.5 Appendix . . . 42

3 Arbitrage situations in markets trading American and co-terminal European options **46**
3.1 Problem setting . . . 46
3.2 Setup . . . 49
3.3 No-arbitrage bounds on option prices . . . 51
3.4 Construction of the price functions . . . 54
3.4.1 Computation of the prices for strikes in $\mathbb{K}(\mathcal{P}_0^*)$ . . . 54
3.4.2 Price-misspecifications and their corrections . . . 55
3.4.3 Expansion of the initial set of prices . . . 58

3.4.4 Computation of the prices for strikes in $\mathbb{K}(\mathcal{P}_i^*)$ 58
3.4.5 Modification to the price corrections 59
3.5 Algorithm 61
3.6 Arbitrage situations 64
3.6.1 Sub- and super-replicating strategies 64
3.6.2 Situation I: Violation of the no-arbitrage conditions by the prices of traded options 70
3.6.3 Situation II: Violation of $\mathbf{e}_{i,p} \geq E_{lb}^{rhs}(K_{i,p}, \mathcal{P}_0^*)$ 74
3.6.4 Situation III: Violation of $A_{ub}(K_{i,p}, \mathcal{P}_{i,p-1}) \geq \mathbf{a}_{i,p}$ 81
3.7 Admissibility of the price functions 86
3.7.1 Violation of the upper bound under $\mathcal{P}_1^*$ 88
3.7.2 Violation of convexity under $\mathcal{P}_1^*$ 95
3.7.3 Violations under $\mathcal{P}_i^*$, $i \geq 2$ 114
3.8 Convergence of the algorithm 117
3.9 Conclusion 120
3.10 Appendix 121
3.10.1 Properties of the Legendre-Fenchel condition 121
3.10.2 General properties of the price functions 123
3.10.3 General properties of the revised price functions 131
3.10.4 Properties of the price functions when a violation of $\overline{A} \geq A$ occurs 133
3.10.5 Properties of the price functions under $\mathcal{P}_1^*$ when a violation of $\overline{A} \geq A$ occurs 137
3.10.6 Properties of the price functions under $\mathcal{P}_1^*$ when a violation of convexity occurs 147
3.10.7 Miscellaneous 162
3.11 Flowchart 167

Bibliography **172**

List of Figures

1-1 Potential picture of $U\delta_m, U\mu_1$ and $U\mu$. 8

1-2 Potential picture of $U\delta_m, U\mu_2$ and $U\mu$. 9

1-3 Call price picture with no-arbitrage bound $(S_0 - Ke^{-rT})_+$. 10

2-1 An example of feasible price functions 23

2-2 Extension of the American price function 26

2-3 The critical time for the next embedding 28

3-1 Payoffs of the super-replicating strategy $P_1^E(K_2; K_1, K_3)$ and the European put option with strike K_2. 65

3-2 Comparison of the payoff of a European option with the respective sub-replicating portfolios $P_2^E(K_3; K_1, K_2)$ and $P_3^E(K_1; K_2, K_3)$. 67

3-3 Flowchart for Algorithm 2 168

3-4 Part 1 of the flowchart for Algorithm 3 169

3-5 Part 2 of the flowchart for Algorithm 3 170

3-6 Flowchart for Algorithm 4 171

List of Algorithms

1 Embedding algorithm 24
2 Option pricing algorithm 61
3 Correction of $A > A_{ub}$ or $E < E_{lb}^{rhs}$ 62
4 Correction of $A > \overline{A}$ 63

Chapter 1

Introduction

Financial markets allow individuals or entities to raise capital, mitigate risk or speculate. One way of transferring risk is to purchase an option. Options are financial derivatives which means that their value depends on the price of an underlying such as a stock or an index. The holder of an option is protected from disadvantageous developments in the price of the underlying asset, as an option gives the holder the right to either buy or sell the underlying at a certain date for a pre-specified price. Options that permit the holder to buy the underlying are termed call options, whereas options that allow the owner to sell the underlying are referred to as put options. Moreover, we have to distinguish between American and European-style options. American options can be exercised at any time up to expiration. European options, in contrast, only at the expiration date.

Although options have been traded over-the-counter for many centuries, the mathematical theory behind the pricing of options was not developed before the 20th century. In his dissertation Bachelier [1900] first derived a pricing formula for European options. The model he used was based on the assumption that the underlying was driven by a Brownian motion with zero drift. His work, however, was largely ignored until its rediscovery in the late 1950s.

A major problem of a model driven by Brownian motion is that the value of the underlying has a positive probability of being negative. To resolve this issue Samuelson [1965] suggested the use of geometric Brownian motion instead of ordinary Brownian motion; that is, he assumed that the underlying price process is given by

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}$$

where S_0 is the current price of the underlying, μ the drift, σ the volatility and W_t a Brownian motion. In this setting Black and Scholes [1973] constructed a dynamic and self-financing trading strategy to hedge financial derivatives and deduced that the initial cost of the hedging portfolio had to be the no-arbitrage price. Despite the fact that some of the modelling assumptions (e.g. the ability to trade continuously and without transaction costs) clearly do not apply to real world markets, the model has been very successful as it provides simple, explicit pricing formulae for many financial derivatives. Extending the ideas of Black and Scholes, Harrison and Kreps [1979] and Harrison and Pliska [1981] showed that the price for any contingent claim Φ can be determined as the discounted expected payoff under the (risk-neutral) equivalent martingale measure, that is

$$V(x) = \mathbb{E}_x[e^{-rt}\Phi(S_t)].$$

Compared to European options it is much harder to find a fair price for American options as the payoff of the option is path-dependent. An exception is the American call option on a non-dividend paying asset for which early exercise is never optimal as demonstrated for example in Björk [2009, p.111-112], implying that its price has to equal the price of the corresponding European option. For the American put option this is not the case and we are required to solve the optimal stopping problem

$$V(x) = \sup_{\tau} \mathbb{E}_x\left[e^{-r\tau}(K - S_\tau)_+\right].$$

In the Black-Scholes model, an explicit solution to this problem for American put options with infinite horizon can be derived, see for example Peskir and Shiryaev [2006, p.377]. If the horizon is finite no closed-form solution is available and we have to resort to numerical methods to find the price.

1.1 Robust pricing of derivatives

A major flaw of the model-based approach is that it exposes us to model risk; that is, the risk that the model in use is not able to capture the real world behaviour of the underlying correctly.

An alternative approach to the pricing of financial derivatives is to identify models consistent with a set of observed prices. Since it is hard to determine the entire set of models, one generally has to be content with finding extremal models. These can then be used to provide upper and lower bounds on the prices of more exotic derivatives. Moreover, we can ask if there exists arbitrage in case the prices are not consistent with any model.

According to the work of Breeden and Litzenberger [1978] the marginal distribution of the underlying at time T can be deduced from the prices of European call (or put) options with maturity T (see Section 1.3 for details). Note further that unlike in the model-based approach a change of measure is not required to price derivatives, as the marginal distribution obtained is already given under the measure used by the market for pricing. If we now denote the call prices as a function of the strike by E_c, then Davis and Hobson [2007] showed that E_c has to satisfy the following conditions to guarantee absence of arbitrage: E_c is non-negative, decreasing and convex, $E_c(0) = S_0$, $E_c'(0+) \geq -1$ and $\lim_{K\to\infty} E_c(K) = 0$, where S_0 is the current price of the underlying.

Without any assumptions on the underlying model, we are able to determine the prices for derivatives depending only on the marginal distribution at time T. For path-dependent options the law, inferred by the call prices for a single maturity T, is not enough to render a unique price. However, Hobson [1998] found that he could construct model-independent upper and lower bounds on the prices of lookback options by studying extremal models that were consistent with the given law at time T. Assuming that the (discounted) price process is a martingale, the Dambis-Dubins-Schwarz Theorem (see Section 1.3) implies that the candidate process is a time change of Brownian motion and we are thus left to find a stopping time such that the stopped Brownian motion has the law induced by the prices of call options at time T. The problem of finding the stopping time at which a Brownian motion has a given law is referred to as the Skorokhod embedding problem, as it was first introduced and solved by Skorokhod [1965]. An extensive survey on the existing solutions of the Skorokhod embedding problem is given in Obłój [2004]. The connections between model-independent option pricing and the Skorokhod embedding problem is discussed in detail in Hobson [2011].

Since Hobson [1998] suggested the use of Skorokhod embedding techniques for the pricing of derivatives, the approach has been applied to a growing number of different pricing problems. Brown et al. [2001] provided price bounds along with a hedging strategy for one-sided barrier options. Davis and Hobson [2007] found no-arbitrage conditions on European call prices for a fixed maturity date and extend the result to the case where call prices for multiple maturities are known. In the papers by Cox and Obłój [2011b,a] robust prices on two-sided barrier options are given, whereas Cox and Wang [2012, 2013] build on results by Dupire [2005] and Carr and Lee [2010] to derive sub and super-hedging strategies for variance options.

The bounds obtained, even though mostly too wide to be used as prices, provide some interesting insights. Oftentimes simple sub- and super-hedges that hold under any model can be deduced from the construction of the bounds. Being semi-static these trading strategies tend to have lower transaction costs than dynamic hedging strategies. Moreover, we can use the bounds to evaluate portfolio positions in extreme market situations in which it would be hard to argue that a specific model holds (see Cox [2014]). It is also possible to deduce structural properties of the option prices from their bounds. For example, in the case of American options the price for a co-terminal European option with the same strike is a lower bound. The difference between the prices then tells us how valuable the early exercise feature is under the current model.

1.2 Outline

This thesis is dedicated to the derivation of model-independent no-arbitrage conditions on American put options.

Chapter 1.3. In the Preliminaries we discuss the connection between model-free price bounds on derivatives and the Skorokhod embedding problem in detail. We introduce the 'Chacon-Walsh' solution to the Skorokhod embedding problem, which we will use in Chapter 2 to argue that given prices are consistent with a model if certain no-arbitrage conditions hold.

Chapter 2. The main result in this chapter concerns necessary and sufficient conditions for the absence of arbitrage in markets trading American and co-terminal European put options: specifically, we give four conditions which we show to be necessary and sufficient. Since Davis and Hobson [2007] provide no-arbitrage conditions for European put options, we are only interested in finding conditions on the prices of American options in terms of the European prices.

In Section 2.2 simple trading strategies are used to prove the existence of arbitrage whenever one of the conditions is violated. Moreover, we argue in Section 2.3 that these conditions are also sufficient in the case where only finitely many American and European options are traded. To this end we develop a recursive algorithm that generates a market model for any (finite) set of prices satisfying the no-arbitrage conditions. The algorithm will divide the price functions in each iteration into two new pairs of functions that can be interpreted as independent sets of American and co-terminal European option prices with a later start date. At the same time, we can extend the underlying price process up to the current splitting time. Ultimately, the problem will be reduced to a setting in which the price functions can be represented by a trivial model and we obtain a price process that reproduces the given American and European option prices.

Chapter 3. Based on the result in Theorem 2.3.10 we know that the conditions given in Lemma 2.2.1 and Theorem 2.2.3 guarantee the absence of model-independent arbitrage in markets trading only in finitely many American and co-terminal European put options. It is not enough, however, to determine whether these conditions are satisfied by the piecewise linear interpolations between the prices of the traded options. Thus we will address in Chapter 3 the problem of finding a suitable algorithm for the construction of American and European price functions complying with the no-arbitrage conditions. Moreover, we will be able to give explicit arbitrage portfolios should the algorithm fail to produce admissible price functions.

1.3 Preliminaries

We begin with a more detailed discussion on the connection between the problem of finding model-independent option price bounds and solutions to the Skorokhod embedding problem. The following result on which this approach is based is due to Breeden and Litzenberger [1978] and states that the marginal distribution at a fixed time T can be computed from the European call option prices with maturity T.

Lemma 1.3.1. *Suppose that European call options with maturity T are traded in the market at any strike $K \in (0, \infty)$. Let us furthermore assume that their prices are computed as the discounted expected payoff under the probability measure $\mathbb{Q}$, that is, for any $K \in (0, \infty)$*

$$C(K) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)_+\right].$$

Then we have

$$\mathbb{Q}(S_T > K) = e^{rT}\left|\frac{\partial}{\partial K}C(K+)\right|$$

and under the assumption that C is twice differentiable

$$\mathbb{Q}(S_T \in dK) = e^{rT}\frac{\partial^2}{\partial K^2}C(K)$$

has to hold.

Under the assumption that the underlying price process is a martingale, the following theorem implies that the candidate process for the underlying can be represented as a time change of Brownian motion with a given distribution at a stopping time. For a proof of this result we refer the reader to Karatzas and Shreve [1998, p.174-175].

Theorem 1.3.2. *(Dambis-Dubins-Schwarz) Let $M = \{M_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a continuous local martingale that satisfies $lim_{t\to\infty}\langle M\rangle_t = \infty$ $\mathbb{P}$-a.s. Define, for each $0 \leq s < \infty$, the stopping time $T(s) = \inf\{t \geq 0; \langle M\rangle_t > s\}$.*

Then the time-changed process

$$B_s = M_{T(s)}, \mathcal{G} = \mathcal{F}_{T(s)}; 0 \leq s < \infty$$

is a standard one-dimensional Brownian motion. In particular, the filtration $\{\mathcal{G}\}$ satisfies the usual conditions and we have $\mathbb{P}$-a.s. $M_t = B_{\langle M\rangle_t}$ for $0 \leq t < \infty$.

From this we can conclude that $\langle M\rangle_T$ is a solution to the Skorokhod embedding problem. More importantly, it is possible to use a solution τ to the Skorokhod embedding problem, $B_\tau \sim \mu$, to obtain a martingale

$$M_t = B_{\frac{t}{T-t}\wedge\tau}$$

with $M_T \sim \mu$.

1.3.1 The potential picture

One type of approach to generate solutions to the Skorokhod embedding problem is to use the 1-1 correspondence between a probability measure with finite first moment and its potential. For this purpose we define the potential, using the notation in Obłój [2004], and point out some immediate consequences of the definition.

Definition 1.3.3. *Denote by $\mathcal{M}^1$ the set of probability measures on $\mathbb{R}$ with finite first moment, that is $\mu \in \mathcal{M}^1$ iff $\int |x|\mu(dx) < \infty$. Let $\mathcal{M}^1_m$ denote the subset of measures with expectation equal to m. The one-dimensional potential operator U acting from $\mathcal{M}^1$ into the space of continuous, non-positive functions, $U : \mathcal{M}^1 \to C(\mathbb{R}, \mathbb{R}^-)$, is defined through $U\mu(x) = -\int_{\mathbb{R}} |x-y|\mu(dy)$ and we will refer to $U\mu$ as the potential of μ.*

Moreover, we will use the notation $\mu_n \Rightarrow \mu$ to indicate that the sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to the measure μ. Following Obłój [2004] we present important properties of the potential, for which proofs can be found in Chacon [1977] and Chacon and Walsh [1976].

Proposition 1.3.4. *For a probability measure $\mu \in \mathcal{M}^1_m$, $m \in \mathbb{R}$, the potential of μ, $U\mu$, satisfies the following properties:*

(i) *$U\mu$ is concave and Lipschitz-continuous with parameter 1.*

(ii) *$U\mu(x) \le U\delta_m(x) = -|x-m|$ and for $\nu \in \mathcal{M}^1$ the inequality $U\nu \le U\mu$ implies $\nu \in \mathcal{M}^1_m$.*

(iii) *For $\mu, \nu \in \mathcal{M}^1_m$, $\lim_{|x|\to\infty} |U\mu(x) - U\nu(x)| = 0$.*

(iv) *For $\mu_n \in \mathcal{M}^1_m$, $n \in \mathbb{N}$, $\mu_n \Rightarrow \mu$ if and only if $U\mu_n(x) \to U\mu(x)$ pointwise for all $x \in \mathbb{R}$.*

(v) *Consider a Brownian motion with initial law $B_0 \sim \nu$. Denoting the exiting time of an interval $[a,b]$ by $T_{a,b} = \inf\{t \ge 0 : B_t \notin [a,b]\}$ and setting $\rho \sim B_{T_{a,b}}$ it follows that $U\rho|_{(-\infty,a]\cup[b,\infty)} = U\nu|_{(-\infty,a]\cup[b,\infty)}$ and that $U\rho$ is linear on $[a,b]$.*

(vi) *For any $x \in \mathbb{R}$, $\mu((-\infty,x]) = \frac{1}{2}(1 - (U\mu)'(x+))$ and $\mu((-\infty,x)) = \frac{1}{2}(1 - (U\mu)'(x-))$.*

1.3.2 The Chacon-Walsh solution to the Skorokhod embedding problem

The results in the previous section were used by Chacon and Walsh [1976] to construct the following solution to the Skorokhod embedding problem. Suppose we want to embed a probability measure $\mu \in \mathcal{M}_m^1$. The idea is to create a sequence of probability measures $(\mu_n)_{n\in\mathbb{N}}$ with mean m such that their potentials converge pointwise to the potential of μ. We can then conclude from (iv) in Proposition 1.3.4 that the measures μ_n have to converge weakly to the measure μ and the stopping time embedding the target distribution μ into Brownian motion will be given by a limiting procedure of the stopping times embedding the distributions μ_n, $n \in \mathbb{N}$.

Let us begin by pointing out that, according to Proposition 1.3.4 (ii), the potential of μ has to satisfy the inequality $U\mu(x) \leq U\delta_m(x)$ for any $x \in \mathbb{R}$. We can then choose (for any non-trivial measure μ) an arbitrary $x_1 \in \mathbb{R}$ for which $U\mu(x_1) < U\delta_m(x_1)$ and determine the tangent at x_1 to the function $U\mu$ (see Figure 1-1). This tangent, given

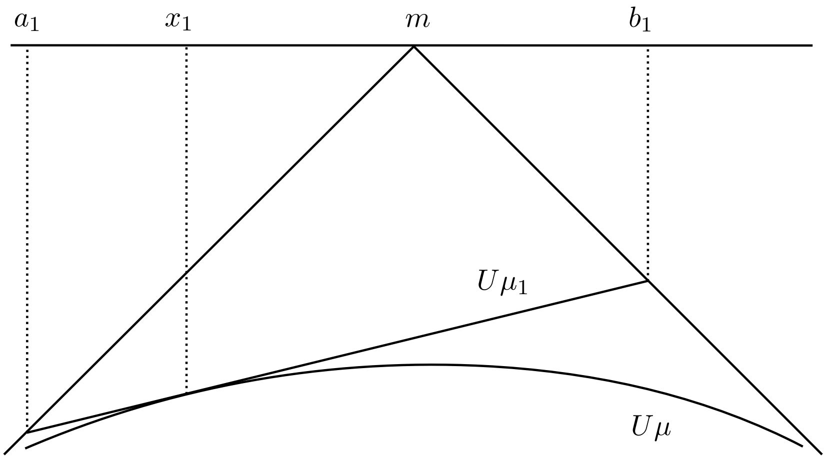

Figure 1-1: Potential picture of $U\delta_m, U\mu_1$ and $U\mu$.

by $t_1(x) = (U\mu)'(x_1)(x - x_1) + U\mu(x_1)$, will intersect with the potential $U\delta_m$ in two points a_1 and b_1, say, where $a_1 < m < b_1$. Moreover, it allows us to define for any $x \in \mathbb{R}$ a new potential

$$U\mu_1(x) = U\delta_m(x)1_{\{x\in(-\infty,a_1)\cup(b_1,\infty)\}} + t_1(x)1_{\{x\in[a_1,b_1]\}}$$

belonging to a probability measure μ_1 with mean m that satisfies $U\mu(x) \leq U\mu_1(x) \leq U\delta_m(x)$ for any $x \in \mathbb{R}$. Since the potential $U\mu_1$ is a piecewise linear function that only has kinks at a_1 and b_1 we can conclude from Proposition 1.3.4 (vi) that the corresponding measure μ_1 consists only of two atoms, one at a_1 and the other one at b_1. This distribution can be easily embedded into Brownian motion, as it can be interpreted as the first exiting time of the interval $[a_1, b_1]$ by a Brownian motion starting in m at time zero.

In the next step we will choose a second point x_2, $x_2 \neq x_1$, at which we compute the tangent t_2 to the function $U\mu$. This time, however, we determine the points a_2 and b_2 where t_2 intersects with the potential $U\mu_1$ instead of $U\delta_m$ (see Figure 1-2). Note

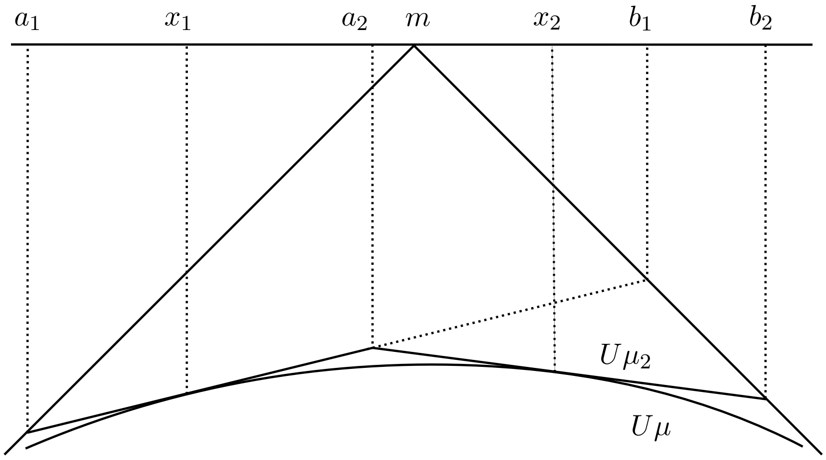

Figure 1-2: Potential picture of $U\delta_m, U\mu_2$ and $U\mu$.

that in the case where $x_2 > x_1$ we will have $a_2 < b_1 < b_2$, while for $x_2 < x_1$ we find that $a_2 < a_1 < b_2$. As before, we can interpret the function

$$U\mu_2(x) = U\mu_1(x)1_{\{x \notin [a_2, b_2]} + t_2(x)1_{\{x \in [a_2, b_2]\}}$$

as the potential of a measure μ_2. This measure will have 3 atoms and can be embedded using the stopping time $T_{a_1,b_1} + T_{a_2,b_2} \cdot \theta_{T_{a_1,b_1}}$, where θ is the standard shift operator.

Iterating this procedure will yield a sequence of potentials $(U\mu_n)_{n\in\mathbb{N}}$ that converges pointwise to the potential $U\mu$, as any concave function can be represented as the infimum over a countable set of linear functions (see Williams [2010, §6.6]). Moreover, we know from Proposition 1.3.4 (iv) that the measures μ_n converge weakly to the measure μ. The stopping time embedding the measure μ is therefore obtained as the limit (as $n \to \infty$) of

$$T_{a_1,b_1} + T_{a_2,b_2} \cdot \theta_{T_{a_1,b_1}} + T_{a_3,b_3} \cdot \theta_{T_{a_2,b_2}} + ... + T_{a_n,b_n} \cdot \theta_{T_{a_{n-1},b_{n-1}}}.$$

1.3.3 Connecting the potential of a measure to option prices

In this section we highlight the 1-1 correspondence between prices of European call option with maturity T and the potential of the marginal distribution of the underlying at time T.

Proposition 1.3.5. *Suppose the prices for European call options with maturity T are determined as the discounted expected payoff under the probability measure μ with mean S_0e^{rT}. Denoting the potential of the measure μ by $U\mu$ and the current price of the underlying by S_0 the following equality*

$$U\mu(x) = e^{rT}(S_0 - 2C(x)) - x$$

has to hold.

Proof. We begin by noting that

$$|x-y| = (x-y)_+ + (y-x)_+. \tag{1.1}$$

Let us now replace the variable y by the random variable Y which we assume to have distribution μ. Taking expectations and multiplying by -1 the equation in (1.1) becomes

$$\begin{aligned} U\mu(x) &= -\int_{-\infty}^{x}(x-y)\mu(dy) - C(x)e^{rT} \\ &= -\int_{-\infty}^{\infty}(x-y)\mu(dy) - \int_{x}^{\infty}(y-x)\mu(dy) - C(x)e^{rT} \\ &= e^{rT}(S_0 - 2C(x)) - x. \end{aligned}$$ □

Due to Lemma 1.3.1 and Proposition 1.3.5 it is possible to use the call price picture (see Figure 1-3) to construct solutions to the Skorokhod embedding problem whenever the law is given by European call prices. This way we can construct solutions to the Skorokhod embedding problem in a financial setting.

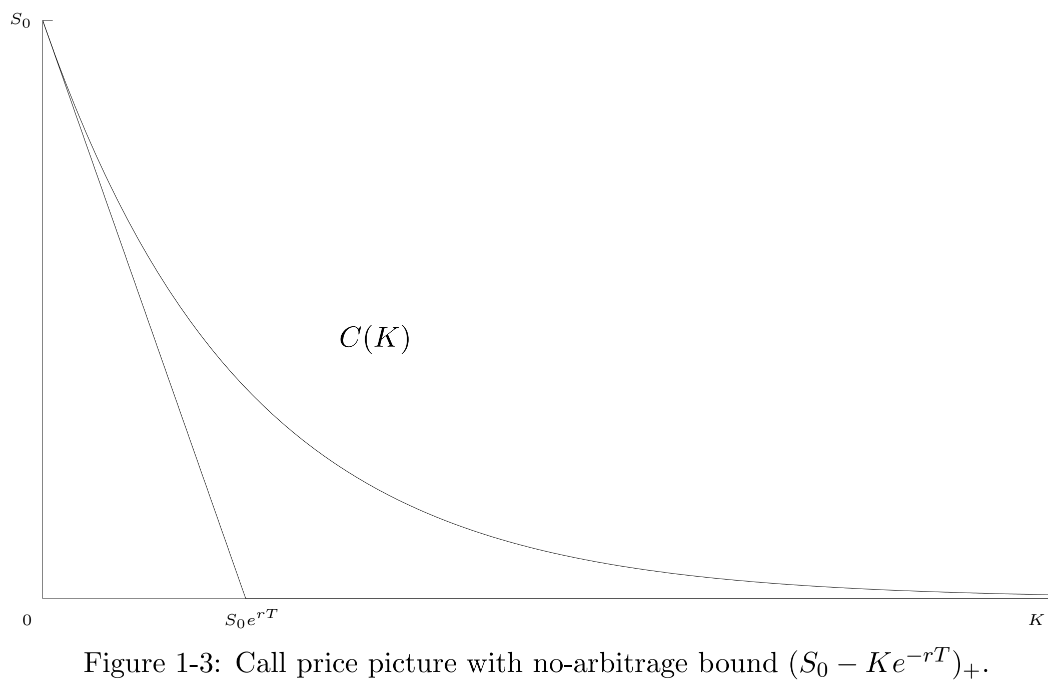

Figure 1-3: Call price picture with no-arbitrage bound $(S_0 - Ke^{-rT})_+$.

Moreover, put-call parity, a model-independent feature of European option prices linking put prices P and call prices C via $C(K) - P(K) = S_0 - e^{-rT}K$, allows us to generate solutions to the Skorokhod embedding problem in the put price picture. The difference between the call and put picture being that the put prices are increasing in

K and that the lower no-arbitrage bound is given by $(Ke^{-rT} - S_0)_+$.

1.3.4 Arbitrage in the model-free setting

Since we are interested in drawing conclusions about derivative prices that hold under a wide class of models we do not specify a probability measure. This, in turn, implies that we cannot use the standard definition of arbitrage any longer. It is therefore necessary that we provide a different type of arbitrage, one that is independent of the probability measure. For that purpose we will introduce model-independent arbitrage, as defined in Davis and Hobson [2007]. To do that, we first have to explain what a semi-static portfolio is.

Definition 1.3.6. *A portfolio is semi-static if it involves a fixed position in traded options at time zero and if the position in the underlying asset can only be modified at finitely many times.*

Definition 1.3.7. *There is a model-independent arbitrage if we can form a semi-static portfolio in the underlying asset and the options such that the initial portfolio value is strictly negative, but all subsequent cash-flows are non-negative.*

The lack of model-independent arbitrage, however, does not imply that there exists a model consistent with given prices. To guarantee this, we require the absence of a second type of arbitrage, termed weak arbitrage by Davis and Hobson [2007].

Definition 1.3.8. *There is a weak arbitrage opportunity if there is no model-independent arbitrage, but, given the null sets of the model, there is a semi-static portfolio such that the initial portfolio value is non-positive, but all sub-sequent cash-flows are non-negative, and the probability of a positive cash-flow is non-zero.*

In the following example we will demonstrate the difference between model- independent arbitrage and weak arbitrage.

Example 1.3.9. *Suppose that European put options with strike K_i are traded at price P_i, $i = 1, 2$ and that $K_1 < K_2$. If $P_1 > P_2$ there exists model-independent arbitrage, as we can make an initial profit selling short a European option with strike K_1 and purchasing a European option with strike K_2 while at maturity the payoff of the option with strike K_2 will dominate the payoff of the option with strike K_1.*

In the case where both options trade for the same price the portfolio no longer has negative cost and thus a model-independent arbitrage portfolio does not exists. However, in a model where $\mathbb{P}(S_T < K_2) > 0$ the same portfolio has a non-zero probability of a positive cash-flow. In a model where $\mathbb{P}(S_T < K_2) = 0$ this portfolio has no chance of giving a positive payoff, then again we can simply sell a European option with strike K_1 to make a profit, as the option will not be exercised at maturity. We have thus shown that there exists weak arbitrage if both options have the same price.

Chapter 2

Model-independent no-arbitrage conditions on American put options

(This work has appeared in Cox and Hoeggerl [2013])

We consider the pricing of American put options in a model-independent setting: that is, we do not assume that asset prices behave according to a given model, but aim to draw conclusions that hold in any model. We incorporate market information by supposing that the prices of European options are known.

In this setting, we are able to provide conditions on the American put prices which are necessary for the absence of arbitrage. Moreover, if we further assume that there are finitely many European and American options traded, then we are able to show that these conditions are also sufficient. To show sufficiency, we construct a model under which both American and European options are correctly priced at all strikes simultaneously. In particular, we need to carefully consider the optimal stopping strategy in the construction of our process.

2.1 Introduction

The standard approach to pricing contingent claims is to postulate a model and to determine the prices as the discounted expected payoffs under some equivalent risk-neutral measure. A major problem with this approach is that no model can capture the real world behaviour of asset prices fully and this leaves us prone to model risk. An alternative to the model-based approach is to try to ask: *when are observed prices consistent with some model?* When there is no model which is consistent with observed prices, it can often then be shown that then there exists an arbitrage which works under all models. Since these properties hold independently of any model, we shall refer to such notions as being model-independent.

The basis of the model-independent approach, which we follow and which can be traced back to the insights of Breeden and Litzenberger [1978], is to suppose European call options are sufficiently liquidly traded that they are no longer considered as being priced under a model, but are obtained exogenously from the market. According to Breeden and Litzenberger [1978] call prices for a fixed maturity date T can then be used to recover the marginal distribution of the underlying at time T. This way contingent claims depending only on the distribution at the fixed time T can be priced without having made any assumptions on the underlying model. Hobson [1998] first observed that, by considering the possible martingales which are consistent with the inferred law, one can often infer extremal properties of the class of possible price processes, and then use these to deduce bounds on the prices of other options on the same underlying when using the European option prices as hedging instrument. This approach has been extended in recent years to pricing various path-dependent options using Skorokhod embedding techniques. Hobson [1998], for example, determined how to hedge lookback options. Brown et al. [2001] showed how to hedge barrier options. Davis and Hobson [2007] determined the range of traded option prices for European calls, whereas Cox and Obłój [2011a,b] found robust prices on double touch and no-touch barrier options, and Cox and Wang [2012] have extended results of Dupire [2005] and Carr and Lee [2010] regarding options on variance. We refer to Hobson [2011] for an overview of this literature. Recently, Galichon et al. [2011] applied the Kantorovich duality to transform the problem of superhedging under volatility uncertainty to an optimal transportation problem, where they managed to recover the results from Hobson [1998] for lookback options.

In this paper, we will be interested in the prices of American put options, and in particular, whether a given set of American put prices and co-terminal European put prices are consistent with the absence of model-independent arbitrage. Our only financial assumptions are that we can buy and sell both types of derivatives initially at the given prices, and that we can trade in the underlying frictionlessly a discrete number of times. Under these conditions, we are able to give a set of simple conditions on the prices which, if violated, guarantee the existence of an arbitrage under *any* model for the asset prices. In addition, we show that these conditions are sufficient in the restricted setting where only finitely many European and American options trade. Specifically, given prices which satisfy our conditions, we are able to produce a model and a pricing measure that reproduce these prices. Clearly, the restriction to a finite number of traded options is not a significant restriction for practical purposes.

Several authors have considered arbitrage conditions on American options in the model-free setting. Closely related to our work is the work of Ekström and Hobson [2009], who determine a time-homogeneous stock price process consistent with given perpetual option prices, and the subsequent generalisation to a wider class of optimal stopping problems by Hobson and Klimmek [2011], however both these papers work under the assumption that the price process lies in the class of time-homogenous diffusions, an assumption that we do not make. Also of relevance is a working paper of Neuberger [2009], who found arbitrage bounds for a single American option with a finite horizon through a linear programming approach. Neuberger takes as given the prices of European options at all maturities, rather than a single maturity as we do, and is able to relate the range of arbitrage-free prices to solutions of a linear programming problem. Although we only consider prices with a single common maturity date, the conclusions we provide are more concrete. Finally, Shah [2006] has obtained an upper and lower bound on an American put option with fixed strike from given American put options with the same maturity, but different strikes. He does not consider the impact of co-terminal European options, and his resulting conditions are therefore easily shown to be satisfied by some model in a one-step procedure.

The main results in this paper therefore concern necessary and sufficient conditions for the absence of arbitrage in quoted co-terminal European and American options: specifically, we are able to give four conditions which we show are necessary and together are sufficient. It is well known (e.g. Davis and Hobson [2007] or Carr and Madan [2005]) which conditions must be placed on European put options for the absence of model-independent arbitrage, so we are interested only in conditions on the American options in terms of the European prices. Three of the conditions are not too surprising: there are known upper and lower bounds, and the American prices must be increasing and convex. However we also establish a fourth condition in terms of the value and the gradient of the European and American options, which we have not found elsewhere in the literature. This condition also has a natural representation in terms of the Legendre-Fenchel transform.

To establish that our conditions are necessary for the absence of model-independent arbitrage, we show that there exists a simple strategy that creates an arbitrage should any of the conditions be violated. It turns out to be much harder to show that our conditions are sufficient: to do this, it is necessary for us to specialise to the case where there are only finitely many traded options, and in this setting, we are able to construct a model under which all options are correctly priced. This requires us both to construct a price process, and to keep track of the value function of an optimal stopping problem. The description of this process will comprise a large amount of the content of this chapter. While this approach is in spirit close to many of the papers which exploit Skorokhod embedding technologies (e.g. Cox and Obłój [2011a,b], Cox and Wang [2012], Hobson [1998]), there are also a number of differences: specifically, that we do not use a time-changed Brownian motion, nor do we attempt to construct an 'extremal' embedding; rather, the embedding step will form a fairly small part of the description of our overall construction.

The construction of the process which attains a given set of prices is described by means of an algorithm: from a set of possible American and European put prices, we shall describe how the prices may be 'split' into two new pairs of functions, which can then be considered as independent sets of European and American prices at a later time. By repeated splitting, we are able to show that the problem eventually reduces to a trivial model which we can describe easily. From this recursive procedure, we are able to reconstruct a process which satisfies all our required conditions. It will turn out that the price process we recover is fairly simple: the price will grow at the interest rate until a non-random time, at which the price jumps to one of two fixed levels. This splitting continues until the maturity date, when it jumps to a final position.

The conditions that we derive should be of interest both for theoretical and practical purposes. They are important for market makers and speculators alike, as a violation of the conditions represents a clear misspecification in the prices under any model, allowing for arbitrage which can be realised using a simple semi-static trading strategy. Our conditions also present simple consistency checks that can be applied to verify that the output of any numerical procedure is valid, and to extrapolate prices which are not quoted from existing market data. In addition, the results we present can also be used as a mechanism to provide an estimate of model-risk associated with a particular position in a set of American options.

The rest of the chapter is organised as follows. In Section 2.2 we discuss the necessary conditions and show that a violation of any of these conditions leads to model-independent arbitrage. In Section 2.3 we will then argue that for any given set of prices A and E that satisfy the necessary conditions there exists a model and a viable price process, hence the conditions also have to be sufficient for the absence of model-independent arbitrage. The Appendix contains some additional proofs that would have only impaired the reading fluency of the paper.

2.2 Necessary conditions for the American put price function A

Assume we are given an underlying asset S which does not pay dividends and which may be traded frictionlessly. In addition, we may hold cash which accrues interest at a constant rate $r > 0$. Furthermore, we will be able to trade options on the underlying at given prices at time 0 only, and these options will always have a common maturity date T.

As we are interested in model-independent behaviour we do not begin by specifying a model or probability measure. It is therefore not immediately clear what arbitrage or the absence of arbitrage means. Along the lines of Davis and Hobson [2007], we say that there exists model-independent arbitrage if we can construct a semi-static portfolio in the underlying and the options that has strictly negative initial value and only non-negative subsequent cashflows. Further we consider a portfolio to be semi-static if it involves holding a position in the options and the underlying, where the position in the options was fixed at the initial time and the position in the underlying can only be altered finitely many times by a self-financing strategy.

There are situations where no model-independent arbitrage opportunities exist, but where we still can find a semi-static portfolio such that the initial portfolio value is non-positive, all subsequent cashflows are non-negative and the probability of a positive cashflow is non-zero, if only the null sets of the underlying model are known. These trading strategies were termed weak arbitrage in Davis and Hobson [2007].

We will consider two cases, one where we are given European put option prices at a finite number of strikes and one where we are given a European price function E for all strikes $K \geq 0$. When there are only finitely many option prices given we shall assume that the European Call prices satisfy the conditions given in Theorem 3.1 of Davis and Hobson [2007] — that is, that there is neither a model-independent, nor a weak arbitrage. It follows from the absence of model-independent arbitrage that Put-Call parity has to hold.

To obtain a European put price function E from the given option prices $E(K_1)$, $E(K_2)$,..., $E(K_n)$ we proceed as follows. First, we note that European put options with strike zero have to satisfy $E(0) = 0$, as their payoff will always be zero. Furthermore, we have that the given option prices satisfy $E(K_i) \geq e^{-rT}K_i - S_0$ for all $i = 1, ..., n$. We will now argue that the case where $E(K_i) > e^{-rT}K_i - S_0$ for all $i = 1, ..., n$ can be reduced to the case where $E(K_n) = e^{-rT}K_n - S_0$ holds. To this end, let us assume that $E(K_i) > e^{-rT}K_i - S_0$ for all $i = 1, ..., n$. It is then possible to extend the set of strikes by a final strike K_{n+1} for which we set $E(K_{n+1}) = e^{-rT}K_{n+1} - S_0$. In order for the European option prices to satisfy the no-arbitrage conditions below, it is necessary that the last strike K_{n+1} is chosen such that

$$K_{n+1} \geq \frac{(E(K_n) + S_0)K_{n-1} - (E(K_{n-1}) + S_0)K_n}{E(K_n) - E(K_{n-1}) - e^{-rT}K_n + e^{-rT}K_{n-1}},$$

where the term on the right hand-side is the strike at which the linear function

$$(E(K_n) - E(K_{n-1}))(K - K_{n-1})/(K_n - K_{n-1}) + E(K_{n-1})$$

intersects with the lower bound $e^{-rT}K - S_0$. We can therefore assume, without loss of generality, that we are always given a set of European prices where the right-most price lies on the lower bound $e^{-rT}K - S_0$. The European put option prices

$E(K_0), E(K_1), ..., E(K_{n+1})$ can then be extended to a continuous function on the positive reals by interpolating linearly between the given option prices on $[0, K_{n+1}]$ and setting $E(K) = e^{-rT}K - S_0$ for any $K \geq K_{n+1}$.

From Davis and Hobson [2007] we can then derive the following conditions on the European put price function E that have to be satisfied for any positive strike K to guarantee the absence of model-independent arbitrage.

Lemma 2.2.1. *Suppose the prices of European put options with maturity T are given for a finite number of strikes $K_1, ..., K_n$. Denote the European put option prices as a function of the strike K by E, where E is constructed as explained above. Then the European put prices are free of model-independent and weak arbitrage opportunities if and only if the following conditions are satisfied:*

1. *The European put price function E is increasing and convex in K.*

2. *The function $(e^{-rT}K - S_0)_+$ is a lower bound for E.*

3. *The function $e^{-rT}K$ is an upper bound for E.*

4. *For any $K \geq 0$ with $E(K) > e^{-rT}K - S_0$ we have $E'(K+) < e^{-rT}$.*

Here S_0 is the current price of the underlying asset.

In the situation where European put prices are given for all positive strikes we can replace the fourth condition of Lemma 2.2.1 by $|E(K) - (e^{-rT}K - S_0)| \to 0$ as $K \to \infty$ under the assumption that there is no *weak free lunch without vanishing risk* (for details see Cox and Obłój [2011a]).

Returning to the situation where there are finitely many strikes given we can conclude due to Breeden and Litzenberger [1978] that these conditions are sufficient to imply the existence of a probability measure μ on $\mathbb{R}^+$ such that

$$E(K) = \int (e^{-rT}K - x)_+ \mu(dx).$$

In addition, the following result has to hold.

Lemma 2.2.2. *If there exists a probability measure μ on $\mathbb{R}^+$ such that $\int x\mu(dx) = S_0$ and $E(K) = \int (e^{-rT}K - x)_+\mu(dx)$, then the European put price function E satisfies the conditions of Lemma 2.2.1.*

Proof. The first condition follows from the fact that μ is a probability measure and that the integrand $(e^{-rT}K - x)_+$ of E is increasing and convex. The lower bound is obtained by applying Jensen's inequality to the convex function $x \mapsto (e^{-rT}K - x)_+$, whereas the upper bound follows from $(e^{-rT}K - x)_+ \leq e^{-rT}K$ as μ is only defined on $\mathbb{R}^+$.

In the case of the fourth condition we will prove the contrapositive. Note that $E'(K+) = e^{-rT} \int \mathbf{1}_{[0,e^{-rT}K]}(x)\mu(dx)$. Since μ is a probability measure and we assume that there exists a K^* with $E'(K^*+) \geq e^{-rT}$ we can conclude that $\mu([0, e^{-rT}K^*]) = 1$, hence for any $K \geq K^*$ we must have

$$E(K) = \int (e^{-rT}K - x)\mu(dx) = e^{-rT}K - S_0,$$

which completes the proof. □

Under these assumptions we are now able to state the main result of this section, Theorem 2.2.3, which will give us conditions on A that necessarily have to be fulfilled for A to be an arbitrage-free American put price function, assuming we are given the prices of co-terminal European put options satisfying the conditions above.

Theorem 2.2.3. *If A is an arbitrage-free American put price function then it must satisfy the following conditions:*

(i) The American put price function A is increasing and convex in K.

(ii) For any $K \geq 0$ we have

$$A'(K+)K - A(K) \geq E'(K+)K - E(K).$$

(iii) The function $\max\{E(K), K - S_0\}$ is a lower bound for $A(K)$.

(iv) The function $E(e^{rT}K)$ is an upper bound for $A(K)$.

With the exception of *(ii)*, these properties are not too surprising: it is well known that the American put price must be convex and increasing, and it is also clear that the price of the American option must dominate both the corresponding European option, and its immediate exercise value. The upper bound given in *(iv)* appears to date back to Margrabe [1978]. Although he works in the Black-Scholes setting, his arguments hold also in the general case under consideration here.

Remark 2.2.4. *(i) Recall that the Legendre-Fenchel transform of a function $f : \mathbb{R} \to \mathbb{R}$ is given by $f^*(k) = \sup_{x\in\mathbb{R}}\{kx - f(x)\}$, so we can rewrite the second condition of Theorem 2.2.3 as*

$$A^*(A'(K+)) \geq E^*(E'(K+)) \tag{2.1}$$

for all $K \geq 0$. This can be seen by rewriting $f^*(k) = -\inf_{x\in\mathbb{R}}\{f(x) - kx\}$ and noting that the function f is given for $x \geq 0$, and is non-negative, increasing and convex in our case.

(ii) It follows directly from condition (ii) of Theorem 2.2.3 that the early exercise premium $A - E$ *has to be increasing, as* $A'(K) - E'(K) \geq \frac{A(K)-E(K)}{K}$ *is positive. However, these statements are not equivalent, and there exist examples where the early-exercise premium is increasing, and the other necessary conditions are satisfied, but condition (ii) of the theorem fails.*

Proof of Theorem 2.2.3. We will prove each statement separately using model-independent arbitrage arguments. To see that the American put price function A has to be increasing in the strike K we will assume the contrary so that we have $A(K_1) > A(K_2)$ for any two positive strikes $K_1 < K_2$. We can then make an initial profit of $A(K_1) - A(K_2)$ by short selling an American put option with strike K_1 and buying an American put option with strike K_2. To guarantee that any subsequent cashflow is positive we only have to close out the long position when the American with strike K_1 is exercised, leaving us with $K_2 - K_1 > 0$. We can then conclude that the function $A(K)$ has to be increasing in K, since there would be an arbitrage opportunity otherwise.

As in the case before we will prove that the function A has to be convex by assuming that $\alpha A(K_1) + (1-\alpha)A(K_2) < A(\alpha K_1 + (1-\alpha)K_2)$ for some $\alpha \in [0,1]$ and $K_1 < K_2$ holds. This way a portfolio consisting of a short position in an American put option with strike $\alpha K_1 + (1-\alpha)K_2$ and a long position of α units in an American put option with strike K_1 and $(1-\alpha)$ units in an American put option with strike K_2 has strictly negative initial cost. If we close out the long positions when the counterparty in the short contract exercises we have at the time of exercise, denoted τ, at least

$$\alpha(K_1 - S_\tau) + (1-\alpha)(K_2 - S_\tau) + (S_\tau - (\alpha K_1 + (1-\alpha)K_2)) = 0.$$

Therefore absence of arbitrage implies that $A(K)$ has to be convex in K.

As proved in Lemma 2.5.1 we have that the condition in (ii) is equivalent to

$$\frac{1}{\epsilon}(A(K+\epsilon) - A(K)) - \frac{1}{K}A(K) \geq \frac{1}{\epsilon}(E(K+\epsilon) - E(K)) - \frac{1}{K}E(K) \tag{2.2}$$

for all $K \geq 0$ and any $\epsilon > 0$. Suppose the condition in (2.2) is violated, then we can make an initial profit by selling $\frac{1}{\epsilon}$ units of $E(K+\epsilon)$ and $\frac{K+\epsilon}{K\epsilon}$ units of $A(K)$, while buying $\frac{1}{\epsilon}$ units of $A(K+\epsilon)$ and $\frac{K+\epsilon}{K\epsilon}$ units of $E(K)$.

Suppose now that the shorted American was exercised at time τ, where we then also exercised the long American to obtain at maturity T a cashflow of

$$\frac{1}{\epsilon}\left[(e^{r(T-\tau)}(K+\epsilon) - S_T) - (K+\epsilon-S_T)_+\right] \\ - \frac{K+\epsilon}{K\epsilon}\left[(e^{r(T-\tau)}K - S_T) - (K-S_T)_+\right],$$

which is equal to

$$\begin{cases} \frac{1}{K}S_T & , S_T \geq K+\epsilon \\ \frac{K+\epsilon}{K\epsilon}(S_T-K) & , S_T \in [K, K+\epsilon] \\ 0 & , S_T \leq K, \end{cases}$$

implying arbitrage. If the shorted American is not exercised, exercising the long American at maturity will cover the short position in the European.

To obtain the upper bound we suppose $E(e^{rT}K) < A(K)$. We sell the American option with strike K, and buy the European with strike $e^{rT}K$, making an initial profit of $A(K) - E(e^{rT}K)$. If the shorted American is not exercised we are guaranteed a positive cashflow from the long position in the European. In the case where the American is exercised at time τ it generates a cashflow $(S_T - Ke^{r(T-\tau)})$ at maturity. Further we receive the amount $(e^{rT}K - S_T)_+$ from the European option. In the case where $S_T < e^{rT}K$ we have

$$(e^{rT}K - S_T) + (S_T - Ke^{r(T-\tau)}) = e^{rT}K(1-e^{-r\tau}) > 0.$$

Whereas for $S_T \geq e^{rT}K$ the European put $E(e^{rT}K)$ has 0 payoff, but by the assumption on K the American put now gives us $(S_T - Ke^{r(T-\tau)}) > 0$.

Analogously we can show that the lower bound has to hold and we have therefore proved all the statements of the theorem. □

Remark 2.2.5. *The upper and lower bounds on the American put price, given in* (*iii*) *and* (*iv*) *of Theorem 2.2.3 respectively, can also be seen to be tight, that is, there exist models that attain the bounds as American put price function. In the case of the lower bound the following underlying price process satisfies* $A(K) = \max\{(K-S_0)_+, E(K)\}$. *Set*

$$S_t = \begin{cases} e^{-r(T-t)}\mathbb{E}Y & , t \in [0,T) \\ Y & , t = T \end{cases}$$

where Y *is an integrable random variable with distribution* μ. *This process grows at the interest rate up to* T, *where it jumps to its final distribution* Y. *The discounted price process* $e^{-rt}S_t$ *is by definition a martingale with respect to its natural filtration* $\mathcal{F}_t^S$. *As the process grows at the interest rate between the times* 0 *and maturity* T *we know that the payoff obtained by exercising at time zero will always exceed the payoff for exercising at any time* $t \in (0,T)$, *hence the only possible stopping times are* 0 *and* T, *which gives* $A(K) = \max\{(K-S_0)_+, E(K)\}$.

In the case of the upper bound the following price process $(S_t)_{t\geq 0}$ *has* $E(e^{rT}K)$ *as*

American price function. Set

$$S_t = \begin{cases} e^{-rT}\mathbb{E}Y & , t = 0 \\ e^{-r(T-t)}Y & , t \in (0,T] \end{cases}$$

where again $r > 0$ is the interest rate, T the maturity date and Y the integrable final distribution. We consider the natural filtration generated by S_t and note that although this does not satisfy the usual conditions, S_t is nevertheless a martingale with respect to this filtration and if we consider the sequence of stopping times $\tau_n = \frac{1}{n}$, we get

$$\begin{aligned} A(K) &\geq \lim_{n\to\infty} e^{-rT}\mathbb{E}\left[(e^{rT-r/n}K - Y)_+\right] \\ &= e^{-rT}\mathbb{E}\left[(e^{rT}K - S_T)_+\right] \\ &= E(e^{rT}K), \end{aligned}$$

as required.

2.3 Sufficiency of the conditions on the American put price function A

In order to show that the necessary conditions in Theorem 2.2.3 are also sufficient for the absence of model-independent arbitrage it is enough to determine for any given set of American and European put prices a market model such that the European and American put option prices satisfy $e^{-rT}\mathbb{E}(K - S_T)_+ = E(K)$ and

$$\sup_{0\leq\tau\leq T} \mathbb{E}e^{-r\tau}(K - S_\tau)_+ = A(K), \tag{2.3}$$

where the supremum is taken over all stopping times τ taking values between 0 and T.* A market model consists of a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ and an underlying price process $(S_t)_{0\leq t\leq T}$ where $(e^{-rt}S_t)_{t\geq 0}$ is an $\mathcal{F}_t$-martingale under $\mathbb{P}$.

In general it appears to be a harder task to show that the conditions of Theorem 2.2.3 are also sufficient, particularly if it is assumed that a continuum of option strikes trade. Consequently, we shall consider a slightly restricted setup (although one that is still practically very relevant): henceforth we will assume that we are given American and European prices for a finite number of strikes, from which we can extrapolate general functions A and E for which the conditions of Theorem 2.2.3 and Lemma 2.2.1 hold.†

*Karatzas [1988] showed that $\sup_{0\leq\tau\leq T}\mathbb{E}e^{-r\tau}(K - S_\tau)_+$ is the fair price for an American option with strike K and maturity T.

†Given a finite set of traded options which are derived from some model, it is not the case that

Contrary to the embedding problem considered in Buehler [2006], Cousot [2007] or Davis and Hobson [2007], where marginals for multiple fixed times are given, the definition of the American put option requires us to incorporate the American prices into $(S_t)_{t\geq 0}$ at the unknown optimal stopping time τ^* before the European prices are embedded at maturity T.

Suppose the piecewise linear functions A and E satisfy the conditions of Theorem 2.2.3 and Lemma 2.2.1 and are given as follows. In the case of the European price function E we will use the 1-1 correspondence between European put options with maturity T and the marginal distribution at time T given in Breeden and Litzenberger [1978] to characterise E using $\mu = p_1\delta_{K_1^E} + ... + p_n\delta_{K_n^E}$ with $\mathbb{E}^{\mu}(X) = S_0 e^{rT}$ where $r > 0$ is the interest rate.

The function A is given for a finite number of strikes $K_1^A, ..., K_m^A$ and interpolated linearly between them. Furthermore, we can assume without loss of generality that American options with strike zero are traded at zero cost. Additionally, we know that $A(K) = K - S_0$ has to hold for (at least) all strikes $K \geq K_n^E e^{-rT}$, as we have by the definition of μ that the upper bound $E(e^{rT}K)$ coincides for these strikes with the lower bound given by $K - S_0$. Thus we can conclude that the American price function lies on the lower bound for all strikes above $K_n^E e^{-rT}$ and therefore that the price for American options with strike K_m^A lies on the lower bound, i.e. that the price for American options at the final strike K_m^A satisfies $A(K_m^A) = K_m^A - S_0$. We can then write the functions A and E as

$$
\begin{aligned}
A(K) &= \max\{0, s_1^A(K - S_d^1), ..., s_{m-1}^A(K - S_d^{m-1}), K - S_0\} \\
E(K) &= \max\{0, s_1^E(K - K_1^E) + d_1, ..., s_{n-1}^E(K - K_{n-1}^E) + d_{n-1}, \\
&\qquad\qquad e^{-rT}K - S_0\},
\end{aligned} \tag{2.4}
$$

where the linear pieces are listed in the order they appear along the x-axis. In Figure 2-1 below the general setting is depicted, where the given European and American prices as functions of the strike K are denoted by E and A respectively.

The idea now is to construct the process $(S_t)_{t\geq 0}$ such that in each step a linear piece of A is incorporated by assigning probability masses p_d and p_u to two suitably chosen points S_d and S_u, respectively. The order in which the pieces are incorporated is determined by their *critical times*, which we will define below. For example, if, in the first step, we incorporate the American prices, corresponding to the linear piece $s_j^A(K - S_d^j)$, at time t_c^*, then the underlying price process will have as American put

their linear interpolation will satisfy the conditions of Theorem 2.2.3 and Lemma 2.2.1 automatically, however it seems plausible that there should be some larger set of strikes which do. Indeed, we believe that, given a set of traded option prices, either we can construct a piecewise linear extension satisfying the conditions of Theorem 2.2.3 or there exists model-independent arbitrage. However, this is a non-trivial result and we leave a formal proof to subsequent work.

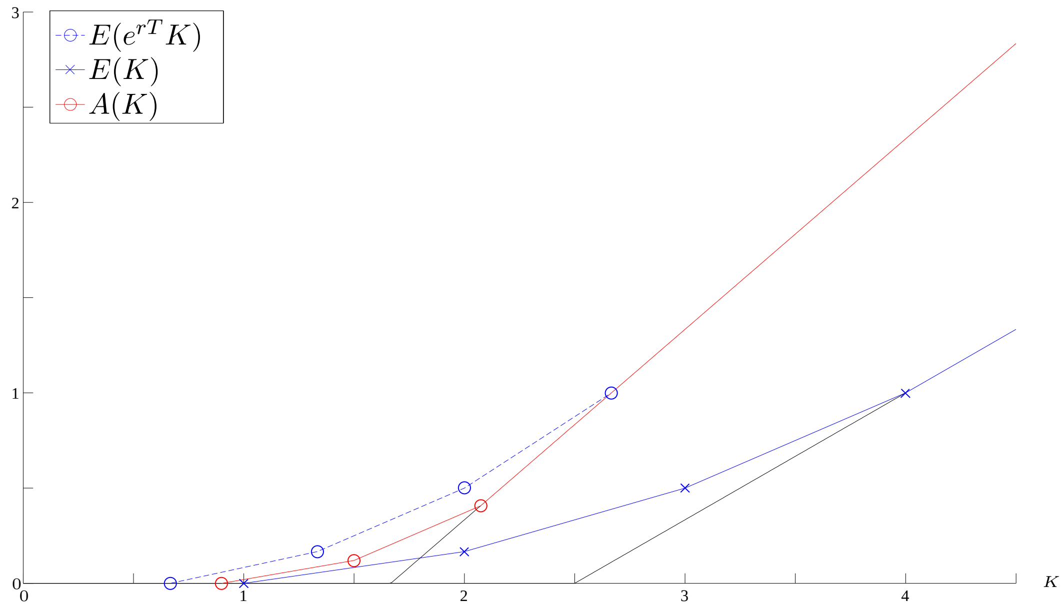

Figure 2-1: An example of feasible American and European put prices as functions of the strike. Also displayed is the upper bound $E(e^{rT}K)$ and the functions $(K - S_0)_+$ and $(e^{-rT}K - S_0)_+$.

price surface the function $\max\{0, s_j^A(K - S_d^j), K - S_0\}$ if stopping is only allowed up to the time t_c^*. We will call this procedure of incorporating the linear pieces of the American put price surface as *embedding* the linear pieces of the American. A similar procedure will be used to embed the European prices. After we embedded a linear piece of A in this manner we will consider two new pictures P_1 and P_2. These pictures will portray prices for American and European options for any positive strike that start at the last embedding time and mature at time T where the price of the underlying asset at the starting time is assumed to be S_d or S_u, respectively. In the sequel we will refer to these new pictures as subpictures. The reason for this being that for each fixed strike the prices for American options in the new pictures can be compounded to yield the continuation value of the option in the original picture. In addition, the marginal distribution at maturity T in the original picture can be recovered as a weighted combination of the marginal distributions at time T in the subpictures, where the respective weights are given by p_d and p_u. We can thus retrieve the prices for European options in the original picture by summing up the weighted prices in the subpictures as will be discussed in Proposition 2.3.5. Moreover, we will show that the price functions in the subpictures satisfy again the conditions of Theorem 2.2.3 and Lemma 2.2.1. Furthermore, the special choice of the critical times allows us to treat the subpictures separately. Since the number of linear pieces remaining in the subpictures P_1 and P_2 is reduced by one in each step and the European E can be embedded at maturity T, we can argue inductively that the algorithm embeds A and E in finitely many steps.

2.3.1 Algorithm

In this section we outline the algorithm which embeds the functions A and E, where each step will be explained in more detail in the subsequent sections.

Algorithm 1 Embedding algorithm

1: Set $t^*_{old} = 0$, $S_0 = \mathbb{E}^\mu(e^{-rT} S_T)$.

2: Modify A beyond $\tilde{K} = \inf\{K \geq 0 : A(K) = K - S_0\}$ by extending the linear piece $s^A_{m-1}(K - S_d^{m-1})$ up to the first atom of E where the necessary condition

$$A'(K+)K - A(K) \geq E'(K+)K - E(K)$$

from Theorem 2.2.3 is violated. From this strike on $A'(K+)$ is determined such that this condition is fulfilled with equality. Denote this extension by $\tilde{A}$ and the number of linear pieces of $\tilde{A}$ by $N_{\tilde{A}}$.

3: Compute the critical time t^*_c and the critical strike K^*, determining the linear piece $s^A_k(K - S^k_d)$ of $\tilde{A}$, where $k \in \{1, ..., N_{\tilde{A}}\}$, that should be embedded next.

4: Embed $s^A_k(K - S^k_d)$ by assigning probability mass p_d to S_d and p_u to S_u at time $t^* = t^*_{old} + t^*_c$, where $p_d = e^{rt^*_c} s^A_k$, $S_d = S^k_d$, $p_u = 1 - p_d$, $S_u = \frac{S_0 e^{rt^*_c} - p_d S^k_d}{p_u}$. For $t^*_{old} < t < t^*$ set $S_t = e^{r(t - t^*_{old})} S_0$. If $t^*_c = 0$, replace the jump to S_0 at time t^* by jumps to S_d and S_u. Update $t^*_{old} = t^*$.

5: Split μ into μ_1 and μ_2, the given European prices E into E_1 and E_2 and the given function A into A_1 and A_2.

6: If $A_1 \neq E_1 \vee (K - S_d)_+$ set $A = A_1$, $E = E_1$ and $S_0 = S_d$ then go to 2., otherwise embed E_1 at T.

7: If $A_2 \neq E_2 \vee (K - S_u)_+$ set $A = A_2$, $E = E_2$ and $S_0 = S_u$ then go to 2., otherwise embed E_2 at T.

2.3.2 Existence and calculation of the critical time

In this section we will construct a method to determine the critical time t^*_c, which will tell us when to embed the next linear piece of the given function A. The actual jump of S then occurs at $t^* = t^*_{old} + t^*_c$, where t^*_{old} is the time where the parent was embedded or 0 in the first step.

As we want to interpret the function A for a fixed strike $K \geq 0$ as the American put option price on an unknown underlying price process S, we intend to split the function A at t^* into two independent functions A_1 and A_2 that can again be interpreted as American put option prices, where the underlying price process then starts at time t^* in S_d or S_u respectively.

It follows that the contract length for the European put price functions E_1 and E_2 has to be modified to $(T - t^*)$. This directly affects the upper bound $\overline{A}$ given by

$$\overline{A}(K, t) = E(e^{r((T - t^*_{old}) - t)}K) \tag{2.5}$$

for $0 \leq t \leq T - t^*_{old}$, which will play a crucial role in finding the critical time t^*_c.

Furthermore, we have the problem that A only provides information on the underlying S up to the strike K^A_m above which exercising A immediately is optimal. This information is not enough to reconstruct A_1 and A_2 independently forcing us to generate additional information on the underlying S by extending A beyond K^A_m. As long as this extension still satisfies the necessary conditions in Theorem 2.2.3 this extension will not affect the American put prices with respect to the underlying S, since $K - S_0$ will dominate these payoffs for $K \geq K^A_m$.

By extending A linearly beyond K^A_m, only correcting the slope $A'(K+)$ when in an atom of E, where the condition $A'(K+)K - A(K) \geq E'(K+)K - E(K)$ is violated, we obtain

$$\tilde{A}(K) = \begin{cases} A(K) & , 0 \leq K \leq K^A_m \\ s^A_{m-1}(K - S^{m-1}_d) & , K^A_m \leq K \leq K^E_p \\ \tilde{A}'(K^E_i+)(K - K^E_i) + \tilde{A}(K^E_i) & , K^E_i \leq K \leq K^E_{i+1} \\ \tilde{A}'(K^E_{N_E}+)(K - K^E_{N_E}) + \tilde{A}(K^E_{N_E}) & , K \geq K^E_{N_E}, \end{cases} \tag{2.6}$$

where $i = p, ..., N_E - 1$, $\tilde{A}'(K^E_i+) = E'(K^E_i+) + \frac{\tilde{A}(K^E_i) - E(K^E_i)}{K^E_i}$ and K^E_p is the first atom of E after K^A_m where condition (ii) of Theorem 2.2.3 is violated (Fig. 2-2). Further set $N_{\tilde{A}} = N_A + N_E - p$.

Lemma 2.3.1. *Suppose the functions A and E given by (2.4) satisfy the conditions of Theorem 2.2.3 and Lemma 2.2.1. Then A can be extended as in (2.6) to $\tilde{A}$, where $\tilde{A}$ and E satisfy again the conditions of Theorem 2.2.3, except that $\tilde{A}$ no longer has $K - S_0$ as lower bound.*

Proof. Let us start by pointing out that the condition

$$\tilde{A}'(K+)K - \tilde{A}(K) \geq E'(K+)K - E(K)$$

is trivially fulfilled for all $K \geq 0$ by the choice of the extension $\tilde{A}$.

To see that $\tilde{A}$ is bounded below by E remember that this is fulfilled up to K^A_m by the assumptions on A and E. Hence for E to exceed $\tilde{A}$ between K^A_m and K^E_p we would need $E'(K+) > \tilde{A}'(K+)$ for some K which can be ruled out, since we know that $\tilde{A}'(K+)K - \tilde{A}(K) \geq E'(K+)K - E(K)$ and $\tilde{A} \geq E$ holds in K^A_m. For $K \geq K^E_p$ we can argue inductively for each of the intervals, since the condition already has to hold in

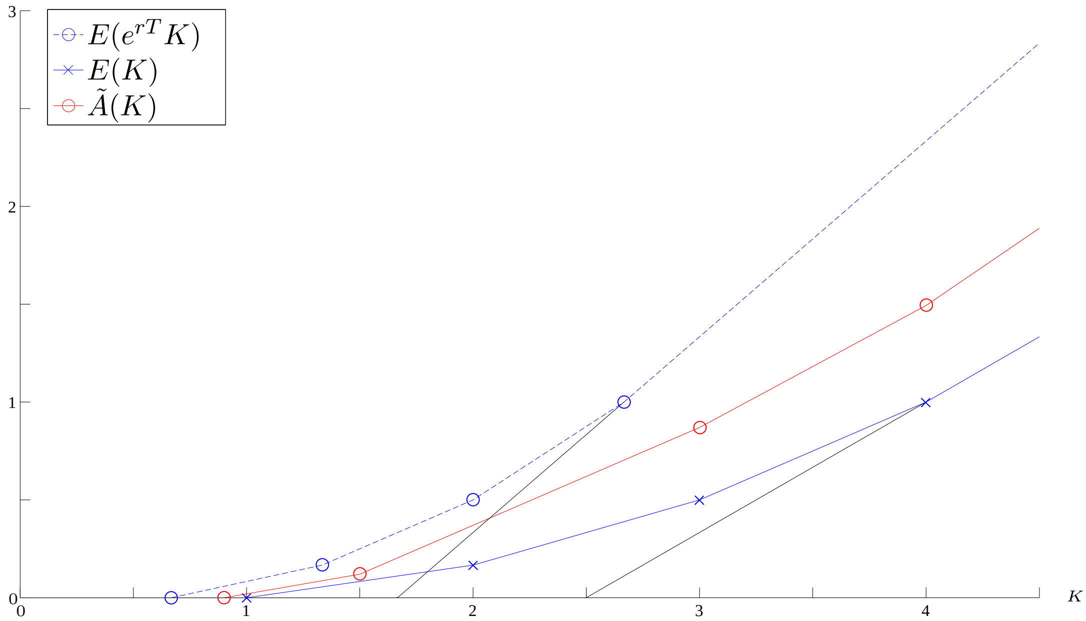

Figure 2-2: Extension of the American price function A, given in Figure 2-1, to $\tilde{A}$

the respective left endpoint of the interval K_i^E, $i = p, ..., n-1$.

Next we will show that $\tilde{A}(K)$ is bounded above by $\overline{A}(K, 0)$, which was defined in (2.5). Note that $\overline{A}(K, 0) \geq A(K)$ for all $K \geq 0$ has to hold as we assumed that the functions A and E satisfy the necessary conditions for $t^*_{old} = 0$. We will now show that we actually have $\tilde{A}(K) \leq A(K)$ for all $K \geq 0$. Up to K_p^E this is trivially fulfilled by definition of A and $\tilde{A}$. From K_p^E onwards we have that $\tilde{A}'(K+)K - \tilde{A}(K) = E'(K+)K - E(K)$ and therefore $A'(K+)K - A(K) \geq \tilde{A}'(K+)K - \tilde{A}(K)$ has to hold for all $K \geq 0$ by the assumptions on A and E in Theorem 2.2.3. Using the fact that $A(K_p^E) \geq \tilde{A}(K_p^E)$ we can then conclude that we must have $A'(K_p^E+) \geq \tilde{A}'(K_p^E+)$. This allows us now to argue inductively and in the same way as for the lower bound to obtain that $\overline{A}(K, 0) \geq \tilde{A}(K)$.

That $\tilde{A}$ is increasing for all $K \geq 0$ is an immediate consequence of the facts that $\tilde{A} \geq E$ and that $E' \geq 0$ as

$$\tilde{A}'(K_i^E+) = E'(K_i^E+) + \frac{\tilde{A}(K_i^E) - E(K_i^E)}{K_i^E} \geq 0 \tag{2.7}$$

for $i \geq p$.

To prove that $\tilde{A}$ is convex it is enough to show that the slope of $\tilde{A}$ is increasing for any strike $K \geq K_p^E$, as we know already that A is convex. Note that we can write

$$\begin{aligned}\tilde{A}(K_{i+1}^E) &= \tilde{A}'(K_i^E+)(K_{i+1}^E - K_i^E) + \tilde{A}(K_i^E)\\ E(K_{i+1}^E) &= E'(K_i^E+)(K_{i+1}^E - K_i^E) + E(K_i^E),\end{aligned}$$

since both $\tilde{A}$ and E are piecewise linear functions. Further we have that

$$E'(K_i^E+)K_i^E - E(K_i^E) = \tilde{A}'(K_i^E+)K_i^E - \tilde{A}(K_i^E)$$

for strikes $K_i^E \geq K_p^E$. Taking into account the definition of the slope of $\tilde{A}$ given in (2.7) we can then conclude that for $i \geq p$ we have

$$\begin{aligned}\tilde{A}'(K_{i+1}^E+) &= E'(K_{i+1}^E+) + \frac{\tilde{A}(K_{i+1}^E) - E(K_{i+1}^E)}{K_{i+1}^E} \\ &= \tilde{A}'(K_i^E+) + (E'(K_{i+1}^E+) - E'(K_i^E+)),\end{aligned}$$

where we furthermore used that

$$\tilde{A}(K_{i+1}^E) = \tilde{A}'(K_i^E+)(K_{i+1}^E - K_i^E) + \tilde{A}(K_i^E)$$

and

$$E(K_{i+1}^E) = E'(K_i^E+)(K_{i+1}^E - K_i^E) + E(K_{i+1}^E).$$

Since E' is increasing it follows that $\tilde{A}'$ has to be increasing as well and thus $\tilde{A}$ has to be convex again. □

To determine a suitable critical time t_c^*, where the next linear piece of A is embedded, we recall two important properties that we want to be fulfilled. First of all the underlying price process S has to be a martingale and secondly we want the two subpictures, obtained by splitting at time t^* in the *critical strike* K^*, to be disjoint. In this context we refer to the subpictures as being disjoint when the points S_d and S_u are being assigned the exact amount of mass required to embed all the linear pieces in the respective subpicture at time t^*, allowing us to consider them separately.

We choose the critical time t_c^* to be the first time t, where waiting any longer would result in $\overline{A}(K, t+\epsilon) < \tilde{A}(K)$ for some $K > 0$ and any $\epsilon > 0$, where $\overline{A}(K,t)$ denotes the upper bound on $\tilde{A}$ (see Fig. 2-3 below). We will show in Lemma 2.3.2 that the critical time t_c^* exists and is finite. Furthermore, we will see that the aforementioned properties are then satisfied.

Before we show the existence of the critical time t_c^* note that the last linear piece of A, given by $K - s_{old}^*$ (where s_{old}^* is the starting point of the asset in the original picture) is already incorporated in the model, since the underlying process S starts at t_{old}^* in s_{old}^*. In particular, exercising the American option at time t_{old}^*, when the process is at s_{old}^* will give payoff $K - s_{old}^*$ at time t_{old}^*. Therefore the linear piece $K - s_{old}^*$ can be omitted when looking for the critical time. Recall that, we will say that a linear piece of the American is embedded whenever we incorporate the prices along the line into the model by jumping mass p_d to S_d (and p_u to S_u).

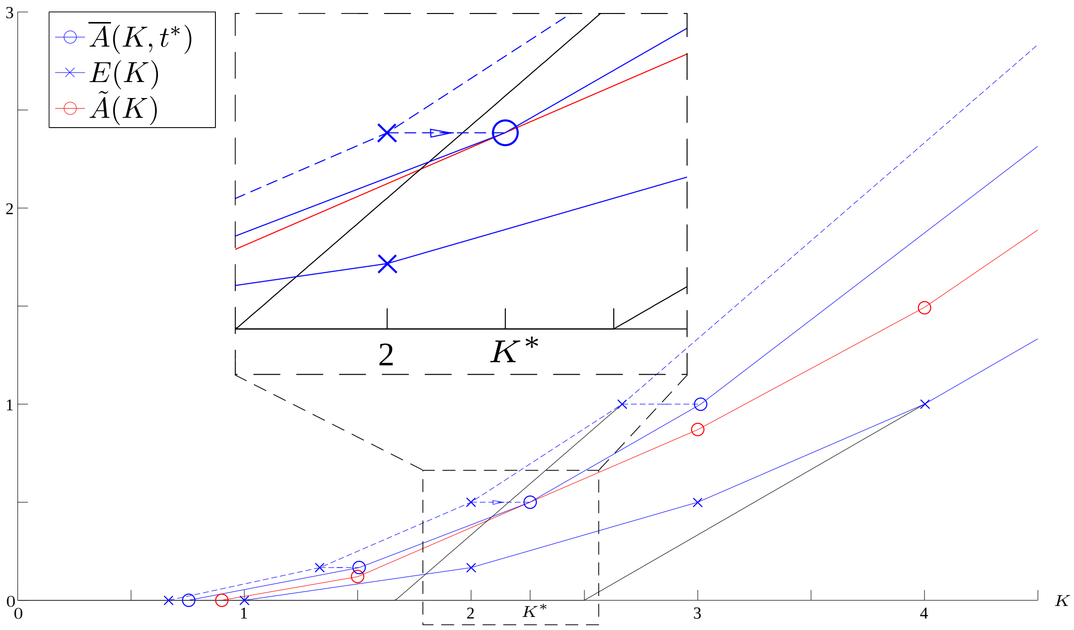

Figure 2-3: As t increases, the function $\overline{A}(K,t) = E(e^{r((T-t^*_{old})-t)}K)$ moves to the right. In this example, the critical time t^*_c, which embeds the next piece of A, occurs at K^* as $\overline{A}(K^*, t^*_c) = \tilde{A}(K^*)$.

Lemma 2.3.2. *Suppose the given functions A and E satisfy the necessary conditions of Theorem 2.2.3 and Lemma 2.2.1, where A is extended to $\tilde{A}$ as in (2.6) and the European put price function E with contract length $T - t^*_{old}$ is given by the marginal distribution $\mu = p_1\delta_{K_1^E} + ... + p_n\delta_{K_n^E}$ with maturity T. Assume also that the upper bound $\overline{A}$ is given by $\overline{A}(K,t) = E(e^{r(T-t^*_{old}-t)}K)$.*

Then we have that the critical time t^*_c exists and is bounded by $T - t^*_{old}$. It is attained when a kink of the upper bound $\overline{A}$ first hits $\tilde{A}$ and can be written as

$$\begin{aligned} t^*_c &= \inf_{i,j} t^{i,j} \\ &= \inf_{i,j} \inf\{t \geq 0 : \overline{A}(u_i, t) < f_j(u_i)\}, \end{aligned} \tag{2.8}$$

where $u_i = K_i^E e^{-r(T-t^*_{old}-t)}$, $i \in \{1, ..., n\}$, $j \in \{1, ..., N_{\tilde{A}}\}$ and f_j is the j-th linear piece of $\tilde{A}$.

Proof. It is a simple consequence of the convexity of the functions $\overline{A}$ and $\tilde{A}$ that the critical time t^*_c, if it exists, occurs whenever a kink of the upper bound $\overline{A}$ intersects with $\tilde{A}$. Hence the critical time t^*_c, should it exist, is given by (2.8). Since we know that $\overline{A}(K,0) \geq \tilde{A}(K) \geq E(K)$ and

$$\overline{A}(K,t) = E(e^{r((T-t^*_{old})-t)}K) \to E(K)$$

for all $K \geq 0$, as $t \to (T - t^*_{old})$, the representation in (2.8) guarantees the existence of an i and j such that $t^{i,j} \leq T - t^*_{old}$ given that $\tilde{A}$ has not been embedded completely yet. ☐

It follows that the critical strike K^*, where we will split the picture, is given by the time-t^* value of the largest atom of $\overline{A}$ which intersects at the critical time t^*_c with the function $\tilde{A}$, i.e.

$$K^* = \sup\{K \geq 0 : \overline{A}(K, t^*_c) = \tilde{A}(K)\}. \tag{2.9}$$

The following lemma will give us now a simple way of determining $\inf_i t^{i,j}$, where $i \in \{1, ..., N_E\}$ and $j \in \{1, ..., N_{\tilde{A}}\}$, thereby highlighting the close connection between the necessary condition

$$\tilde{A}'(K+)K - \tilde{A}(K) \geq E'(K+)K - E(K) \tag{2.10}$$

and the embedding time t^* for a fixed linear piece f_j of $\tilde{A}$.

Proposition 2.3.3. *Suppose the given functions A and E satisfy the necessary conditions of Theorem 2.2.3 and Lemma 2.2.1, where A is extended to $\tilde{A}$ as in (2.6) and the European put price function E with contract length $T - t^*_{old}$ uses the marginal distribution $\mu = p_1\delta_{K^E_1} + ... + p_n\delta_{K^E_n}$ with maturity T.*

Consider the function $f_j(K) = s^A_j K - d$ appearing as the j-th linear piece of $\tilde{A}$ and assume that the European price function E coincides in $[K^E_i, K^E_{i+1}]$ with $g_i(K) = s^E_i K - d_i$ then we have

$$\inf_i t^{i,j} = \frac{1}{r}\ln\left(\frac{s^E_{i^*}}{s^A_j} + \frac{d - d_{i^*}}{s^A_j}\frac{1}{K^E_{i^*}}\right) + (T - t^*_{old}), \tag{2.11}$$

where $i^* = \min\{1 \leq i \leq n : f'_j(K^E_i+)K^E_i - f_j(K^E_i) < g'_i(K^E_i+)K^E_i - g_i(K^E_i)\}$ or equivalently $i^* = \min\{1 \leq i \leq n : d < d_i\}$.

Proof. The i-th linear piece $g_{t,i}(K)$ of the upper bound $\overline{A}(K,t)$ at time t in the interval $[K^E_i e^{-r((T-t^*_{old})-t)}, K^E_{i+1}e^{-r((T-t^*_{old})-t)}]$ is given by

$$g_{t,i}(K) = g_i(e^{r((T-t^*_{old})-t)}K).$$

To determine for any fixed linear piece $g_{t,i}(K)$, $i \in \{1, ..., n\}$ the time $t^{i,j}$ when the atom $K_i^E e^{-r((T-t_{old}^*)-t)}$ intersects with f_j we rewrite $g_{t,i}(K)$ as follows

$$g_{t,i}(K) = s_i^E e^{r(T-t_{old}^*)}K - d_i - s_i^E(e^{r(T-t_{old}^*)} - e^{r((T-t_{old}^*)-t)})K,$$

which can then be interpreted as a clockwise rotation about the fixed point $(0, -d_i)$, as t increases. In addition we know the strike $\hat{K}$ where the atom $K_i^E e^{-r((T-t_{old}^*)-t)}$ has to hit f_j, since the value of $g_{t,i}(K)$ remains unchanged in the atom $K_i^E e^{-r((T-t_{old}^*)-t)}$ over time, as we have $\overline{A}(K,t) = E(e^{r((T-t_{old}^*)-t)}K)$. This allows us to obtain the candidate time

$$t^{i,j} = \frac{1}{r}\ln\left(\frac{s_i^E}{s_j^A} + \frac{d-d_i}{s_j^A}\frac{1}{K_i^E}\right) + (T - t_{old}^*)$$

by setting $g_{t,i}(\hat{K}) = g_i(K_i^E)$ and solving for t.

This result now tells us that $t^{i,j}$ is a decreasing function of K_i^E for $d > d_i$ as s_i^E,s_j^A, r and T are all positive constants, implying that for the two consecutive atoms K_i^E and K_{i+1}^E, lying on the same linear piece g_i, the right atom K_{i+1}^E will give a smaller candidate time. As K_{i+1}^E is also the left-side endpoint of the next linear piece we can conclude by induction that as long as a linear piece g_k, $k \geq i$, still satisfies $d > d_k$ its right-side endpoint will attain a smaller candidate time than any atom before.

Analogously we see that for $d < d_i$ the function $t^{i,j}$ is increasing in K_i^E. Hence the critical time has to be attained in the atom $K_{i^*}^E$, which is the rightmost atom still lying on a linear piece g_k satisfying $d \geq d_k$, but at the same time is the first atom lying on a linear piece g_{k+1} where $d < d_{k+1}$. The existence of this atom $K_{i^*}^E$ is guaranteed by the fact that $d_n = S_0$, whereas $d < S_0$ for any linear piece of A that is not embedded yet. □

Remark 2.3.4. *(i) This result implies that the critical time for a fixed linear piece f_i of A is attained when the kink of $\overline{A}$ meets $\tilde{A}$, where the kink corresponds to the European strike at which the Legendre-Fenchel condition between f_i and E is violated for the first time. Note that this is not a contradiction to the Legendre-Fenchel condition of Theorem 2.2.3, it simply means that the kink of $\overline{A}$ responsible for the critical time lies to the right of the interval where $A = f_i$.*

(ii) As it is possible that the upper bound $\overline{A}$ intersects with $\tilde{A}$ at the critical time t_c^* in a kink of $\tilde{A}$ we need to specify which of the two linear pieces of $\tilde{A}$ we will embed. Proposition 2.3.3 tells us now that we have to take the right-hand side linear piece given by $\tilde{A}'(K^*+)(K - K^*) + \tilde{A}(K^*)$.

2.3.3 The splitting procedure

After we determined the embedding time $t^* = t^*_{old} + t^*_c$ and the critical strike K^* we will divide the functions A and E into two separate parts A_1, A_2, and E_1, E_2 respectively, such that A_i, E_i $i \in \{1,2\}$ satisfy again all the conditions in Theorem 2.2.3 and from which it will be possible to recover the initial functions A and E.

Splitting of the European put option prices E

To obtain E_1 and E_2 from E we have to split μ, the marginal distribution given at maturity T. Since the critical strike K^* is given in time-t^* value, the respective atom of μ, where we have to split, is $K^* e^{r(T-t^*)}$. The following lemma will show how to split μ into μ_1 and μ_2 and how to recover E from E_1 and E_2.

Proposition 2.3.5. *Assume the given functions A and E satisfy the necessary conditions of Theorem 2.2.3 and Lemma 2.2.1, where A is extended to $\tilde{A}$ as in (2.6) and the European put price function E with contract length $T - t^*_{old}$ is given by the marginal distribution $\mu = p_1 \delta_{K^E_1} + ... + p_n \delta_{K^E_n}$ with mean $\mathbb{E}^\mu(X) = e^{r(T-t^*_{old})} S_0$ at maturity T. Suppose further that the critical time t^*_c is given by (2.8) and the associated critical strike K^* by (2.9). At the time $t^* = t^*_{old} + t^*_c$ the linear piece $s^A_k(K - S^k_d)$ of $\tilde{A}$ is embedded by assigning the probability mass p_d to S_d and p_u to S_u, where p_d, S_d, p_u and S_u are given in Section 2.3.1. For the time between the jumps set the underlying price process $S_t = \mathbb{E}^\mu(X) e^{-r(T-t)}$, where $t^*_{old} < t < t^*$.*

Then we can write $\mu = p_d \mu_1 + p_u \mu_2$, where μ_1 and μ_2 are given by

$$\mu_1 = p_d^{-1} \left[\mu\big|_{[K^E_1, K^* e^{r(T-t^*)})} + (p_d - \mathbb{P}(S_T < K^* e^{r(T-t^*)}))\delta_{K^* e^{r(T-t^*)}} \right] \tag{2.12}$$

and

$$\mu_2 = p_u^{-1} \left[(\mathbb{P}(S_T \leq K^* e^{r(T-t^*)}) - p_d)\delta_{K^* e^{r(T-t^*)}} + \mu\big|_{(K^* e^{r(T-t^*)}, K^E_n]} \right], \tag{2.13}$$

and satisfy $\mathbb{E}^{\mu_1}(e^{-r(T-t^*)} X_1) = S_d$ and $\mathbb{E}^{\mu_2}(e^{-r(T-t^*)} X_2) = S_u$. Dividing the distribution μ into μ_1 and μ_2 the European put option E with maturity T can be written as

$$E^\mu(K) = e^{-rt^*_c} \left[p_d E^{\mu_1}_1(K) + p_u E^{\mu_2}_2(K) \right], \tag{2.14}$$

where $E^{\mu_1}_1$ and $E^{\mu_1}_2$ are European put options starting at t^*, having maturity T and satisfying the conditions in Lemma 2.2.1.

Proof. Firstly, let us show that the mass that is placed in $K^* e^{r(T-t^*)}$ for either of the two distributions μ_1 and μ_2 is non-negative. Without loss of generality we can assume that $K^*e^{r(T-t^*)}$ is the l-th atom of μ and that the linear piece we just embedded was $f_k = s^A_k(K - S^k_d)$. If we set $\hat{K}_1 = \max\{K^E_{l-1}, K^A_k\}$, where $K^A_k = \inf\{K \geq 0 : \tilde{A}(K) = s^A_k(K - S^k_d)\}$ then we have for the upper bound $\overline{A}$, which is given by $\overline{A}(K,t) = E(e^{r((T-t^*_{old})-t)}K)$ that

$$\overline{A}(K^*, t^*_c) = \overline{A}'(K^*-, t^*_c)(K^* - \hat{K}_1) + \overline{A}(\hat{K}_1, t^*_c)$$

and at the same time for the extended American $\tilde{A}$ from (2.6) that

$$\tilde{A}(K^*) = \tilde{A}'(K^*-)(K^* - \hat{K}_1) + \tilde{A}(\hat{K}_1).$$

By the definition of t^*_c we see that $\overline{A}(K^*, t^*_c) = \tilde{A}(K^*)$. Combining this with the fact that we must have $\overline{A}(\hat{K}_1, t^*_c) \geq \tilde{A}(\hat{K}_1)$ at the critical time t^*_c we can conclude that $\tilde{A}'(K^*-) \geq \overline{A}'(K^*-, t^*_c)$. Then again, we can use that $p_d = e^{rt^*_c}\tilde{A}'(K^*-)$ and that

$$\overline{A}'(K^*-, t^*_c) = E'(e^{r((T-t^*_{old})-t^*_c)}K^*-)e^{r((T-t^*_{old})-t^*_c)}$$

to see that $p_d \geq E'(e^{r((T-t^*_{old})-t^*_c)}K^*-)e^{r((T-t^*_{old}))} = \mathbb{P}(S_T < K^*e^{r(T-t^*)})$. To show the other inequality, set $\hat{K}_2 = \min\{K^E_{l+1}, K^A_{k+1}\}$, where $K^A_{k+1} = \sup\{K \geq 0 : \tilde{A}(K) = s^A_k(K - S^k_d)\}$, and note that $\tilde{A}'(K+) \geq \tilde{A}'(K-)$ as $\tilde{A}$ is convex. We can then argue analogously to above that $p_d \leq \mathbb{P}(S_T \leq K^*e^{r(T-t^*)})$, where the inequality turns around as we have now $K^* \leq \hat{K}_2$.

By the martingale property of $(S_t)_{t^*_{old} \leq t \leq t^*}$ we have

$$\mathbb{E}^\mu(X) = S_0 e^{r(T-t^*_{old})} = S_0 e^{rt^*_c} e^{r(T-t^*)} = (p_d S_d + p_u S_u)e^{r(T-t^*)}. \tag{2.15}$$

At the same time we can write

$$\mathbb{E}^\mu(X) = p_d \mathbb{E}^{\mu_1}(X_1) + p_u \mathbb{E}^{\mu_2}(X_2), \tag{2.16}$$

since we clearly have $\mu = p_d\mu_1 + p_u\mu_2$. Equating now (2.15) and (2.16) we obtain $\mathbb{E}^{\mu_1}(X_1) = S_d e^{r(T-t^*)}$ and $\mathbb{E}^{\mu_2}(X_2) = S_u e^{r(T-t^*)}$.

We can then conclude that $E^\mu(K) = e^{-rt^*_c}\left[p_d E^{\mu_1}_1(K) + p_u E^{\mu_2}_2(K)\right]$, as we know that $\mu = p_d\mu_1 + p_u\mu_2$ and that E_1 and E_2 have contract length $(T - t^*)$. From the last two statements and Lemma 2.2.2 it follows now directly that E_1 and E_2 satisfy the conditions of Lemma 2.2.1. $\square$

Splitting of the American put option prices A

In the case of the European put option prices E the existence of a 1-1 correspondence between E and μ allows us to split the function E by dividing μ. For the American put option prices A this 1-1 correspondence to the marginal distribution at a fixed deterministic time does not exist, since the time when it is optimal to exercise the option depends on the path of the underlying. We therefore need a different method to split A that still allows us to recover the original function A from the two new functions A_1 and A_2. The idea behind the specific choice of split in (2.19) is that we want to separate the already embedded immediate exercise from the continuation value in each step.

Proposition 2.3.6. *Assume the given functions A and E satisfy the necessary conditions of Theorem 2.2.3 and Lemma 2.2.1, where A is extended to $\tilde{A}$ as in (2.6) and the European put price function E with contract length $T - t^*_{old}$ is given by the marginal distribution $\mu = p_1\delta_{K_1^E} + ... + p_n\delta_{K_n^E}$ with mean $\mathbb{E}^\mu(X) = e^{r(T-t^*_{old})}S_0$ at maturity T. Suppose further that the time of the next jump t^*, the critical time t^*_c and the associated critical strike K^* were determined as in Section 2.3.2 at which point the linear piece $s_k^A(K - S_d^k)$ of $\tilde{A}$ is embedded by jumping the mass p_d to S_d and p_u to S_u, where p_d, S_d, p_u and S_u are given in Section 2.3.1. For the time between the jumps the underlying price process is set to be $S_t = \mathbb{E}^\mu(X)e^{-r(T-t)}$, where $t^*_{old} < t < t^*$.*

Then the function A can be split into

$$A_1(K) = e^{rt^*_c}p_d^{-1}\max\{0, f_1, f_2, ..., f_k\} \tag{2.17}$$

and

$$A_2(K) = e^{rt^*_c}p_u^{-1}\left[\max\{f_k, f_{k+1}, ..., f_{N_{\tilde{A}}}, e^{-rt^*_c}K - S_0\} - f_k\right], \tag{2.18}$$

where $f_i = s_i^A(K - S_d^i)$, $i = 1, ..., N_{\tilde{A}}$, are the given piecewise linear functions of $\tilde{A}$ and

$$A(K) = \max\{K - S_0, e^{-rt^*_c}(p_d A_1(K) + p_u A_2(K))\}. \tag{2.19}$$

The functions A_1 and E_1 as well as the functions A_2 and E_2 will then satisfy the necessary conditions of Theorem 2.2.3 again.

Proof. To see that (2.19) is satisfied we note that for $0 \le K \le K^*$ we have $A_2(K) = 0$ and therefore by the definition of A we have $A(K) = \max\{K - S_0, e^{-rt^*_c}p_d A_1(K)\}$ in that interval. For $K \ge K^*$ we have $A_1(K) = e^{rt^*_c}p_d^{-1}f_k(K)$ and

$$A_2(K) = e^{rt^*_c}p_u^{-1}((\tilde{A}(K) \vee (e^{-rt^*_c}K - S_0)) - f_k(K))$$

and therefore

$$\begin{aligned} A(K) &= \max\{K - S_0, \tilde{A} \vee (e^{-rt^*_c}K - S_0)\} \\ &= \max\{K - S_0, e^{-rt^*_c}(p_d A_1(K) + p_u A_2(K))\}, \end{aligned} \tag{2.20}$$

which holds true by the definition of A and $\tilde{A}$ and the fact that $K - S_0$ dominates $e^{-rt_c^*}K - S_0$ for all $K \geq 0$.

We then have to check that the necessary conditions from Theorem 2.2.3 are satisfied in the left hand-side picture P_1, where our new American is now A_1 and the new European is E_1. To see that A_1 has to be increasing, we can argue that the linear extension of an increasing function is again increasing and the multiplication by a positive constant does not change that. Also we have that A_1 has to be a convex function as it is the maximum over linear functions.

Let us now show that A_1 and E_1 satisfy

$$A_1'(K+)K - A_1(K) \geq E_1'(K+)K - E_1(K) \tag{2.21}$$

for all $K \geq 0$. In the case where $K \leq K^*$ we have $A_1(K) = e^{rt_c^*}p_d^{-1}A(K)$ and $E_1(K) = e^{rt_c^*}p_d^{-1}E(K)$. Since the original functions A and E satisfy this condition and $e^{rt_c^*}p_d^{-1} > 0$ the functions A_1 and E_1 inherit this property.

Next we show that the condition (2.21) also holds for $K \geq K^*$. From Lemma 2.5.2 in the appendix we know that it is enough to check the condition for the atoms of E_1. Starting out with the last atom $K^*e^{r(T-t^*)}$ of E_1 we have

$$\begin{aligned} A_1'(K^*e^{r(T-t^*)}+)K^*e^{r(T-t^*)} - A_1(K^*e^{r(T-t^*)}) &= A_1'(K^*+)K^* - A_1(K^*) \\ &= K^* - A_1(K^*), \end{aligned}$$

where the fact that A_1 is linearly extended beyond K^* gives the first equality and the fact that $A_1'(K^*+) = e^{rt_c^*}p_d^{-1}f_k' = 1$ gives the second equality. Then again, since $A_1(K^*) = E_1(K^*e^{r(T-t^*)})$ and $E_1'(K^*e^{r(T-t^*)}+) = e^{-r(T-t^*)}$ we see that

$$K^* - A_1(K^*) = E_1'(K^*e^{r(T-t^*)}+)K^*e^{r(T-t^*)} - E_1(K^*e^{r(T-t^*)}).$$

This shows that the condition is fulfilled for $K = K^*e^{r(T-t^*)}$, but since E_1 is a convex function it follows that $E_1'(K+)K - E_1(K)$ is increasing in K, which readily implies for $K^* \leq K \leq K^*e^{r(T-t^*)}$

$$\begin{aligned} E_1'(K+)K - E_1(K) &\leq E_1'(K^*e^{r(T-t^*)}+)K^*e^{r(T-t^*)} - E_1(K^*e^{r(T-t^*)}) \\ &\leq A_1'(K^*e^{r(T-t^*)})+)K^*e^{r(T-t^*)} - A_1(K^*e^{r(T-t^*)}) \\ &= A_1'(K+)K - A_1(K), \end{aligned}$$

where the last equality is due to the fact that A_1 is linear beyond K^*. Hence we must have $A_1'(K+)K - A_1(K) \geq E_1'(K+)K - E_1(K)$ for all $K \geq 0$.

To see that E_1 is a lower bound on A_1 we use that $E_1(K) = e^{rt_c^*}p_d^{-1}E(K)$ and $A_1(K) = e^{rt_c^*}p_d^{-1}A(K)$ for $0 \leq K \leq K^*$. As $e^{rt_c^*}p_d^{-1}$ is positive and we have $A(K) \geq$

$E(K)$ in the original picture we obtain $A_1(K) \geq E_1(K)$ for $0 \leq K \leq K^*$. For $K \geq K^*$ we know already that the condition $A_1'(K+)K - A_1(K) \geq E_1'(K+)K - E_1(K)$ has to hold. Combined with the fact that $A_1(K^*) \geq E_1(K^*)$ we obtain that $A_1'(K^*+) \geq E_1'(K^*+)$, which then implies that we must have $A_1(K) \geq E_1(K)$ for as long as the slope of E_1 does not change. By induction on the atoms of μ_1 to the right of K^* we obtain that E_1 is a lower bound on A_1 for all strikes $K \geq 0$.

To show that $\overline{A}_1(K, t_c^*) = E_1(e^{r(T-t^*)}K)$ is an upper bound on A_1 we distinguish the two cases $0 \leq K \leq K^*$ and $K \geq K^*$. In the first case we can use again that $E_1(K) = e^{rt_c^*}p_d^{-1}E(K)$ and that $A_1(K) = e^{rt_c^*}p_d^{-1}A(K)$, which then only has to be combined with $E(e^{r(T-t^*)}K) \geq A(K)$ to obtain the result. The second case follows using the definition of the time t^*, where we have $E(e^{r(T-t^*)}K) \geq A(K)$ and $E(e^{r(T-t^*)}K^*) = A(K^*)$ implying $E_1(e^{r(T-t^*)}K^*) = A_1(K^*)$. Since the last atom of μ_1 is $K^*e^{r(T-t^*)}$ we can conclude that $E_1'(K+) = e^{-r(T-t^*)}$ for any $K \geq K^*e^{r(T-t^*)}$. Then again $\overline{A}_1'(K, t_c^*) = E_1'(e^{r(T-t^*)}K)e^{r(T-t^*)}$, which is 1 and therefore coincides with $A_1'(K)$ for $K \geq K^*$. Hence we showed that $\overline{A}_1(K, t_c^*) \geq A_1(K)$ for all strikes $K \geq 0$.

To be able to split the initial picture into the two subpictures P_1 and P_2 we have to show that the necessary conditions from Theorem 2.2.3 also hold in the right hand-side picture P_2. To see that A_2 is an increasing function we note that $\max\{f_k, ..., f_{m-1}\}$ is increasing, since each linear piece f_i with $i = 1, ..., m-1$ is increasing. Subtracting f_k, does not affect the monotonicity since we have $f_k' \leq f_i'$ for all $i = k, ..., m-1$ as they are ordered by appearance. To obtain A_2 we only have to consider $e^{rt_c^*}p_u^{-1}\max\{\max\{0, f_{k+1} - f_k, ..., f_{m+n-p} - f_k\}, e^{-rt_c^*}K - S_0 - f_k\}$, which is again increasing as the maximum over increasing functions. Further it follows immediately that A_2 has to be convex, since it is the maximum over linear functions multiplied by the positive constant $e^{rt_c^*}p_u^{-1}$. It only remains to show that the condition

$$A_2'(K+)K - A_2(K) \geq E_2'(K+)K - E_2(K) \tag{2.22}$$

holds for all $K \geq 0$. For $0 \leq K \leq \min\{K^*e^{r(T-t^*)}, K_m^A\}$, where $K_m^A = \inf\{K \geq 0 : A(K) = K - S_0\}$, the condition is trivially fulfilled, since the left hand-side is non-negative by the monotonicity and convexity of A_2 and E_2 is constantly 0 there. Lemma 2.5.3 shows that the condition is also fulfilled for $K \geq K_m^A$ as $\hat{A}$ is the extension of A_2 from (2.6), which leaves the case $\min\{K^*e^{r(T-t^*)}, K_m^A\} < K \leq K_m^A$. For $\min\{K^*e^{r(T-t^*)}, K_m^A\} < K \leq K_m^A$ we can write $A_2(K) = p_u^{-1}(e^{rt_c^*}A(K) - p_dA_1(K))$ and $E_2(K) = p_u^{-1}(e^{rt_c^*}E(K) - p_dE_1(K))$. Hence the condition (2.22) simplifies to

$$\begin{aligned} &e^{rt_c^*}(A'(K+)K - A(K)) - p_d(A_1'(K+)K - A_1(K)) \\ &\qquad \geq e^{rt_c^*}(E'(K+)K - E(K)) - p_d(E_1'(K+)K - E_1(K)). \end{aligned} \tag{2.23}$$

Then again we know from the necessary conditions on A and E that $A'(K+)K - A(K) \geq$

$E'(K+)K - E(K)$. Combining this with the fact that for $K \geq K^* e^{r(T-t^*)}$ we have $A_1(K) = K - S_d$ and $E_1(K) = e^{-r(T-t^*)}K - S_d$ we obtain $A_1'(K+)K - A_1(K) = E_1'(K+)K - E_1(K) = S_d$. Therefore the condition has to hold for all strikes $K \geq 0$.

We still have to show that E_2 is a lower bound on A_2. Consider first the case $0 \leq K \leq e^{r(T-t^*)}K^*$, where we know that $E_2(K) = 0$ as the support of μ_2 begins in $e^{r(T-t^*)}K^*$. Since A_2 is given as the maximum over finitely many linear functions and 0 we can immediately conclude that we must have $A_2(K) \geq E_2(K)$ for all strikes $0 \leq K \leq e^{r(T-t^*)}K^*$. In the case where $K \geq e^{r(T-t^*)}K^*$ we know already that $A_2(e^{r(T-t^*)}K^*) \geq E_2(e^{r(T-t^*)}K^*)$ and since we showed that $A_2'(K+)K - A_2(K) \geq E_2'(K+)K - E_2(K)$ has to hold for all $K \geq 0$ we can conclude that $A_2'(e^{r(T-t^*)}K^*+) \geq E_2'(e^{r(T-t^*)}K^*+)$. Hence we have $A_2(K) \geq E_2(K)$ for all strikes where the right hand-side derivative of E_2 remains unchanged. This way we can show by induction on the atoms of E_2 that we must have $A_2(K) \geq E_2(K)$ for all strikes $K \geq 0$.

Finally we are left with showing that $\overline{A}_2$, given by $\overline{A}_2(K) = E_2(e^{r(T-t^*)}K)$ is an upper bound on A_2. As this is trivially fulfilled for $K < K^*$ it is enough to consider $K \geq K^*$. To this end we note that we must have

$$\overline{A}_2(K,0) = p_u^{-1}(e^{rt_c^*}\overline{A}(K,t_c^*) - p_d\overline{A}_1(K,0))$$

by the definition of $\overline{A}_2$ and the representation of E by E_1 and E_2 in Proposition 2.3.5. We can then rewrite $\overline{A}_2 \geq A_2$ as

$$\begin{aligned} p_u^{-1}(e^{rt_c^*}\overline{A}(K,t_c^*) - p_d\overline{A}_1(K,0)) \geq& \\ & e^{rt_c^*}p_u^{-1}(\max\{f_k, f_{k+1}, ..., f_{N_{\tilde{A}}}, e^{-rt_c^*}K - S_0\} - f_k). \quad (2.24) \end{aligned}$$

We can now use the fact that $\overline{A}_1(K,0) = E_1(e^{r(T-t^*)}K)$ and since we have $K \geq K^*$ we obtain further that $p_d\overline{A}_1(K,0) = K - S_d$, which equals exactly $e^{rt_c^*}f_k$. Hence the inequality in (2.24) reduces to

$$\overline{A}(K,t_c^*) \geq \max\{f_k, f_{k+1}, ..., f_{m+n-p}, e^{-rt_c^*}K - S_0\}$$

or equivalently $\overline{A}(K,t_c^*) \geq \max\{\tilde{A}, e^{-rt_c^*}K - S_0\}$, which has to hold as we know that $\overline{A}$ is an upper bound on $\tilde{A}$ with

$$\begin{aligned} \overline{A}(K,t_c^*) &= E(e^{r(T-t^*)}K) \\ &= \overline{A}(Ke^{-rt_c^*}, 0) \\ &\geq Ke^{-rt_c^*} - S_0. \end{aligned}$$

where the last inequality is because $\overline{A}$ is initially an upper bound on A. Hence $\overline{A}_2$ is an upper bound on A_2 for all strikes $K \geq 0$. □

This result shows that the initial picture can be divided into the two subpictures P_1 and P_2, where each of these pictures satisfies again the necessary conditions of Theorem 2.2.3. Note that the splitting of the function $\tilde{A}$ as in (2.19) can be interpreted as separating the immediate exercise from the continuation value. The additional term $(e^{-rt_c^*}K - S_0) - f_k$ in A_2 represents the immediate payoff in S_u at time t^*, since

$$\begin{aligned} e^{-rt_c^*}K - S_0 - f_k &= e^{-rt_c^*}K - S_0 - e^{-rt_c^*}p_d(K - S_d) \\ &= e^{-rt_c^*}p_u(K - S_u), \end{aligned}$$

where the last equality is obtained by using the definition of S_u.

Remark 2.3.7. *It is possible that two or more kinks of the function $\overline{A}(K,t)$ intersect with different linear pieces of $\tilde{A}$ at the same critical time. From the definition of the critical strike K^* in (2.9) it follows that the algorithm will embed the rightmost linear piece first by jumping mass to, say, S_d^{old} and S_u^{old}. In the left hand-side subpicture the critical time for at least one linear piece of $\tilde{A}$ then has to be zero. To embed that piece we need to jump immediately to S_d^{new} and S_u^{new}. This is done by removing the jump to S_d^{old} at the critical time and replacing it by jumps to S_d^{new} and S_u^{new}. The underlying price process can then jump to S_d^{new}, S_u^{new} or S_u^{old} at the critical time. In this way any finite number of linear pieces can be embedded at once, if necessary.*

The representation in (2.19) then has to be extended to allow the embedding of multiple linear pieces at once. Suppose the algorithm embeds k linear pieces, where the j-th linear piece is embedded by jumping mass p_d^j to S_d^j and p_u^j to S_u^j for $j = 1, ..., k$. The representation for $A(K)$ is then given by

$$A(K) = \max\{K - S_0, e^{-rt_c^*}(\tilde{p}_d^k A_1^k(K) + \sum_{j=1}^{k} \tilde{p}_u^j A_2^j(K))\}, \tag{2.25}$$

where $\tilde{p}_d^k = \prod_{i=1}^k p_d^i$ and $\tilde{p}_u^j = p_u^j \prod_{i=1}^{j-1} p_d^i$ for $j = 1, ..., k$.

2.3.4 Convergence of the Algorithm

After having defined the splitting procedure we are now able to state the following proposition, which will then allow us to argue that the embedding algorithm only needs a finite number of steps to produce an admissible price process that has A and E as its American and European put option prices respectively.

Proposition 2.3.8. *Assume the functions A and E are given by (2.4) and satisfy the necessary conditions of Theorem 2.2.3 and Lemma 2.2.1, where A is extended to $\tilde{A}$ as in (2.6) and the European put price function E is given by the marginal distribution $\mu = p_1\delta_{K_1^E} + ... + p_n\delta_{K_n^E}$ at maturity T. Suppose that of the $N_{\tilde{A}}$ linear pieces of $\tilde{A}$*

the linear pieces added to the American put price function A by (2.6) are given by f_i, $i = m, ..., N_{\tilde{A}}$, then f_i, $i = m, ..., N_{\tilde{A}}$ are all embedded together at maturity T.

Proof. Let us assume, without loss of generality, that for the rightmost linear piece f_{m-1} of the original American A there exists at least one strike $K \in [K^A_{m-1}, K^A_m]$ for which $f'_{m-1}(K+)K - f_{m-1}(K) > E'(K+)K - E(K)$, otherwise consider the first linear piece of A to the left of f_{m-1} where this condition is satisfied with respect to the correct interval. This assumption ensures that f_{m-1} is not embedded together with the pieces f_i, $i = m, ..., m+n-p$ at maturity. By the definition of $\tilde{A}$ we have for $i = m, ..., m+n-p-1$ that

$$f'_i(K^E_i+)K^E_i - f_i(K^E_i) = E'(K^E_i+)K^E_i - E(K^E_i),$$

but

$$f'_i(K^E_{i+1}+)K^E_{i+1} - f_i(K^E_{i+1}) < E'(K^E_{i+1}+)K^E_{i+1} - E(K^E_{i+1}),$$

where we have to consider the last linear piece of A separately. Combined with Remark 2.3.4 we can then conclude that the linear pieces f_i, $i = m-1, ..., m+n-p-1$, of $\tilde{A}$ attain their critical time in the right-side endpoint K^E_{i+1} of their respective interval, as it is the first European strike at which the Legendre-Fenchel condition does not hold anymore.

Then again we know from the definition of $\tilde{A}$ and $\overline{A}$ that the linear piece of $\overline{A}$ on which $K^E_i e^{-rT}$ and $K^E_{i+1}e^{-rT}$ lie will coincide with f_i for any $i = m, ..., m+n-p-1$ at its critical time t^*_c, as the two linear functions agree for the strikes $K = 0$ and $K = K^E_{i+1}e^{-r(T-t^*_c)}$. Hence the critical time attained in K^E_{i+1} will coincide with the time obtained by K^E_i. The convexity of $\tilde{A}$ then guarantees that the linear piece f_{i-1} will have a smaller critical time than f_i for all $i = m, ..., m+n-p-1$, as $\overline{A}(K^E_i, t)$ — the kink in $\overline{A}$ responsible for the critical time of f_i — will hit f_{i-1} before hitting f_i. Analogously, we obtain that the last linear piece of A will be embedded the last.

Suppose for now that we are embedding f_{m-1} as first linear piece of A then the American A_2 in P_2 has to coincide with the European E_2, as the strikes where the slopes change are $K = K^E_p, ..., K^E_{m+n-p}$ for both functions and Lemma 2.5.3 from the appendix ensures that we have

$$A'_2(K+)K - A_2(K) = E'_2(K+)K - E_2(K)$$

for all $K \geq 0$.

Finally we still have to rule out that embedding another linear piece f_k, $k < m-1$, first could cause us to embed f_i, $i > m-1$, before f_{m-1}. This can be achieved by using Lemma 2.5.3, noting that the extension of A_2 is obtained by transforming $\tilde{A}$ as in (2.18) omitting to take the maximum with $e^{-rt^*}K - S_0$. The extension will then again be convex by Proposition 2.3.6 allowing us to conduct the same line of argument as above. $\square$

This proposition allows us now to determine how the left and right hand-side subpictures must appear after we have embedded the last linear piece of the original function A that did not coincide (partially) with a linear piece of E. By the definition of the algorithm in Section 2.3.1 and Proposition 2.3.8 we know that only pieces of the original function A are passed down to the left hand-side picture, since none of the linear pieces of the extension are embedded before maturity T. Hence we have $A_1 = E_1 \vee (K - S_d)_+$, where E_1 appears as the American and European prices could coincide on an interval.

Similarly we obtain $A_2 = E_2 \vee (K - S_u)_+$ for the right hand-side picture: having embedded the last linear piece of the original A that did not coincide with a linear piece of E, we are only left with the linear pieces added by step 2 of the algorithm in Section 2.3.1 or a piece coinciding with a linear piece of E. Then again, Lemma 2.5.3 guarantees that all these linear pieces satisfy $A'(K+)K - A(K) = E'(K+)K - E(K)$ for any $K \geq 0$ and that the linear pieces of A and E change at the same strikes implying that they have to coincide. Therefore we have $A_2 = E_2 \vee (K - S_u)_+$.

The following corollary to Proposition 2.3.8 will provide us with an upper bound on the number of steps necessary to embed the given functions A and E.

Corollary 2.3.9. *Suppose the given functions A and E satisfy the necessary conditions of Theorem 2.2.3 and Lemma 2.2.1 and that the number of linear pieces of A is given by N_A, then the total number of steps necessary to embed A and E is bounded above by $2N_A + 1$.*

Proof. Using Proposition 2.3.8 we know that after N_A steps we finished embedding all the original linear pieces of A and are left with at most $N_A + 1$ subpictures. We also see that all the new linear pieces added to $\tilde{A}$ are embedded at maturity by the reasoning above. In each of the subpictures we therefore either have $A_1 = E_1 \vee (K - S_d)_+$ or $A_2 = E_2 \vee (K - S_u)_+$. Hence we are only left with embedding linear pieces of E, which can be done in a single step at maturity T for each of the $N_A + 1$ subpictures. Therefore we can conclude that the whole algorithm has to terminate after at most $2N_A + 1$ steps. $\square$

We are now able to state the major theorem of this paper, which will show that the conditions given in Theorem 2.2.3 are not only necessary for the absence of arbitrage, but indeed sufficient.

Theorem 2.3.10. *Suppose we are given American and European price functions that are piecewise linear and satisfy the conditions given in Theorem 2.2.3 and Lemma 2.2.1. Using Algorithm 1 a model $(\mathbb{Q}, (S_t)_{t \in [0,T]})$ can be constructed such that the discounted* *underlying price process* $(e^{-rt}S_t)_{t\in[0,T]}$ *is a martingale with* $e^{-rT}\mathbb{E}^{\mathbb{Q}}(K-S_T)_+ = E(K)$ *and* $\sup_{0\le\tau\le T}\mathbb{E}^{\mathbb{Q}}[e^{-r\tau}(K-S_\tau)_+] = A(K)$ *for all strikes* $K \ge 0$ *and stopping times* τ *taking values in* $[0,T]$.

Proof. By construction, the underlying S is a martingale, since in each step where we are embedding a linear piece of A we choose the upper node S_u by the martingale property and the process S grows between the jumps at the interest rate. Further we know from Proposition 2.3.5 that $\mathbb{E}^{\mu_1}(e^{-r(T-t^*)}X) = S_d$ and $\mathbb{E}^{\mu_2}(e^{-r(T-t^*)}X) = S_u$, guaranteeing that the martingale property is preserved in the last embedding step in each subpicture.

To see that the European put option prices on the underlying S coincide with the given prices E we recall from Proposition 2.3.5 that the sum of the marginal distributions at maturity T in the subpictures coincides with the distribution implied by E at maturity T.

Finally we still need to show that the American put option prices on the underlying S agree with the given prices A. To this end we first show that it cannot be optimal to exercise between jumps. For a fixed path of the underlying S we have for $t_1 < t < t_2$, where t_1 and t_2 are jump-times for this path, that $e^{-rt_1}K > e^{-rt}K$, and as $e^{-rt_1}S_{t_1} = e^{-rt}S_t$ we obtain $e^{-rt_1}(K-S_{t_1})_+ \ge e^{-rt}(K-S_t)_+$. Hence optimal exercise can only occur at the actual jump times t_j.

If we represent a node $\underline{m}$ by $\underline{m} = a_1a_2...a_k$, where $a_i \in \mathbb{N}$ and $k \in \mathbb{N}$ then we obtain the j-th child of $\underline{m}$ by $\underline{n} = \underline{m}j$. Let us then denote the time at which the child $\underline{n}$ is created by $t(\underline{n})$. The number of children of the node $\underline{m}$ will be denoted by $c(\underline{m})$ and the asset price at that node is $s(\underline{m})$. We can also find the height $h(\underline{n})$ which is the maximum number of splits possible to reach maturity. This can be defined by

$$h(\underline{m}) = \begin{cases} 0 & , \text{ if } t(\underline{m}) = T \\ 1 + \max_{k\le c(\underline{m})}\{h(\underline{m}k)\} & , \text{ otherwise} \end{cases}$$

and corresponds to the maximum number of embedding steps needed after node $\underline{m}$.

Let $A(K, t(\underline{n}), \underline{n})$ be the price which is obtained by following the transformation of A by the algorithm in Section 2.3.1 up to the subpicture, where we just jumped to the node $\underline{n}$. If we can show now that the value of the American put option in each node $\underline{n}$ and for each strike K, denoted by

$$v(K, t(\underline{n}), \underline{n}) = \sup_{0\le\tau\le T-t(\underline{n})} \mathbb{E}[e^{-r\tau}(K-S_\tau)_+ | S_{t(\underline{n})} = s(\underline{n})] \tag{2.26}$$

coincides with the price given by $A(K, t(\underline{n}), \underline{n})$, then we will have shown that

$$\sup_{0\le\tau\le T} \mathbb{E}[e^{-r\tau}(K - S_\tau)_+] = A(K) \tag{2.27}$$

has to hold. By the Dynamic Programming Principle (Theorem 21.7 in Björk [2009]), the optimal stopping problem in (2.26) can be rewritten as the Bellman equation

$$v(K, t(\underline{n}), \underline{n}) = \max\{(K - s(\underline{n}))_+, \sum_{k=1}^{c(\underline{n})} e^{-r(t(\underline{n}k)-t(\underline{n}))} p_k(\underline{n}) v(K, t(\underline{n}k), \underline{n}k)\}, \tag{2.28}$$

where $p_k(\underline{n})$ is the probability of being at $s(\underline{n}k)$ at time $t(\underline{n}k)$ given we are at node $\underline{n}$. Using (2.28) we can now prove that

$$v(K, t(\underline{n}), \underline{n}) = A(K, t(\underline{n}), \underline{n}) \tag{2.29}$$

by induction on the height of the node $\underline{n}$. For a node of height 1 we know from step 6, or 7 resp., of the algorithm in Section 2.3.1 that $A(K, t(\underline{n}), \underline{n}) = (K - s(\underline{n}))_+ \vee E(K)$, where E is the European with contract length $(T - t(\underline{n}))$ and marginal distribution given by the direct children of the node $\underline{n}$ and their transition probabilities. Hence the value of E agrees with the second expression on the right hand-side of (2.28) and therefore we have that for nodes of height 1 the equation in (2.29) is satisfied.

Suppose now that we know $v(K, t(\underline{n}), \underline{n}) = A(K, t(\underline{n}), \underline{n})$ for all nodes up to a height h. Then again, we must have $v(K, t(\underline{n}), \underline{n}) = A(K, t(\underline{n}), \underline{n})$ for nodes $\underline{n}$ of height $h+1$, as the definition of the given prices for nodes of height $h+1$ in (2.25) is the maximum over the immediate exercise at that node, $(K - s(\underline{n}))_+$, and

$$e^{-r(t(\underline{n}1)-t(\underline{n}))}(p_1(\underline{n})v(K, t(\underline{n}1), \underline{n}1) + ... + p_{c(\underline{n})}(\underline{n})v(K, t(\underline{n}c(\underline{n})), \underline{n}c(\underline{n}))).$$

This coincides with the continuation value in the Bellman equation, as each node has by construction exactly $c(\underline{n})$ direct children. Hence we conclude by induction that the American put option prices on the underlying S have to coincide with the given prices A. □

2.4 Conclusion

In this paper we presented no-arbitrage conditions on American put option prices in a model-independent setting, where our only financial assumptions were that we can buy and sell both types of derivatives initially at the given prices, and that we can trade in the underlying frictionlessly at a discrete number of times.

Any violation of the conditions of Theorem 2.2.3 implies the existence of a simple arbitrage strategy. More importantly, we also showed that there always exists a model under which the discounted expected payoffs coincide with the given American and European prices whenever all the conditions are satisfied.

We believe that the results of this paper can be applied in many different ways. Market makers and speculators alike could use the conditions of Theorem 2.2.3 to find misspecifications in the market prices. Simple trading strategies, provided in the proof of Theorem 2.2.3, can then be used to generate arbitrage. Furthermore the necessary conditions present a way of verifying the plausibility of prices obtained by numerical procedures or to extrapolate non-quoted prices from existing market data. Additionally, the results presented in this paper can be used to get an estimate for the model-risk associated with a particular position in the set of American options.

Lastly we think that the results of this paper lead to the following interesting and unanswered questions. Are the conditions of Theorem 2.2.3 also sufficient in a generalised setting where the American and European prices are given as continuous (and convex) functions? What conclusions can be made about the range of prices for portfolios consisting of long and short positions in American put options with different strikes? Is it possible to say something about the exercise behaviour of the long positions with respect to the exercise behaviour of the short positions (c.f. Henderson et al. [2013], who consider a related problem for portfolios of American put options)? What are conditions for the absence of model-independent arbitrage in a market trading American and European put options, where European option prices are known for different maturity dates? How do the conditions on the option prices change if the underlying is allowed to pay dividends?

2.5 Appendix

Lemma 2.5.1. *Suppose the given functions A and E satisfy the necessary conditions (i), (iii) and (iv) of Theorem 2.2.3 and Lemma 2.2.1, then the following conditions are all equivalent:*

(i) $\forall K \geq 0 : \forall \epsilon > 0 :$

$$\frac{A(K+\epsilon) - A(K)}{\epsilon} K - A(K) \geq \frac{E(K+\epsilon) - E(K)}{\epsilon} K - E(K). \tag{2.30}$$

(ii) There exists an $\tilde{\epsilon} = \tilde{\epsilon}(K)$ such that (2.30) holds for all positive ϵ less than $\tilde{\epsilon}$.

(iii) $\forall K \geq 0 : A'(K+)K - A(K) \geq E'(K+)K - E(K).$

Remark. *Any of the conditions in Lemma 2.5.1 above implies that for traded strikes*

$K^E_j \leq K^A_i \leq K^E_{j'} \leq K^A_{i'}$ the discretized version

$$\frac{A(K^A_{i'}) - A(K^A_i)}{K^A_{i'} - K^A_i} K^A_i - A(K^A_i) \geq \frac{E(K^E_{j'}) - E(K^E_j)}{K^E_{j'} - K^E_j} K^E_j - E(K^E_j) \tag{2.31}$$

has to hold. The market exhibits model-independent arbitrage whenever the condition is violated. This follows from the convexity of the function A.

Proof of Lemma 2.5.1. The implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are trivially fulfilled, since the set of ϵ for which we consider the inequality is in each case a subset of the set of ϵ from the statement above.

We then only have to show $(iii) \Rightarrow (i)$ to prove equivalence between the 3 statements. Note further that it is enough to consider the case $K > 0$, since for $K = 0$ we have $A(K) = E(K) = 0$. If we suppose now that the condition $A'(K+)K - A(K) \geq E'(K+)K - E(K)$ holds for $K > 0$ then we can show that $\frac{A(K)-E(K)}{K}$ has to be increasing on any compact interval $[a, b] \subset (0, \infty)$. To prove this we use Theorem 1 from Miller and Vyborny [1986] implying that it is enough to show that $\frac{A(K)-E(K)}{K}$ is continuous on $[a, b]$ and that for all $K \in (a, b)$ the right sided derivative exists and is non negative. Since we know that A and E are convex functions on $(0, \infty)$ we know that their right sided derivatives exist and that $\frac{A(K)-E(K)}{K}$ is continuous on any subinterval $[a, b] \subset (0, \infty)$. Let us consider now the right side derivative of $\frac{A(K)-E(K)}{K}$ given by

$$\begin{aligned}
\partial_+ \frac{A(K) - E(K)}{K} &= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \left(\frac{(A(K+\epsilon) - E(K+\epsilon))}{K+\epsilon} - \frac{(A(K) - E(K))}{K} \right) \\
&= \frac{1}{K^2} (A'(K+)K - A(K)) - (E'(K+)K - E(K)),
\end{aligned}$$

which is non-negative as we have $A'(K+)K - A(K) \geq E'(K+)K - E(K)$. Hence $\frac{A(K)-E(K)}{K}$ is increasing and we can therefore write

$$\begin{aligned}
\epsilon \frac{A(K) - E(K)}{K} &\leq \int_K^{K+\epsilon} \frac{A(u) - E(u)}{u} du \\
&\leq \int_K^{K+\epsilon} (A'(u+) - E'(u+)) du \\
&= A(K+\epsilon) - E(K+\epsilon) - (A(K) - E(K)),
\end{aligned}$$

where the integral in the second line is well defined as a convex function is differentiable almost everywhere. The inequality in the second line is obtained by the assumption $A'(K+)K - A(K) \geq E'(K+)K - E(K)$. We have therefore shown that $\frac{A(K+\epsilon)-A(K)}{\epsilon} K - A(K) \geq \frac{E(K+\epsilon)-E(K)}{\epsilon} K - E(K)$ has to hold for any $\epsilon > 0$ and $K \geq 0$. □

Lemma 2.5.2. *Assume the piecewise linear functions A and E satisfy the necessary conditions (i),(iii) and (iv) of Theorem 2.2.3 and Lemma 2.2.1. Suppose further* *that their kinks are in $K_1^A, ..., K_m^A$ and $K_1^E, ..., K_n^E$ respectively. Then the condition $A'(K+)K - A(K) \geq E'(K+)K - E(K)$ holds for all strikes $K \geq 0$ if and only if it holds in the kinks of E.*

Proof. We only have to show that it is enough to have the condition fulfilled in all strikes K_i^E, $i = 1, ..., m$, since the other implication is trivially fulfilled. Suppose now the condition is fulfilled in K_i^E and choose a strike $K \in [K_i^E, K^A]$, where $K^A = \min_{j=1,...,m}\{K_j^A : K_i^E < K_j^A < K_{i+1}^E\}$. This way we have

$$\begin{aligned}
A'(K+)K - A(K) &\geq A'(K_i^E+)K - A(K) \\
&= A'(K_i^E+)K - A'(K_i^E+)(K - K_i^E) - A(K_i^E) \\
&= A'(K_i^E+)K_i^E - A(K_i^E) \\
&\geq E'(K_i^E+)K_i^E - E(K_i^E),
\end{aligned}$$

where the first inequality holds since $A'(K^A+) \geq A'(K^A-)$. To obtain the equality in the second line we simply use the fact that for any K in that interval we can write $A(K) = A'(K_i^E+)(K - K_i^E) + A(K_i^E)$ and for the last inequality that the condition is known to hold in K_i^E. But then again we can rewrite

$$\begin{aligned}
E'(K_i^E+)K_i^E - E(K_i^E) &= E'(K_i^E+)K - E'(K_i^E+)(K - K_i^E) - E(K_i^E) \\
&= E'(K+)K - E(K).
\end{aligned}$$

Hence we have $A'(K+)K - A(K) \geq E'(K+)K - E(K)$. This leaves us to show that for any strike $K \in (K^A, K_{i+1}^E)$ the condition is fulfilled, but we can use the same argument now, inductively on the kinks of A between K^A and K_{i+1}^E, we have that the condition has to hold for any strike $K \in [K_i^E, K_{i+1}^E)$. Since the strike K_i^E was taken arbitrarily we know that the condition has to hold for all strikes $K \in [K_1^E, \infty)$. Then again for any strike prior to K_1^E the condition is trivially fulfilled, since we know that A is increasing and convex and therefore has to satisfy $A'(K+)K - A(K) \geq 0$. □

Lemma 2.5.3. *Assume the functions A and E given by (2.4) satisfy the necessary conditions of Theorem 2.2.3 and Lemma 2.2.1, where A is extended to $\tilde{A}$ as in (2.6) and the European put price function E with contract length $(T - t^*_{old})$ is given by the marginal distribution $\mu = p_1\delta_{K_1^E} + ... + p_n\delta_{K_n^E}$ with mean $\mathbb{E}^\mu(X) = e^{r(T-t^*_{old})}S_0$ at maturity T. Suppose further that the time of the next jump t^* and the associated critical strike K^* were determined as in Section 2.3.2 at which point the linear piece $s_k^A(K - S_d^k)$ of $\tilde{A}$, denoted by f_k, is embedded.*

Moreover we assume that E_2 is given by Proposition 2.3.5 and define $\hat{A}(K) = e^{rt_c^*}p_u^{-1}\left(\max\{f_m, ..., f_{N_{\tilde{A}}}\} - f_k\right)$ then it follows that

$$\hat{A}'(K+)K - \hat{A}(K) = E_2'(K+)K - E_2(K)$$

for $K \geq K_p^E$, where K_p^E and $N_{\tilde{A}}$ are defined in Section 2.3.1.

Remark. *The result of Lemma 2.5.3 shows that we obtain the extension in the right hand-side sub-picture P_2 by transforming the extension in the original picture P, as the functions $\hat{A}$ and $\tilde{A}_2$ coincide in K_p^E and the Legendre-Fenchel condition is satisfied with equality for $K \geq K_p^E$.*

Proof of Lemma 2.5.3. We know already from Lemma 2.5.2 that it is enough to check the condition in the atoms of E_2, which by the definition of E_2 in Proposition 2.3.5 coincide with the ones of E. Consider therefore K_j^E, where $j \in \{p, ..., n\}$ and assume, without loss of generality, that the American A to the right of K_j^E is given by f_i then we have

$$\hat{A}'(K_j^E+)K_j^E - \hat{A}(K_j^E) = \\ e^{rt_c^*} p_u^{-1} [(f_j'(K_j^E+)K_j^E - f_j(K_j^E)) - (f_k'(K_j^E+)K_j^E - f_k(K_j^E))].$$

Furthermore we can use Proposition 2.3.5 to write

$$E_2(K_j^E) = p_u^{-1}(e^{rt_c^*} E(K_j^E) - p_d E_1(K_j^E)),$$

since Proposition 2.3.3 guarantees that we have $K_j^E \geq K^* e^{r(T-t^*)}$. As we are only considering strikes where $A'(K+)K - A(K) = E'(K+)K - E(K)$ holds, we get that the equation $\hat{A}'(K_j^E+)K_j^E - \hat{A}(K_j^E) = E_2'(K_j^E+)K_j^E - E_2(K_j^E)$ reduces to

$$e^{rt_c^*}(f_k'(K_j^E+)K_j^E - f_k(K_j^E)) = p_d(E_1'(K_j^E+)K_j^E - E_1(K_j^E)).$$

This equality has to hold though, since we know that $f_k(K) = e^{-rt_c^*} p_d(K - S_d)$ and $E_1(K) = e^{-r(T-t^*)}K - S_d$ for $K \geq K^* e^{r(T-t^*)}$. $\square$

Chapter 3

Arbitrage situations in markets trading American and co-terminal European options

We consider a market in which American and co-terminal European put options are traded for finitely many strikes. From the given prices it is then either possible to construct American and European put price functions satisfying the conditions given in Theorem 2.2.3 of Chapter 2 or to construct a portfolio generating model-independent arbitrage.

3.1 Problem setting

Suppose that both American put options and co-terminal European put options are each traded at finitely many strikes in the market. We are then interested in investigating arbitrage opportunities in this market that hold under any model.

In the paper by Davis and Hobson [2007] no-arbitrage conditions for markets trading only in European call options are provided. These can be translated into the following conditions for the absence of arbitrage in markets trading in European put options as has been pointed out already in Section 2.2.

Lemma 3.1.1. *Suppose the prices of European put options with maturity T are given for a set of finitely many strikes $\mathbb{K}_0^E$ and extended to the European put price function E as in Lemma 2.2.1 in Chapter 2. The current price of the underlying asset is denoted by S_0. Then the European put prices are free of model-independent and weak arbitrage opportunities if and only if the following conditions are satisfied:*

1. *The European put price function E is increasing and convex in the strike K.*

2. *The function $(e^{-rT}K - S_0)_+$ is a lower bound for E.*

3. The function $e^{-rT}K$ *is an upper bound for* E.

4. For any $K \geq 0$ *with* $E(K) > e^{-rT}K - S_0$ *we have* $E'(K+) < e^{-rT}$.

Given that the European put option prices are free of arbitrage, the price function for co-terminal American put options needs to comply with the following conditions to guarantee absence of arbitrage according to Theorem 2.3.10 in Chapter 2.

Theorem 3.1.2. *Suppose we are given American and European price functions* A *and* E *that are piecewise linear and that each of the corresponding options has maturity* T. *If the European price function satisfies the conditions given in Lemma 3.1.1 while the American price function satisfies*

(i) A *is increasing and convex in* K,

(ii) For all $K \geq 0$ *we have*

$$A'(K+)K - A(K) \geq E'(K+)K - E(K),$$

(iii) The function $\max\{E(K), K - S_0\}$ *is a lower bound for* $A(K)$,

(iv) The function $\overline{A}(K) := E(e^{rT}K)$ *is an upper bound for* $A(K)$,

then there exists a model $(\mathbb{Q}, (S_t)_{t\in[0,T]})$ *such that the discounted underlying price process* $(e^{-rt}S_t)_{t\in[0,T]}$ *is a martingale with* $e^{-rT}\mathbb{E}^{\mathbb{Q}}(K - S_T)_+ = E(K)$ *and*

$$\sup_{0\leq\tau\leq T} \mathbb{E}^{\mathbb{Q}}[e^{-r\tau}(K - S_\tau)_+] = A(K)$$

for all strikes $K \geq 0$ *and stopping times* τ *taking values in* $[0, T]$.

It is not enough, however, to determine whether the conditions of Lemma 3.1.1 and Theorem 3.1.2 are satisfied by the functions A^{lin} and E^{lin}, obtained by interpolating linearly between the traded option prices, as more sophisticated functions A and E may exist that satisfy them. Thus we will address in the sequel the problem of finding a suitable algorithm for the construction of American and European price functions complying with the no-arbitrage conditions. Moreover, we will show that there exists arbitrage in the market should the algorithm fail to produce admissible price functions.

As we consider markets where American and European options are traded at (possibly) different strikes, we will replace condition (ii) of Theorem 3.1.2 by the following equivalent condition: for any combination of strikes $K_i, K_{i'} \in \mathbb{K}_0^A$ and $K_j, K_{j'} \in \mathbb{K}_0^E$ with $K_j < K_{j'}$, $K_i < K_{i'}$, $K_j \leq K_i$ and $K_{j'} \leq K_{i'}$ we must have

$$\frac{A(K_{i'}) - A(K_i)}{K_{i'} - K_i}K_i - A(K_i) \geq \frac{E(K_{j'}) - E(K_j)}{K_{j'} - K_j}K_j - E(K_j). \tag{3.1}$$

That the two conditions are equivalent follows immediately from Lemma A.1 in Chapter 2 together with the convexity of the price functions A and E. Furthermore, Proposition 3.10.1 in the appendix provides an arbitrage portfolio, consisting only of traded options, if (3.1) is violated.

We will refer to the second condition in Theorem 3.1.2 subsequently as the Legendre-Fenchel condition, since it can be expressed using the homonymous transform. To see this, recall that the Legendre-Fenchel transform of a convex function $f : \mathbb{R} \to \mathbb{R}$ is given by $f^*(k) = \sup_{x\in\mathbb{R}}\{kx - f(x)\}$. We can thus rewrite the condition as $A^*(A'(K+)) \geq E^*(E'(K+))$. Moreover, we would like to point out that we will frequently use the term convex conjugate to refer to the Legendre-Fenchel transform.

Before we continue, we will make some assumptions on the market and its participants. The first assumption is mild and related to the behaviour of market participants.

Assumption 3.1.3. *Any market participant prefers more money to less and will act accordingly.*

In particular, we will use the argument that no one would purchase an American put option for more than its immediate exercise value if he or she intended to sell it off immediately again.

In the remaining assumptions we will restrict ourselves to a subset of the markets trading American and co-terminal European put options.*

Assumption 3.1.4. *There exists at least one in-the-money American put option in the market that trades at its intrinsic value.*

Assumption 3.1.5. *There exists at least one European put option in the market that trades at its non-zero lower bound. That is, European options for some strike $K > e^{rT}S_0$ are traded in the market at $e^{-rT}K - S_0$, assuming that the current value of the underlying is given by S_0.*

Notation 3.1.6. *The set of markets trading in American and co-terminal European put options satisfying Assumption 3.1.3, Assumption 3.1.4 and Assumption 3.1.5 will be denoted by $\mathcal{M}$.*

Using this notation the result we are interested in showing can be written as follows.

Theorem 3.1.7. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are given by $\mathcal{P}_0^* \in \mathcal{M}$. Then either the algorithm provided in Section 3.5 will construct American and European price functions satisfying the no-arbitrage conditions of Lemma 3.1.1 and Theorem 3.1.2 or there exists arbitrage in the market.*

*This restriction is not necessary and by a slight modification of the algorithm the result can be extended to general markets. However, the proof of the result for general markets is considerably more technical.

We will further improve the presentation by introducing the following notation.

Notation 3.1.8. *The functions we consider are piecewise linear, thus the slope between two neighbouring strikes x_k and x_{k+1}, where $x_{k+1} > x_k$, is constant and given by $f'(x_k+) = (f(x_{k+1}) - f(x_k))/(x_{k+1} - x_k)$. Let us denote the y-intercept of the tangent by $cc(f, x_k, x_{k+1})$, that is*

$$cc(f, x_k, x_{k+1}) = f'(x_k+)(0 - x_k) + f(x_k) \tag{3.2}$$

and note that $cc(f, x_k, x_{k+1}) = -f^*(f'(x_k+))$.

Notation 3.1.9. *This notation can further be extended to the linear interpolation between the prices of two different functions f and g at the strikes x_k and x_{k+1}, respectively. For that purpose we write*

$$cc(f, g, x_k, x_{k+1}) = f(x_k) - x_k(g(x_{k+1}) - f(x_k))/(x_{k+1} - x_k).$$

The rest of this chapter is organised as follows. In Section 3.2 we discuss the setup in more detail and introduce some notation. We continue in Section 3.3 by deriving no-arbitrage bounds on the prices of individual options. These bounds are then used in the next section to construct price functions. In addition, possible price-misspecifications are highlighted and corrections are suggested. The final algorithm is then presented in Section 3.5. In Section 3.6 we proceed by identifying the different situations in which the algorithm is unable to construct admissible price functions and show that in each of these situations there has to exist arbitrage in the market. Section 3.7 is dedicated to proving that a set of prices that is admissible up to a strike ξ_i can be extended by the algorithm to a set of prices admissible up to the next strike ξ_{i+1} if the algorithm does not stop due to an arbitrage. In Section 3.8 we argue that the algorithm converges and that either the resulting price functions satisfy the conditions given in Lemma 3.1.1 and Theorem 3.1.2 or that there exists arbitrage in the market. Section 3.9 concludes the chapter.

3.2 Setup

Suppose now that American options trade for a finite set of strikes $0 < K_1^A < K_2^A < ... < K_{m_1}^A < \infty$ and that the price for an American option with strike K_i^A, $i \in \{1, ..., m_1\}$, is denoted by $\hat{\mathbf{a}}_i$. Similarly, we assume that European options are traded in the market for a finite set of strikes $0 < K_1^E < K_2^E < ... < K_{m_2}^E < \infty$ and we denote the price for a European option with strike K_j^E, $j \in \{1, ..., m_2\}$, by $\hat{\mathbf{e}}_j$. Note further that a put option with strike zero cannot have a positive payoff and thus its price has to be 0. Hence, we can always assume that both American and European options with strike zero are traded in the market at price 0. Using $K_0^A = K_0^E = 0$, we introduce the ordered set of prices

$$\mathcal{P}_0^* = \{(\mathbf{0}, K_0^A), (\hat{\mathbf{a}}_1, K_1^A), ..., (\hat{\mathbf{a}}_{m_1}, K_{m_1}^A); (\mathbf{0}, K_0^E), (\hat{\mathbf{e}}_1, K_1^E), ..., (\hat{\mathbf{e}}_{m_2}, K_{m_2}^E)\}$$

and the sets of strikes

$$\begin{aligned}\mathbb{K}^A(\mathcal{P}_0^*) &= \{K_0^A, K_1^A, K_2^A, ..., K_{m_1}^A\},\\ \mathbb{K}^E(\mathcal{P}_0^*) &= \{K_0^E, K_1^E, K_2^E, ..., K_{m_2}^E\},\\ \mathbb{K}(\mathcal{P}_0^*) &= \mathbb{K}^A(\mathcal{P}_0^*) \cup \mathbb{K}^E(\mathcal{P}_0^*).\end{aligned}$$

As we aim to provide price functions satisfying the no-arbitrage conditions we will keep track of their construction using sets like

$$\begin{aligned}\mathcal{P} &= \{\mathcal{P}^A; \mathcal{P}^E\} \\ &= \{(\mathbf{a}_0, {}^AK_0), ..., (\mathbf{a}_{n_1}, {}^AK_{n_1}); (\mathbf{e}_0, {}^EK_0), ..., (\mathbf{e}_{n_2}, {}^EK_{n_2})\},\end{aligned} \tag{3.3}$$

that no longer consist only of the prices of traded options. Without loss of generality we furthermore assume that the strikes in $\mathcal{P}$ are ordered, that is ${}^AK_0 < {}^AK_1 < ... < {}^AK_{n_1}$ and ${}^EK_0 < {}^EK_1 < ... < {}^EK_{n_2}$. Moreover, we write

$$\begin{aligned}\mathbb{K}^A(\mathcal{P}) &= \{{}^AK_0, {}^AK_1, ..., {}^AK_{n_1}\},\\ \mathbb{K}^E(\mathcal{P}) &= \{{}^EK_0, {}^EK_1, ..., {}^EK_{n_2}\}\end{aligned}$$

and

$$\mathbb{K}(\mathcal{P}) = \mathbb{K}^A(\mathcal{P}) \cup \mathbb{K}^E(\mathcal{P}).$$

In addition, we will denote the largest strike at which the price of an American option exceeds its intrinsic value by $K_{l_1(\mathcal{P})}$ where

$$l_1(\mathcal{P}) = \underset{0 \le i \le n_1}{\arg\max}\{\mathbf{a}_i > {}^AK_i - S_0\}. \tag{3.4}$$

Analogously, we denote the largest strike at which the price of a European option exceeds the lower bound $e^{-rT}K - S_0$ by $K_{l_2(\mathcal{P})}$ where

$$l_2(\mathcal{P}) = \underset{0 \le j \le n_2}{\arg\max}\{\mathbf{e}_j > {}^EK_j e^{-rT} - S_0\}. \tag{3.5}$$

Note that for $\mathcal{P} \in \mathcal{M}$, we must have $l_1(\mathcal{P}) < n_1$ and $l_2(\mathcal{P}) < n_2$.

To obtain price functions we will interpolate linearly between the respective prices in $\mathcal{P}$, that is, we define for any strike $K \geq 0$

$$A(K,\mathcal{P}) = \begin{cases} (K - {}^AK_{n_1}) + \mathbf{a}_{n_1}, & \text{if } K \geq {}^AK_{n_1} \\ \frac{\mathbf{a}_{i+1}-\mathbf{a}_i}{{}^AK_{i+1}-{}^AK_i}(K - {}^AK_i) + \mathbf{a}_i, & \text{if } K \in [{}^AK_i, {}^AK_{i+1}] \end{cases} \tag{3.6}$$

where $i \in \{0, 1, ..., n_1 - 1\}$ and analogously for any strike $K \geq 0$

$$E(K,\mathcal{P}) = \begin{cases} (K - {}^EK_{n_2}) + \mathbf{e}_{n_1}, & \text{if } K \geq {}^EK_{n_2} \\ \frac{\mathbf{e}_{j+1}-\mathbf{e}_j}{{}^EK_{j+1}-{}^EK_j}(K - {}^EK_j) + \mathbf{e}_j, & \text{if } K \in [{}^EK_j, {}^EK_{j+1}] \end{cases} \tag{3.7}$$

where $j \in \{0, 1, ..., n_2 - 1\}$. Note that the price functions are extended like this beyond the respective final strike to accommodate the fact that both the American and European price function will coincide with its respective lower bound for large strikes.

Furthermore, it follows from the definition $\overline{A}(K,\mathcal{P}) = E(Ke^{rT},\mathcal{P})$ that the upper bound is given as the linear interpolation between the prices $\overline{\mathbf{a}}_j$ at ${}^EK_je^{-rT}$, where $\overline{\mathbf{a}}_j = \mathbf{e}_j$ and $0 \leq j \leq n_2$.

3.3 No-arbitrage bounds on option prices

To guarantee that the price functions we construct are admissible the prices have to lie within the range implied by the no-arbitrage conditions. Let us start by examining the bounds on the European price function provided by convexity. Suppose we want to find the upper bound on the price of a European option with strike K given the set of prices $\mathcal{P}^E$, then for $0 \leq K < {}^EK_{n_2}$ there exists $j = \arg\max_{0 \leq j' \leq n_2}\{{}^EK_{j'} \in \mathbb{K}^E(\mathcal{P}) : {}^EK_{j'} \leq K\}$ and the upper bound is given by

$$E_{ub}(K,\mathcal{P}) = \frac{\mathbf{e}_{j+1} - \mathbf{e}_j}{{}^EK_{j+1} - {}^EK_j}(K - {}^EK_j) + \mathbf{e}_j. \tag{3.8}$$

In the absence of any restriction on the upper price for a European option with strike $K \geq {}^EK_{n_2}$, we set $E_{ub}(K,\mathcal{P}) = e^{-rT}K - S_0$.

Consider now the lower bound on a European option implied by convexity. In this case we obtain two individual bounds, one from each side. Let us first discuss the left hand-side lower bound given the set of prices $\mathcal{P}^E$. As we need to know at least the prices of two European options to the left, the left hand-side lower bound is not defined on $[{}^EK_0, {}^EK_1)$, however, we know that the price for a European put option cannot be negative and we can therefore use 0 on that interval. For any strike $K \geq {}^EK_1$ there exists $j_1 = \arg\max_{1 \leq j' \leq n_2}\{{}^EK_{j'} \in \mathbb{K}^E(\mathcal{P}) : {}^EK_{j'} \leq K\}$ and we write

$$E_{lb}^{lhs}(K,\mathcal{P}) = \frac{\mathbf{e}_{j_1} - \mathbf{e}_{j_1-1}}{{}^EK_{j_1} - {}^EK_{j_1-1}}(K - {}^EK_{j_1}) + \mathbf{e}_{j_1}. \tag{3.9}$$

Similarly, we obtain for $K \leq {}^E K_{n_2-1}$ the right hand-side lower bound

$$E_{lb}^{rhs}(K, \mathcal{P}) = \frac{\mathbf{e}_{j_2+1} - \mathbf{e}_{j_2}}{{}^E K_{j_2+1} - {}^E K_{j_2}}(K - {}^E K_{j_2}) + \mathbf{e}_{j_2}. \tag{3.10}$$

where $j_2 = \arg\min_{1 \leq j' \leq n_2 - 1}\{{}^E K_{j'} \in \mathbb{K}^E(\mathcal{P}) : {}^E K_{j'} > K\}$. For any strike $K \geq {}^E K_{n_2-1}$ we will use the universal lower bound $e^{-rT}K - S_0$.

Since we can only rule out a violation of convexity if both these bounds hold, we can conclude that the lower bound on the price of a European option with strike K is given by

$$E_{lb}(K, \mathcal{P}) = \max\{E_{lb}^{lhs}(K, \mathcal{P}), E_{lb}^{rhs}(K, \mathcal{P})\}. \tag{3.11}$$

Analogously, we can deduce from the convexity of the American price function that the option prices have to lie between the following no-arbitrage bounds. The upper bound on the price of an American option with strike $K \in [{}^A K_0, {}^A K_{n_1})$ is given by

$$A_{ub}(K, \mathcal{P}) = \frac{\mathbf{a}_{i+1} - \mathbf{a}_i}{{}^A K_{i+1} - {}^A K_i}(K - {}^A K_i) + \mathbf{a}_i, \tag{3.12}$$

where $i = \arg\max_{0 \leq i' \leq n_1}\{{}^A K_{i'} \in \mathbb{K}^A(\mathcal{P}) : {}^A K_{i'} \leq K\}$. For the price of an American option with strike $K \geq {}^A K_{n_1}$ the upper bound is set to be $A_{ub}(K, \mathcal{P}) = K - S_0$.

The left hand-side lower bound on the price of an American option with strike K is either given by 0 for any strike $K \in [{}^A K_0, {}^A K_1)$ or by

$$A_{lb}^{lhs}(K, \mathcal{P}) = \frac{\mathbf{a}_{i_1} - \mathbf{a}_{i_1-1}}{{}^A K_{i_1} - {}^A K_{i_1-1}}(K - {}^A K_{i_1}) + \mathbf{a}_{i_1}, \tag{3.13}$$

for $K \geq {}^A K_1$ where $i_1 = \arg\max_{1 \leq i' \leq n_1}\{{}^A K_{i'} \in \mathbb{K}^A(\mathcal{P}) : {}^A K_{i'} \leq K\}$. Likewise, the convexity provides the following right hand-side lower bound on the price of an American option with strike $K \in [{}^A K_0, {}^A K_{n_1-1})$

$$A_{lb}^{rhs}(K, \mathcal{P}) = \frac{\mathbf{a}_{i_2+1} - \mathbf{a}_{i_2}}{{}^A K_{i_2+1} - {}^A K_{i_2}}(K - {}^A K_{i_2}) + \mathbf{a}_{i_2}. \tag{3.14}$$

where $i_2 = \arg\min_{1 \leq i' \leq n_1 - 1}\{{}^A K_i' \in \mathbb{K}^A(\mathcal{P}) : {}^A K_{i'} > K\}$. For any strike $K \geq {}^A K_{n_1-1}$ we can use the universal lower bound $K - S_0$. The lower bound is then defined to be

$$A_{lb}(K, \mathcal{P}) = \max\{A_{lb}^{lhs}(K, \mathcal{P}), A_{lb}^{rhs}(K, \mathcal{P})\}. \tag{3.15}$$

The second condition in Theorem 3.1.2, the Legendre-Fenchel condition, yields an additional constraint on the upper bound of European option prices. For that purpose let us assume that the Legendre-Fenchel condition, provided in (3.1), holds with equality. Suppose, moreover, that ${}^A K_i, {}^A K_{i'} \in \mathbb{K}^A(\mathcal{P})$ and ${}^E K_j, {}^E K_{j'} \in \mathbb{K}^E(\mathcal{P})$

satisfy ${}^E K_j = {}^E K_{j'-1}$, ${}^E K_j = {}^A K_i$ and ${}^E K_{j'} = {}^A K_{i'}$, then we obtain

$$E_{lf}({}^E K_{j'}, \mathcal{P}) = \mathbf{a}_{j'} - \frac{{}^E K_{j'}}{{}^E K_{j'-1}}[\mathbf{a}_{j'-1} - \mathbf{e}_{j'-1}]. \tag{3.16}$$

for ${}^E K_{j'-1} > 0$. In the case where the strike ${}^E K_{j'-1} = 0$ the Legendre-Fenchel condition does not provide additional information and thus we will set $E_{lf}({}^E K_{j'}, \mathcal{P}) = \mathbf{a}_{j'}$ which always has to hold.

Likewise, the Legendre-Fenchel condition gives the following lower bound on the price of an American option with strike ${}^A K_{i'}$

$$A_{lf}({}^A K_{i'}, \mathcal{P}) = \mathbf{e}_{i'} + \frac{{}^A K_{i'}}{{}^A K_{i'-1}}[\mathbf{a}_{i'-1} - \mathbf{e}_{i'-1}]. \tag{3.17}$$

whenever ${}^A K_{i'-1} > 0$ and we will set $A_{lf}({}^A K_{i'}, \mathcal{P}) = \mathbf{e}_{i'}$ for ${}^A K_{i'-1} = 0$.

To guarantee that condition (iv) of Theorem 3.1.2 holds an additional right hand-side lower bound on the price of an American option with strike K has to be introduced. Setting $i_1 = \min\{1 \leq i' \leq n_1 : {}^A K_{i'} > K\}$, the convexity of the American price function implies that

$$A(K, \mathcal{P}) \geq \frac{\overline{A}(K_q, \mathcal{P}) - \mathbf{a}_{i_1}}{K_q - {}^A K_{i_1}}(K - {}^A K_{i_1}) + \mathbf{a}_{i_1}$$

for any $K_q \in ({}^A K_{i_1}, {}^A K_{i_1+1}]$, where $1 \leq i_1 \leq n_1$ and ${}^A K_{n_1+1} = \infty$. As we mentioned already, the upper bound $\overline{A}$ will be piecewise linear by construction and we can therefore conclude that it is enough to consider the lower bounds obtained by the kinks of $\overline{A}$. We will see below that the number of lower bounds can be reduced further to the ones implied by the kinks in $\overline{A}$ at strikes of the type ${}^E K_j e^{-rT} \in ({}^A K_{i_1}, {}^A K_{i_1+1}]$, where $0 \leq j \leq m_2$, $1 \leq i_1 \leq n_1$ and ${}^A K_{n_1+1} = \infty$. Denoting this set of strikes by S_{i_1}, the following ancillary lower bound for the price of an American put option with strike $K \in ({}^A K_{i_1-1}, {}^A K_{i_1})$, $1 \leq i_1 \leq n_1$, is obtained

$$A_{lb}^{\overline{A},r}(K, \mathcal{P}) = \max_{\substack{1 \leq j \leq m_2 \\ K_j^E e^{-rT} \in S_{i_1}}} \left\{ \frac{\overline{\mathbf{a}}_j - \mathbf{a}_{i_1}}{K_j^E e^{-rT} - {}^A K_{i_1}}(K - {}^A K_{i_1}) + \mathbf{a}_{i_1} \right\} \tag{3.18}$$

where we set $A_{lb}^{\overline{A},r}(K, \mathcal{P}) = -\infty$ if $S_{i_1} = \emptyset$ or $K \geq {}^A K_{n_1}$. Analogously, we obtain the left hand-side lower bound for American options with strike K. For that purpose we set $i_2 = \max\{1 \leq i' \leq n_1 : {}^A K_{i'} \leq K\}$. We can then write

$$A_{lb}^{\overline{A},l}(K, \mathcal{P}) = \max_{\substack{1 \leq j \leq m_2 \\ K_j^E e^{-rT} \in S_{i_2-1}}} \left\{ \frac{\mathbf{a}_{i_2} - \overline{\mathbf{a}}_j}{{}^A K_{i_2} - K_j^E e^{-rT}}(K - {}^A K_{i_2}) + \mathbf{a}_{i_2} \right\}, \tag{3.19}$$

where we again set $A_{lb}^{\overline{A},l}(K_p, \mathcal{P}) = -\infty$ if $S_{i_2-1} = \emptyset$ or $K \leq {}^AK_1$. Combining these two bounds we get

$$A_{lb}^{\overline{A}}(K, \mathcal{P}) = \max\{A_{lb}^{\overline{A},l}(K, \mathcal{P}), A_{lb}^{\overline{A},r}(K, \mathcal{P})\}. \tag{3.20}$$

3.4 Construction of the price functions

We aim to generate piecewise linear price functions A and E that satisfy the no-arbitrage conditions and that are consistent with a given set of finitely many option prices $\mathcal{P}_0^* \in \mathcal{M}$. To do this, we move along the strikes in $\mathbb{K}(\mathcal{P}_0^*)$ and compute the price of either the American or European option with that strike. We will keep track of the computed option prices by gradually extending the initial set $\mathcal{P}_0^*$. In particular, having calculated the price for the non-traded option with strike K_p, $p \geq 1$, we define the set of prices $\mathcal{P}_{0,p}$ to be given by

$$\mathcal{P}_{0,p} = \begin{cases} (\mathcal{P}_{0,p-1}^A \cup (\mathbf{a}_p, K_p); \mathcal{P}_{0,p-1}^E), & \text{if } K_p \in \mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_0^*) \\ (\mathcal{P}_{0,p-1}^A; \mathcal{P}_{0,p-1}^E \cup (\mathbf{e}_p, K_p)), & \text{if } K_p \in \mathbb{K}^A(\mathcal{P}_0^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*) \\ \mathcal{P}_{0,p-1}, & \text{if } K_p \in \mathbb{K}^A(\mathcal{P}_0^*) \cap \mathbb{K}^E(\mathcal{P}_0^*). \end{cases}$$

where $\mathcal{P}_{0,0} = \mathcal{P}_0^*$. Note that this has the effect that the first p strikes of $\mathbb{K}^A(\mathcal{P}_{0,p})$ and $\mathbb{K}^E(\mathcal{P}_{0,p})$ coincide.

3.4.1 Computation of the prices for strikes in $\mathbb{K}(\mathcal{P}_0^*)$

To ensure that the algorithm successfully constructs American and European price functions when the market is free of arbitrage, the prices have to be computed so as to yield the widest possible no-arbitrage bounds for the remaining prices. For that purpose we consider the upper and lower bounds derived in the previous section and make the following observations. According to (3.9), the initial left hand-side lower bound for European option prices between two traded strikes will decrease when the prices in the previous interval are increased. To see this consider the strikes ${}^EK_{l-1}, {}^EK_l \in \mathbb{K}(\mathcal{P}_{0,l})$ with ${}^EK_l \in \mathbb{K}^E(\mathcal{P}_0^*)$. If we assume that the strike ${}^EK_l$ corresponds to $K_j^E \in \mathbb{K}^E(\mathcal{P}_0^*)$, we can conclude from (3.9) that an increase in the price $\mathbf{e}_{l-1}$ yields a decreased left hand-side lower bound E_{lb}^{lhs} between the two strikes K_j^E and K_{j+1}^E. Moreover, the upper bound $\overline{A}$ is maximised by maximising E. In addition, the definition of the upper bound E_{lf} in (3.16) shows that decreasing A or increasing E, respectively, in the previous strike results in an increase of E_{lf} in the current strike. Analogously, we see that the lower bound A_{lf}, given by (3.17), will decrease whenever A is decreased or E is increased, respectively, in the previous strike.

Taking all these considerations into account, we conclude that the widest possible no-arbitrage bounds are obtained by maximising the European price function E while minimising the American price function A. We will therefore compute the prices as follows. For $K_p \in \mathbb{K}^A(\mathcal{P}_0^*)\backslash\mathbb{K}^E(\mathcal{P}_0^*)$ the price of a European option with strike K_p is

$$\mathbf{e}_p = \min\left\{E_{ub}(K_p, \mathcal{P}_{0,p-1}), E_{lf}(K_p, \mathcal{P}_{0,p-1})\right\},$$

and for $K_p \in \mathbb{K}^E(\mathcal{P}_0^*)\backslash\mathbb{K}^A(\mathcal{P}_0^*)$ the price of an American option with strike K_p is given by

$$\mathbf{a}_p = \max\left\{A_{lb}(K_p, \mathcal{P}_{0,p-1}), A_{lb}^{\overline{A}}(K_p, \mathcal{P}_{0,p-1}), A_{lf}(K_p, \mathcal{P}_{0,p-1})\right\}.$$

3.4.2 Price-misspecifications and their corrections

Computing the prices for American and European put options like this does not guarantee that $\mathbf{e}_p \geq E_{lb}(K_p, \mathcal{P}_0^*)$, $\mathbf{a}_p \leq A_{ub}(K_p, \mathcal{P}_{0,p-1})$ or $A(K_p e^{-rT}, \mathcal{P}_{0,p}) \leq \overline{\mathbf{a}}_p$, respectively. We thus need to argue that either a violation of any of these conditions can be resolved or there exists arbitrage in the market.

Suppose first that at strike $K_p \in \mathbb{K}(\mathcal{P}_0^*)\backslash\mathbb{K}^E(\mathcal{P}_0^*)$ the price for a European option with strike K_p is determined to be $\mathbf{e}_p$ and that $\mathbf{e}_p < E_{lb}(K_p, \mathcal{P}_0^*)$. In order to obtain a European price function complying with the no-arbitrage conditions the price $\mathbf{e}_p$ has to be increased to at least $E_{lb}(K_p, \mathcal{P}_0^*)$. This, however, will cause a violation of the upper bound $E_{lf}(K_p, \mathcal{P}_{0,p-1})$. To allow the algorithm to choose a valid price we are therefore required to amend the computed prices prior to the strike K_p. For that purpose we will introduce a second algorithm that will start in the strike K_p and work backwards computing revised prices $\mathbf{a}_k^n$ and $\mathbf{e}_k^n$ for $k \leq p$. In particular, the algorithm begins with the revised option prices $\mathbf{a}_p^n = \mathbf{a}_p$ and $\mathbf{e}_p^n = E_{lb}(K_p, \mathcal{P}_0^*)$ and the price set

$$\mathcal{P}^{rev} = ((\mathbf{a}_p^n, K_p); (\mathbf{e}_p^n, K_p)).$$

It then determines the prices of non-traded options to the left such that the Legendre-Fenchel condition holds with equality between neighbouring strikes. Since the Legendre-Fenchel condition is a transitive property, according to Proposition 3.10.2, we can compute the option prices using $\mathbf{a}_q^n = \mathbf{a}_q$ and

$$\mathbf{e}_q^n = \mathbf{a}_q^n - \frac{K_q}{K_p}[\mathbf{a}_p^n - \mathbf{e}_p^n]$$

for $K_q \in \mathbb{K}(\mathcal{P}_0^*)\backslash\mathbb{K}^E(\mathcal{P}_0^*)$ and $\mathbf{e}_q^n = \mathbf{e}_q$ and

$$\mathbf{a}_q^n = \mathbf{e}_q^n + \frac{K_q}{K_p}[\mathbf{a}_p^n - \mathbf{e}_p^n]$$

for $K_q \in \mathbb{K}(\mathcal{P}_0^*)\backslash\mathbb{K}^A(\mathcal{P}_0^*)$. The price set $\mathcal{P}^{rev}$ is then extended in each step to $\mathcal{P}^{rev} =$

$\mathcal{P}^{rev} \cup ((\mathbf{a}_q^n, K_q); (\mathbf{e}_q^n, K_q))$. From this set of prices the revised price functions A^n and E^n are readily obtained by

$$A^n(K, \mathcal{P}^{rev}) = \frac{\mathbf{a}_{i+1}^n - \mathbf{a}_i^n}{K_{i+1} - K_i}(K - K_i) + \mathbf{a}_i^n$$

for $K \in [K_i, K_{i+1}]$ where $K_i, K_{i+1} \in \mathbb{K}^A(\mathcal{P}^{rev})$ and by

$$E^n(K, \mathcal{P}^{rev}) = \frac{\mathbf{e}_{j+1}^n - \mathbf{e}_j^n}{K_{j+1} - K_j}(K - K_j) + \mathbf{e}_j^n$$

for $K \in [K_j, K_{j+1}]$ where $K_j, K_{j+1} \in \mathbb{K}^E(\mathcal{P}^{rev})$. The reason for this choice of prices is that it not only ensures that the Legendre-Fenchel condition holds between the revised price functions, but at the same time it allows us to construct an arbitrage portfolio in case the correction of the price functions is not successful.

The revision algorithm will stop in one of two situations: Either a revised price violates a no-arbitrage bound or we arrived at a strike $K_q \in \mathbb{K}(\mathcal{P}_0^*)\backslash\mathbb{K}^A(\mathcal{P}_0^*)$ at which it is possible to introduce an additional price constraint $(\mathbf{a}_q^n, K_q)$ that guarantees that the algorithm can continue with the construction of the price functions beyond K_p. In the first case there exists arbitrage in the market as we will see in Section 3.6. In the second case the algorithm is restarted in strike zero using the new initial set of prices $\mathcal{P}_0^* = ((\mathcal{P}_0^*)^A \cup (\mathbf{a}_q^n, K_q); (\mathcal{P}_0^*)^E)$.

Similarly, we can apply the second algorithm to decide whether or not it is possible to correct a violation of $\mathbf{a}_p \leq A_{ub}(K, \mathcal{P}_{0,p-1})$. Note first that Remark 3.10.43 guarantees that a violation of convexity can be ruled out for strikes $K_p > K_{m_1}^A$. Consider thus the situation where $K_p \in [0, K_{m_1}^A] \cap \mathbb{K}^E(\mathcal{P}_0^*)\backslash\mathbb{K}^A(\mathcal{P}_0^*)$ and the price for an American option with strike K_p is computed to be $\mathbf{a}_p$ with $\mathbf{a}_p > A_{ub}(K, \mathcal{P}_{0,p-1})$. Contrary to the situation where $\mathbf{e}_p < E_{lb}(K_p, \mathcal{P}_0^*)$ we now have to choose the initial starting prices for the second algorithm depending on the type of the strike $K_{p-1} \in \mathbb{K}(\mathcal{P}_0^*)$. If $K_{p-1} \in \mathbb{K}^A(\mathcal{P}_0^*)$, the upper bound is given by the prices of two traded options and we use $\mathbf{e}_p^n = \mathbf{e}_p$ and $\mathbf{a}_p^n = A_{ub}(K_p, \mathcal{P}_0^*)$ as starting prices, since $A_{ub}(K_p, \mathcal{P}_0^*)$ is the maximal price an American option with strike K_p can assume. Then again, if $K_{p-1} \in \mathbb{K}^E(\mathcal{P}_0^*)\backslash\mathbb{K}^A(\mathcal{P}_0^*)$, we are no longer guaranteed that the upper bound $A_{ub}(K_p, \mathcal{P}_{0,p-1})$ is given by the prices of two traded options. If it is not, we need to resort to the Legendre-Fenchel condition to find suitable initial prices that allow the construction of arbitrage portfolios in the cases where the price functions cannot be amended. Specifically, we will set

$$\mathbf{a}_p^n = \frac{\hat{\mathbf{a}}_i - cc(E; K_{p-1}, K_p)}{K_i^A}(K_p - K_i^A) + \hat{\mathbf{a}}_i$$

where $i = \arg\min_{1 \leq i' \leq n_1 - 1}\{K_{i'}^A \in \mathbb{K}^A(\mathcal{P}_0^*) : K_{i'}^A > K_p\}$ and $\mathbf{e}_p^n = \mathbf{e}_p$. The second algorithm is then executed with the appropriate prices $\mathbf{a}_p^n$ and $\mathbf{e}_p^n$.

Finally, we will discuss the situation where a violation of $A(K_pe^{-rT}, \mathcal{P}_{0,p}) \leq \overline{\mathbf{a}}_p$ occurs. We will deduce from the convexity of the American and European price functions that the upper bound holds as long as $\overline{\mathbf{a}}_l \geq A(K_le^{-rT}, \mathcal{P}_{0,l})$ for any strike $K_l \in \mathbb{K}(\mathcal{P}_0^*)$. Moreover, we will see below that it suffices to ensure that $A(K_j^Ee^{-rT}, \mathcal{P}_{0,j}) \leq \overline{\mathbf{a}}_j$ for $K_j^E \in \mathbb{K}^E(\mathcal{P}_0^*)$ to guarantee that the American price function computed by the algorithm complies with the upper bound $\overline{A}$.

In order to explain how a violation of the upper bound can be corrected, we will introduce the concept of a support function, akin to the definition in Davis and Hobson [2007].

Definition 3.4.1. *Suppose $\mathcal{S} = \{(y_i, x_i), i = 0, 1, ..., n\}$ is a set of ordered pairs of non-negative real numbers and increasing in each component. The support function $f : [x_0, x_n] \to \mathbb{R}^+$ of $\mathcal{S}$ is then defined to be the largest increasing and convex function such that $f(x_i) \leq y_i$, $i = 0, 1, ..., n$.*

Suppose now that $K_j^E = \min\{K_{j'} \in \mathbb{K}^E(\mathcal{P}_0^*) : A(K_{j'}e^{-rT}, \mathcal{P}_{0,j'}) > \overline{\mathbf{a}}_{j'}\}$ and that $K_j^Ee^{-rT} \in [K_q, K_{q+1}]$, where $K_q, K_{q+1} \in \mathbb{K}(\mathcal{P}_0^*)$, then it is possible to correct a violation of $\overline{\mathbf{a}}_j \geq A(K_j^Ee^{-rT}, \mathcal{P}_{0,j})$ by replacing the American price function A on $[K_q, K_{q+1}]$ by the support function of the set

$$\mathcal{S} = \{(\mathbf{a}_q, K_q), (\mathbf{a}_{q+1}, K_{q+1})\} \cup \bigcup_{\substack{j' : K_{j'}^Ee^{-rT} \in [K_q, K_{q+1}], \\ K_{j'}^E \in \mathbb{K}^E(\mathcal{P}_0^*)}} (\overline{\mathbf{a}}_{j'}, K_{j'}^Ee^{-rT}). \tag{3.21}$$

To do this, we will use a third algorithm that determines

$$K_{\tilde{q}} = \operatorname*{arg\,min}_{\substack{K_v^Ee^{-rT} \in (K_q, K_{q+1}], \\ K_v^E \in \mathbb{K}^E(\mathcal{P}_0^*)}} \frac{\overline{\mathbf{a}}_v - \mathbf{a}_q}{K_v^Ee^{-rT} - K_q}.$$

We can then update the initial set of prices $\mathcal{P}_0^*$ to

$$(\mathcal{P}_0^*)' = ((\mathcal{P}_0^*)^A \cup (\overline{\mathbf{a}}_{\tilde{q}}, K_{\tilde{q}}); (\mathcal{P}_0^*)^E).$$

Since a violation of the upper bound is corrected by replacing the linear interpolation between the two neighbouring prices by the support function described in (3.21) the prices outside the interval (K_q, K_{q+1}) are unaffected. We can therefore refrain from restarting the algorithm as long as the price sets $\mathcal{P}_{0,s+1}$ are being updated to

$$(\mathcal{P}_{0,s+1})' = ((\mathcal{P}_{0,s})^A \cup (\overline{\mathbf{a}}_{\tilde{q}}, K_{\tilde{q}}); (\mathcal{P}_{0,s})^E \cup (E_{ub}(K_{\tilde{q}}, \mathcal{P}_{0,q}), K_{\tilde{q}}))$$

for any $s \in \{q, ..., p\}$.

3.4.3 Expansion of the initial set of prices

In general, we may have to introduce several auxiliary price constraints before the algorithm succeeds in constructing price functions satisfying the no-arbitrage conditions. This implies that the algorithm may have to be restarted repeatedly and thus we will need to be able to distinguish between the different initial sets. Henceforth, we will denote the initial set for the i-th iteration of the algorithm by $\mathcal{P}_i^*$. Furthermore, the algorithm will start with the set of traded option prices $\mathcal{P}_0^*$ in the first iteration, that is $\mathcal{P}_1^* = \mathcal{P}_0^*$. This guarantees that the set of traded option prices $\mathcal{P}_0^*$ remains unchanged throughout the construction.

Note further that the auxiliary price constraints are not necessarily introduced at traded strikes when correcting a violation of the upper bound. Since we keep introducing new constraints to the set of initial prices, we need to distinguish between the strikes of the different iterations. To this end, we will denote the j-th strike of $\mathbb{K}(\mathcal{P}_i^*)$ by $K_{i,j}$. Moreover, we obtain $\mathbb{K}^{aux}(\mathcal{P}_i^*)$, the set of auxiliary strikes for the i-th iteration, by $\mathbb{K}^{aux}(\mathcal{P}_i^*) = \mathbb{K}^A(\mathcal{P}_i^*) \backslash \mathbb{K}^A(\mathcal{P}_0^*)$.

We will further denote the strike at which a violation of convexity occurs during the i-th iteration by K_i^{vc}. Further we will use K_i^{aux} to denote the strike at which the algorithm stops revising option prices and introduces an auxiliary price constraint to correct the violation of convexity at K_i^{vc}.

In addition, we will use $\mathbb{K}_1^{aux}(\mathcal{P}_i^*)$ to denote the set of auxiliary strikes at which constraints were introduced to correct violations of convexity in the first $i-1$ iterations of the algorithm. Similarly, we use $\mathbb{K}_2^{aux}(\mathcal{P}_i^*)$ for the set of auxiliary strikes at which constraints were introduced to correct violations of the upper bound in the first i iterations.

3.4.4 Computation of the prices for strikes in $\mathbb{K}(\mathcal{P}_i^*)$

Due to the introduction of additional strikes to the initial set the computation of the prices for non-traded options has to be extended to auxiliary strikes. Given the initial set of prices $\mathcal{P}_i^*$, the algorithm will thus move along the strikes in $\mathbb{K}(\mathcal{P}_i^*)$ and calculate option prices as follows. For $K_{i,p} \in \mathbb{K}(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_i^*)$ the price of a European option with strike $K_{i,p}$ is

$$\mathbf{e}_{i,p} = \min\left\{E_{lf}(K_{i,p}, \mathcal{P}_{i,p-1}), E_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})\right\}, \tag{3.22}$$

and for $K_{i,p} \in \mathbb{K}(\mathcal{P}_i^*) \backslash \mathbb{K}^A(\mathcal{P}_i^*)$ the price of an American option with strike $K_{i,p}$ is given by

$$\mathbf{a}_{i,p} = \max\left\{A_{lb}(K_{i,p}, \mathcal{P}_{i,p-1}), A_{lb}^{\overline{A}}(K_{i,p}, \mathcal{P}_{i,p-1}), A_{lf}(K_{i,p}, \mathcal{P}_{i,p-1})\right\}. \tag{3.23}$$

3.4.5 Modification to the price corrections

The possible violations of the no-arbitrage conditions in the i-th iteration are now given by $\mathbf{e}_{i,p} < E_{lb}(K_{i,p}, \mathcal{P}_0^*)$, $\mathbf{a}_{i,p} > A_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})$ or $A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p}) > \overline{\mathbf{a}}_{i,p}$. To correct a violation of convexity, we start with the price set $\mathcal{P}_i^{rev} = ((\mathbf{a}_{i,p}^n, K_{i,p}); (\mathbf{e}_{i,p}^n, K_{i,p}^n))$, where $\mathbf{a}_{i,p}^n$ and $\mathbf{e}_{i,p}^n$ are described in Section 3.4.2, and move backwards along the strikes in $\mathbb{K}(\mathcal{P}_i^*)$ computing the revised prices $\mathbf{a}_{i,q}^n = \mathbf{a}_{i,q}$ and

$$\mathbf{e}_{i,q}^n = \mathbf{a}_{i,q}^n - \frac{K_{i,q}}{K_{i,p}}[\mathbf{a}_{i,p}^n - \mathbf{e}_{i,p}^n] \tag{3.24}$$

for $K_{i,q} \in \mathbb{K}(\mathcal{P}_i^*)\backslash\mathbb{K}^E(\mathcal{P}_i^*)$ or the prices $\mathbf{e}_{i,q}^n = \mathbf{e}_{i,q}$ and

$$\mathbf{a}_{i,q}^n = \mathbf{e}_{i,q}^n + \frac{K_{i,q}}{K_{i,p}}[\mathbf{a}_{i,p}^n - \mathbf{e}_{i,p}^n] \tag{3.25}$$

for $K_{i,q} \in \mathbb{K}(\mathcal{P}_i^*)\backslash\mathbb{K}^A(\mathcal{P}_i^*)$. In each step the price set $\mathcal{P}_i^{rev}$ is then updated to $\mathcal{P}_i^{rev} = \mathcal{P}_i^{rev} \cup ((\mathbf{a}_{i,q}^n, K_{i,q}); (\mathbf{e}_{i,q}^n, K_{i,q}))$. From this set of prices the revised price functions A^n and E^n are readily obtained by

$$A^n(K, \mathcal{P}_i^{rev}) = \frac{\mathbf{a}_{i,j+1}^n - \mathbf{a}_{i,j}^n}{K_{i,j+1} - K_{i,j}}(K - K_{i,j}) + \mathbf{a}_{i,j}^n \tag{3.26}$$

for $K \in [K_{i,j}, K_{i,j+1}]$ where $K_{i,j}, K_{i,j+1} \in \mathbb{K}^A(\mathcal{P}_i^{rev})$, $\mathbf{a}_{i,j}^n, \mathbf{a}_{i,j+1}^n \in (\mathcal{P}^{rev})^A$ and by

$$E^n(K, \mathcal{P}^{rev}) = \frac{\mathbf{e}_{i,j+1}^n - \mathbf{e}_{i,j}^n}{K_{i,j+1} - K_{i,j}}(K - K_{i,j}) + \mathbf{e}_{i,j}^n \tag{3.27}$$

for $K \in [K_{i,j}, K_{i,j+1}]$ where $K_{i,j}, K_{i,j+1} \in \mathbb{K}^E(\mathcal{P}_i^{rev})$, $\mathbf{e}_{i,j}^n, \mathbf{e}_{i,j+1}^n \in (\mathcal{P}_i^{rev})^E$.

Observe further that, according to Proposition 3.10.41, each auxiliary price constraint introduced by the algorithm corresponds to the price of a super-replicating portfolio for the American option with the respective strike. We thus have to take the following additional no-arbitrage bounds into account when revising option prices in the i-th iteration. Consider first the bounds implied by auxiliary price constraints of type 1. For $K \in [K_u^A, K_{u+1}^A)$ with $0 \leq u < m_1$ and $K_{m_1+1}^A = \infty$ we obtain

$$A_{lb}^{t_1,r}(K, \mathcal{P}_i^*) = \max_{\substack{1 \leq s \leq m_1 \\ K_s \in (K_{u+1}^A, K_{u+2}^A) \cap \mathbb{K}_1^{aux}(\mathcal{P}_i^*)}} \left\{ \frac{\mathbf{a}_s - \hat{\mathbf{a}}_{u+1}}{K_s - K_{u+1}^A}(K - K_{u+1}^A) + \hat{\mathbf{a}}_{u+1} \right\}$$

as right hand-side lower bound. We will further set $A_{lb}^{t_1,r}(K, \mathcal{P}_i^*) = -\infty$ whenever $(K_{u+1}^A, K_{u+2}^A) \cap \mathbb{K}_1^{aux}(\mathcal{P}_i^*) = \emptyset$. Analogously, the left hand-side lower bound for American options with strike $K \in (K_u^A, K_{u+1}^A]$ with $1 \le u \le m_1$ is given by

$$A_{lb}^{t_1,l}(K, \mathcal{P}_i^*) = \max_{\substack{1 \le s \le m_1 \\ K_s \in (K_{u-1}^A, K_u^A) \cap \mathbb{K}_1^{aux}(\mathcal{P}_i^*)}} \left\{ \frac{\hat{\mathbf{a}}_u - \mathbf{a}_s}{K_u^A - K_s}(K - K_u^A) + \hat{\mathbf{a}}_u \right\}$$

where we again set $A_{lb}^{t_1,l}(K, \mathcal{P}_i^*) = -\infty$ if $(K_{u-1}^A, K_u^A) \cap \mathbb{K}_1^{aux}(\mathcal{P}_i^*) = \emptyset$. Combining these two bounds we get

$$A_{lb}^{t_1}(K, \mathcal{P}_i^*) = \max\{A_{lb}^{t_1,l}(K, \mathcal{P}_i^*), A_{lb}^{t_1,r}(K, \mathcal{P}_i^*)\}.$$

Similarly, a lower bound with respect to the auxiliary prices of type 2 could be defined. Note, however, that this bound is always dominated by $A_{lb}^{\overline{A}}$ as $(K_{u-1}^A, K_u^A) \cap \mathbb{K}_2^{aux}(\mathcal{P}_i^*) \subset S_{u-1}$ and thus we will refrain from using this bound entirely.

The algorithm then stops in one of the following situations: Either a revised price violates a no-arbitrage condition and there exists arbitrage as we will see in Section 3.6 or the algorithm comes across a strike $K_{i,q} \in \mathbb{K}(\mathcal{P}_i^*) \backslash \mathbb{K}^A(\mathcal{P}_i^*)$ at which it is possible to introduce an auxiliary price constraint $(\mathbf{a}_{i,q}^n, K_{i,q})$. In the latter case the algorithm is restarted with the new initial set $\mathcal{P}_{i+1}^* = ((\mathcal{P}_i^*)^A \cup (\mathbf{a}_{i,q}^n, K_{i,q}); (\mathcal{P}_0^*)^E)$. Moreover, if the algorithm reaches $K_{i,q} \in \mathbb{K}^{aux}(\mathcal{P}_i^*)$ it removes the previously introduced auxiliary constraint, computes a revised price and proceeds correcting prices as described above.

In the situation where $\overline{\mathbf{a}}_{i,p} < A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p})$ and $K_{i,p}e^{-rT} \in (K_{i,q}, K_{i,q+1}]$ for $K_{i,q}, K_{i,q+1} \in \mathbb{K}(\mathcal{P}_i^*)$, the algorithm determines

$$K_{i,\tilde{q}} = \operatorname*{arg\,min}_{\substack{K_v^E e^{-rT} \in (K_{i,q}, K_{i,q+1}], \\ K_v^E \in \mathbb{K}^E(\mathcal{P}_1^*)}} \frac{\overline{\mathbf{a}}_v - \mathbf{a}_{i,q}}{K_v^E e^{-rT} - K_{i,q}} \tag{3.28}$$

and updates the initial set of prices $\mathcal{P}_i^*$ to

$$(\mathcal{P}_i^*)' = ((\mathcal{P}_i^*)^A \cup (\overline{\mathbf{a}}_{i,\tilde{q}}, K_{i,\tilde{q}}); (\mathcal{P}_i^*)^E). \tag{3.29}$$

Having introduced the auxiliary price constraint $(\overline{\mathbf{a}}_{i,\tilde{q}}, K_{i,\tilde{q}})$ we also have to update the price sets $\mathcal{P}_{i,s+1}$ to

$$(\mathcal{P}_{i,s+1})' = ((\mathcal{P}_{i,s})^A \cup (\overline{\mathbf{a}}_{i,\tilde{q}}, K_{i,\tilde{q}}); (\mathcal{P}_{i,s})^E \cup (E_{ub}(K_{i,\tilde{q}}, \mathcal{P}_{i,q}))) \tag{3.30}$$

for any $s \in \{q, ...p\}$. We then continue the algorithm by computing the price for non-traded options with strike $K_{i,p+2}$.

3.5 Algorithm

We will now present the algorithm that either constructs admissible price functions or highlights an arbitrage opportunity in the market. To this end, we will start the algorithm with the initial price set $\mathcal{P}_1^*$ by setting $i = 1$. Moreover, we set $\mathbb{K}_1^{aux}(\mathcal{P}_1^*) = \emptyset$ and $\mathbb{K}_2^{aux}(\mathcal{P}_1^*) = \emptyset$. Note also that we provide a flowchart of the following algorithm in Section 3.11 of the appendix.

Algorithm 2 Option pricing algorithm

% Initialisation step
Set $p = 1$.

% Computation of option prices
if $K_{i,p} \in \mathbb{K}(\mathcal{P}_i^*) \backslash \mathbb{K}^A(\mathcal{P}_i^*)$ **then**
  Compute $\mathbf{a}_{i,p}$ as in (3.23).
  Set $\mathcal{P}_{i,p} = (\mathcal{P}_{i,p-1}^A \cup (\mathbf{a}_{i,p}, K_{i,p}); \mathcal{P}_{i,p-1}^E)$.
else if $K_{i,p} \in \mathbb{K}^A(\mathcal{P}_i^*)$ **then**
  Compute $\mathbf{e}_{i,p}$ as in (3.22).
  if $K_{i,p} \in \mathbb{K}(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_i^*)$ **then**
    Set $\mathcal{P}_{i,p} = (\mathcal{P}_{i,p-1}^A; \mathcal{P}_{i,p-1}^E \cup (\mathbf{e}_{i,p}, K_{i,p}))$.
  end if
end if

% Check whether a necessary condition is violated
if $K_{i,p} \in \mathbb{K}^A(\mathcal{P}_i^*)$ **and** $\mathbf{e}_{i,p} < E_{lb}^{rhs}(K_{i,p}, \mathcal{P}_0^*)$ **then**
  Stop. Start Algorithm 3 using $\mathbf{e}_{i,p}^n = E_{lb}^{rhs}(K_{i,p}, \mathcal{P}_0^*)$, $\mathbf{a}_{i,p}^n = \mathbf{a}_{i,p}$ and

$$\mathcal{P}_i^{rev} = ((\mathbf{a}_{i,p}^n, K_{i,p}); (\mathbf{e}_{i,p}^n, K_{i,p})).$$

else if $K_{i,p} \in \mathbb{K}(\mathcal{P}_i^*) \backslash \mathbb{K}^A(\mathcal{P}_i^*)$ **and** $\mathbf{a}_{i,p} > A_{ub}(K_{i,p}, \mathcal{P}_0^*)$ **then**
  Stop. Start Algorithm 3 using $\mathbf{a}_{i,p}^n = A_{ub}(K_{i,p}, \mathcal{P}_0^*)$, $\mathbf{e}_{i,p}^n = \mathbf{e}_{i,p}$ and

$$\mathcal{P}_i^{rev} = ((\mathbf{a}_{i,p}^n, K_{i,p}); (\mathbf{e}_{i,p}^n, K_{i,p})).$$

else if $K_{i,p} \in \mathbb{K}(\mathcal{P}_i^*) \backslash \mathbb{K}^A(\mathcal{P}_i^*)$ **and** $\mathbf{a}_{i,p} > A_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})$ **then**
  Stop. Start Algorithm 3 using

$$\mathbf{a}_{i,p}^n = \frac{\hat{\mathbf{a}}_j - cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p})}{K_j^A}(K_{i,p} - K_j^A) + \hat{\mathbf{a}}_j,$$

  where $j = \arg\min_{1 \le j' \le m_1} \{K_{j'}^A \in \mathbb{K}^A(\mathcal{P}_0^*) : K_{j'}^A > K_{i,p}\}$, $\mathbf{e}_{i,p}^n = \mathbf{e}_{i,p}$ and $\mathcal{P}_i^{rev} = ((\mathbf{a}_{i,p}^n, K_{i,p}); (\mathbf{e}_{i,p}^n, K_{i,p}))$.
else if $K_{i,p} \in \mathbb{K}(\mathcal{P}_i^*)$ **and** $A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p}) > \overline{\mathbf{a}}_{i,p}$ **then**
  Start Algorithm 4.
end if

25: % Repeat these steps for the other strikes
26: If $K_{i,p} < K^E_{m_2}$: Set $p = p + 1$ and go to line 5.

Algorithm 3 Correction of $A > A_{ub}$ or $E < E^{rhs}_{lb}$

1: % Initialisation step
2: Set $q = p - 1$.
3:
4: % Backwards calculation of option prices
5: **if** $K_{i,q} \in \mathbb{K}(\mathcal{P}^*_i) \backslash \mathbb{K}^E(\mathcal{P}^*_i)$ **then**
6:     Set $\mathbf{e}^n_{i,q} = \mathbf{a}_{i,q} - \frac{K_{i,q}}{K_{i,p}}[\mathbf{a}^n_{i,p} - \mathbf{e}^n_{i,p}]$, $\mathbf{a}^n_{i,q} = \mathbf{a}_{i,q}$ and

$$\mathcal{P}^{rev}_i = \mathcal{P}^{rev}_i \cup ((\mathbf{a}^n_{i,q}, K_{i,q}); (\mathbf{e}^n_{i,q}, K_{i,q})).$$

7: **else if** $K_{i,q} \in \mathbb{K}^E(\mathcal{P}^*_i)$ **then**
8:     Set $\mathbf{a}^n_{i,q} = \mathbf{e}_{i,q} + \frac{K_{i,q}}{K_{i,p}}[\mathbf{a}^n_{i,p} - \mathbf{e}^n_{i,p}]$, $\mathbf{e}^n_{i,q} = \mathbf{e}_{i,q}$ and

$$\mathcal{P}^{rev}_i = \mathcal{P}^{rev}_i \cup ((\mathbf{a}^n_{i,q}, K_{i,q}); (\mathbf{e}^n_{i,q}, K_{i,q})).$$

9: **end if**
10:
11: % Check for arbitrage
12: **if** $\mathbf{a}^n_{i,q} < A_{lb}(K_{i,q}, \mathcal{P}^*_0)$ **then**
13:     Stop, arbitrage!
14: **else if** $\mathbf{a}^n_{i,q} < A^{\bar{A}}_{lb}(K_{i,q}, \mathcal{P}^*_0)$ **then**
15:     Stop, arbitrage!
16: **else if** $\mathbf{a}^n_{i,q} < A^{t_1}_{lb}(K_{i,q}, \mathcal{P}^*_i)$ **then**
17:     Stop, arbitrage!
18: **else if** $\mathbf{a}^n_{i,q} < 0$ **then**
19:     Stop, arbitrage!
20: **else if** $\mathbf{e}^n_{i,q} > E_{ub}(K_{i,q}, \mathcal{P}^*_0)$ **then**
21:     Stop, arbitrage
22: **end if**
23:
24: % Stopping condition
25: **if** ($\mathbf{a}_{i,q} = A^{lhs}_{lb}(K_{i,q}, \mathcal{P}_{i,q-1})$ **and** $A^{lhs}_{lb}(K_{i,q}, \mathcal{P}_{i,q-1}) > A_{lf}(K_{i,q}, \mathcal{P}_{i,q-1})$) **or** $K_{i,q} \in \mathbb{K}^{aux}_1(\mathcal{P}^*_i)$ **then**
26:     **if** $K_{i,q} \in \mathbb{K}^{aux}_1(\mathcal{P}^*_i)$ **then**
27:         Remove $(\mathbf{a}_{i,q}, K_{i,q})$ from $\mathcal{P}^*_i$ and $K_{i,q}$ from $\mathbb{K}^{aux}_1(\mathcal{P}^*_i)$.
28:     **end if**
29:     Set $j = 1$.
30:     **while** $(K_{i,q-j} \notin \mathbb{K}^A(\mathcal{P}^*_i))$ **do**
31:         $j = j + 1$.
32:     **end while**

for $s = 0 : j-1$ **do**
  if $cc(A, A^n; K_{i,q-j}, K_{i,q-s}; \mathcal{P}_{i,p}) \leq cc(E; K_{i,q-s-1}, K_{i,q-s}; \mathcal{P}_{i,p})$ **then**
    Stop! Set $\mathbb{K}_1^{aux}(\mathcal{P}_{i+1}^*) = \mathbb{K}_1^{aux}(\mathcal{P}_i^*) \cup K_{i,q-s}$.
    Set $\mathbb{K}_2^{aux}(\mathcal{P}_{i+1}^*) = \mathbb{K}_2^{aux}(\mathcal{P}_i^*)$.
    Restart Algorithm 2 using $i' = i+1$ and

$$\mathcal{P}_{i'}^* = ((\mathcal{P}_i^*)^A \cup (\mathbf{a}_{i,q-s}^n, K_{i,q-s}); (\mathcal{P}_0^*)^E).$$

  else
    if $s < j-1$ **then**
      Set $\mathbf{e}_{i,q-s-1}^n = \mathbf{e}_{i,q-s-1}$,

$$\mathbf{a}_{i,q-s-1}^n = \mathbf{e}_{i,q-s-1} + \frac{K_{i,q-s-1}}{K_{i,q-s}}[\mathbf{a}_{i,q-s}^n - \mathbf{e}_{i,q-s}^n],$$

      and

$$\mathcal{P}_i^{rev} = \mathcal{P}_i^{rev} \cup ((\mathbf{a}_{i,q-s-1}^n, K_{i,q-s-1}); (\mathbf{e}_{i,q-s-1}^n, K_{i,q-s-1})).$$

    else
      Set $\mathbf{e}_{i,q-j}^n = \mathbf{a}_{i,q-j} - \frac{K_{i,q-j}}{K_{i,q-s}}[\mathbf{a}_{i,q-s}^n - \mathbf{e}_{i,q-s}^n]$.
      Stop, there exists arbitrage as $\mathbf{e}_{i,q-j}^n > E_{ub}(K_{i,q-j}, \mathcal{P}_0^*)$.
    end if
  end if
end for
end if

% Repeat these steps
Set $q = q-1$ and go to line 5.

Algorithm 4 Correction of $A > \overline{A}$

% Initialisation step
Set $q_1 = \arg\max_{1 \leq j' \leq p}\{K_{i,j'} \in \mathbb{K}(\mathcal{P}_i^*) : K_{i,j'} < K_{i,p}e^{-rT}\}$ and

$$K_{i,q_2} = \underset{K_{i,v}e^{-rT} \in (K_{i,q_1}, K_{i,q_1+1}] \cap \mathbb{K}^E(\mathcal{P}_i^*)e^{-rT}}{\arg\min} \frac{\overline{\mathbf{a}}_{i,v} - \mathbf{a}_{i,q_1}}{K_{i,v}e^{-rT} - K_{i,q_1}}.$$

% Introduction of additional constraint
Update the initial set of prices $\mathcal{P}_i^*$ to

$$(\mathcal{P}_i^*)' = ((\mathcal{P}_i^*)^A \cup (\overline{\mathbf{a}}_{i,q_2}, K_{i,q_2}); (\mathcal{P}_0^*)^E)$$

and set

$$(\mathcal{P}_{i,s+1})' = ((\mathcal{P}_{i,s})^A \cup (\overline{\mathbf{a}}_{i,q_2}, K_{i,q_2}); (\mathcal{P}_{i,s})^E \cup (E_{ub}(K_{i,q_2}, \mathcal{P}_{i,q_1}), K_{i,q_2}))$$

for any $s \in \{q_1, ..., p\}$.

Set $\mathbb{K}_2^{aux}((\mathcal{P}_i*)') = \mathbb{K}_2^{aux}(\mathcal{P}_i^*) \cup K_{i,q_2}$.

% Exit

Resume Algorithm 2 with $\mathcal{P}_i^* = (\mathcal{P}_i^*)'$.

3.6 Arbitrage situations

In this section we will identify the arbitrage portfolios for the different situations in which the algorithm is unable to construct admissible price functions. As the options violating the no-arbitrage conditions are not necessarily traded in the market, we are required to find suitable sub- and super-replicating strategies for these situations first.

3.6.1 Sub- and super-replicating strategies

Let us start by giving the definition of semi-static sub- and super-replicating portfolios as found in Hobson [2011, p.9].

Definition 3.6.1. *The portfolio P_1 is a semi-static super-replicating portfolio for the portfolio P_2 if P_1 is a semi-static portfolio and $P_1 \geq P_2$ almost surely. Analogously, the portfolio P_1 is a semi-static sub-replicating portfolio for the portfolio P_2 if P_1 is a semi-static portfolio and $P_1 \leq P_2$ almost surely.*

We can then distinguish between the strategies that exploit either the convexity of the price functions or the Legendre-Fenchel condition.

Convexity-based strategies

In the situation where the European price function violates convexity, the following well-known sub- and super-replicating portfolios exist (see for example Laurent and Leisen [2000, p.8]).

Lemma 3.6.2. *Consider the three strikes K_1, K_2 and K_3, where $K_1 < K_2 < K_3$. Suppose further that European put options with maturity T are traded at the strikes K_1 and K_3 for $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_3$, respectively. Then the portfolio $P_1^E(K_2; K_1, K_3)$, consisting of*

- $\frac{K_3-K_2}{K_3-K_1}$ *units of the European option with strike K_1*
- $\frac{K_2-K_1}{K_3-K_1}$ *units of the European option with strike K_3,*

super-replicates a co-terminal European put option with strike K_2 at cost $\alpha_1 \cdot \hat{\mathbf{e}}_1 + (1 - \alpha_1) \cdot \hat{\mathbf{e}}_3$, where $\alpha_1 = (K_3 - K_2)/(K_3 - K_1)$.

Proof. To see that portfolio $P_1^E(K_2; K_1, K_3)$ super-replicates the payoff of a European option with strike K_2 it suffices to compare their payoffs at maturity T. Depending on the terminal value of the underlying the portfolio $P_1^E(K_2; K_1, K_3)$ pays

$$\begin{cases} 0, & \text{if } S_T \geq K_3 \\ \frac{K_2-K_1}{K_3-K_1} \cdot (K_3 - S_T), & \text{if } S_T \in [K_1, K_3] \\ K_2 - S_T, & \text{if } S_T \leq K_1, \end{cases}$$

Since this payoff dominates $(K_2 - S_T)_+$, we can conclude that $P_1^E(K_2; K_1, K_3)$ super-replicates the European put option with strike K_2 (see Figure 3-1).

Moreover, we know that the price for European options with strike K_1 is given by $\hat{\mathbf{e}}_1$, while we have to pay $\hat{\mathbf{e}}_3$ for a European options with strike K_3. We can therefore conclude that the portfolio $P_1^E(K_2; K_1, K_3)$ can be purchased for $\alpha_1 \cdot \hat{\mathbf{e}}_1 + (1 - \alpha_1) \cdot \hat{\mathbf{e}}_3$, where $\alpha_1 = (K_3 - K_2)/(K_3 - K_1)$. $\square$

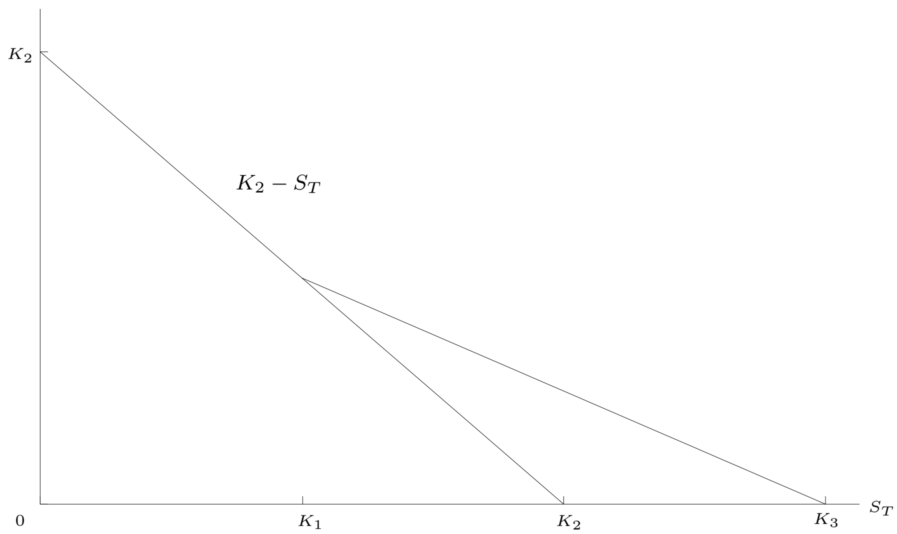

Figure 3-1: Payoffs of the super-replicating strategy $P_1^E(K_2; K_1, K_3)$ and the European put option with strike K_2.

Remark 3.6.3. *As the price of the portfolio $P_1^E(K_2; K_1, K_3)$ is given by $\alpha_1 \cdot \hat{\mathbf{e}}_1 + (1 - \alpha_1) \cdot \hat{\mathbf{e}}_3$, where $\alpha_1 \in (0, 1)$, it follows that the price of the super-replicating portfolio is obtained by interpolating linearly between the given prices $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_3$.*

If we suppose in addition that the European option with strike K_2 is traded for $\hat{\mathbf{e}}_2$, where $\hat{\mathbf{e}}_2 > \alpha_1 \cdot \hat{\mathbf{e}}_1 + (1 - \alpha_1) \cdot \hat{\mathbf{e}}_3$, then it is possible to generate arbitrage by purchasing a butterfly spread. That is, we go long the super-replicating portfolio $P_1^E(K_2; K_1, K_3)$ while short selling the European option with strike K_2.

Similarly, the sub-replicating portfolios for European put options are obtained by linear extrapolation.

Lemma 3.6.4. *Consider the three strikes K_1, K_2 and K_3, where $K_1 < K_2 < K_3$. Suppose further that European put options with maturity T are traded at the strikes K_1 and K_2 for $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$, respectively, then the portfolio $P_2^E(K_3; K_1, K_2)$, consisting of*

- $-\frac{K_3-K_2}{K_2-K_1}$ *units of the European option with strike K_1*
- $\frac{K_3-K_1}{K_2-K_1}$ *units of the European option with strike K_2.*

sub-replicates a co-terminal European put option with strike K_3 at cost $\alpha_2 \cdot \hat{\mathbf{e}}_1 + (1 - \alpha_2) \cdot \hat{\mathbf{e}}_2$, where $\alpha_2 = -(K_3 - K_2)/(K_2 - K_1)$.

If we assume instead that European put options with maturity T are traded at the strikes K_2 and K_3 for $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$, resp., then the portfolio $P_3^E(K_1; K_2, K_3)$, consisting of

- $\frac{K_3-K_1}{K_3-K_2}$ *units of the European option with strike K_2*
- $-\frac{K_2-K_1}{K_3-K_2}$ *units of the European option with strike K_3.*

sub-replicates a co-terminal European put option with strike K_1 at cost $\alpha_3\hat{\mathbf{e}}_2 + (1-\alpha_3) \cdot \hat{\mathbf{e}}_3$, where $\alpha_3 = (K_3 - K_1)/(K_3 - K_2)$.

Proof. Comparing the payoff of portfolio $P_2^E(K_3; K_1, K_2)$ given by

$$\begin{cases} 0, & \text{if } S_T \geq K_2 \\ \frac{K_3-K_1}{K_2-K_1} \cdot (K_2 - S_T), & \text{if } S_T \in [K_1, K_2] \\ K_3 - S_T, & \text{if } S_T \leq K_1, \end{cases}$$

to $(K_3 - S_T)_+$, the payoff of the European option with strike K_3, we see that the portfolio $P_2^E(K_3; K_1, K_2)$ is a sub-replicating portfolio (see Figure 3-2).

Similarly, we conclude that the portfolio $P_3^E(K_1; K_2, K_3)$ with payoff

$$\begin{cases} 0, & \text{if } S_T \geq K_3 \\ -\frac{K_2-K_1}{K_3-K_2} \cdot (K_3 - S_T), & \text{if } S_T \in [K_2, K_3] \\ K_1 - S_T, & \text{if } S_T \leq K_2, \end{cases}$$

sub-replicates the European put option with strike K_1 (see Figure 3-2). It follows, moreover, without further ado that the prices of the sub-replicating portfolios are given by $\alpha_2 \cdot \hat{\mathbf{e}}_1 + (1 - \alpha_2) \cdot \hat{\mathbf{e}}_2$ and $\alpha_3\hat{\mathbf{e}}_2 + (1 - \alpha_3) \cdot \hat{\mathbf{e}}_3$, respectively. □

In contrast to European options, we have to take the early exercise feature into account when super-replicating American options. We are therefore required to supply (at least) one specific exercising strategy that guarantees a payoff dominating the option's payoff irrespective of the option's exercise time. To this end, we will adjust portfolio P_1^E to the current setting and provide a suitable exercising strategy.

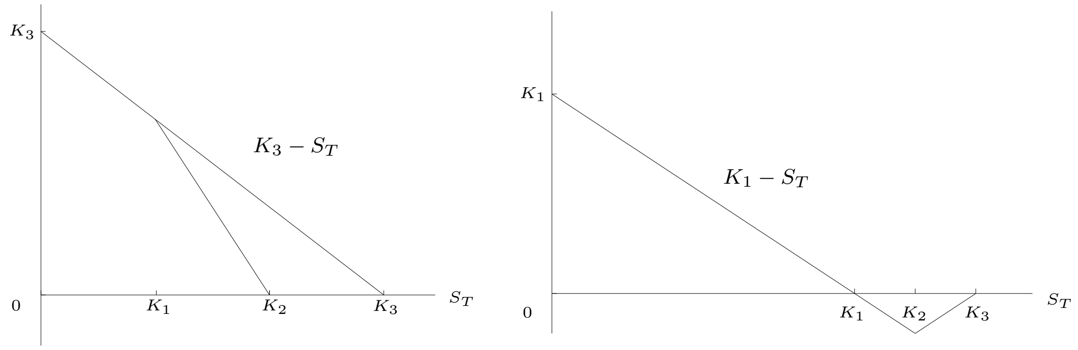

Figure 3-2: Comparison of the payoff of a European option with the respective sub-replicating portfolios $P_2^E(K_3; K_1, K_2)$ and $P_3^E(K_1; K_2, K_3)$.

Corollary 3.6.5. *Consider the three strikes K_1, K_2 and K_3, where $K_1 < K_2 < K_3$. Suppose further that American put options with maturity T are traded at the strikes K_1 and K_3 for $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_3$, respectively. Then the portfolio $P_1^A(K_2; K_1, K_3)$, consisting of*

- $\frac{K_3-K_2}{K_3-K_1}$ *units of the American option with strike K_1*
- $\frac{K_2-K_1}{K_3-K_1}$ *units of the American option with strike K_3,*

super-replicates a co-terminal American put option with strike K_2 if it is exercised simultaneously with the American option with strike K_2. Moreover, the super-replicating portfolio $P_1^A(K_2; K_1, K_3)$ can be purchased in the market for $\alpha \cdot \hat{\mathbf{a}}_1 + (1-\alpha) \cdot \hat{\mathbf{a}}_3$, where $\alpha = (K_3 - K_2)/(K_3 - K_1)$.

Proof. To see that the portfolio $P_1^A(K_2; K_1, K_3)$ super-replicates an American option with strike K_2, we consider the payoff of the auxiliary portfolio $P(K_1, K_2, K_3)$, consisting of a long position in the portfolio $P_1^A(K_2; K_1, K_3)$ and a short position of 1 unit of the American option with strike K_2, at the time of exercise τ. If $\tau < T$ then the payoff is given by

$$\begin{cases} 0, & \text{if } S_\tau \geq K_3 \\ \frac{K_2-K_1}{K_3-K_1} \cdot (K_3 - S_\tau), & \text{if } S_\tau \in [K_2, K_3] \\ \frac{K_3-K_2}{K_3-K_1} \cdot (S_\tau - K_1), & \text{if } S_\tau \in [K_1, K_2] \\ 0, & \text{if } S_\tau \leq K_1, \end{cases}$$

which is non-negative regardless of the evolution of the underlying price process. Similarly we can conclude from Proposition 3.6.2 that $P_1^A(K_2; K_1, K_3)$ has to be non-negative whenever the American options where not exercised prior to maturity T. It follows that the portfolio $P_1^A(K_2; K_1, K_3)$ super-replicates an American option with strike K_2 if exercised correctly. The cost of the super-replicating portfolio is then given by $\alpha\hat{\mathbf{a}}_1 + (1-\alpha) \cdot \hat{\mathbf{a}}_3$. ☐

Remark 3.6.6. *Note that it is not possible to sub-replicate American options applying the analog modifications to the portfolios in Lemma 3.6.4. This is due to the fact that the payoff of a sub-replicating portfolio has to be dominated by the American option no matter the exercise strategies used. In particular, the portfolios in Lemma 3.6.4 each contain a short position in American options for which the choice of exercising is not ours.*

LF-based strategies

The following super-replicating portfolios are closely related to the Legendre-Fenchel condition and will play an important role in creating the arbitrage strategies in the following sections.

Proposition 3.6.7. *Suppose European put options with strike K_1 and strike K_2 and maturity T are traded in the market at $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$, respectively. Moreover, co-terminal American options with strike K_2 are traded for $\hat{\mathbf{a}}_2$. The portfolio $P_1^{LF}(K_1, K_2)$, consisting of*

- 1 *unit of European with strike K_1*
- $\frac{K_1}{K_2}$ *units of the American with strike K_2*
- $-\frac{K_1}{K_2}$ *units of the European with strike K_2,*

then super-replicates the American put option with strike K_1 at cost $\hat{\mathbf{e}}_1 + \frac{K_1}{K_2}[\hat{\mathbf{a}}_2 - \hat{\mathbf{e}}_2]$ if the position in the American option with strike K_2 is exercised simultaneously with the American option with strike K_1.

Similarly, we can assume that American put options with strike K_1 and K_2 are traded in the market for $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$, respectively. Suppose further that co-terminal European put options with strike K_1 are traded in the market for $\hat{\mathbf{e}}_1$. The portfolio $P_2^{LF}(K_1, K_2)$, consisting of

- $\frac{K_2}{K_1}$ *units of European with strike K_1,*
- $-\frac{K_2}{K_1}$ *units of American with strike K_1*
- 1 *unit of American options with strike K_2*

then super-replicates the European put option with strike K_2 at cost $\hat{\mathbf{a}}_2 - \frac{K_2}{K_1}[\hat{\mathbf{a}}_1 - \hat{\mathbf{e}}_1]$ if we exercise the American option with strike K_2 simultaneously with the American option at strike K_1.

Proof. In the first case we consider the payoff of the auxiliary portfolio $P_1(K_1, K_2)$, consisting of a long position in the super-replicating portfolio $P_1^{LF}(K_1, K_2)$ and a short position of one American option with strike K_1. If the American with strike K_1 is exercised at the time τ, where $\tau < T$, then the payoff is given by

$$\begin{cases} (1 - \frac{K_1}{K_2}) \cdot S_T, & \text{if } S_T \geq K_2 \\ S_T - K_1, & \text{if } S_T \in [K_1, K_2] \\ 0, & \text{if } S_T \leq K_1. \end{cases}$$

Otherwise the American option with strike K_1 is not exercised prior to maturity and its payoff is matched by the payoff of the European option with strike K_1. Analogously, the American option with strike K_2 is not exercised before maturity, according to the exercise strategy. Hence, the value of the portfolio at time T is zero. Since the payoff is non-negative regardless of both the value of the underlying at time T and the time of exercise, we can conclude that the portfolio $P_1^{LF}(K_1, K_2)$ super-replicates the American option with strike K_1.

Similarly, we note that the payoff of the portfolio $P_2(K_1, K_2)$, given by a long position in the super-replicating portfolio $P_2^{LF}(K_1, K_2)$ and a short position in the European option with strike K_2 is given by

$$\begin{cases} (\frac{K_2}{K_1} - 1) \cdot S_T, & \text{if } S_T \geq K_2 \\ (\frac{S_T}{K_1} - 1) \cdot K_2, & \text{if } S_T \in [K_1, K_2] \\ 0, & \text{if } S_T \leq K_1 \end{cases}$$

when the exercise time for the American options is strictly before maturity, otherwise the payoff is zero. Since the payoff is non-negative, we can conclude that the portfolio $P_2^{LF}(K_1, K_2)$ can be used to super-replicate a European option with strike K_2. Additionally, we see immediately that the cost of the super-replicating portfolios is given by $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{e}}_2$, respectively. $\square$

The next result shows how to derive a super-replicating strategy using the Legendre-Fenchel condition between three strikes.

Proposition 3.6.8. *Consider the three strikes K_1, K_2 and K_3, where $K_1 < K_2 < K_3$. Suppose European options with strike K_1 and strike K_2 and maturity T are traded in the market at $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$, respectively. Moreover, co-terminal American options with strike K_3 are traded for $\hat{\mathbf{a}}_3$. Then the portfolio $P_3^{LF}(K_2; K_1, K_2, K_3)$, consisting of*

- $\frac{K_2}{K_3} \cdot \frac{K_3 - K_2}{K_2 - K_1}$ *units of European options with strike K_1*
- $-\frac{K_1}{K_3} \cdot \frac{K_3 - K_2}{K_2 - K_1}$ *units of European options with strike K_2*
- $\frac{K_2}{K_3}$ *units of American options with strike K_3,*

super-replicates the payoff of an American option with strike K_2 if the position in American options with strike K_3 is exercised simultaneously with the American option with strike K_2.

Moreover, the cost of the portfolio is given by $\alpha \cdot cc(E; K_1, K_2) + (1-\alpha) \cdot \hat{\mathbf{a}}_3$, where

$$cc(E; K_1, K_2) = \hat{\mathbf{e}}_1 - \frac{\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_1}{K_2 - K_1} K_1$$

and $\alpha = 1 - K_2/K_3$.

Proof. Consider the payoff of the auxiliary portfolio $P(K_1, K_2, K_3)$, consisting of a long position in the super-replicating portfolio $P_3^{LF}(K_2; K_1, K_2, K_3)$ and a short position in the American option with strike K_2. If the American option with strike K_2 is exercised at time τ, where $\tau < T$, the total payoff at maturity is given by

$$\begin{cases} (1 - \frac{K_2}{K_3}) \cdot S_T, & \text{if } S_T \geq K_2 \\ \frac{K_2}{K_3} \cdot \frac{K_3 - K_2}{K_2 - K_1} \cdot (S_T - K_1), & \text{if } S_T \in [K_1, K_2] \\ 0, & \text{if } S_T \leq K_1. \end{cases}$$

In the case where the American options were not exercised prior to maturity the payoff of the portfolio is given by

$$\begin{cases} 0, & \text{if } S_T \geq K_3 \\ \frac{K_2}{K_3} \cdot (K_3 - S_T), & \text{if } S_T \in [K_2, K_3] \\ \frac{K_2}{K_3} \cdot \frac{K_3 - K_2}{K_2 - K_1} \cdot (S_T - K_1), & \text{if } S_T \in [K_1, K_2] \\ 0, & \text{if } S_T \leq K_1. \end{cases}$$

As, in both cases, the payoff is non-negative regardless of the value of the underlying at maturity T, we can conclude that the portfolio $P_3^{LF}(K_2; K_1, K_2, K_3)$ super-replicates the American option with strike K_2. Furthermore, we see that the cost for this portfolio is given by $\alpha \cdot cc(E; K_1, K_2) + (1-\alpha) \cdot \hat{\mathbf{e}}_3$. $\square$

3.6.2 Situation I: Violation of the no-arbitrage conditions by the prices of traded options

The first type of arbitrage opportunities that we would like to discuss arises when the prices of traded options violate any of the no-arbitrage conditions.

Proposition 3.6.9. *Suppose we are given a set of traded prices $\mathcal{P}_0^* \in \mathcal{M}$ and that the price functions $A(K, \mathcal{P}_0^*)$ and $E(K, \mathcal{P}_0^*)$ up to $K_{m_1}^A$ and $K_{m_2}^E$ are given by (3.6) and (3.7), respectively. In addition, we assume that the current price of the underlying is given by S_0. Then there exists either model-independent or weak arbitrage if any of the* *following conditions is violated:*

(i) *The function* $E(K, \mathcal{P}_0^*)$ *satisfies the conditions of Lemma 3.1.1 on* $[0, \infty)$.

(ii) *The function* $A(K, \mathcal{P}_0^*)$ *is increasing and convex on* $[0, \infty)$.

(iii) *For any strike* $K_i^A \in \mathbb{K}^A(\mathcal{P}_0^*)$ *we have* $\hat{\mathbf{a}}_i \geq K_i^A - S_0$ *and* $A'(K_i^A+, \mathcal{P}_0^*) \leq 1$. *In particular,* $A'(K_i^A+, \mathcal{P}_0^*) < 1$ *has to hold whenever the strike* $K_i^A \leq K_{l_1(\mathcal{P}_0^*)}$.

(iv) *For strikes* $K_i^A, K_{i'}^A \in \mathbb{K}^A(\mathcal{P}_0^*)$ *and* $K_j^E, K_{j'}^E \in \mathbb{K}^E(\mathcal{P}_0^*)$, *with* $K_j^E < K_{j'}^E$, $K_i^A < K_{i'}^A$, $K_j^E \leq K_i^A$ *and* $K_{j'}^E \leq K_{i'}^A$ *the Legendre-Fenchel condition holds (see (3.1)).*

(v) *For any strike* $K_j^E \in [0, K_{m_1}^A] \cap \mathbb{K}^E(\mathcal{P}_0^*)$ *we have*

$$\overline{\mathbf{a}}_j \geq \max\{A_{lb}(K_j^E e^{-rT}, \mathcal{P}_0^*), K_j^E e^{-rT} - S_0\}.$$

(vi) *For any strike* $K_i^A \in [0, K_{m_2}^E] \cap \mathbb{K}^A(\mathcal{P}_0^*)$ *the inequality* $\hat{\mathbf{a}}_i \geq E_{lb}(K_i^A, \mathcal{P}_0^*)$ *holds.*

(vii) *For any strike* $K_j^E \in [0, K_{m_1}^A] \cap \mathbb{K}^E(\mathcal{P}_0^*)$ *we must have* $A(K_j^E, \mathcal{P}_0^*) \geq \hat{\mathbf{e}}_j$.

(viii) *For any strike* $K_i^A \in [0, K_{m_2}^E e^{-rT}] \cap \mathbb{K}^A(\mathcal{P}_0^*)$ *the inequality* $\overline{A}(K_i^A, \mathcal{P}_0^*) \geq \hat{\mathbf{a}}_i$ *holds.*

(ix) *For* $K_i^A \in \mathbb{K}^A(\mathcal{P}_0^*)$ *and* $K_j^E \in \mathbb{K}^E(\mathcal{P}_0^*)$ *with* $K_j^E e^{-rT} \in [0, K_i^A)$ *we must have*

$$\frac{\hat{\mathbf{a}}_i - \overline{\mathbf{a}}_j}{K_i^A - K_j^E e^{-rT}} \leq 1.$$

Proof. Due to Lemma 2.2.1 in Chapter 2 we know that there either exists model-independent or weak arbitrage if (i) does not hold. Similarly, we can use Theorem 2.2.3 in Chapter 2 to argue that A has to be increasing and convex and that $K - S_0$ is a lower bound on the price for American options with strike K. We now want to argue that $A'(K_i^A+, \mathcal{P}_0^*) \leq 1$ for $K_i^A \in \mathbb{K}^A(\mathcal{P}_0^*)$. Suppose for contradiction that the slope between the strikes K_i^A and K_{i+1}^A is given by

$$\frac{\hat{\mathbf{a}}_{i+1} - \hat{\mathbf{a}}_i}{K_{i+1}^A - K_i^A} > 1, \tag{3.31}$$

then the portfolio $P_{sl}^A(K_i^A, K_{i+1}^A)$, consisting of $K_{i+1}^A - K_i^A$ units of cash and one unit of American option with strike K_i^A, can be used to super-replicate the payoff of an American option with strike K_{i+1}^A. To see this, we consider a long position in $P_{sl}^A(K_i^A, K_{i+1}^A)$ while shorting an American option with strike K_{i+1}^A. At maturity T the total payoff is then at least

$$(e^{rT} - e^{r(T-\tau)})(K_{i+1}^A - K_i^A) \tag{3.32}$$

in the case where the shorted American with strike K_{i+1}^A is exercised at time $\tau < T$, as we will exercise the American with strike K_i^A simultaneously. In the case where the options have not been exercised prior to maturity the American options can only be exercised at maturity and the payoff depending on the value of the underlying is given by

$$\begin{cases} e^{rT}(K_{i+1}^A - K_i^A), & \text{for } S_T \geq K_{i+1}^A \\ (e^{rT}-1)(K_{i+1^A} - K_i^A) + (S_T - K_i^A), & \text{for } S_T \in [K_i^A, K_{i+1}^A] \\ (e^{rT}-1)(K_{i+1}^A - K_i^A), & \text{for } S_T \leq K_i^A, \end{cases} \tag{3.33}$$

This implies that the difference between the payoff of the portfolio $P_{sl}^A(K_i^A, K_{i+1}^A)$ and the payoff of an American option with strike K_{i+1}^A is non-negative. Since the cost of purchase of a long position in the portfolio $P_{sl}^A(K_i^A, K_{i+1}^A)$ and a short position in the American option with strike K_{i+1}^A is strictly negative we can conclude that there has to exist model-independent arbitrage whenever (3.31) holds.

Consider now the special case where $K_i^A \leq K_{l_1(\mathcal{P}_0^*)}$ and $\hat{\mathbf{a}}_{i+1} - \hat{\mathbf{a}}_i = K_{i+1}^A - K_i^A$. It then follows immediately that $\hat{\mathbf{a}}_{i+1} > K_{i+1}^A - S_0$ has to hold. Moreover, the cost of the super-replicating portfolio $P_{sl}^A(K_i^A, K_{i+1}^A)$ coincides with the cost of the American option with strike K_{i+1}^A. Hence, the cost of the arbitrage portfolio consisting of a long position in $P_{sl}^A(K_i^A, K_{i+1}^A)$ and a short position in an American option with strike K_{i+1}^A is zero. To see that there exists model-independent arbitrage in this case we are thus required to show that all subsequent cash-flows are strictly positive. Observe, however, that according to (3.32) we can only guarantee a non-negative payoff if the American options are exercised at $\tau = 0$. Then again, immediate exercise can be ruled out as an American option with a price strictly larger than the lower bound will never be exercised immediately according to the Assumption 3.1.3. Hence, there exists model-independent arbitrage in the market if $\hat{\mathbf{a}}_{i+1} - \hat{\mathbf{a}}_i = K_{i+1}^A - K_i^A$ for $K_i^A \leq K_{l_1(\mathcal{P}_0^*)}$. We can therefore conclude that $A'(K_i^A+, \mathcal{P}_0^*) \leq 1$ has to hold for any strike $K_i^A \in \mathbb{K}^A(\mathcal{P}_0^*)$. In addition, we showed that $A'(K_i^A+, \mathcal{P}_0^*) < 1$ holds whenever the strike $K_i^A \leq K_{l_1(\mathcal{P}_0^*)}$. A proof that there exists arbitrage in the market if condition (iv) is violated can be found in Proposition 3.10.1.

Next we will argue that condition (v) has to hold in any market free of arbitrage. Let us first consider the situation where $K_j^E e^{-rT} \in \mathbb{K}^A(\mathcal{P}_0^*)$ and $K_j^E \in \mathbb{K}^E(\mathcal{P}_0^*)$. In that case we can make an initial profit by selling an American option with strike $K_j^E e^{-rT}$, while buying a European option with strike K_j^E. As the European option with strike K_j^E super-replicates the payoff of the American option with strike $K_j^E e^{-rT}$ we are furthermore guaranteed that this portfolio has a non-negative payoff and we have thus shown that there has to exist arbitrage in the market.

Let us now assume that the lower bound for American options with strike $K_j^E e^{-rT}$

is given by $A_{lb}^{lhs}(K_j^E e^{-rT}, \mathcal{P}_0^*)$ which is induced by the prices $\hat{\mathbf{a}}_i$ and $\hat{\mathbf{a}}_{i+1}$. Suppose for a moment that American options with strike $K_j^E e^{-rT}$ are traded in the market at price $\overline{\mathbf{a}}_j$, then the American price function cannot be convex between the strikes K_i^A, K_{i+1}^A and $K_j^E e^{-rT}$. We could therefore generate arbitrage by holding the super-replicating portfolio $P_1^A(K_{i+1}^A; K_i^A, K_j^E e^{-rT})$ while shorting an American option with strike K_{i+1}^A as the cost of the arbitrage portfolio is strictly negative. According to Theorem 2.2.3 in Chapter 2, it is, moreover, possible to super-replicate an American option with strike $K_j^E e^{-rT}$ using the traded European option with strike K_j^E. Replacing the position in American options with strike $K_j^E e^{-rT}$ in the arbitrage portfolio above by an equivalent position in European options with strike K_j^E we can conclude that there has to exist arbitrage in the market whenever there is a strike $K_j^E e^{-rT}$ with $\overline{\mathbf{a}}_j < A_{lb}^{lhs}(K_j^E e^{-rT}, \mathcal{P}_0^*)$. An analogous argument shows that the market cannot be free of arbitrage if $\overline{\mathbf{a}}_j < A_{lb}^{rhs}(K_j^E e^{-rT}, \mathcal{P}_0^*)$ for $K_j^E e^{-rT} \in [0, K_{m_1-1}^A]$.

We are thus left to show that there exists arbitrage whenever $K_j^E e^{-rT} - S_0 > \overline{\mathbf{a}}_j$ for some $K_j^E \in \mathbb{K}^E(\mathcal{P}_0^*)$. Since $\overline{\mathbf{a}}_j = \hat{\mathbf{e}}_j$, this inequality implies that the European option with strike K_j^E is below its lower bound and we can thus conclude from Lemma 3.1.1 that there has to exist arbitrage. Analogously, we conclude that a violation of condition (vi) implies the existence of an arbitrage, as an American option can be used to super-replicate the corresponding European option.

Suppose next that condition (vii) is violated, then there exists a strike $K_j^E \in [0, K_{m_1}^A] \cap \mathbb{K}^E(\mathcal{P}_0^*)$ such that $A(K_j^E, \mathcal{P}_0^*) < \hat{\mathbf{e}}_j$. If we assume that $K_j^E \in \mathbb{K}^E(\mathcal{P}_0^*) \cap \mathbb{K}^A(\mathcal{P}_0^*)$, then a portfolio consisting of one American option with strike K_j^E and a short position of one European option with strike K_j^E has strictly negative cost and can be used to generate arbitrage. To offset the payment of the short position we only have to hold the American option until maturity where it is exercised whenever the price of the underlying is below the strike.

Consider now the situation where $K_j^E \in \mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_0^*)$ and suppose that

$$A(K_j^E, \mathcal{P}_0^*) = \frac{\hat{\mathbf{a}}_{l+1} - \hat{\mathbf{a}}_l}{K_{l+1}^A - K_l^A}(K_j^E - K_l^A) + \hat{\mathbf{a}}_l$$

for some $l \in \{0, ..., m_1\}$. In this situation we can super-replicate an American option with strike K_j^E using portfolio $P_1^A(K_j^E; K_l^A, K_{l+1}^A)$ from Corollary 3.6.5. At the same time we know that a European option with strike K_j^E will sub-replicate the American option with strike K_j^E. Holding a portfolio consisting of a long position in $P_1^A(K_j^E; K_l^A, K_{l+1}^A)$ and a shorted European option with strike K_j^E is thus guaranteed to have a non-negative payoff regardless of the price development of the underlying. Since we assumed that $A(K_j^E, \mathcal{P}_0^*) < \hat{\mathbf{e}}_j$, this portfolio will have strictly negative initial cost and we thus showed that there exists model-independent arbitrage in the market whenever condition (vii) is violated. In the same way, we can show that condition (viii) has to hold for $K_i^A \leq K_{m_2}^E$, as an American option with strike K_i^A can be super-replicated using a European option with strike $K_i^A e^{rT}$, which in turn can be super-replicated at cost $\overline{A}(K_i^A, \mathcal{P}_0^*)$ using portfolio $P_1^E(K_i^A e^{rT}; K_j^E, K_{j+1}^E)$ for $K_i^A \in [K_j^E, K_{j+1}^E]$ and $j \in \{0, ..., m_2 - 1\}$.

We are thus left to argue that condition (ix) has to hold. Let us assume for the moment that American options with strike $K_j^E e^{-rT}$ are traded. It then follows that

$$\frac{\hat{\mathbf{a}}_i - A(K_j^E e^{-rT}, \mathcal{P}_0^*)}{K_i^A - K_j^E e^{-rT}} > 1$$

has to hold and according to (iii) there would exist model-independent arbitrage in the market. Although American options with strike $K_j^E e^{-rT}$ are not necessarily traded in the market, there still has to exist arbitrage if $K_j^E \in \mathbb{K}^E(\mathcal{P}_0^*)$ as we have seen in Theorem 2.2.3 of Chapter 2 that an American option with strike $K_j^E e^{-rT}$ can be super-replicated using a European option with strike K_j^E.

We can therefore conclude that a violation of any of the condition above means that there has to exist arbitrage in the market. $\square$

Definition 3.6.10. *A set of prices $\mathcal{P}$ complying with the conditions of Proposition 3.6.9 is said to satisfy the Standing Assumptions.*

Notation 3.6.11. *Suppose we are given a set of prices $\mathcal{P} \in \mathcal{M}$ satisfying the Standing Assumptions, then we will write $\mathcal{P} \in \overline{\mathcal{M}}$.*

The following result is a direct consequence of condition (ix) of the Standing Assumptions.

Corollary 3.6.12. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. The price for American put options with strike $K \in (K_{m_2}^E e^{-rT}, \infty) \cap \mathbb{K}^A(\mathcal{P}_0^*)$ is then given by $K - S_0$.*

3.6.3 Situation II: Violation of $\mathbf{e}_{i,p} \geq E_{lb}^{rhs}(K_{i,p}, \mathcal{P}_0^*)$

In this section we will discuss the arbitrage situations that occur when the algorithm fails to correct a violation of $\mathbf{e}_{i,p} < E_{lb}^{rhs}(K_{i,p}, \mathcal{P}_0^*)$ at the strike $K_{i,p} \in \mathbb{K}(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_i^*)$ using Algorithm 3. In that case there exists either a strike $K_{i,q} \in \mathbb{K}(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_i^*)$ with $\mathbf{e}_{i,q}^n > E_{ub}(K_{i,q}, \mathcal{P}_0^*)$ or a strike $K_{i,q} \in \mathbb{K}^E(\mathcal{P}_i^*)$ with

$$\mathbf{a}_{i,q}^n < \max\{A_{lb}(K_{i,q}, \mathcal{P}_0^*), A_{lb}^{\overline{A}}(K_{i,q}, \mathcal{P}_0^*)\}.$$

Generally, the construction of the arbitrage portfolios is based on the idea of finding a sub- and a super-replicating strategy for the non-traded option with strike $K_{i,q}$.

Since the convexity property of the corresponding price function is violated, the sub-replicating portfolio will be more expensive than the super-replicating portfolio and we could generate arbitrage by taking a long position in the super-replicating portfolio, while short selling the sub-replicating portfolio.

However, not all the positions in the arbitrage portfolio will be traded options and we are thus required to replace the long positions by a super-replicating portfolio and the short positions by a sub-replicating portfolio. If the sub- and super-replicating portfolios have the same price as the portfolio they replace, we can conclude that the modified portfolio has negative cost of purchase and non-negative terminal value and thus generates arbitrage in the market.

In particular, we will use the fact that the Legendre-Fenchel condition holds with equality between the strikes $K_{i,q}$ and $K_{i,p}$ to replace the position in the non-traded option with strike $K_{i,q}$ by a portfolio consisting of the traded option with strike $K_{i,q}$ as well as a long position in American options with strike $K_{i,p}$ and a short position of European options with strike $K_{i,p}$. The short position in the European options with strike $K_{i,p}$ can then be sub-replicated using the portfolio P_3^E and we obtain a portfolio consisting only of traded options.

Note, moreover, that we are restricted to the portfolios of Section 3.6.1 during the construction of the arbitrage portfolio and thus it is important to find a suitable initial portfolio. It is this restriction that makes the derivation of the arbitrage portfolios in some cases seem slightly artificial.

In the following proposition, the strikes $K_{i,q}$ and $K_{i,p}$ correspond to $K_{u_1}^A \in \mathbb{K}^A(\mathcal{P}_0^*)$ and $K_{u_2}^A \in \mathbb{K}^A(\mathcal{P}_0^*)\backslash\mathbb{K}^E(\mathcal{P}_0^*)$, respectively.

Proposition 3.6.13. *Suppose that American and co-terminal European options are traded in the market and that their prices are provided by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Then we can generate model-independent arbitrage in the market if there exist strikes $K_{u_1}^A \in \mathbb{K}^A(\mathcal{P}_0^*)$ and $K_{u_2}^A \in \mathbb{K}^A(\mathcal{P}_0^*)\backslash\mathbb{K}^E(\mathcal{P}_0^*)$ with $K_{u_1}^A < K_{u_2}^A$, such that*

$$\hat{\mathbf{a}}_{u_1} - \frac{K_{u_1}^A}{K_{u_2}^A} \cdot [\hat{\mathbf{a}}_{u_2} - E_{lb}^{rhs}(K_{u_2}^A, \mathcal{P}_0^*)] > E_{ub}(K_{u_1}^A, \mathcal{P}_0^*).$$

Specifically, if we set $v = \arg\min_{1 \le j' \le m_2 - 1}\{K_{j'}^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K_{j'}^E \ge K_{u_2}^A\}$ and $w = \arg\max_{0 \le j' \le m_2}\{K_{j'}^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K_{j'}^E \le K_{i,q}\}$, then we can generate arbitrage using the portfolio $P(E_{lb}, E_{ub}; K_{u_2}^A, K_{u_1}^A, \mathcal{P}_0^*)$, consisting of a long position of

- $\frac{K_{w+1}^E - K_{u_1}^A}{K_{w+1}^E - K_w^E}$ *units of the European option with strike K_w^E,*
- $\frac{K_{u_1}^A - K_w^E}{K_{w+1}^E - K_w^E}$ *units of the European option with strike K_{w+1}^E,*
- $\frac{K_{u_1}^A}{K_{u_2}^A}$ *units of the American option with strike $K_{u_2}^A$,*

- $\frac{K_{u_1}^A}{K_{u_2}^A} \cdot \frac{K_v^E - K_{u_2}^A}{K_{v+1}^E - K_v^E}$ *units of the European option with strike* K_{v+1}^E

and a short position of

- 1 *unit of the American option with strike* $K_{u_1}^A$*,*
- $\frac{K_{u_1}^A}{K_{u_2}^A} \cdot \frac{K_{v+1}^E - K_p^A}{K_{v+1}^E - K_v^E}$ *units of the European option with strike* K_v^E*.*

if the position in the American option with strike $K_{u_2}^A$ *is exercised simultaneously with the American option with strike* $K_{u_1}^A$*.*

Proof. Let us assume for the moment that both European options with strike $K_{u_1}^A$ and $K_{u_2}^A$ are traded in the market and that their prices are given by

$$\hat{\mathbf{a}}_{u_1} - \frac{K_{u_1}^A}{K_{u_2}^A} \cdot [\hat{\mathbf{a}}_{u_2} - E_{lb}^{rhs}(K_{u_2}^A, \mathcal{P}_0^*)]$$

and $E_{lb}^{rhs}(K_{u_2}^A, \mathcal{P}_0^*)$, respectively. This would imply that the Legendre-Fenchel condition holds with equality between the strikes $K_{u_1}^A$ and $K_{u_2}^A$. We could then super-replicate a European option with strike $K_{u_2}^A$ for $E_{lb}^{rhs}(K_{u_2}^A, \mathcal{P}_0^*)$ using the portfolio $P_2^{LF}(K_{u_1}^A, K_{u_2}^A)$. Moreover, it is possible to sub-replicate a European option with strike $K_{u_2}^A$ using the portfolio $P_3^E(K_{u_2}^A; K_v^E, K_{v+1}^E)$ at cost $E_{lb}^{rhs}(K_{u_2}^A, \mathcal{P}_0^*)$. Hence, we are interested in taking a long position in the super-replicating portfolio $P_2^{LF}(K_{u_1}^A, K_{u_2}^A)$ while short selling the sub-replicating portfolio $P_3^E(K_{u_2}^A; K_v^E, K_{v+1}^E)$.

Since European options with strike $K_{u_1}^A$ are not actually traded, we have to replace their long position by a super-replicating portfolio. According to Proposition 3.6.2, we can use the portfolio $P_1^E(K_{u_1}^A; K_w^E, K_{w+1}^E)$ costing $E_{ub}(K_{u_1}^A, \mathcal{P}_0^*)$ to do so. As we assumed that

$$\hat{\mathbf{a}}_{u_1} - \frac{K_{u_1}^A}{K_{u_2}^A} \cdot [\hat{\mathbf{a}}_{i,p} - E_{lb}^{rhs}(K_{u_2}^A, \mathcal{P}_0^*)] > E_{ub}(K_{u_1}^A, \mathcal{P}_0^*),$$

we can conclude that the portfolio $P(E_{lb}, E_{ub}; K_{u_2}^A, K_{u_1}^A; \mathcal{P}_0^*)$ generates arbitrage if exercised correctly. ☐

We continue by investigating the situation where an American option with strike $K_{i,q} \in \mathbb{K}^E(\mathcal{P}_0^*)$, can be super-replicated for less than $A_{lb}(K_{i,q}, \mathcal{P}_0^*)$. To this end, we assume that the strikes $K_{i,q}$ and $K_{i,p}$ correspond to $K_{u_1}^E \in \mathbb{K}^E(\mathcal{P}_0^*)$ and $K_{u_2}^A \in \mathbb{K}^A(\mathcal{P}_0^*) \backslash K^E(\mathcal{P}_0^*)$, respectively.

Proposition 3.6.14. *Suppose that American and co-terminal European options are traded in the market and that their prices are provided by* $\mathcal{P}_0^* \in \overline{\mathcal{M}}$*. Then we can generate model-independent arbitrage in the market if there exist strikes* $K_{u_1}^E \in \mathbb{K}^E(\mathcal{P}_0^*)$

and $K^A_{u_2} \in \mathbb{K}^A(\mathcal{P}^*_0)\backslash K^E(\mathcal{P}^*_0)$ *with* $K^E_{u_1} < K^A_{u_2}$*, such that*

$$\hat{\mathbf{e}}_{u_1} + \frac{K^E_{u_1}}{K^A_{u_2}} \cdot [\hat{\mathbf{a}}_{u_2} - E^{rhs}_{lb}(K^A_{u_2}, \mathcal{P}^*_0)] < A^{lhs}_{lb}(K^E_{u_1}, \mathcal{P}^*_0). \tag{3.34}$$

Specifically, if we set $v = \arg\min_{1\le j'\le m_2-1}\{K^E_{j'} \in \mathbb{K}^E(\mathcal{P}^*_0) : K^E_{j'} \ge K^A_{u_2}\}$ *and* $w = \arg\max_{0\le j'\le m_1}\{K^A_{j'} \in \mathbb{K}^A(\mathcal{P}^*_0) : K^A_{j'} \le K^E_{u_1}\}$*, then we can generate arbitrage using the portfolio* $P(E_{lb}, A^{lhs}_{lb}; K^A_{u_2}, K^E_{u_1}; \mathcal{P}^*_0)$*, consisting of a long position of*

- $\frac{K^E_{u_1}-K^A_w}{K^E_{u_1}-K^A_{w-1}}$ *units of American options with strike* K^A_{w-1}*,*
- $\frac{K^A_w-K^A_{w-1}}{K^E_{u_1}-K^A_{w-1}}$ *units of European options with strike* $K^E_{u_1}$*,*
- $\frac{K^E_{u_1}}{K^A_{u_2}} \cdot \frac{K^A_w-K^A_{w-1}}{K^E_{u_1}-K^A_{w-1}}$ *units of American options with strike* $K^A_{u_2}$*,*
- $\frac{K^E_{u_1}}{K^A_{u_2}} \cdot \frac{K^A_w-K^A_{w-1}}{K^E_{u_1}-K^A_{w-1}} \cdot \frac{K^E_v-K^A_{u_2}}{K^E_{v+1}-K^E_v}$ *units of European option with strike* K^E_{v+1}

and a short position of

- 1 *unit of the American option with strike* K^A_w*,*
- $\frac{K^E_{u_1}}{K^A_{u_2}} \cdot \frac{K^A_w-K^A_{w-1}}{K^E_{u_1}-K^A_{w-1}} \cdot \frac{K^E_{v+1}-K^A_{u_2}}{K^E_{v+1}-K^E_v}$ *units of European option with strike* K^E_v*,*

if we exercise the positions in the American options with strike K^A_{w-1} *and* $K^A_{u_2}$ *simultaneously with the American option with strike* K^A_w*.*

Similarly, if

$$\hat{\mathbf{e}}_{u_1} + \frac{K^E_{u_1}}{K^A_{u_2}} \cdot [\hat{\mathbf{a}}_{u_2} - E^{rhs}_{lb}(K^A_{u_2}, \mathcal{P}^*_0)] < A^{rhs}_{lb}(K^E_{u_1}, \mathcal{P}^*_0)$$

then we can generate arbitrage using the portfolio $P(E_{lb}, A^{rhs}_{lb}; K^A_{u_2}, K^E_{u_1}; \mathcal{P}^*_0)$*, consisting of a long position of*

- $\frac{K^A_{w+2}-K^A_{w+1}}{K^A_{w+2}-K^E_{u_1}}$ *units of European options with strike* $K^E_{u_1}$*,*
- $\frac{K^A_{w+1}-K^E_{u_1}}{K^A_{w+2}-K^E_{u_1}}$ *units of American options with strike* K^A_{w+2}
- $\frac{K^E_{u_1}}{K^A_{u_2}} \cdot \frac{K^A_{w+2}-K^A_{w+1}}{K^A_{w+2}-K^E_{u_1}}$ *units of American options with strike* $K^A_{u_2}$
- $\frac{K^E_{u_1}}{K^A_{u_2}} \cdot \frac{K^A_{w+2}-K^A_{w+1}}{K^A_{w+2}-K^E_{u_1}} \cdot \frac{K^E_v-K^A_{u_2}}{K^E_{v+1}-K^E_v}$ *units of European option with strike* K^E_{v+1}

and a short position of

- 1 *unit of the American option with strike* K^A_{w+1}*,*

- $\frac{K_{u_1}^E}{K_{u_2}^A} \cdot \frac{K_{w+2}^A - K_{w+1}^A}{K_{w+2}^A - K_{u_1}^E} \cdot \frac{K_{v+1}^E - K_{u_2}^A}{K_{v+1}^E - K_v^E}$ *units of European option with strike* K_v^E,

if the positions in the American options with strike K_{w+2}^A *and* $K_{u_2}^A$ *are exercised simultaneously with the American option at strike* K_{w+1}^A.

Proof. Let us assume for now that American options with strike $K_{u_1}^E$ are traded in the market for the price

$$\hat{\mathbf{e}}_{u_1} + \frac{K_{u_1}^E}{K_{u_2}^A} \cdot [\hat{\mathbf{a}}_{u_2} - E_{lb}^{rhs}(K_{u_2}^A, \mathcal{P}_0^*)].$$

We could then super-replicate an American option with strike K_w^A using the portfolio $P_1^A(K_w^A; K_{w-1}^A, K_{u_1}^E)$ at cost

$$\alpha \cdot \hat{\mathbf{a}}_{w-1} + (1-\alpha) \cdot \left[\hat{\mathbf{e}}_{u_1} + \frac{K_{u_1}^E}{K_{u_2}^A} \cdot [\hat{\mathbf{a}}_{u_2} - E_{lb}^{rhs}(K_{u_2}^A, \mathcal{P}_0^*)]\right],$$

where $\alpha = (K_{u_1}^E - K_w^A)/(K_{u_1}^E - K_{w-1}^A)$. Hence, we would obtain an arbitrage portfolio by holding $P_1^A(K_w^A; K_{w-1}^A, K_{u_1}^E)$ while short selling an American option with strike K_w^A, as the set-up cost is given by

$$\alpha \cdot \hat{\mathbf{a}}_{w-1} + (1-\alpha) \cdot \left[\hat{\mathbf{e}}_{u_1} + \frac{K_{u_1}^E}{K_{u_2}^A} \cdot [\hat{\mathbf{a}}_{u_2} - E_{lb}^{rhs}(K_{u_2}^A, \mathcal{P}_0^*)]\right] - \hat{\mathbf{a}}_w$$

which is strictly negative due to (3.34).

Since American options with strike $K_{u_1}^E$ are not actually traded in the market, this arbitrage portfolio cannot be generated. We will therefore super-replicate an American option with strike $K_{u_1}^E$ at cost

$$\hat{\mathbf{e}}_{u_1} + \frac{K_{u_1}^E}{K_{u_2}^A} \cdot [\hat{\mathbf{a}}_{u_2} - E_{lb}^{rhs}(K_{u_2}^A, \mathcal{P}_0^*)]$$

in two steps. Suppose for the moment that European options with strike $K_{u_2}^A$ are traded in the market at $E_{lb}^{rhs}(K_{u_2}^A, \mathcal{P}_0^*)$, then we can purchase the super-replicating portfolio $P_1^{LF}(K_{u_1}^E, K_{u_2}^A)$ for $\hat{\mathbf{e}}_{u_1} + \frac{K_{u_1}^E}{K_{u_2}^A} \cdot [\hat{\mathbf{a}}_{u_2} - E_{lb}^{rhs}(K_{u_2}^A, \mathcal{P}_0^*)]$ in the market.

However, as European options with strike $K_{u_2}^A$ are not traded, it follows that the portfolio $P_1^{LF}(K_{u_1}^E, K_{u_2}^A)$ contains a short position in those options that cannot be acquired. To ensure that we can super-replicate the payoff of an American option with strike $K_{u_1}^E$, we thus have to find a sub-replicating portfolio for the European option with strike $K_{u_2}^A$. According to Proposition 3.6.4, the portfolio $P_3^E(K_{u_2}^A; K_{i,v}^E, K_{v+1}^E)$ will sub-replicate the European option with strike $K_{u_2}^A$ at cost $E_{lb}^{rhs}(K_{u_2}^A, \mathcal{P}_0^*)$. Hence, we can conclude that the portfolio $P(E_{lb}, A_{lb}^{lhs}; K_{u_2}^A, K_{u_1}^E; \mathcal{P}_0^*)$ generates arbitrage, as it has non-negative payoff regardless of the evolution of the underlying and a strictly negative cost of

$$\alpha \cdot \hat{\mathbf{a}}_{w-1} + (1-\alpha) \cdot \left[\hat{\mathbf{e}}_{u_1} + \frac{K^E_{u_1}}{K^A_{u_2}} \cdot [\hat{\mathbf{a}}_{u_2} - E^{rhs}_{lb}(K^A_{u_2}, \mathcal{P}^*_0)] \right] - \hat{\mathbf{a}}_w,$$

where $\alpha = (K^E_{u_1} - K^A_w)/(K^E_{u_1} - K^A_{w-1})$.

To obtain the arbitrage portfolio $P(E_{lb}, A^{rhs}_{lb}; K^E_{u_1}, K^A_{u_2}; \mathcal{P}^*_0)$ in the case where

$$\hat{\mathbf{e}}_{u_1} + \frac{K^E_{u_1}}{K^A_{u_2}} \cdot [\hat{\mathbf{a}}_{u_2} - E^{rhs}_{lb}(K^A_{u_2}, \mathcal{P}^*_0)] < A^{rhs}_{lb}(K^E_{u_1}, \mathcal{P}^*_0)$$

we proceed analogously. The only difference to the arbitrage portfolio in the previous case is that we now aim to super-replicate an American option with strike K^A_{w+1} using the portfolio $P^A_1(K^A_{w+1}; K^E_{u_1}, K^A_{w+2})$, which includes a long position in the non-traded American option with strike $K^E_{u_1}$. □

Remark 3.6.15. *Note that we can rule out the case where the left hand-side lower bound $A^{lhs}_{lb}(K^E_{u_1}, \mathcal{P}^*_0)$ is given by zero, as a negative price for American options with strike $K^E_{u_1}$ would imply that European options are traded for a strictly negative price in the market.*

We are thus left to show that there exists arbitrage whenever an American option with strike $K_{i,q} \in \mathbb{K}^E(\mathcal{P}^*_0)$, can be super-replicated in the market for less than $A^{\overline{A}}_{lb}(K_{i,q}, \mathcal{P}^*_0)$. To do so, we suppose again that the strikes $K_{i,q}$ and $K_{i,p}$ correspond to $K^E_{u_1} \in \mathbb{K}^E(\mathcal{P}^*_0)$ and $K^A_{u_2} \in \mathbb{K}^A(\mathcal{P}^*_0) \backslash K^E(\mathcal{P}^*_0)$, respectively.

Proposition 3.6.16. *Suppose that American and co-terminal European options are traded in the market and that their prices are provided by $\mathcal{P}^*_0$. Then we can generate model-independent arbitrage in the market if there exist strikes $K^E_{u_1} \in \mathbb{K}^E(\mathcal{P}^*_0)$ and $K^A_{u_2} \in \mathbb{K}^A(\mathcal{P}^*_0) \backslash K^E(\mathcal{P}^*_0)$ with $K^E_{u_1} < K^A_{u_2}$, such that*

$$\hat{\mathbf{e}}_{u_1} + \frac{K^E_{u_1}}{K^A_{u_2}} \cdot [\hat{\mathbf{a}}_{u_2} - E^{rhs}_{lb}(K^A_{u_2}, \mathcal{P}^*_0)] < A^{\overline{A},l}_{lb}(K^E_{u_1}, \mathcal{P}^*_0).$$

Specifically, let us write $v = \arg\min_{1 \le j' \le m_2 - 1}\{K^E_{j'} \in \mathbb{K}^E(\mathcal{P}^*_0) : K^E_{j'} \geq K^A_{u_2}\}$ and $w = \arg\max_{0 \le j' \le m_1}\{K^A_{j'} \in \mathbb{K}^A(\mathcal{P}^*_0) : K^A_{j'} \leq K^E_{u_1}\}$. Suppose

$$A^{\overline{A},l}_{lb}(K^E_{u_1}, \mathcal{P}^*_0) = \frac{\hat{\mathbf{a}}_w - \overline{\mathbf{a}}_j}{K^A_w - K^E_j e^{-rT}} \cdot (K^E_{u_1} - K^A_w) + \hat{\mathbf{a}}_w,$$

where $K^E_j e^{-rT} \in S_{w-1}$, then we can use portfolio $P(E_{lb}, A^{\overline{A},l}_{lb}; K^A_{u_2}, K^E_{u_1}; \mathcal{P}^*_0)$, consisting of a long position of

- $\frac{K^E_{u_1} - K^A_w}{K^E_{u_1} - K^E_j e^{-rT}}$ *units of European options with strike K^E_j*

- $\frac{K_w^A - K_j^E e^{-rT}}{K_{u_1}^E - K_j^E e^{-rT}}$ *units of European options with strike* $K_{u_1}^E$
- $\frac{K_{u_1}^E}{K_{u_2}^A} \cdot \frac{K_w^A - K_j^E e^{-rT}}{K_{u_1}^E - K_j^E e^{-rT}}$ *units of American options with strike* $K_{u_2}^A$
- $\frac{K_{u_1}^E}{K_{u_2}^A} \cdot \frac{K_w^A - K_j^E e^{-rT}}{K_{u_1}^E - K_j^E e^{-rT}} \cdot \frac{K_v^E - K_{u_2}^A}{K_{v+1}^E - K_v^E}$ *units of European option with strike* K_{v+1}^E

and a short position of

- 1 *unit of American options with strike* K_w^A
- $\frac{K_{u_1}^E}{K_{u_2}^A} \cdot \frac{K_w^A - K_j^E e^{-rT}}{K_{u_1}^E - K_j^E e^{-rT}} \cdot \frac{K_{v+1}^E - K_{u_2}^A}{K_{v+1}^E - K_v^E}$ *units of European option with strike* K_v^E

to generate arbitrage in the market if the position in the American options with strike $K_{u_2}^A$ *is exercised simultaneously with the American option at strike* K_w^A.

Similarly, suppose

$$\hat{\mathbf{e}}_{u_1} + \frac{K_{u_1}^E}{K_{u_2}^A} \cdot [\hat{\mathbf{a}}_{u_2} - E_{lb}^{rhs}(K_{u_2}^A, \mathcal{P}_0^*)] < A_{lb}^{\overline{A},r}(K_{u_1}^E, \mathcal{P}_0^*),$$

where

$$A_{lb}^{\overline{A},r}(K_{u_1}^E, \mathcal{P}_0^*) = \frac{\overline{\mathbf{a}}_j - \hat{\mathbf{a}}_{w+1}}{K_j^E e^{-rT} - K_{w+1}^A} \cdot (K_{u_1}^E - K_{w+1}^A) + \hat{\mathbf{a}}_{w+1}$$

with $K_j^E e^{-rT} \in S_{w+1}$, *then we can use the portfolio* $P(E_{lb}, A_{lb}^{\overline{A},r}; K_{u_2}^A, K_{u_1}^E; \mathcal{P}_0^*)$, *consisting of a long position of*

- $\frac{K_j^E e^{-rT} - K_{w+1}^A}{K_j^E e^{-rT} - K_{u_1}^E}$ *units of European options with strike* $K_{u_1}^E$
- $\frac{K_{w+1}^A - K_{u_1}^E}{K_j^E e^{-rT} - K_{u_1}^E}$ *units of European options with strike* K_j^E
- $\frac{K_{u_1}^E}{K_{u_2}^A} \cdot \frac{K_j^E e^{-rT} - K_{w+1}^A}{K_j^E e^{-rT} - K_{u_1}^E}$ *units of American options with strike* $K_{u_2}^A$
- $\frac{K_{u_1}^E}{K_{u_2}^A} \cdot \frac{K_j^E e^{-rT} - K_{w+1}^A}{K_j^E e^{-rT} - K_{u_1}^E} \cdot \frac{K_v^E - K_{u_2}^A}{K_{v+1}^E - K_v^E}$ *units of European option with strike* K_{v+1}^E

and a short position of

- 1 *unit of American options with strike* K_{w+1}^A
- $\frac{K_{u_1}^E}{K_{u_2}^A} \cdot \frac{K_j^E e^{-rT} - K_{w+1}^A}{K_j^E e^{-rT} - K_{u_1}^E} \cdot \frac{K_{v+1}^E - K_{u_2}^A}{K_{v+1}^E - K_v^E}$ *units of European option with strike* K_v^E

to generate arbitrage if we exercise the position in the American option with strike $K_{u_2}^A$ *simultaneously with the American option with strike* K_{w+1}^A.

Proof. Assume for a moment that both American options with strike $K_j^E e^{-rT}$ and $K_{u_1}^E$ are traded in the market and that their respective prices are $\overline{\mathbf{a}}_j$ and $\hat{\mathbf{e}}_{u_1} + \frac{K_{u_1}^E}{K_{u_2}^A} \cdot [\hat{\mathbf{a}}_{u_2} - E_{lb}^{rhs}(K_{u_2}^A, \mathcal{P}_0^*)]$. In this case the portfolio $P_1^A(K_w^A; K_j^E e^{-rT}, K_{u_1}^E)$ could be used to super-replicate an American option with strike K_w^A. Moreover, we could generate arbitrage by holding a portfolio consisting of a long position in $P_1^A(K_w^A; K_j^E e^{-rT}, K_{u_1}^E)$ and a short position of one American option with strike K_w^A, as its cost is given by

$$\alpha \cdot \overline{\mathbf{a}}_j + (1-\alpha) \cdot \left[\hat{\mathbf{e}}_{u_1} + \frac{K_{u_1}^E}{K_{u_2}^A} \cdot [\hat{\mathbf{a}}_{u_2} - E_{lb}^{rhs}(K_{u_2}^A, \mathcal{P}_0^*)]\right] - \hat{\mathbf{a}}_w,$$

where $\alpha = (K_{u_1}^E - K_w^A)/(K_{u_1}^E - K_j^E e^{-rT})$, and thus strictly negative. However, American options are neither traded at strike $K_j^E e^{-rT}$ nor at strike $K_{u_1}^E$ in the market and we are forced to find a super-replicating portfolio for each one of them. Since we discussed already how to super-replicate an American option with strike $K_{u_1}^E$ using the portfolios $P_2^{LF}(K_{u_1}^E, K_{u_2}^A)$ and $P_3^E(K_{u_2}^A; K_v^E, K_{v+1}^E)$, we are only left to find a super-replicating portfolio for the payoff of an American option with strike $K_j^E e^{-rT}$ with cost $\overline{\mathbf{a}}_j$. In Theorem 2.2.3 of Chapter 2, we argued that a European option with strike K_j^E can be used to super-replicate the payoff of an American option with strike $K_j^E e^{-rT}$. In particular, we know from the definition of the upper bound $\overline{A}$ that the cost for a European option with strike K_j^E matches $\overline{\mathbf{a}}_j$. We can therefore conclude that the portfolio $P(E_{lb}, A_{lb}^{\overline{A},l}; K_{u_2}^A, K_{u_1}^E; \mathcal{P}_0^*)$ generates arbitrage in the market. The derivation of the arbitrage portfolio $P(E_{lb}, A_{lb}^{\overline{A},r}; K_{u_2}^A, K_{u_1}^E; \mathcal{P}_0^*)$ follows analogously. ☐

3.6.4 Situation III: Violation of $A_{ub}(K_{i,p}, \mathcal{P}_{i,p-1}) \geq \mathbf{a}_{i,p}$

We will now present the arbitrage portfolios that can be used in case there exists a strike $K_{i,p} \in \mathbb{K}^E(\mathcal{P}_i^*) \backslash \mathbb{K}^A(\mathcal{P}_0^*)$ where $\mathbf{a}_{i,p} > A_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})$. Recall that in this situation the revised price for American options with strike $K_{i,p}$ depends on whether or not the upper bound A_{ub} is given by the prices of two traded options. If this is the case it is possible to super-replicate the American option with strike $K_{i,p}$ using portfolio P_1^A. Otherwise we will use portfolio P_3^{LF} to super-replicate the option. Setting the revised price for American options with strike $K_{i,p}$ equal to the cost of the respective super-replicating portfolio then allows us to generate arbitrage whenever $\mathbf{e}_{i,q}^n > E_{ub}(K_{i,q}, \mathcal{P}_0^*)$ for $K_{i,q} \in \mathbb{K}^A(\mathcal{P}_0^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*)$ or $\mathbf{a}_{i,q}^n < \max\{A_{lb}(K_{i,q}, \mathcal{P}_0^*), A_{lb}^{\overline{A}}(K_{i,q}, \mathcal{P}_0^*)\}$ for $K_{i,q} \in \mathbb{K}^E(\mathcal{P}_0^*)$ occurs at strike $K_{i,q}$, where $K_{i,q} < K_{i,p}$.

Note that we will only discuss the situation where $\mathbf{a}_{i,p} > A_{ub}(K_{i,p}, \mathcal{P}_0^*)$, as the arbitrage portfolios in the second case is obtained by replacing P_1^A by P_3^{LF} in the arbitrage portfolio.

In the following proposition the strikes $K_{i,q}$ and $K_{i,p}$ correspond to $K_{u_1}^A \in \mathbb{K}^A(\mathcal{P}_0^*)$ and $K_{u_2}^E \in \mathbb{K}^E(\mathcal{P}_0^*) \backslash K^A(\mathcal{P}_0^*)$, respectively.

Proposition 3.6.17. *Suppose that American and co-terminal European options are traded in the market and that their prices are provided by $\mathcal{P}_0^*$. Then we can generate model-independent arbitrage in the market if there exist strikes $K_{u_1}^A \in \mathbb{K}^A(\mathcal{P}_0^*)$ and $K_{u_2}^E \in \mathbb{K}^E(\mathcal{P}_0^*)\backslash\mathbb{K}^A(\mathcal{P}_0^*)$ with $K_{u_1}^A < K_{u_2}^E$, such that*

$$\hat{\mathbf{a}}_{u_1} - \frac{K_{u_1}^A}{K_{u_2}^E} \cdot [A_{ub}(K_{u_2}^E, \mathcal{P}_0^*) - \hat{\mathbf{e}}_{u_2}] > E_{ub}(K_{u_1}^A, \mathcal{P}_0^*).$$

Specifically, if we set $v = \arg\max_{1\leq j'\leq m_1-1}\{K_{j'}^A \in \mathbb{K}^A(\mathcal{P}_0^*) : K_{j'}^A \leq K_{u_2}^E\}$ and $w = \arg\max_{0\leq j'\leq m_2}\{K_{j'}^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K_{j'}^E \leq K_{u_1}^A\}$, then we can generate arbitrage using the portfolio $P(A_{ub}, E_{ub}; K_{u_2}^E, K_{u_1}^A, \mathcal{P}_0^*)$, consisting of a long position of

- $\frac{K_{w+1}^E - K_{u_1}^A}{K_{w+1}^E - K_w^E}$ *units of the European option with strike K_w^E,*
- $\frac{K_{u_1}^A - K_w^E}{K_{w+1}^E - K_w^E}$ *units of the European option with strike K_{w+1}^E,*
- $\frac{K_{u_1}^A}{K_{u_2}^E} \cdot \frac{K_{v+1}^A - K_{u_2}^E}{K_{v+1}^A - K_v^A}$ *units of the American option with strike K_v^A*
- $\frac{K_{u_1}^A}{K_{u_2}^E} \cdot \frac{K_{u_2}^E - K_v^A}{K_{v+1}^A - K_v^A}$ *units of the American option with strike K_{v+1}^A*

and a short position of

- 1 *unit of the American option with strike $K_{u_1}^A$,*
- $\frac{K_{u_1}^A}{K_{u_2}^E}$ *units of the European option with strike $K_{u_2}^E$.*

if the American options with strike K_v^A and K_{v+1}^A are exercised simultaneously with the American option with strike $K_{u_1}^A$.

Proof. If we assume that both European options with strike $K_{u_1}^A$ and American options with strike $K_{u_2}^E$ are traded in the market for

$$\hat{\mathbf{a}}_{u_1} - \frac{K_{u_1}^A}{K_{u_2}^E} \cdot [A_{ub}(K_{u_2}^E, \mathcal{P}_0^*) - \hat{\mathbf{e}}_{u_2}]$$

and $A_{ub}(K_{u_2}^E, \mathcal{P}_0^*)$, respectively, we can super-replicate a (traded) American option with strike $K_{u_1}^A$ using the portfolio $P_1^{LF}(K_{u_1}^A, K_{u_2}^E)$ for $\hat{\mathbf{a}}_{u_1}$. However, as neither European options with strike $K_{u_1}^A$ nor American options with strike $K_{u_2}^E$ are actually traded, we have to find a super-replicating portfolio for either position. In the case of the European option with strike $K_{u_1}^A$, we can use the portfolio $P_1^E(K_{u_1}^A; K_w^E, K_{w+1}^E)$. Moreover, we have that the super-replicating portfolio can be purchased in the market for $E_{ub}(K_{u_1}^A, \mathcal{P}_0^*)$, which is strictly less than

$$\hat{\mathbf{a}}_{u_1} - \frac{K_{u_1}^A}{K_{u_2}^E} \cdot [A_{ub}(K_{u_2}^E, \mathcal{P}_0^*) - \hat{\mathbf{e}}_{u_2}].$$

An American option with strike $K^E_{u_2}$ can then be super-replicated using the portfolio $P^A_1(K^E_{u_2}; K^A_v, K^A_{v+1})$ at cost $A_{ub}(K^E_{u_2}, \mathcal{P}^*_0)$. Combined this implies that we can find a super-replicating portfolio for an American option with strike $K^A_{u_1}$ for strictly less than $\hat{\mathbf{a}}_{u_1}$, the amount it is traded for in the market. Hence, we can generate arbitrage using the portfolio $P(A_{ub}, E_{ub}; K^E_{u_2}, K^A_{u_1}, \mathcal{P}^*_0)$. $\square$

We continue by investigating the arbitrage portfolios for the situation where American options with strike $K_{i,q}$ can be super-replicated for strictly less than $A_{lb}(K_{i,q}, \mathcal{P}^*_0)$, where $K_{i,q} \in \mathbb{K}^E(\mathcal{P}^*_0)$. To do so, we assume that the strikes $K_{i,q}$ and $K_{i,p}$ correspond to $K^E_{u_1} \in \mathbb{K}^E(\mathcal{P}^*_0)$ and $K^E_{u_2} \in \mathbb{K}^E(\mathcal{P}^*_0) \backslash \mathbb{K}^A(\mathcal{P}^*_0)$, respectively.

Proposition 3.6.18. *Suppose that American and co-terminal European options are traded in the market and that their prices are provided by $\mathcal{P}^*_0$. Then we can generate model-independent arbitrage in the market if there exist strikes $K^E_{u_1} \in \mathbb{K}^E(\mathcal{P}^*_0)$, $K^E_{u_2} \in \mathbb{K}^E(\mathcal{P}^*_0) \backslash \mathbb{K}^A(\mathcal{P}^*_0)$ with $K^E_{u_1} < K^E_{u_2}$, such that*

$$\hat{\mathbf{e}}_{u_1} + \frac{K^E_{u_1}}{K^E_{u_2}} \cdot [A_{ub}(K^E_{u_2}, \mathcal{P}^*_0) - \hat{\mathbf{e}}_{u_2}] < A^{lhs}_{lb}(K^E_{u_1}, \mathcal{P}^*_0).$$

Specifically, if we set $v = \arg\max_{1 \le j' \le m_1}\{K^A_{j'} \in \mathbb{K}^A(\mathcal{P}^*_0) : K^A_{j'} \le K^E_{u_2}\}$ and $w = \arg\max_{0 \le j' \le m_1}\{K^A_{j'} \in \mathbb{K}^A(\mathcal{P}^*_0) : K^A_{j'} \le K^E_{u_1}\}$, then we can generate arbitrage using the portfolio $P(A_{ub}, A^{lhs}_{lb}; K^E_{u_2}, K^E_{u_1}, \mathcal{P}^*_0)$, consisting of a long position of

- $\frac{K^E_{u_1} - K^A_w}{K^E_{u_1} - K^A_{w-1}}$ *units of the American option with strike K^A_{w-1},*
- $\frac{K^A_w - K^A_{w-1}}{K^E_{u_1} - K^A_{w-1}}$ *units of the European option with strike $K^E_{u_1}$,*
- $\frac{K^E_{u_1}}{K^E_{u_2}} \cdot \frac{K^E_{u_1} - K^A_w}{K^E_{u_1} - K^A_{w-1}} \cdot \frac{K^A_{v+1} - K^E_{u_2}}{K^A_{v+1} - K^A_v}$ *units of the American option with strike K^A_v.*
- $\frac{K^E_{u_1}}{K^E_{u_2}} \cdot \frac{K^E_{u_1} - K^A_w}{K^E_{u_1} - K^A_{w-1}} \cdot \frac{K^E_{u_2} - K^A_v}{K^A_{v+1} - K^A_v}$ *units of the American option with strike K^A_{v+1}*

and a short position of

- 1 *unit of the American option with strike K^A_w,*
- $\frac{K^E_{u_1}}{K^E_{u_2}} \cdot \frac{K^E_{u_1} - K^A_w}{K^E_{u_1} - K^A_{w-1}}$ *units of the European option with strike $K^E_{u_2}$.*

if we exercise the positions in the American options with strike K^A_{w-1}, K^A_v and K^A_{v+1} simultaneously with the American option with strike K^A_w.

Similarly, if

$$\hat{\mathbf{e}}_{u_1} + \frac{K^E_{u_1}}{K^E_{u_2}} \cdot [A_{ub}(K^E_{u_2}, \mathcal{P}^*_0) - \hat{\mathbf{e}}_{u_2}] < A^{rhs}_{lb}(K^E_{u_1}, \mathcal{P}^*_0).$$

then we can generate arbitrage using the portfolio $P(A_{ub}, A_{lb}^{rhs}; K_{u_2}^E, K_{u_1}^E; \mathcal{P}_0^*)$, *consisting of a long position of*

- $\frac{K_{w+1}^A - K_{w+1}^A}{K_{w+1}^A - K_{u_1}^E}$ *units of the European option with strike* $K_{u_1}^E$,
- $\frac{K_{w+1}^A - K_{u_1}^A}{K_{w+1}^A - K_{u_1}^E}$ *units of the American option with strike* K_{w+1}^A
- $\frac{K_{u_1}^E}{K_{u_2}^E} \cdot \frac{K_{w+1}^A - K_{w+1}^A}{K_{w+1}^A - K_{u_1}^E} \cdot \frac{K_{v+1}^A - K_{u_2}^E}{K_{v+1}^A - K_v^A}$ *units of the American option with strike* K_v^A
- $\frac{K_{u_1}^E}{K_{u_2}^E} \cdot \frac{K_{w+1}^A - K_{w+1}^A}{K_{w+1}^A - K_{u_1}^E} \cdot \frac{K_{u_2}^E - K_v^A}{K_{v+1}^A - K_v^A}$ *units of the American option with strike* K_{v+1}^A

and a short position of

- 1 *unit of the American option with strike* K_{w+1}^A,
- $\frac{K_{u_1}^E}{K_{u_2}^E} \cdot \frac{K_{w+1}^A - K_{w+1}^A}{K_{w+1}^A - K_{u_1}^E}$ *units of the European option with strike* $K_{u_2}^E$.

if we exercise the American options with strike K_{w+1}^A, K_v^A *and* K_{v+1}^A *simultaneously with the American option with strike* K_{w+1}^A.

Proof. We assume for the moment that American options with strike $K_{u_1}^E$ are traded at cost

$$\hat{\mathbf{e}}_{u_1} + \frac{K_{u_1}^E}{K_{u_2}^E} \cdot [A_{ub}(K_{u_2}^E, \mathcal{P}_0^*) - \hat{\mathbf{e}}_{u_2}],$$

then we can super-replicate a (traded) American option with strike K_w^A using the portfolio $P_1^A(K_w^A; K_{w+1}^A, K_{u_1}^E)$ for

$$\alpha_1 \cdot \hat{\mathbf{a}}_{w-1} + (1 - \alpha_1) \cdot \hat{\mathbf{e}}_{u_1} + \frac{K_{u_1}^E}{K_{u_2}^E} \cdot [A_{ub}(K_{u_2}^E, \mathcal{P}_0^*) - \hat{\mathbf{e}}_{u_2}],$$

where $\alpha_1 = (K_{u_1}^E - K_w^A)/(K_{u_1}^E - K_{w-1}^A)$. According to the assumptions, the super-replicating portfolio would cost strictly less than $\hat{\mathbf{a}}_w$. However, American options with strike $K_{u_1}^E$ are not actually traded and thus we have to super-replicate that position. To this end, let us suppose that American options with strike $K_{u_2}^E$ are traded at $A_{ub}(K_{u_2}^E, \mathcal{P}_0^*)$, then we can super-replicate an American option with strike $K_{u_1}^E$ for $\hat{\mathbf{e}}_{u_1} + \frac{K_{u_1}^E}{K_{u_2}^E} \cdot [A_{ub}(K_{u_2}^E, \mathcal{P}_0^*) - \hat{\mathbf{e}}_{u_2}]$ using the portfolio $P_1^{LF}(K_{u_1}^E, K_{u_2}^E)$.

However, American options with strike K_{u_1} are not traded and therefore have to be super-replicated. This can be achieve for $A_{ub}(K_{u_2}^E, \mathcal{P}_0^*)$, using the portfolio $P_1^A(K_{u_2}^E; K_v^A, K_{v+1}^A)$. Hence, we found a super-replicating strategy for the American option with strike K_w^A that can be purchased in the market for strictly less than $\hat{\mathbf{a}}_w$. We can thus conclude that the portfolio $P(A_{ub}, A_{lb}^{lhs}; K_{u_2}^E, K_{u_1}^E; \mathcal{P}_0^*)$ generates arbitrage in the market.

Similarly, we consider the super-replicating portfolio $P_1^A(K_{w+1}^A; K_{u_1}^E, K_{w+1}^A)$ for the traded American option with strike K_w^A in the second case. Under the assumption that American options with strike $K_{u_1}^E$ trade for

$$\hat{\mathbf{e}}_{u_1} + \frac{K_{u_1}^E}{K_{u_2}^E} \cdot [A_{ub}(K_{u_2}^E, \mathcal{P}_0^*) - \hat{\mathbf{e}}_{u_2}],$$

this portfolio can be purchased for strictly less than $\hat{\mathbf{a}}_{w+1}$. As this is not the case, we super-replicate the American option with strike $K_{u_1}^E$ as above. It follows that the portfolio $P(A_{ub}, A_{lb}^{rhs}; K_{u_2}^E, K_{u_1}^E; \mathcal{P}_0^*)$ generates arbitrage if the American options with strike K_v^A and K_{v+1}^A are exercised correctly. □

We are thus left to argue that there exists arbitrage in the market whenever American options with strike $K_{i,q}$ can be super-replicated for less than $A_{lb}^{\overline{A}}(K_{i,q}, \mathcal{P}_0^*)$. To do so, we assume that the strikes $K_{i,q}$ and $K_{i,p}$ correspond to $K_{u_1}^E \in \mathbb{K}^E(\mathcal{P}_0^*)$ and $K_{u_2}^E \in \mathbb{K}^E(\mathcal{P}_0^*) \backslash K^A(\mathcal{P}_0^*)$, respectively. Note further that the derivation for the arbitrage portfolios is analogous to the previous cases and thus will be omitted here.

Proposition 3.6.19. *Suppose that American and co-terminal European options are traded in the market and that their prices are provided by $\mathcal{P}_0^*$. Then we can generate model-independent arbitrage in the market if there exist strikes $K_{u_1}^E \in \mathbb{K}^E(\mathcal{P}_0^*)$, $K_{u_2}^E \in \mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_0^*)$ with $K_{u_1}^E < K_{u_2}^E$, such that*

$$\hat{\mathbf{e}}_{u_1} + \frac{K_{u_1}^E}{K_{u_2}^E} \cdot [A_{ub}(K_{u_2}^E, \mathcal{P}_0^*) - \hat{\mathbf{e}}_{u_2}] < A_{lb}^{\overline{A},l}(K_{u_1}^E, \mathcal{P}_0^*).$$

Specifically, let us write $v = \arg\max_{1 \le j' \le m_1} \{K_{j'}^A \in \mathbb{K}^A(\mathcal{P}_0^*) : K_{j'}^A \le K_{u_2}^E\}$ and $w = \arg\max_{0 \le j' \le m_1} \{K_{j'}^A \in \mathbb{K}^A(\mathcal{P}_0^*) : K_{j'}^A \le K_{u_1}^E\}$, Suppose

$$A_{lb}^{\overline{A},l}(K_{u_1}^E, \mathcal{P}_0^*) = \frac{\hat{\mathbf{a}}_w - \overline{\mathbf{a}}_j}{K_w^A - K_j^E e^{-rT}} \cdot (K_{u_1}^E - K_w^A) + \hat{\mathbf{a}}_w,$$

where $K_j^E e^{-rT} \in S_{w-1}$, then we can use the portfolio $P(A_{ub}, A_{lb}^{\overline{A},l}; K_{u_2}^E, K_{u_1}^E, \mathcal{P}_0^*)$, consisting of a long position of

- $\frac{K_{u_1}^E - K_w^A}{K_{u_1}^E - K_j^E e^{-rT}}$ *units of the European option with strike K_j^E,*
- $\frac{K_w^A - K_j^E e^{-rT}}{K_{u_1}^E - K_j^E e^{-rT}}$ *units of the European option with strike $K_{u_1}^E$,*
- $\frac{K_{u_1}^E}{K_{u_2}^E} \cdot \frac{K_{u_1}^E - K_w^A}{K_{u_1}^E - K_j^E e^{-rT}} \cdot \frac{K_{v+1}^A - K_{u_2}^E}{K_{v+1}^A - K_v^A}$ *units of the American option with strike K_v^A,*
- $\frac{K_{u_1}^E}{K_{u_2}^E} \cdot \frac{K_{u_1}^E - K_w^A}{K_{u_1}^E - K_j^E e^{-rT}} \cdot \frac{K_{u_2}^E - K_v^A}{K_{v+1}^A - K_v^A}$ *units of the American option with strike K_{v+1}^A*

and a short position of

- 1 *unit of the American option with strike* K_w^A,
- $\frac{K_{u_1}^E}{K_{u_2}^E} \cdot \frac{K_{u_1}^E - K_w^A}{K_{u_1}^E - K_j^E e^{-rT}}$ *units of the European option with strike* $K_{u_2}^E$

to generate arbitrage if we exercise the positions in the American options with strike K_v^A *and* K_{v+1}^A *simultaneously with the American option with strike* K_w^A.

Similarly, suppose

$$\hat{\mathbf{e}}_{u_1} + \frac{K_{u_1}^E}{K_{u_2}^E} \cdot [A_{ub}(K_{u_2}^E, \mathcal{P}_0^*) - \hat{\mathbf{e}}_{u_2}] < A_{lb}^{\overline{A},r}(K_{u_1}^E, \mathcal{P}_0^*),$$

where

$$A_{lb}^{\overline{A},r}(K_{u_1}^E, \mathcal{P}_0^*) = \frac{\overline{\mathbf{a}}_j - \hat{\mathbf{a}}_{w+1}}{K_j^E e^{-rT} - K_{w+1}^A} \cdot (K_{u_2}^E - K_{w+1}^A) + \hat{\mathbf{a}}_{w+1},$$

with $K_j^E e^{-rT} \in S_{w+1}$, *then we can use the portfolio* $P(A_{ub}, A_{lb}^{\overline{A},r}; K_{u_2}^E, K_{u_1}^E; \mathcal{P}_0^*)$, *consisting of a long position of*

- $\frac{K_j^E e^{-rT} - K_{w+1}^A}{K_j^E e^{-rT} - K_{u_1}^E}$ *units of the European option with strike* $K_{u_1}^E$,
- $\frac{K_{w+1}^A - K_{u_1}^E}{K_j^E e^{-rT} - K_{u_1}^E}$ *units of the European option with strike* K_j^E
- $\frac{K_{u_1}^E}{K_{u_2}^E} \cdot \frac{K_j^E e^{-rT} - K_{w+1}^A}{K_j^E e^{-rT} - K_{u_1}^E} \cdot \frac{K_{v+1}^A - K_{u_2}^E}{K_{v+1}^A - K_v^A}$ *units of the American option with strike* K_v^A
- $\frac{K_{u_1}^E}{K_{u_2}^E} \cdot \frac{K_j^E e^{-rT} - K_{w+1}^A}{K_j^E e^{-rT} - K_{u_1}^E} \cdot \frac{K_{u_2}^E - K_v^A}{K_{v+1}^A - K_v^A}$ *units of the American option with strike* K_{v+1}^A

and a short position of

- 1 *unit of the American option with strike* K_{w+1}^A,
- $\frac{K_{u_1}^E}{K_{u_2}^E} \cdot \frac{K_j^E e^{-rT} - K_{w+1}^A}{K_j^E e^{-rT} - K_{u_1}^A}$ *units of the European option with strike* $K_{u_2}^E$.

if we exercise the American options with strike K_v^A *and* K_{v+1}^A *simultaneously with the American option with strike* K_{w+1}^A.

3.7 Admissibility of the price functions

We are left to argue that the algorithm constructs admissible price functions A and E from a given set of prices $\mathcal{P}_0^*$ whenever none of the arbitrage situations of Section 3.6 occurs.

To this end, we will introduce the term of ξ-admissibility, which we will use to incorporate all the important properties of the price functions up to strike ξ.

Definition 3.7.1. *Suppose we are given a strike* $\xi \in (0, \infty)$. *We say that a set of prices* $\mathcal{P}$ *is a* ξ*-admissible* $\mathcal{P}_0^*$*-extension if:*

(I) $(\hat{\mathbf{a}}_i, K_i^A) \in \mathcal{P}^A$ *for* $i = 1, ..., m_1$ *and* $(\hat{\mathbf{e}}_j, K_j^E) \in \mathcal{P}^E$ *for* $j = 1, ..., m_2$.

(II) $\mathbb{K}(\mathcal{P}) \subset [0, \infty)$.

(III) $(\mathbb{K}^A(\mathcal{P}_0^*) \cup \mathbb{K}^E(\mathcal{P}_0^*)) \cap (0, \xi] \subseteq (\mathbb{K}^A(\mathcal{P}) \cap \mathbb{K}^E(\mathcal{P})) \cap (0, \xi] = (\mathbb{K}^A(\mathcal{P}) \cup \mathbb{K}^E(\mathcal{P})) \cap (0, \xi]$, *i.e. all traded options are priced correctly up to* ξ *under* $\mathcal{P}$ *and if one type of option is priced at a strike so is the other.*

(IV) The European price function $E(K, \mathcal{P})$ *defined in (3.7) satisfies the conditions in Lemma 3.1.1 for any strike* $K \geq 0$

(i) $E(K, \mathcal{P})$ *is increasing and convex.*

(ii) $E(K, \mathcal{P}) \geq (e^{-rT}K - S_0)_+$.

(iii) $E(K, \mathcal{P}) \leq e^{-rT}$.

(iv) If $E(K, \mathcal{P}) > e^{-rT}K - S_0$, *then* $E'(K+, \mathcal{P}) < e^{-rT}$.

(V) The American price function $A(K, \mathcal{P})$ *defined in (3.6) satisfies the conditions in Theorem 3.1.2 for any strike* $K \geq 0$

(i) $A(K, \mathcal{P})$ *is increasing and convex.*

(ii) $A(K, \mathcal{P}) \geq (K - S_0)_+$.

(iii) $A(K, \mathcal{P}) \geq E(K, \mathcal{P})$ *for* $K \in (0, \xi]$.

(iv) $A(K, \mathcal{P}) \leq \overline{A}(K, \mathcal{P})$ *for* $K \in (0, e^{-rT}\xi]$.

(VI) For $K_i, K_{i'} \in [0, \xi) \cup \mathbb{K}^A(\mathcal{P})$ *and* $K_j, K_{j'} \in [0, \xi) \cup \mathbb{K}^E(\mathcal{P})$ *with* $K_i < K_{i'}$, $K_j < K_{j'}$, $K_j \leq K_i$ *and* $K_{j'} \leq K_{i'}$ *the Legendre-Fenchel condition holds, i.e.*

$$\frac{A(K_{i'}) - A(K_i)}{K_{i'} - K_i} K_i - A(K_i) \geq \frac{E(K_{j'}) - E(K_j)}{K_{j'} - K_j} K_j - E(K_j) \tag{3.35}$$

where we suppressed the dependence of the price functions on the set of prices $\mathcal{P}$. *This simplifies for* $K \in (0, \xi)$ *to*

$$A'(K+, \mathcal{P})K - A(K, \mathcal{P}) \geq E'(K+, \mathcal{P})K - E(K, \mathcal{P}).$$

Using the definition of ξ-admissibility we are now able to show that the price functions generated by the algorithm satisfy the no-arbitrage conditions whenever the market is free of arbitrage.

For this purpose we begin by considering a possible violation of the no-arbitrage conditions when the initial set of prices is given by $\mathcal{P}_1^*$ as this allows us to rule out the existence of auxiliary price constraints of type 1.

3.7.1 Violation of the upper bound under $\mathcal{P}_1^*$

We start by analysing the situation in which a violation of the upper bound occurs. To this end, we assume that the algorithm computed $\mathcal{P}_{1,p-1}$ a $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension. In the next step the algorithm calculates the price for the non-traded option with strike $K_{1,p}$ according to either (3.22) or (3.23) depending on the type of the strike $K_{1,p}$. The algorithm then stops due to a violation of $\overline{\mathbf{a}}_{1,p} \geq A(K_{1,p}e^{-rT}, \mathcal{P}_{1,p})$, where we assume that $K_{1,p}e^{-rT} \in (K_{1,q}, K_{1,q+1}]$ for $K_{1,q}, K_{1,q+1} \in \mathbb{K}(\mathcal{P}_1^*)$. In addition, let us write

$$u = \arg\max\{K_i^A \in \mathbb{K}^A(\mathcal{P}_0^*) : K_i^A < K_{1,p}e^{-rT}\}.$$

We are then left to argue that the algorithm will successfully correct this violation without affecting the prices of options outside of $(K_{1,q}, K_{1,q+1})$. To see that this is the case we have to distinguish between the two cases where either $[K_u^A, K_{u+1}^A] \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ or $[K_u^A, K_{u+1}^A] \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) \neq \emptyset$

Case I: $[K_u^A, K_{u+1}^A] \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$

In this case the violation of the upper bound at $K_{1,p}e^{-rT}$ is the first violation of a no-arbitrage condition in $[K_u^A, K_{u+1}^A]$. Before we can argue that Algorithm 4 successfully corrects the violation, we need to show the existence of a strike $K_{j'}^E e^{-rT} \in (K_{1,q}, K_{1,q+1})$ as the auxiliary price constraint will be introduced at such a strike.

Proposition 3.7.2. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

The algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of $\overline{\mathbf{a}}_{1,p} \geq A(K_{1,p}e^{-rT}, \mathcal{P}_{1,p})$, where $K_{1,p}e^{-rT} \in (K_{1,q}, K_{1,q+1}]$ for $K_{1,q}, K_{1,q+1} \in [K_u^A, K_{u+1}^A] \cap \mathbb{K}(\mathcal{P}_1^*)$. If we assume further that $[K_u^A, K_{1,p}e^{-rT}] \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ and

$$j' = \arg\min\{K_s^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K_s^E \geq K_{1,p}\},$$

then we have $K_{j'}^E e^{-rT} \in (K_{1,q}, K_{1,q+1})$.

Proof. Let us assume first that $K_{1,q+1} \in \mathbb{K}^A(\mathcal{P}_0^*)$. Proposition 3.10.18 shows that $[K_{1,p}e^{-rT}, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$. We can furthermore argue that $\mathbb{K}_1^{aux}(\mathcal{P}_1^*) = \emptyset$ as the algorithm has not been restarted yet. It follows that Proposition 3.10.17 can be applied, which states that $K_{j'}^E e^{-rT} < K_{1,q+1}$. The definition of j', moreover, implies that $K_{j'}^E e^{-rT} > K_{1,q}$ and thus $K_{j'}^E e^{-rT} \in (K_{1,q}, K_{1,q+1})$ has to hold.

Suppose now that $K_{1,q+1} \in \mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_0^*)$, then we can deduce from $[K_u^A, K_{u+1}^A] \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ that $K_{1,q+1} \in (\mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*))$. Hence, we can apply Proposition 3.10.24 to see that

$$\mathbf{a}_{1,q+1} = \max\{A_{lb}^{rhs}(K_{1,q+1}, \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{1,q+1}, \mathcal{P}_0^*)\}.$$

We know, moreover, from Proposition 3.10.5 that

$$\mathbf{a}_{1,s} = \max\{A_{lb}^{rhs}(K_{1,s}, \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{1,s}, \mathcal{P}_0^*)\}$$

for any strike $K_{1,s} \in [K_{1,q+1}, K_{u+1}^A)$. If we assume for contradiction that $K_{j'}^E e^{-rT} \geq K_{1,q+1}$, then Proposition 3.10.17 guarantees that $K_{j'}^E e^{-rT} \in [K_{1,q+1}, K_{u+1}^A)$. Since the American price function is obtained by interpolating linearly between the option prices in $\mathcal{P}_{1,p}^A$, we must have

$$A(K_{j'}^E e^{-rT}, \mathcal{P}_{1,p}) = \max\{A_{lb}^{rhs}(K_{j'}^E, \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{j'}^E, \mathcal{P}_0^*)\}.$$

It follows that $\overline{\mathbf{a}}_{j'} < \max\{A_{lb}^{rhs}(K_{j'}^E, \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{j'}^E, \mathcal{P}_0^*)\}$ which is a contradiction to $(viii)$ of the Standing Assumptions. It follows that $K_{j'}^E e^{-rT} \in (K_{1,q}, K_{1,q+1})$. $\square$

We can now show that under $\mathcal{P}_1^*$ Algorithm 4 successfully corrects a violation of the upper bound in $K_{1,p}e^{-rT}$ if $[K_u^A, K_{u+1}^A] \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$. Note also that henceforth we will be using $K_l(\mathcal{P}_{i,p})$ to refer to the l-th strike in $\mathbb{K}(\mathcal{P}_{i,p})$.

Proposition 3.7.3. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

The algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of $\overline{\mathbf{a}}_{1,p} \geq A(K_{1,p}e^{-rT}, \mathcal{P}_{1,p})$, where $K_{1,p}e^{-rT} \in (K_{1,q}, K_{1,q+1}]$ for $K_{1,q}, K_{1,q+1} \in [K_u^A, K_{u+1}^A] \cap \mathbb{K}(\mathcal{P}_1^*)$. If we assume that $[K_u^A, K_{1,p}e^{-rT}] \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ and that Algorithm 4 extended the price set $\mathcal{P}_{1,p}$ to

$$\mathcal{P}_{1,p+1} = ((\mathcal{P}_{1,p})^A \cup (\overline{\mathbf{a}}_j, K_j^E e^{-rT}); (\mathcal{P}_{1,p})^E \cup (E_{ub}(K_j^E e^{-rT}, \mathcal{P}_{1,p}), K_j^E e^{-rT})),$$

then $\mathcal{P}_{1,p+1}$ has to be a $K_{1,p+1}$-admissible $\mathcal{P}_0^*$-extension.

Proof. We begin by arguing that the auxiliary price constraint $(\overline{\mathbf{a}}_j, K_j^E e^{-rT})$ lies within the no-arbitrage bounds inferred by the set of prices $\mathcal{P}_{1,p}$. Note also that $j \geq j'$ for $j' = \arg\min\{K_s \in \mathbb{K}^E(\mathcal{P}_0^*) : K_s \geq K_{1,p}\}$

First we show that $\overline{\mathbf{a}}_j < A_{ub}(K_j^E e^{-rT}, \mathcal{P}_{1,p})$. To this end, we start by arguing that $\overline{\mathbf{a}}_{j'} < A_{ub}(K_{j'}^E e^{-rT}, \mathcal{P}_{1,p})$. According to Proposition 3.10.18, we must have $[K_u^A, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$. We can thus apply Proposition 3.10.14 to see that the prices $\overline{\mathbf{a}}_{1,p-1}$, $\overline{\mathbf{a}}_{1,p}$ and $\overline{\mathbf{a}}_j$ are co-linear. Proposition 3.10.22 further argues that $\mathbf{a}_{1,q} \leq \overline{A}(K_{1,q}, \mathcal{P}_{1,p})$ has to hold. Combined with the fact that $\mathcal{P}_{1,p-1}$ is a $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension we can conclude that $A(K,\mathcal{P}_{1,p}) \leq \overline{A}(K,\mathcal{P}_{1,p})$ for any strike $K \leq \max\{K_{1,q}, K_{1,p-1}e^{-rT}\}$. In addition, we know that the price for European options with strike $K_j^E e^{-rT}$ is set to be $E_{ub}(K_j^E e^{-rT},\mathcal{P}_{1,p})$ and thus the upper bound remains unchanged after the introduction of the auxiliary price constraint. Taking into account the assumption that $\overline{\mathbf{a}}_{1,p} < A(K_{1,p}e^{-rT},\mathcal{P}_{1,p})$, we get $\overline{\mathbf{a}}_{j'} < A(K_{j'}^E e^{-rT},\mathcal{P}_{1,p})$. Since the price functions are obtained by interpolating linearly between the given option prices we must therefore have $\overline{\mathbf{a}}_{j'} < A_{ub}(K_{j'}^E e^{-rT},\mathcal{P}_{1,p})$.

Let us assume now that $j > j'$, then we can deduce from (3.28) that

$$\frac{\overline{\mathbf{a}}_j - \mathbf{a}_{1,q}}{K_j^E e^{-rT} - K_{1,q}} \leq \frac{\overline{\mathbf{a}}_{j'} - \mathbf{a}_{1,q}}{K_{j'}^E e^{-rT} - K_{1,q}}.$$

Having argued already that $\overline{\mathbf{a}}_{j'} < A_{ub}(K_{j'}^E e^{-rT},\mathcal{P}_{1,p})$, we can now conclude that $\overline{\mathbf{a}}_j < A_{ub}(K_j^E e^{-rT},\mathcal{P}_{1,p})$ has to hold as well.

The argument that $\overline{\mathbf{a}}_j \geq A_{lb}^{lhs}(K_j^E e^{-rT},\mathcal{P}_{1,p})$ is given in Proposition 3.10.23 as the term on the right hand-side in (3.64) corresponds to the left hand-side lower bound on $[K_{1,q}, K_{1,q+1}]$.

To see that $\overline{\mathbf{a}}_j \geq \max\{A_{lb}^{rhs}(K_j^E e^{-rT},\mathcal{P}_{1,p}), A_{lb}^{\overline{A},r}(K_j^E e^{-rT},\mathcal{P}_{1,p})\}$ we have to distinguish between the two cases where $K_{1,q+1} \in \mathbb{K}^A(\mathcal{P}_0^*)$ or $K_{1,q+1} \in (\mathbb{K}^E(\mathcal{P}_1^*)\backslash\mathbb{K}^A(\mathcal{P}_0^*))$. In the first case we know that the price for an American option with strike $K_{1,s} \in (K_{u+1}^A, K_{u+2}^A)$ has to satisfy $\mathbf{a}_{1,s} \geq A_{lb}^{\overline{A},l}(K_{1,s},\mathcal{P}_0^*)$ and thus $\overline{\mathbf{a}}_j \geq A_{lb}^{rhs}(K_j^E e^{-rT},\mathcal{P}_{1,p})$. The Standing Assumptions, moreover, guarantee in (v) that $\overline{\mathbf{a}}_j \geq A_{lb}^{\overline{A},r}(K_j^E e^{-rT},\mathcal{P}_0^*)$ and thus

$$\overline{\mathbf{a}}_j \geq \max\{A_{lb}^{rhs}(K_j^E e^{-rT},\mathcal{P}_{1,p}), A_{lb}^{\overline{A},r}(K_j^E e^{-rT},\mathcal{P}_{1,p})\}$$

has to hold.

In the second case we can apply Proposition 3.10.24 to show that

$$\mathbf{a}_{1,q+1} = \max\{A_{lb}^{rhs}(K_{1,q+1},\mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{1,q+1},\mathcal{P}_0^*)\}.$$

Hence, the right hand-side lower bound $A_{lb}^{rhs}(K_j^E e^{-rT},\mathcal{P}_{1,p})$ is given by the prices of two traded options. We can thus conclude from the Standing Assumptions that

$$\overline{\mathbf{a}}_j \geq \max\{A_{lb}^{rhs}(K_j^E e^{-rT},\mathcal{P}_{1,p}), A_{lb}^{\overline{A},r}(K_j^E e^{-rT},\mathcal{P}_{1,p})\}.$$

Before we show that the Legendre-Fenchel condition holds after the auxiliary price constraint $(\overline{\mathbf{a}}_j, K_j^E e^{-rT})$ was introduced, we would like to point out that the entire argument below is given with respect to the augmented price set $\mathcal{P}_{1,p+1}$ obtained by adding in the new price constraint $(\overline{\mathbf{a}}_j, K_j^E e^{-rT})$ and shifting all prices at strikes greater than $K_j^E e^{-rT}$ to the right by one. This means that the strike at which the violation occurred is no longer given by $K_{1,p}$, but is now $K_{1,p+1}$. Note, however, that the same conclusion cannot be made for any of the strikes $K_{1,s}$, $s \leq p-1$, as we may have $K_{1,q} = K_{1,s}$. For clarification we will therefore refer to these strikes in the sequel as $K_s(\mathcal{P}_{1,p})$ and to the new ones by $K_s(\mathcal{P}_{1,p+1})$.

Suppose first that $j = j'$, then we know from above that $A(K, \mathcal{P}_{1,p+1}) \leq \overline{A}(K, \mathcal{P}_{1,p+1})$ for any strike $K \leq \max\{K_{1,q}, K_{p-1}(\mathcal{P}_{1,p})e^{-rT}\}$. Taking further into account that $A(K_j^E e^{-rT}, \mathcal{P}_{1,p+1}) = \overline{\mathbf{a}}_j$ and that the function $\overline{A}$ is linear on $[K_{p-1}(\mathcal{P}_{1,p})e^{-rT}, K_{j'}^E e^{-rT}]$ according to Remark 3.10.16 we can conclude that

$$\begin{aligned} cc(A; K_{1,q+1}, K_{1,q+2}; \mathcal{P}_{1,p+1}) &\leq cc(A; K_{1,q}, K_{1,q+1}; \mathcal{P}_{1,p+1}) \\ &\leq cc(\overline{A}; K_{1,p}e^{-rT}, K_{1,p+1}e^{-rT}; \mathcal{P}_{1,p+1}) \\ &= cc(E; K_{1,p}, K_{1,p+1}; \mathcal{P}_{1,p+1}) \\ &\leq cc(E; K_{1,q+1}, K_{1,q+2}; \mathcal{P}_{1,p+1}) \\ &= cc(E; K_{1,q}, K_{1,q+1}; \mathcal{P}_{1,p+1}) \end{aligned}$$

where the equality in the third line has to hold according to Proposition 3.10.4 and the inequality in the penultimate line is due to the convexity of the European price function up to $K_{p+1}(\mathcal{P}_{1,p+1})$ which still has to hold after adding in the price $E_{ub}(K_j^E e^{-rT}, \mathcal{P}_{1,q})$. Hence, the Legendre-Fenchel condition has to hold on $[K_{1,q}, K_{q+2}(\mathcal{P}_{1,p+1})]$.

An analog argument can be used to show that the Legendre-Fenchel condition holds on $[K_{1,q}, K_{q+2}(\mathcal{P}_{1,p+1})]$ when $j > j'$, as

$$\frac{\overline{\mathbf{a}}_j - \mathbf{a}_{1,q}}{K_j^E e^{-rT} - K_{1,q}} \leq \frac{\overline{\mathbf{a}}_{v'} - \mathbf{a}_{1,q}}{K_{v'}^E e^{-rT} - K_{1,q}}$$

for any strike $K_{v'}^E e^{-rT} \in (K_{1,q}, K_{q+2}(\mathcal{P}_{1,p+1}))$ with $K_{v'}^E \in \mathbb{K}^E(\mathcal{P}_0^*)$ guarantees that

$$\overline{\mathbf{a}}_{v'} \geq \frac{\overline{\mathbf{a}}_j - \mathbf{a}_{1,q}}{K_j^E e^{-rT} - K_{1,q}}(K_{v'}^E e^{-rT} - K_{1,q}) + \mathbf{a}_{1,q} \tag{3.36}$$

and thus

$$cc(A; K_{1,q}, K_{1,q+1}; \mathcal{P}_{1,p+1}) \leq cc(\overline{A}; K_{j-1}^E e^{-rT}, K_j^E e^{-rT}; \mathcal{P}_0^*).$$

Finally, we will argue that the American price function $A(\cdot, \mathcal{P}_{1,p+1})$ will not exceed the upper bound $\overline{A}$ in any strike up to $K_{1,p+1}e^{-rT}$. Note first that the added price $E_{ub}(K_j^E e^{-rT}, \mathcal{P}_{1,q})$ has no effect on the European price function and thus the upper bound $\overline{A}$ remains unchanged as well. Recall next that $A(K, \mathcal{P}_{1,p+1}) \leq \overline{A}(K, \mathcal{P}_{1,p+1})$ for any strike $K \leq K_{1,q}$. We are thus left to argue that the upper bound $\overline{A}$ also holds on $[K_{1,q}, K_{1,p+1}e^{-rT}]$. By construction we have $A(K_j^E e^{-rT}, \mathcal{P}_{1,p+1}) = \overline{\mathbf{a}}_j$. We can then use (3.36) together with the convexity of the European price function on $[0, K_{1,p+1}]$ to argue that $A(K, \mathcal{P}_{1,p+1}) \leq \overline{A}(K, \mathcal{P}_{1,p+1})$ for any strike $K \in [K_{1,q}, K_{1,p+1}e^{-rT}]$ and therefore we have shown that $\mathcal{P}_{1,p+1}$ is a $K_{1,p+1}$-admissible $\mathcal{P}_0^*$-extension. $\square$

Case II: $[K_u^A, K_{u+1}^A] \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) \neq \emptyset$

Suppose that the last auxiliary price constraint was introduced at the strike $K_{1,\tilde{q}} \in [K_u^A, K_{u+1}^A] \cap \mathbb{K}^{aux}(\mathcal{P}_1^*)$. Since we assumed that the initial set of prices is given by $\mathcal{P}_1^*$, a previous violation of convexity can be ruled out and thus we must have $K_{1,\tilde{q}} \in \mathbb{K}_2^{aux}(\mathcal{P}_1^*)$.

To see that Algorithm 4 successfully corrects a violation of the upper bound in this setting we first show again the existence of a strike $K_{j'}^E e^{-rT} \in (K_{1,q}, K_{1,q+1})$.

Proposition 3.7.4. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

The algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of $\overline{\mathbf{a}}_{1,p} \geq A(K_{1,p}e^{-rT}, \mathcal{P}_{1,p})$, where $K_{1,p}e^{-rT} \in (K_{1,q}, K_{1,q+1}]$ for $K_{1,q}, K_{1,q+1} \in [K_u^A, K_{u+1}^A] \cap \mathbb{K}(\mathcal{P}_1^*)$. If we assume further that $[K_u^A, K_{1,p}e^{-rT}] \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) \neq \emptyset$ and

$$j' = \arg\min\{K_s^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K_s^E \geq K_{1,p}\},$$

then we have $q = \tilde{q}$ and $K_{j'}^E e^{-rT} \in (K_{1,\tilde{q}}, K_{1,\tilde{q}+1})$.

Proof. We begin by noting that according to Proposition 3.10.18 $K_{1,p}e^{-rT} > K_{1,\tilde{q}}$ and thus $K_{j'}^E e^{-rT} > K_{1,\tilde{q}}$ as well. Hence, we are left to show that $K_{j'}^E e^{-rT} < K_{1,\tilde{q}+1}$. To do so, we will argue that either $K_{1,\tilde{q}+1} \in \mathbb{K}^A(\mathcal{P}_1^*)$ or

$$\mathbf{a}_{1,\tilde{q}+1} = \max\{A_{lb}^{rhs}(K_{1,\tilde{q}+1}, \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{1,\tilde{q}+1}, \mathcal{P}_0^*)\}$$

for $K_{1,\tilde{q}+1} \in \mathbb{K}^E(\mathcal{P}_1^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*)$. In the first case we can immediately rule out that $K_{1,\tilde{q}+1} \in \mathbb{K}^{aux}(\mathcal{P}_1^*)$ according to Proposition 3.10.18 and thus $K_{1,\tilde{q}+1} \in \mathbb{K}^A(\mathcal{P}_0^*)$. Moreover, Proposition 3.10.17 guarantees that $K_{j'}^E e^{-rT} < K_{u+1}^A$ and therefore $K_{j'}^E e^{-rT} \in (K_{1,\tilde{q}}, K_{1,\tilde{q}+1})$ for $K_{1,\tilde{q}+1} \in \mathbb{K}^A(\mathcal{P}_1^*)$.

Hence, we are left to consider the situation where $K_{1,\tilde{q}+1} \in \mathbb{K}^E(\mathcal{P}_1^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*)$. To see that also in this case $K_{j'}^E e^{-rT} \in (K_{1,\tilde{q}}, K_{1,\tilde{q}+1})$ holds, we will proceed as follows. Let us assume for contradiction that $K_{j'}^E e^{-rT} \geq K_{1,\tilde{q}+1}$, then we can deduce from Proposition 3.10.17 that $K_{j'}^E e^{-rT} \in [K_{1,\tilde{q}+1}, K_{u+1}^A)$. Suppose further that the first violation of the upper bound in $[K_u^A, K_{u+1}^A]$ was corrected by introducing the auxiliary constraint $(\overline{\mathbf{a}}_{1,r}, K_{1,r})$. We can then conclude that $r \leq \tilde{q}$ and that according to Proposition 3.10.24 the price for American options with strike $K_{1,r+1}$ must be given by $\mathbf{a}_{1,r+1} = \max\{A_{lb}^{rhs}(K_{1,r+1}, \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{1,r+1}, \mathcal{P}_0^*)\}$. Applying Proposition 3.10.5 we can further conclude that

$$A(K, \mathcal{P}_{1,p}) = \max\{A_{lb}^{rhs}(K, \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K, \mathcal{P}_0^*)\}$$

for any strike $K \in [K_{1,r+1}, K_{u+1}^A]$. This, however, yields a contradiction to $(viii)$ of the Standing Assumptions, as this guarantees that

$$\overline{\mathbf{a}}_{j'} \geq \max\{A_{lb}^{rhs}(K_{j'}e^{-rT}, \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{j'}e^{-rT}, \mathcal{P}_0^*)\}.$$

We can therefore conclude that $K_{j'}^E e^{-rT} \in (K_{1,\tilde{q}}, K_{1,\tilde{q}+1})$ and $q = \tilde{q}$. □

Remark 3.7.5. *Suppose that the first time a violation of the upper bound occurs on $[K_u^A, K_{u+1}^A]$ is at $K_{1,p_1}e^{-rT} \in [K_{1,q}, K_{1,q+1}]$, then Proposition 3.7.4 readily implies that any further violation of the upper bound between K_u^A and K_{u+1}^A at a strike $K_{1,p_l}e^{-rT} \in [K_u^A, K_{u+1}^A]$ has to satisfy $K_{1,p_l}e^{-rT} < K_{1,q+1}$.*

Proposition 3.7.6. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

The algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of $\overline{\mathbf{a}}_{1,p} \geq A(K_{1,p}e^{-rT}, \mathcal{P}_{1,p})$, where $K_{1,p}e^{-rT} \in (K_{1,q}, K_{1,q+1}]$ for $K_{1,q}, K_{1,q+1} \in [K_u^A, K_{u+1}^A] \cap \mathbb{K}(\mathcal{P}_1^*)$. If we assume that $[K_u^A, K_{1,p}e^{-rT}] \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) \neq \emptyset$ and that Algorithm 4 extended the price set $\mathcal{P}_{1,p}$ to

$$\mathcal{P}_{1,p+1} = ((\mathcal{P}_{1,p})^A \cup (\overline{\mathbf{a}}_j, K_j^E e^{-rT}); (\mathcal{P}_{1,p})^E \cup (E_{ub}(K_j^E e^{-rT}, \mathcal{P}_{1,p}), K_j^E e^{-rT})),$$

then $\mathcal{P}_{1,p+1}$ has to be a $K_{1,p+1}$-admissible $\mathcal{P}_0^*$-extension.

Proof. We begin by showing that the auxiliary price constraint $(\overline{\mathbf{a}}_j, K_j^E e^{-rT})$ lies within the no-arbitrage bounds inferred by the set of prices $\mathcal{P}_{1,p}$. Moreover, we will use $j' = \arg\min\{K_s^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K_s^E \geq K_{1,p}\}$ and thus have $j \geq j'$.

To see that $\overline{\mathbf{a}}_j < A_{ub}(K_j^E e^{-rT}, \mathcal{P}_{1,p})$ holds we proceed analogously to the argument in the proof of Proposition 3.7.3. The only difference in the argument is the way we show that $\mathbf{a}_{1,q} \leq \overline{A}(K_{1,q}, \mathcal{P}_{1,p})$. In the current setting we can use Proposition 3.7.4 to guarantee that $K_{1,q} \in \mathbb{K}_2^{aux}(\mathcal{P}_1^*)$ and thus $\mathbf{a}_{1,q} = \overline{A}(K_{1,q}, \mathcal{P}_0^*)$ has to hold.

Next we will show that $\overline{\mathbf{a}}_j \geq A_{lb}^{lhs}(K_j^E e^{-rT}, \mathcal{P}_{1,p})$. To do so, we will assume for contradiction that $\overline{\mathbf{a}}_j < A_{lb}^{lhs}(K_j^E e^{-rT}, \mathcal{P}_{1,p})$. We can thus conclude that

$$\frac{\mathbf{a}_{1,q} - \mathbf{a}_{1,q-1}}{K_{1,q} - K_{1,q-1}} \geq \frac{\overline{\mathbf{a}}_j - \mathbf{a}_{1,q-1}}{K_j^E e^{-rT} - K_{1,q-1}} \tag{3.37}$$

has to hold for $K_{1,q-1} \in \mathbb{K}(\mathcal{P}_1^*)$. Applying Proposition 3.7.4 we further see that

$K_{1,q} \in \mathbb{K}_2^{aux}(\mathcal{P}_1^*)$. Combined with the fact that $K_j^E e^{-rT} \in (K_{1,q}, K_{1,q+1})$ and $K_j^E \in \mathbb{K}^E(\mathcal{P}_0^*)$ this yields a contradiction, as the algorithm would have chosen the constraint $(\overline{\mathbf{a}}_j, K_j^E e^{-rT})$ over $(\overline{A}(K_{1,q}, \mathcal{P}_0^*), K_{1,q})$ previously. Hence, we can conclude that $\overline{\mathbf{a}}_j \geq A_{lb}^{lhs}(K_j^E e^{-rT}, \mathcal{P}_{1,p})$ has to hold.

The argument showing that

$$\overline{\mathbf{a}}_j \geq \max\{A_{lb}^{rhs}(K_j^E e^{-rT}, \mathcal{P}_{i,p}), A_{lb}^{\overline{A},r}(K_j^E e^{-rT}, \mathcal{P}_{i,p})\}$$

can be taken directly from Proposition 3.7.3 as the right hand-side lower bound is not affected by any auxiliary price constraints to the left. So can the proofs that the Legendre-Fenchel condition holds and that the American price function will not exceed the upper bound $\overline{A}$. We can thus conclude that $\mathcal{P}_{1,p+1}$ is a $K_{1,p+1}$-admissible $\mathcal{P}_0^*$-extension. □

The price for a European option at a strike in $\mathbb{K}_2^{aux}(\mathcal{P}_1^*)$

In the following proposition we argue that the price for a European option with strike $K_j^E e^{-rT}$ has to be given by $E_{ub}(K_j^E e^{-rT}, \mathcal{P}_{1,p})$ to guarantee the admissibility of the price functions.

Proposition 3.7.7. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

The algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of $\overline{\mathbf{a}}_{1,p} \geq A(K_{1,p}e^{-rT}, \mathcal{P}_{1,p})$, where $K_{1,p}e^{-rT} \in (K_{1,q}, K_{1,q+1}]$ for $K_{1,q}, K_{1,q+1} \in [K_u^A, K_{u+1}^A] \cap \mathbb{K}(\mathcal{P}_1^*)$. If we assume that Algorithm 4 extended the price set $\mathcal{P}_{1,p}$ to

$$\mathcal{P}_{1,p+1} = ((\mathcal{P}_{1,p})^A \cup (\overline{\mathbf{a}}_j, K_j^E e^{-rT}); (\mathcal{P}_{1,p})^E \cup (E_{ub}(K_j^E e^{-rT}, \mathcal{P}_{1,p}), K_j^E e^{-rT}))$$

in order to correct the violation and that $K_{1,s} = K_j^E$, then $\mathcal{P}_{1,s}$ has to be a $K_{1,s}$-admissible $\mathcal{P}_0^*$-extension.

Proof. We argued already in Proposition 3.7.3 and Proposition 3.7.6, respectively, that the set $\mathcal{P}_{1,p+1}$ is a $K_{1,p+1}$-admissible $\mathcal{P}_0^*$-extension. It is therefore sufficient to rule out both a violation of the upper bound and of convexity on $(K_{1,p+1}, K_j^E]$.

Let us first consider the case where a violation of the upper bound occurs on $(K_{1,p+1}, K_j^E]$. To this end, we set

$$K_{1,r} = \min\{K_{1,\tilde{r}} \in (K_{1,p+1}, K_j^E] \cap \mathbb{K}(\mathcal{P}_1^*) : \overline{\mathbf{a}}_{1,\tilde{r}} < A(K_{1,\tilde{r}}e^{-rT}, \mathcal{P}_{1,\tilde{r}})\}$$

and

$$K_{j'}^E = \min\{K_s^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K_s^E \geq K_{1,r}\}.$$

Observe also that this readily implies that $j' \leq j$. Moreover, we know from Proposition 3.10.14 that the prices $\overline{\mathbf{a}}_{1,r-1}$, $\overline{\mathbf{a}}_{1,r}$ and $\overline{\mathbf{a}}_{j'}$ are co-linear. It thus follows that $\overline{\mathbf{a}}_{j'} < A(K^E_{j'}e^{-rT}, \mathcal{P}_{1,s})$ has to hold. Then again, this is a contradiction to the way the strike K^E_j is chosen in (3.28) and thus a violation of the upper bound on $(K_{1,p+1}, K^E_j]$ can be ruled out.

We are therefore left to argue that a violation of convexity can be ruled out on $(K_{1,p+1}, K^E_j]$. This, however, follows directly from Proposition 3.10.25. We can thus conclude that $\mathcal{P}_{1,s}$ is a $K_{1,s}$-admissible $\mathcal{P}^*_0$-extension. $\square$

3.7.2 Violation of convexity under $\mathcal{P}^*_1$

There are two possible violations of convexity that may occur during the construction of the price functions. On the one hand it is possible that the price for an American option with strike $K_{1,p} \in \mathbb{K}(\mathcal{P}^*_1)\backslash\mathbb{K}^A(\mathcal{P}^*_1)$ exceeds the upper bound $A_{ub}(K_{1,p}, \mathcal{P}_{1,p-1})$ if the price is computed to be $\mathbf{a}_{1,p} = A_{lf}(K_{1,p}, \mathcal{P}_{1,p-1})$. On the other hand the algorithm may compute the price for a European option with strike $K_{1,p} \in \mathbb{K}(\mathcal{P}^*_1)\backslash\mathbb{K}^E(\mathcal{P}^*_1)$ to be $\mathbf{e}_{1,p} = E_{lf}(K_{1,p}, \mathcal{P}_{1,p-1})$ such that we have $\mathbf{e}_{1,p} < E_{lb}(K_{1,p}, \mathcal{P}_{1,p-1})$. Although the starting prices for Algorithm 3 depend on the type of the violation the revision process does not and thus we can discuss both cases at once.

To this end, let us assume that the first violation of convexity occurs at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}^*_1)$ and that Algorithm 3 works backwards through the strikes in $\mathbb{K}(\mathcal{P}^*_1)$ until it reaches $K_{1,q} \in (K^A_u, K^A_{u+1}) \cap \mathbb{K}^E(\mathcal{P}^*_1)$ where it stops and introduces the auxiliary price constraint $(\mathbf{a}^n_{1,q}, K_{1,q})$. We will then argue in this section that the prices obtained by restarting the algorithm with the new initial set

$$P^*_2 = ((\mathcal{P}_1)^A \cup (\mathbf{a}^n_{1,q}, K_{1,q}); (\mathcal{P}^*_0)^E)$$

will be a K^{vc}_1-admissible $\mathcal{P}^*_0$-extension. To do so, we will discuss first how the price functions constructed from the initial set $\mathcal{P}^*_2$ would look like if the algorithm was to ignore any new violations of the upper bound $\overline{A}$. That is, Algorithm 2 is executed normally, but a violation of the upper bound does not start Algorithm 4. Instead Algorithm 2 continues with the computation of option prices. It makes sense to consider these price functions as we have seen in Section 3.7.1 that a correction of the upper bound has no effect on the other option prices under $\mathcal{P}^*_1$. Note also that in this case the algorithm will not introduce any auxiliary price constraints and thus the enumeration of the strikes between the price set $\mathcal{P}_{1,p}$ and $\mathcal{P}_{2,p}$ will remain unchanged.

Proposition 3.7.8. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}^*_0 \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}^*_1$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}^*_0$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

Algorithm 2 stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of convexity. Algorithm 3 computes revised prices for non-traded options with strikes $K_{1,s} \in [K_{1,q}, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*)$. At the strike $K_{1,q} \in (K_u^A, K_{u+1}^A) \cap \mathbb{K}^E(\mathcal{P}_0^*)$ it stops and defines the new initial set of prices $\mathcal{P}_2^*$ by $\mathcal{P}_2^* = ((\mathcal{P}_1^*)^A \cup (\mathbf{a}_{1,q}^n, K_{1,q}); (\mathcal{P}_0^*)^E)$. Algorithm 2 is then restarted with the initial set $\mathcal{P}_2^*$. If violations of the upper bound $\overline{A}$ are ignored by Algorithm 2 the price functions are given by

$$\mathbf{a}_{2,s} = \begin{cases} \mathbf{a}_{1,s}, & \text{if } K_{2,s} < K_{2,r} \\ A_{lb}^{rhs}(K_{2,s}, \mathcal{P}_2^*), & \text{if } K_{2,s} \in [K_{2,r}, K_u^A) \\ A_{ub}(K_{2,s}, \mathcal{P}_2^*), & \text{if } K_{2,s} \in [K_u^A, K_{2,q}) \\ \mathbf{a}_{1,s}^n, & \text{if } K_{2,s} \in [K_{2,q}, K_{2,p}] \end{cases} \tag{3.38}$$

and

$$\mathbf{e}_{2,s} = \begin{cases} \mathbf{e}_{1,s}, & \text{if } K_{2,s} \le K_{2,q} \\ \mathbf{e}_{1,s}^n, & \text{if } K_{2,s} \in [K_{2,q}, K_{2,p}] \end{cases} \tag{3.39}$$

for

$$K_{2,r} = \min\{K_{2,s} \in (K_{2,w}, K_u^A) \cap \mathbb{K}(\mathcal{P}_2^*) : A_{lb}^{rhs}(K_{2,s}, \mathcal{P}_2^*) > \mathbf{a}_{1,s}\}$$

and

$$K_{2,w} = \max\{K \in \mathbb{K}^A(\mathcal{P}_2^*) : K < K_u^A\}. \tag{3.40}$$

Remark 3.7.9. *Note that the left hand-side lower bound $A_{lb}^{lhs}(\cdot, \mathcal{P}_{1,q-1})$ in the strike $K_{1,q}$ has to be strictly positive as there exists arbitrage in the market otherwise according to Remark 3.6.15. If $[0, K_1^A] \cap \mathbb{K}^E(\mathcal{P}_0^*) = \emptyset$, we can, moreover, use (vi) of the Standing Assumptions to argue that $\mathbf{e}_{1,1} = E_{lf}(K_{1,1}, \mathcal{P}_{1,0}) = \hat{\mathbf{a}}_1 \ge E_{lb}^{rhs}(K_{1,1}, \mathcal{P}_0^*)$ and thus a violation of convexity in K_1^A can be ruled out as well. It follows that the strike $K_u^A = \max\{K_s^A \in \mathbb{K}^A(\mathcal{P}_0^*) : K_s^A < K_{1,q}\}$ satisfies $K_u^A > 0$.*

Proof. Observe that the existence of the strike $K_{2,r}$ is guaranteed due to Proposition 3.10.39. We then begin by showing that both the computed American and European price functions remain unchanged for any strike $K \in [0, K_{2,r}) \cap \mathbb{K}(\mathcal{P}_2^*)$. To this end, we note that the price sets $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ differ only by the auxiliary price constraint $(\mathbf{a}_{1,q}^n, K_{1,q})$ as any auxiliary price constraint introduced to correct a violation of the upper bound during the first iteration is added to $\mathcal{P}_1^*$. Hence, a change in the price functions can only be caused by either the new constraint $(\mathbf{a}_{1,q}^n, K_{1,q})$ or by the algorithm pricing European options with strikes in $\mathbb{K}_2^{aux}(\mathcal{P}_2^*)$ differently to (3.30). We will now argue that the first time the auxiliary price constraint $(\mathbf{a}_{1,q}^n, K_{1,q})$ affects the pricing of American options is on the interval $(K_{2,w}, K_u^A)$. This follows from the fact that the auxiliary price constraint $(\mathbf{a}_{1,q}^n, K_{1,q})$ appears for the first time in the pricing formula for American options when the right hand-side lower bound is given by

$$A_{lb}^{rhs}(K, \mathcal{P}_2^*) = \frac{\mathbf{a}_{1,q}^n - \hat{\mathbf{a}}_u}{K_{1,q} - K_u^A}(K - K_u^A) + \hat{\mathbf{a}}_u$$

which is the case on $(K_{2,w}, K_u^A)$. Taking into account the definition of the strike $K_{2,r}$ we can conclude that the prices for American options with strike $K_{2,s} \in [0, K_{2,r}) \cap \mathbb{K}(\mathcal{P}_2^*)$ are not affected by the auxiliary price constraint $(\mathbf{a}_{1,q}^n, K_{1,q})$.

According to Proposition 3.10.40, the algorithm will determine the price for European options with strike $K_{2,l} \in (0, K_{2,w}] \cap \mathbb{K}_2^{aux}(\mathcal{P}_2^*)$ to be $E_{ub}(K_{2,l}, \mathcal{P}_{2,l-1})$. Hence, we can conclude that $\mathbf{a}_{2,s} = \mathbf{a}_{1,s}$ and $\mathbf{e}_{2,s} = \mathbf{e}_{1,s}$ for any strike $K_{2,s} \in [0, K_{2,r}) \cap \mathbb{K}(\mathcal{P}_2^*)$.

We proceed by showing that $\mathbf{a}_{2,s} = A_{lb}^{rhs}(K_{2,s}, \mathcal{P}_2^*)$ and $\mathbf{e}_{2,s} = \mathbf{e}_{1,s}$ for any strike $K_{2,s} \in [K_{2,r}, K_u^A)$. According to Proposition 3.10.39 the strike

$$K_{2,r} = \min\{K_{2,s} \in (K_{2,w}, K_u^A) \cap \mathbb{K}(\mathcal{P}_2^*) : A_{lb}^{rhs}(K_{2,s}, \mathcal{P}_2^*) > \mathbf{a}_{1,s}\}$$

exists and by its definition we must have $\mathbf{a}_{2,r} = A_{lb}^{rhs}(K_{2,r}, \mathcal{P}_2^*)$. Note further that the definition of $K_{2,w}$ in (3.40) takes into account the strikes of the auxiliary price constraints already introduced. Combined with the fact that Algorithm 2 ignores possible violations of the upper bound in the second iteration we are guaranteed that $[K_{2,w}, K_u^A] \cap \mathbb{K}^{aux}(\mathcal{P}_2^*) = \emptyset$. We can thus deduce from $K_{2,r} > K_{2,w}$ that $[K_{2,r}, K_u^A] \cap \mathbb{K}^{aux}(\mathcal{P}_2^*) = \emptyset$ has to hold as well. We can therefore apply Proposition 3.10.5 to see that $\mathbf{a}_{2,s} = A_{lb}^{rhs}(K_{2,s}, \mathcal{P}_2^*)$ for any strike $K_{2,s} \in [K_{2,r}, K_u^A)$. Moreover, we can deduce from the definition of $K_{2,w}$ that $(K_{2,w}, K_u^A) \cap \mathbb{K}(\mathcal{P}_2^*) \subset \mathbb{K}^E(\mathcal{P}_0^*)$. Hence, the prices for European options with strikes in $(K_{2,w}, K_u^A) \cap \mathbb{K}(\mathcal{P}_2^*)$ are given by $(\mathcal{P}_0^*)^E$ and thus $\mathbf{e}_{2,s} = \mathbf{e}_{1,s}$ for any strike $K_{2,s} \in (K_{2,w}, K_u^A)$.

Let us continue by investigating the price for American and European options with strike $K_{2,s} \in [K_u^A, K_{2,q})$. Note first that $K_{1,q} \in (K_u^A, K_{1,\tilde{q}}]$ according to Remark 3.10.38. Since we ignore any violation of the upper bound $\overline{A}$ in the second iteration of the algorithm we further know that $K_{2,q} \in (K_u^A, K_{2,\tilde{q}}]$ and according to Proposition 3.10.30 $[K_u^A, K_{2,q}) \cap \mathbb{K}^{aux}(\mathcal{P}_2^*) = \emptyset$. It follows that the price for European options only needs to be computed in K_u^A. To be able to compare the prices for European options in the two iterations we need to determine them first. To this end, suppose that under $\mathcal{P}_{1,p}$ the strike K_u^A corresponds to $K_{1,\tilde{s}}$.

According to Proposition 3.10.32 we know that $\mathbf{a}_{1,\tilde{q}} = A_{lb}^{lhs}(K_{1,\tilde{q}}, \mathcal{P}_{1,p})$ where

$$A_{lb}^{lhs}(K_{1,\tilde{q}}, \mathcal{P}_{1,p}) > A_{lf}(K_{1,\tilde{q}}, \mathcal{P}_{1,p}).$$

Hence,

$$cc(A; K_{1,\tilde{q}-1}, K_{1,\tilde{q}}; \mathcal{P}_{1,p}) < cc(E; K_{1,\tilde{q}-1}, K_{1,\tilde{q}}; \mathcal{P}_{1,p})$$

has to hold. In Proposition 3.10.33 we further argue that the prices $\mathbf{a}_{1,\tilde{s}-1}$, $\mathbf{a}_{1,\tilde{s}}$ and $\mathbf{a}_{1,\tilde{q}}$ are co-linear and thus

$$\begin{aligned} cc(A; K_{1,\tilde{s}-1}, K_{1,\tilde{s}}; \mathcal{P}_{1,p}) &= cc(A; K_{1,\tilde{q}-1}, K_{1,\tilde{q}}; \mathcal{P}_{1,p}) \\ &< cc(E; K_{1,\tilde{q}-1}, K_{1,\tilde{q}}; \mathcal{P}_{1,p}) \\ &\leq cc(E; K_{1,\tilde{s}-1}, K_{1,\tilde{s}}; \mathcal{P}_{1,p}) \end{aligned}$$

follows. This readily implies that the Legendre-Fenchel condition holds with strict inequality on $[K_{1,\tilde{s}-1}, K_{1,\tilde{s}}]$ and thus the price for European options with strike $K_{1,\tilde{s}}$ is given by $\mathbf{e}_{1,\tilde{s}} = E_{ub}(K_{1,\tilde{s}}, \mathcal{P}_{1,\tilde{s}})$.

Let us now determine the price for European options with strike K_u^A which the algorithm computes from the initial price set $\mathcal{P}_2^*$. We then have to distinguish between the two cases where either $K_{2,q} = K_{2,\tilde{s}+1}$ or $K_{2,q} > K_{2,\tilde{s}+1}$. In the first case we know from Remark 3.10.34 that $K_{1,\tilde{s}-1} \in \mathbb{K}^E(\mathcal{P}_0^*)$ has to hold. In addition, we know that Algorithm 3 stops revising option prices in a strike at which European options are traded in the market. Since we assumed that Algorithm 2 ignores any violation of the upper bound and thus refrains from introducing auxiliary price constraints we then also know that $K_{2,\tilde{s}-1}, K_{2,\tilde{s}+1} \in \mathbb{K}^E(\mathcal{P}_0^*)$ has to hold. As we further assumed that Algorithm 3 did not stop due to the existence of an arbitrage we can conclude that $\mathbf{e}_{1,\tilde{s}}^n \leq E_{ub}(K_{1,\tilde{s}}, \mathcal{P}_0^*)$ and thus

$$cc(E; K_{1,\tilde{s}-1}, K_{1,\tilde{s}+1}; \mathcal{P}_{1,p}) \geq cc(E^n; K_{1,\tilde{s}}, K_{1,\tilde{s}+1}; \mathcal{P}_{1,p}).$$

Taking into account that the revised prices are computed such that the Legendre-Fenchel condition holds with equality we furthermore have that

$$cc(E^n; K_{1,\tilde{s}}, K_{1,\tilde{s}+1}; \mathcal{P}_{1,p}) = cc(A, A^n; K_{1,\tilde{s}}, K_{1,\tilde{s}+1}; \mathcal{P}_{1,p}).$$

Since $K_{1,\tilde{s}} = K_u^A$ and $K_{1,\tilde{s}+1} = K_{2,q}$ we can furthermore conclude that

$$cc(A; K_{2,\tilde{s}}, K_{2,\tilde{s}+1}; \mathcal{P}_2^*) = cc(A, A^n; K_{1,\tilde{s}}, K_{1,\tilde{s}+1}; \mathcal{P}_{1,p}).$$

Combined with the fact that $\mathbf{a}_{2,\tilde{s}-1} = A_{lb}^{rhs}(K_{2,\tilde{s}-1}, \mathcal{P}_2^*)$ we obtain that

$$\begin{aligned} cc(E; K_{1,\tilde{s}-1}, K_{1,\tilde{s}+1}; \mathcal{P}_0^*) &\geq cc(E^n; K_{1,\tilde{s}}, K_{1,\tilde{s}+1}; \mathcal{P}_{1,p}) \\ &= cc(A, A^n; K_{1,\tilde{s}}, K_{1,\tilde{s}+1}; \mathcal{P}_{1,p}) \end{aligned}$$

$$\begin{aligned} &= cc(A; K_{2,\tilde{s}}, K_{2,q}; \mathcal{P}_2^*) \\ &= cc(A; K_{2,\tilde{s}-1}, K_{2,\tilde{s}}; \mathcal{P}_{2,\tilde{s}-1}) \end{aligned}$$

and thus $\mathbf{e}_{2,\tilde{s}} = E_{ub}(K_{2,\tilde{s}}, \mathcal{P}_{2,\tilde{s}-1})$ has to hold.

In the second case we have $K_{1,q-1}, K_{1,q} \in \mathbb{K}^E(\mathcal{P}_0^*)$ and according to the stopping condition in line 5 of Algorithm 3

$$cc(A, A^n; K_{1,\tilde{s}}, K_{1,q}; \mathcal{P}_{1,p}) \leq cc(E; K_{1,q-1}, K_{1,q}; \mathcal{P}_{1,p})$$

has to hold. Recall further that we argued in the previous case that $K_{2,\tilde{s}-1}, K_{2,\tilde{s}+1} \in \mathbb{K}^E(\mathcal{P}_0^*)$. The convexity of the European price function $E(\cdot, \mathcal{P}_0^*)$ together with the fact that $K_{2,\tilde{s}-1}, K_{2,\tilde{s}+1} \in \mathbb{K}^E(\mathcal{P}_0^*)$ then implies that

$$cc(E; K_{2,\tilde{s}-1}, K_{2,\tilde{s}+1}; \mathcal{P}_{2,\tilde{s}-1}) \geq cc(E; K_{1,q-1}, K_{1,q}; \mathcal{P}_{1,p}).$$

Analogously to the previous case we also know that $\mathbf{a}_{2,\tilde{s}-1} = A_{lb}^{rhs}(K_{2,\tilde{s}-1}, \mathcal{P}_2^*)$ holds. Hence, we obtain that

$$\begin{aligned} cc(A; K_{2,\tilde{s}-1}, K_{2,\tilde{s}}; \mathcal{P}_{2,\tilde{s}-1}) &= cc(A, A^n; K_{1,\tilde{s}}, K_{1,q}; \mathcal{P}_{1,p}) \\ &\leq cc(E; K_{1,q-1}, K_{1,q}; \mathcal{P}_{1,p}) \\ &\leq cc(E; K_{2,\tilde{s}-1}, K_{2,\tilde{s}+1}; \mathcal{P}_{2,\tilde{s}-1}) \end{aligned}$$

and thus we can conclude again that $\mathbf{e}_{2,\tilde{s}} = E_{ub}(K_{2,\tilde{s}}, \mathcal{P}_{2,\tilde{s}-1})$.

We can now consider the prices for American options with strikes in $[K_u^A, K_{2,q}) \cap \mathbb{K}^E(\mathcal{P}_0^*)$ the algorithm computes from the initial set of prices $\mathcal{P}_2^*$. In the case where $K_{2,q} = K_{2,\tilde{s}+1}$ the set $[K_u^A, K_{2,q}) \cap \mathbb{K}^E(\mathcal{P}_0^*)$ is empty and thus we will assume that $K_{2,q} > K_{2,\tilde{s}+1}$ in the sequel. To see that $\mathbf{a}_{2,s} = A_{ub}(K_{2,s}, \mathcal{P}_{2,s-1})$ for any strike $K_{2,s} \in [K_u^A, K_{2,q}) \cap \mathbb{K}^E(\mathcal{P}_0^*)$, we will assume for contradiction that there exists

$$K_{2,\hat{s}} = \min\{K_{2,s} \in [K_u^A, K_{2,q}) \cap \mathbb{K}^E(\mathcal{P}_0^*) : \mathbf{a}_{2,s} > A_{ub}(K_{2,s}, \mathcal{P}_{2,s-1})\}.$$

According to Proposition 3.10.39 the price for American options with strike $K_{2,\tilde{s}-1}$ is given by $\mathbf{a}_{2,\tilde{s}-1} = A_{lb}^{rhs}(K_{2,\tilde{s}-1}, \mathcal{P}_2^*)$ and thus the upper bound $A_{ub}(K_{2,s}, \mathcal{P}_2^*)$ corresponds to the left hand-side lower bound $A_{lb}^{lhs}(K_{2,s}, \mathcal{P}_{2,\tilde{s}})$ for any strike $K_{2,s} \in [K_u^A, K_{2,q}) \cap \mathbb{K}^E(\mathcal{P}_0^*)$. Hence, we must have

$$\mathbf{a}_{2,\hat{s}} = \max\{A_{lf}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1}), A_{lb}^{\overline{A}}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1}), A_{lb}^{rhs}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1})\}.$$

We will then rule out each of the cases individually. Suppose first that the price for an American option with strike $K_{2,\hat{s}}$ is given by $\mathbf{a}_{2,\hat{s}} = A_{lf}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1})$ and note that the stopping condition in line 5 of Algorithm 3 is given by

$$cc(E; K_{1,q-1}, K_{1,q}; \mathcal{P}_{1,p}) \geq cc(A, A^n; K_{1,\tilde{s}}, K_{1,q}; \mathcal{P}_{1,p}).$$

Note further that

$$cc(A, A^n; K_{1,\tilde{s}}, K_{1,q}; \mathcal{P}_{1,p}) = cc(A_{ub}; K_{2,\tilde{s}}, K_{2,q}; \mathcal{P}_2^*)$$

where

$$A_{ub}(K, \mathcal{P}_2^*) = \frac{\mathbf{a}_{1,q}^n - \mathbf{a}_{2,\tilde{s}}}{K_{1,q} - K_{2,\tilde{s}}}(K - K_{2,\tilde{s}}) + \mathbf{a}_{2,\tilde{s}}$$

for $K \in [K_{2,\tilde{s}}, K_{2,q}]$. Moreover, we obtain from convexity of $E(\cdot, \mathcal{P}_0^*)$ that

$$cc(E; K_{2,\hat{s}-1}, K_{2,\hat{s}}; \mathcal{P}_{2,\hat{s}-1}) \geq cc(E; K_{1,q-1}, K_{1,q}; \mathcal{P}_{1,p}),$$

as $K_{1,q-1}, K_{1,q} \in \mathbb{K}^E(\mathcal{P}_0^*)$ and either $K_{2,\hat{s}-1}, K_{2,\hat{s}} \in \mathbb{K}^E(\mathcal{P}_0^*)$ or $K_{2,\hat{s}-1} = K_{2,\tilde{s}}$ in which case $\mathbf{e}_{2,\hat{s}-1} = E_{ub}(K_{2,\hat{s}-1}, \mathcal{P}_0^*)$. Combined with the assumption that $\mathbf{a}_{2,\hat{s}} = A_{lf}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1})$ these inequalities yield

$$\begin{aligned} cc(A; K_{2,\hat{s}-1}, K_{2,\hat{s}}; \mathcal{P}_{2,\hat{s}-1}) &= cc(E; K_{2,\hat{s}-1}, K_{2,\hat{s}}; \mathcal{P}_{2,\hat{s}-1}) \\ &\geq cc(E; K_{1,q-1}, K_{1,q}; \mathcal{P}_{1,p}) \\ &\geq cc(A, A^n; K_{1,\tilde{s}}, K_{1,q}; \mathcal{P}_{1,p}) \\ &= cc(A_{ub}; K_{2,\tilde{s}}, K_{2,q}; \mathcal{P}_2^*) \end{aligned}$$

which implies that $A_{lf}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1}) \leq A_{ub}(K_{2,\hat{s}}, \mathcal{P}_2^*)$, thereby contradicting the assumption that $\mathbf{a}_{2,\hat{s}} > A_{ub}(K_{2,\hat{s}}, \mathcal{P}_2^*)$.

Suppose now that $\mathbf{a}_{2,\hat{s}} = A_{lb}^{rhs}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1})$, where the right hand-side lower bound is given by

$$\frac{\hat{\mathbf{a}}_{u+1} - \mathbf{a}_{1,q}^n}{K_{u+1}^A - K_{1,q}}(K_{2,\hat{s}} - K_{u+1}^A) + \hat{\mathbf{a}}_{u+1}.$$

We can then deduce from $\mathbf{a}_{2,\hat{s}} > A_{ub}(K_{2,\hat{s}}, \mathcal{P}_2^*)$ that $\mathbf{a}_{1,q}^n > A_{ub}(K_{1,q}, \mathcal{P}_0^*)$. According to Proposition 3.10.35 we further know that $\mathbf{a}_{1,q} > \mathbf{a}_{1,q}^n$ and thus $\mathbf{a}_{1,q} > A_{ub}(K_{1,q}, \mathcal{P}_0^*)$ has to hold. This contradicts the assumption that $\mathcal{P}_{1,p-1}$ is a $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension as $K_{1,q} \leq K_{1,p-1}$ and thus we can conclude that

$$A_{lb}^{rhs}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1}) \leq A_{ub}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1})$$

has to hold.

Let us assume next that $\mathbf{a}_{2,\hat{s}} = A_{lb}^{\overline{A},l}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1})$. In this case there has to exist a strike $K_j^E e^{-rT} \in (K_{2,w}, K_u^A)$ with $K_j^E \in \mathbb{K}^E(\mathcal{P}_0^*)$ such that

$$\begin{aligned}\mathbf{a}_{2,\hat{s}} &= \frac{\hat{\mathbf{a}}_u - \overline{\mathbf{a}}_j}{K_u^A - K_j^E e^{-rT}}(K_{2,\hat{s}} - K_u^A) + \hat{\mathbf{a}}_u \\ &> \frac{\mathbf{a}_{1,q}^n - \hat{\mathbf{a}}_u}{K_{1,q} - K_u^A}(K_{2,\hat{s}} - K_u^A) + \hat{\mathbf{a}}_u \\ &= A_{ub}(K_{2,\hat{s}}, \mathcal{P}_2^*).\end{aligned}$$

It follows that

$$\frac{\hat{\mathbf{a}}_u - \overline{\mathbf{a}}_j}{K_u^A - K_j^E e^{-rT}}(K_{1,q} - K_u^A) + \hat{\mathbf{a}}_u > \mathbf{a}_{1,q}^n.$$

Then again, this would have prompted the algorithm to stop in $K_{1,q}$ due to the existence of an arbitrage. Hence, $\mathbf{a}_{2,\hat{s}} = A_{lb}^{\overline{A},l}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1})$ cannot hold either.

The only possibility left is that the price for American options with strike $K_{2,\hat{s}}$ is given by $\mathbf{a}_{2,\hat{s}} = A_{lb}^{\overline{A},r}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1})$. Then there has to exist a strike $K_j^E e^{-rT} \in (K_{2,q}, K_{u+1}^A)$ with $K_j^E \in \mathbb{K}^E(\mathcal{P}_0^*)$ such that

$$A_{ub}(K_{2,\hat{s}}, \mathcal{P}_2^*) < \frac{\overline{\mathbf{a}}_j - \mathbf{a}_{1,q}^n}{K_j^E e^{-rT} - K_{1,q}}(K_{2,\hat{s}} - K_j^E e^{-rT}) + \overline{\mathbf{a}}_j.$$

It follows that

$$\overline{\mathbf{a}}_j < \frac{\mathbf{a}_{1,q}^n - \hat{\mathbf{a}}_u}{K_{1,q} - K_u^A}(K_j^E e^{-rT} - K_u^A) + \hat{\mathbf{a}}_u$$

holds. Taking into account that $\mathbf{a}_{1,q} > \mathbf{a}_{1,q}^n$ according to Proposition 3.10.35 we furthermore obtain that

$$\overline{\mathbf{a}}_j < \frac{\mathbf{a}_{1,q} - \hat{\mathbf{a}}_u}{K_{1,q} - K_u^A}(K_j^E e^{-rT} - K_u^A) + \hat{\mathbf{a}}_u. \tag{3.41}$$

In contrast we know from Proposition 3.10.33 that $\mathbf{a}_{1,\tilde{s}-1}$, $\mathbf{a}_{1,q-1}$ and $\mathbf{a}_{1,q}$ are co-linear. Since $\mathcal{P}_{1,p-1}$ is a $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension we can further deduce that $\mathbf{a}_{1,\tilde{s}-1} \geq A_{lb}^{\overline{A},r}(K_{1,\tilde{s}-1}, \mathcal{P}_1^*)$. This readily implies that

$$\overline{\mathbf{a}}_j \geq \frac{\mathbf{a}_{1,q} - \hat{\mathbf{a}}_u}{K_{1,q} - K_u^A}(K_j^E e^{-rT} - K_u^A) + \hat{\mathbf{a}}_u$$

and thus contradicts (3.41). Hence, we have shown that the price for American options with strike $K_{2,s} \in [K_u^A, K_{2,q}) \cap \mathbb{K}^E(\mathcal{P}_0^*)$ has to be given by $\mathbf{a}_{2,s} = A_{ub}(K_{2,s}, \mathcal{P}_2^*)$.

We are therefore only left to argue that the prices for options with strikes in $[K_{2,q}, K_{2,p}]$ the algorithm computes from $\mathcal{P}_2^*$ coincide with the revised prices. To this end we will show that depending on the type of strike the price is computed to be either $A_{lf}(K_{2,s}, \mathcal{P}_{2,s-1})$ or $E_{lf}(K_{2,s}, \mathcal{P}_{2,s-1})$. As the prices for both American and European options with strike $K_{2,q}$ coincide with the revised prices it then follows from the Legendre-Fenchel condition that $\mathbf{a}_{2,s} = \mathbf{a}^n_{1,s}$ for $K_{2,s} \in [K_{2,q}, K_{2,p}] \cap (\mathbb{K}^E(\mathcal{P}^*_0)\backslash\mathbb{K}^A(\mathcal{P}^*_2))$ and $\mathbf{e}_{2,s} = \mathbf{e}^n_{1,s}$ for $K_{2,s} \in [K_{2,q}, K_{2,p}] \cap (\mathbb{K}^A(\mathcal{P}^*_2)\backslash\mathbb{K}^E(\mathcal{P}^*_0))$.

To see that the prices coincide we will use induction. In the base step we consider the price for non-traded options with the strike $K_{2,q+1}$. Suppose first that $K_{2,q+1} \in \mathbb{K}^E(\mathcal{P}^*_0)\backslash\mathbb{K}^A(\mathcal{P}^*_2)$ and let us assume for contradiction that the price for American options with strike $K_{2,q+1}$ exceeds $A_{lf}(K_{2,q+1}, \mathcal{P}_{2,q})$. We can then immediately rule out that $\mathbf{a}_{2,q+1}$ is given by $A^{rhs}_{lb}(K_{2,q+1}, \mathcal{P}_{2,q})$ or $A^{\overline{A},r}_{lb}(K_{2,q+1}, \mathcal{P}_{2,q})$ as Proposition 3.10.30 guarantees that $(K_{2,q}, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}^*_2) = \emptyset$ and we know that

$$\mathbf{a}^n_{1,q+1} \geq \max\{A^{rhs}_{lb}(K_{1,q+1}, \mathcal{P}^*_0), A^{\overline{A},r}_{lb}(K_{1,q+1}, \mathcal{P}^*_0)\}.$$

Consider next the case where $\mathbf{a}_{2,q+1} = A^{lhs}_{lb}(K_{2,q+1}, \mathcal{P}_{2,q})$. We must then have that

$$cc(A^{lhs}_{lb}; K_{2,q}, K_{2,q+1}; \mathcal{P}_{2,q}) = cc(A; K_{2,\tilde{s}}, K_{2,q+1}; \mathcal{P}_{2,q}).$$

Moreover, we know that

$$cc(A; K_{2,\tilde{s}}, K_{2,q}; \mathcal{P}_{2,q}) = cc(A, A^n; K_{1,\tilde{s}}, K_{1,q}; \mathcal{P}_{1,p}).$$

Taking into account that

$$cc(A_{lf}; K_{2,q}, K_{2,q+1}; \mathcal{P}_{2,q}) = cc(A^n; K_{1,q}, K_{1,q+1}; \mathcal{P}^{rev}_1)$$

and that $K_{1,q}, K_{1,q+1} \in \mathbb{K}^E(\mathcal{P}^*_0)$, we can deduce that

$$\begin{aligned} cc(A, A^n; K_{1,\tilde{s}}, K_{1,q+1}; \mathcal{P}_{1,p}) &= cc(A; K_{2,\tilde{s}}, K_{2,q}; \mathcal{P}_{2,q}) \\ &= cc(A^{lhs}_{lb}; K_{2,q}, K_{2,q+1}; \mathcal{P}_{2,q}) \\ &< cc(A^n; K_{1,q}, K_{1,q+1}; \mathcal{P}^{rev}_1) \\ &= cc(E^n; K_{1,q}, K_{1,q+1}; \mathcal{P}^{rev}_1) \\ &= cc(E; K_{1,q}, K_{1,q+1}; \mathcal{P}_{1,p}). \end{aligned}$$

We can thus conclude from the stopping condition in line 5 of Algorithm 3 that the algorithm would have stopped already at the strike $K_{1,q+1}$ instead of $K_{1,q}$. As this is not the case we must have $A_{lf}(K_{2,q+1}, \mathcal{P}_{2,q}) \geq A^{lhs}_{lb}(K_{2,q+1}, \mathcal{P}_{2,q})$.

Let us assume last that $\mathbf{a}_{2,q+1} = A^{\overline{A},l}_{lb}(K_{2,q+1}, \mathcal{P}_{2,q})$ where

$$A^{\overline{A},l}_{lb}(K_{2,q+1}, \mathcal{P}_{2,q}) > A_{lf}(K_{2,q+1}, \mathcal{P}_{2,q}).$$

Then there has to exist a strike $K_j^E e^{-rT} \in (K_u^A, K_{2,q})$ with $K_j^E \in \mathbb{K}^E(\mathcal{P}_0^*)$ such that

$$A_{lf}(K_{2,q+1}, \mathcal{P}_{2,q}) < \frac{\mathbf{a}_{1,q}^n - \overline{\mathbf{a}}_j}{K_{1,q} - K_j^E e^{-rT}}(K_{2,q+1} - K_{2,q}) + \mathbf{a}_{1,q}^n$$

We will now show that such a strike cannot exist. According to Proposition 3.10.33 we know that the prices $\mathbf{a}_{1,\tilde{s}-1}$, $\hat{\mathbf{a}}_u$ and $\mathbf{a}_{1,q}$ are co-linear. As the options with strike $K_{1,l}$ were priced by (3.23) it follows that $\mathbf{a}_{1,\tilde{s}-1} \geq A_{lb}^{\overline{A},r}(K_{1,\tilde{s}-1}, \mathcal{P}_1^*)$ and thus

$$\overline{\mathbf{a}}_j \geq \frac{\hat{\mathbf{a}}_u - \mathbf{a}_{1,\tilde{s}-1}}{K_u^A - K_{1,\tilde{s}-1}}(K_j^E e^{-rT} - K_u^A) + \hat{\mathbf{a}}_u.$$

Taking into account that $\mathbf{a}_{2,\tilde{s}-1} > \mathbf{a}_{1,\tilde{s}-1}$ we readily obtain that

$$\overline{\mathbf{a}}_j > \frac{\hat{\mathbf{a}}_u - \mathbf{a}_{2,\tilde{s}-1}}{K_u^A - K_{2,\tilde{s}-1}}(K_j^E e^{-rT} - K_u^A) + \hat{\mathbf{a}}_u.$$

and thus $A_{lb}^{\overline{A},l}(K_{2,q+1}, \mathcal{P}_{2,q}) < A_{lb}^{lhs}(K_{2,q+1}, \mathcal{P}_{2,q})$ has to hold. We argued, however, already that $A_{lb}^{lhs}(K_{2,q+1}, \mathcal{P}_{2,q}) < A_{lf}(K_{2,q+1}, \mathcal{P}_{2,q})$ which implies that $A_{lb}^{\overline{A},l}(K_{2,q+1}, \mathcal{P}_{2,q}) < A_{lf}(K_{2,q+1}, \mathcal{P}_{2,q})$, thereby yielding a contradiction. Hence, we have shown that the price for American options with strike $K_{2,q+1}$ has to be given by $\mathbf{a}_{2,q+1} = A_{lf}(K_{2,q+1}, \mathcal{P}_{2,q})$.

Suppose now that $K_{2,q+1} \in \mathbb{K}^A(\mathcal{P}_2^*)\backslash\mathbb{K}^E(\mathcal{P}_0^*)$ and that the price for European options with strike $K_{2,q+1}$ is given by $\mathbf{e}_{2,q+1} = E_{ub}(K_{2,q+1}, \mathcal{P}_{2,q})$ where $E_{ub}(K_{2,q+1}, \mathcal{P}_{2,q}) < E_{lf}(K_{2,q+1}, \mathcal{P}_{2,q})$. If we assume that the strike $K_j^E = \min\{K_v^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K_v^E > K_{2,q+1}\}$ corresponds to $K_{2,q+n}$ for $n > 1$ we can deduce that

$$\begin{aligned} cc(E; K_{2,q}, K_{2,q+n}; \mathcal{P}_{2,q}) &= cc(E; K_{2,q}, K_{2,q+1}; \mathcal{P}_{2,q}) \\ &> cc(E^n; K_{1,q}, K_{1,q+1}; \mathcal{P}_1^{rev}) \end{aligned}$$

Then again, as $K_{2,q}, K_{2,q+n} \in \mathbb{K}^E(\mathcal{P}_0^*)$ this would imply that $\mathbf{e}_{1,q+1}^n > E_{ub}(K_{1,q+1}, \mathcal{P}_0^*)$ has to hold which cannot be the case as this would have prompted Algorithm 3 to stop at the strike $K_{1,q+1}$ due to the existence of an arbitrage. It follows that the price for European options with strike $K_{2,q+1}$ has to be given by $\mathbf{e}_{2,q+1} = E_{lf}(K_{2,q+1}, \mathcal{P}_{2,q})$.

In the inductive step we assume that the price functions $A(\cdot, \mathcal{P}_{2,p})$ and $E(\cdot, \mathcal{P}_{2,p})$ satisfy the Legendre-Fenchel condition with equality on any sub-interval of $[K_{2,q}, K_{2,\hat{s}-1}]$ for $K_{2,\hat{s}} \leq K_{2,p}$. To argue that the Legendre-Fenchel condition then also has to hold with equality on $[K_{2,\hat{s}-1}, K_{2,\hat{s}}]$ we consider the cases where $K_{2,\hat{s}} \in \mathbb{K}^E(\mathcal{P}_0^*)\backslash\mathbb{K}^A(\mathcal{P}_2^*)$ and $K_{2,\hat{s}} \in \mathbb{K}^A(\mathcal{P}_2^*)\backslash\mathbb{K}^E(\mathcal{P}_0^*)$ separately.

In the first case we assume again that $\mathbf{a}_{2,\hat{s}} > A_{lf}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1})$. Analogously to the base step we can rule out again that

$$\mathbf{a}_{2,\hat{s}} = \max\{A_{lb}^{rhs}(K_{2,\hat{s}}, \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{2,\hat{s}}, \mathcal{P}_0^*)\}.$$

We thus continue by assuming that the algorithm determined the price for American options with strike $K_{2,\hat{s}}$ to be $\mathbf{a}_{2,\hat{s}} = A_{lb}^{lhs}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1})$ where

$$A_{lb}^{lhs}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1}) > A_{lf}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1}).$$

This, however, cannot be the case as Corollary 3.10.36 shows that the revised price functions are convex.

Hence, we are only left with the case where $\mathbf{a}_{2,\hat{s}} = A_{lb}^{\overline{A},l}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1})$. To see that this case can be excluded from consideration as well we have to distinguish between the two situations where either $K_{2,\hat{s}} \in (K_{2,q}, K_{u+1}^A)$ or $K_{2,\hat{s}} \geq K_{u+1}^A$. In the first case the argument from the base step also guarantees that $A_{lb}^{\overline{A},l}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1}) < A_{lf}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1})$ as the American price function is convex. In the second case the left hand-side lower bound is given by $A_{lb}^{\overline{A},l}(K_{2,\hat{s}}, \mathcal{P}_0^*)$. Since $\mathbf{a}_{1,\hat{s}}^n \geq A_{lb}^{\overline{A},l}(K_{2,\hat{s}}, \mathcal{P}_0^*)$ this case can be ruled out as well.

Let us consider now the case where $K_{2,\hat{s}} \in \mathbb{K}^A(\mathcal{P}_2^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*)$ and assume for contradiction that the price for European options with strike $K_{2,\hat{s}}$ is given by $\mathbf{e}_{2,\hat{s}} = E_{ub}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1})$ where $E_{ub}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1}) < E_{lf}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1})$. Then again, we know from Corollary 3.10.36 that the revised price functions are convex which contradicts $E_{ub}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1}) < E_{lf}(K_{2,\hat{s}}, \mathcal{P}_{2,\hat{s}-1})$.

We have therefore shown that the price functions constructed by the algorithm from the initial set of prices $\mathcal{P}_2^*$ are given by (3.38) and (3.39) if we disregard any possible violations of the upper bound. □

Before we can argue that the algorithm using the initial price set $\mathcal{P}_2^*$ computes a K_1^{vc}-admissible $\mathcal{P}_0^*$-extension, we need to analyse the situation in which a violation of the upper bound occurs in more detail.

Proposition 3.7.10. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

Algorithm 2 stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of convexity. Algorithm 3 computes revised prices for non-traded options with strikes $K_{1,s} \in [K_{1,q}, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*)$. At the strike $K_{1,q} \in (K_u^A, K_{u+1}^A) \cap \mathbb{K}^E(\mathcal{P}_0^*)$ it stops and defines the new initial set of prices $\mathcal{P}_2^*$ by $\mathcal{P}_2^* = ((\mathcal{P}_1^*)^A \cup (\mathbf{a}_{1,q}^n, K_{1,q}); (\mathcal{P}_0^*)^E)$.

Using the new initial set $\mathcal{P}_2^*$ Algorithm 2 computes the $K_{2,s-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{2,s-1}$. If the algorithm stops at the strike $K_{2,s} \in [0, K_1^{vc}] \cap \mathbb{K}(\mathcal{P}_2^*)$ due to a violation of $\overline{\mathbf{a}}_{2,s} \geq A(K_{2,s}e^{-rT}, \mathcal{P}_{2,s})$, then $K_{2,s}e^{-rT} > K_{2,r-1}$ for

$$K_{2,r} = \min\{K_{2,l} \in (K_{2,w}, K_u^A) \cap \mathbb{K}(\mathcal{P}_2^*) : A_{lb}^{rhs}(K_{2,l}, \mathcal{P}_2^*) > \mathbf{a}_{1,l}\}$$

and

$$K_{2,w} = \max\{K \in \mathbb{K}^A(\mathcal{P}_2^*) : K < K_u^A\}.$$

Proof. Suppose for contradiction that $K_{2,s}e^{-rT} \in (K_{2,\tilde{q}}, K_{2,\tilde{q}+1})$ where $K_{2,\tilde{q}}, K_{2,\tilde{q}+1} \in [0, K_{2,r-1}] \cap \mathbb{K}(\mathcal{P}_2^*)$. According to Proposition 3.7.8 we know that the price functions remain unchanged up to $K_{2,r-1}$ between the two iterations of the algorithm if no violation of the upper bound occurs. In particular, this means that $A(K, \mathcal{P}_{2,s}) = A(K, \mathcal{P}_{1,p})$ for any strike $K \leq \min\{K_{2,r-1}, K_{2,s}\}$. Taking Proposition 3.10.35 into account we can conclude that the European price function is not decreased on $[0, K_{2,s}]$ and thus $\overline{A}(K_{2,s}e^{-rT}, \mathcal{P}_{2,s}) \geq \overline{A}(K_{1,s}e^{-rT}, \mathcal{P}_{1,p})$ has to hold. We then have to distinguish between the two cases where either $K_{2,s} < K_1^{vc}$ or $K_{2,s} = K_1^{vc}$. In the first case we can use that $\mathcal{P}_{1,p-1}$ is a $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension to obtain that

$$\begin{aligned}
\overline{A}(K_{2,s}e^{-rT}, \mathcal{P}_{2,s}) &\geq \overline{A}(K_{1,s}e^{-rT}, \mathcal{P}_{1,p-1}) \\
&\geq A(K_{1,s}e^{-rT}, \mathcal{P}_{1,s-1}) \\
&= A(K_{2,s}e^{-rT}, \mathcal{P}_{2,s})
\end{aligned}$$

and thus a violation of the upper bound can be ruled out. In the second case we can deduce from the fact that $\mathcal{P}_{2,s-1}$ is a $K_{2,s-1}$-admissible $\mathcal{P}_0^*$-extension that $\overline{\mathbf{a}}_{2,s-1} \geq A(K_{2,s-1}e^{-rT}, \mathcal{P}_{2,s})$. According to Proposition 3.7.8 the Legendre-Fenchel condition has to hold with equality on $[K_{2,s-1}, K_{2,s}]$. Proposition 3.10.7 then readily implies that $\overline{\mathbf{a}}_{2,s} \geq A(K_{2,s}e^{-rT}, \mathcal{P}_{2,s})$. We can thus conclude that $K_{2,s}e^{-rT} > K_{2,r-1}$ has to hold. □

In contrast, a violation of the upper bound at a strike $K_{2,s}e^{-rT} \in (K_{2,r-1}, K_{2,r})$ is possible as the price for American options with strike $K_{2,r}$, $\mathbf{a}_{2,r}$, is increased between the previous and the current iteration of the algorithm and thus the linear interpolation between the prices $\mathbf{a}_{2,r-1}$ and $\mathbf{a}_{2,r}$ may exceed the upper bound. We will thus show that if a violation of the upper bound occurs at a strike $K_{2,s}e^{-rT} \in [K_{2,r-1}, K_{2,r}]$, then there has to exist a discounted European strike $K_j^E e^{-rT} \in [K_{2,r-1}, K_{2,r}]$ for $K_j^E \in \mathbb{K}^E(\mathcal{P}_0^*)$. Recall further that we denote the strike at which Algorithm 3 introduces an auxiliary price to the initial set $\mathcal{P}_2^*$ by K_1^{aux}.

Proposition 3.7.11. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

Algorithm 2 stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of convexity. Algorithm 3 computes revised prices for non-traded options with strikes $K_{1,s} \in [K_{1,q}, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*)$. At the strike $K_{1,q} \in (K_u^A, K_{u+1}^A) \cap \mathbb{K}^E(\mathcal{P}_0^*)$ it stops and defines the new initial *set of prices* $\mathcal{P}_2^*$ *by* $\mathcal{P}_2^* = ((\mathcal{P}_1^*)^A \cup (\mathbf{a}_{1,q}^n, K_{1,q}); (\mathcal{P}_0^*)^E)$.

Using the initial set $\mathcal{P}_2^*$ *Algorithm 2 computes the* $K_{2,s-1}$*-admissible* $\mathcal{P}_0^*$*-extension* $\mathcal{P}_{2,s-1}$. *If the algorithm stops at the strike* $K_{2,s} \in [0, K_1^{vc}] \cap \mathbb{K}(\mathcal{P}_2^*)$ *due to a violation of* $\overline{\mathbf{a}}_{2,s} \geq A(K_{2,s}e^{-rT}, \mathcal{P}_{2,s})$, *where* $K_{2,s}e^{-rT} \in [K_{2,r-1}, K_{2,r}]$, *then we must have for* $K_{j'}^E = \min\{K_l^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K_l^E \geq K_{2,s}\}$ *that* $K_{j'}^E \leq K_1^{aux}$ *and* $K_{j'}^E e^{-rT} \in (K_{2,r-1}, K_{2,r})$.

Proof. We begin by showing that $K_{j'}^E e^{-rT} \leq K_1^{aux}$ has to hold. To this end, we assume for contradiction that $K_{2,s} \in (K_1^{aux}, K_1^{vc}] \cap \mathbb{K}(\mathcal{P}_2^*)$. According to Proposition 3.7.8, we thus have that the Legendre-Fenchel condition holds with equality on $[K_{2,s-1}, K_{2,s}]$ and Proposition 3.10.7 then yields a contradiction to the assumption that $\overline{\mathbf{a}}_{2,s} < A(K_{2,s}e^{-rT}, \mathcal{P}_{2,s})$. Hence, we can conclude that $K_{2,s} \leq K_1^{aux}$ and it thus follows from $K_1^{aux} \in \mathbb{K}^E(\mathcal{P}_0^*)$ that $K_{j'}^E \leq K_1^{aux}$ as well.

To be able to apply Proposition 3.10.17 which shows that $K_{j'}^E e^{-rT} < K_u^A$, we need to argue that $K_{2,r-1} \notin \mathbb{K}_1^{aux}(\mathcal{P}_2^*)$ and that $[K_{2,s}, K_{j'}^E) \cap \mathbb{K}^{aux}(\mathcal{P}_2^*) = \emptyset$. Since $\mathbb{K}_1^{aux}(\mathcal{P}_2^*) = \{K_1^{vc}\}$ and $K_{2,r-1} < K_1^{aux} < K_1^{vc}$ we readily obtain that $K_{2,r-1} \notin \mathbb{K}_1^{aux}(\mathcal{P}_2^*)$. To see that $[K_{2,s}, K_{j'}^E) \cap \mathbb{K}^{aux}(\mathcal{P}_2^*) = \emptyset$ holds we note that according to Proposition 3.10.30 $[K_u^A, K_1^{aux}) \cap \mathbb{K}^{aux}(\mathcal{P}_2^*) = \emptyset$. Combining the definition of $K_{2,w}$ with $K_{2,w} < K_{2,s}$ it follows that $[K_{2,s}, K_{j'}^E) \cap \mathbb{K}^{aux}(\mathcal{P}_2^*) = \emptyset$ and thus we can conclude that $K_{j'}^E e^{-rT} < K_u^A$.

Finally, we will rule out that $K_{j'}^E e^{-rT} \in [K_{2,r}, K_u^A)$ which then guarantees the existence of a discounted European strike $K_{j'}^E e^{-rT} \in (K_{2,r-1}, K_{2,r})$. Note first that since the algorithm stopped in the strike K_1^{aux} due to $\mathbf{a}_{1,q} = A_{lb}^{lhs}(K_{1,q}, \mathcal{P}_{1,p})$, it follows that $\mathbf{a}_{1,q}^n \geq A_{lb}^{\overline{A},l}(K_{1,q}, \mathcal{P}_0^*)$. This, however, readily implies that

$$\overline{\mathbf{a}}_{j'} \geq \frac{\mathbf{a}_{1,q}^n - \hat{\mathbf{a}}_u}{K_{1,q} - K_u^A}(K_{j'}^E e^{-rT} - K_u^A) + \hat{\mathbf{a}}_u.$$

In addition, we must have

$$\frac{\overline{\mathbf{a}}_{2,s} - \overline{\mathbf{a}}_{2,s-1}}{K_{2,s}e^{-rT} - K_{2,s-1}e^{-rT}}(K - K_{2,s}e^{-rT}) + \overline{\mathbf{a}}_{2,s} < \frac{\mathbf{a}_{1,q}^n - \hat{\mathbf{a}}_u}{K_{1,q} - K_u^A}(K - K_u^A) + \hat{\mathbf{a}}_u$$

for $K \geq K_{2,r}$ as $\overline{\mathbf{a}}_{2,s-1} \geq A(K_{2,s-1}e^{-rT}, \mathcal{P}_{2,s})$ but $\mathbf{a}_{2,s} < A(K_{2,s}e^{-rT}, \mathcal{P}_{2,s})$. Since the prices $\overline{\mathbf{a}}_{2,s-1}$, $\overline{\mathbf{a}}_{2,s}$ and $\overline{\mathbf{a}}_{j'}$ are co-linear according to Proposition 3.10.14 this implies that $K_{j'}^E e^{-rT} < K_{2,r}$. Hence, we have shown that $K_{j'}^E e^{-rT} \in [K_{2,r-1}, K_{2,r}]$. □

We will now argue that the price constraint $(\overline{\mathbf{a}}_j, K_j^E e^{-rT})$ determined by Algorithm 4 to correct a violation of the upper bound in $K_{1,s}e^{-rT} \in [K_{r-1}(\mathcal{P}_2^*), K_r(\mathcal{P}_2^*)]$ will lie within its no-arbitrage bounds.

Proposition 3.7.12. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by* $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. *Starting with the initial set of prices* $\mathcal{P}_1^*$ *the algorithm computes the* $K_{1,p-1}$*-admissible* $\mathcal{P}_0^*$*-extension* $\mathcal{P}_{1,p-1}$ *for* $p \geq 1$.

Algorithm 2 stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of convexity. Algorithm 3 computes revised prices for non-traded options with strikes $K_{1,s} \in [K_{1,q}, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*)$. At the strike $K_{1,q} \in (K_u^A, K_{u+1}^A) \cap \mathbb{K}^E(\mathcal{P}_0^*)$ it stops and defines the new initial set of prices $\mathcal{P}_2^*$ by $\mathcal{P}_2^* = ((\mathcal{P}_1^*)^A \cup (\mathbf{a}_{1,q}^n, K_{1,q}); (\mathcal{P}_0^*)^E)$. If Algorithm 2 using the initial set of prices $\mathcal{P}_2^*$ introduces the auxiliary price constraint $(\overline{\mathbf{a}}_j, K_j^E e^{-rT})$ to correct a violation of the upper bound at strike $K_{2,s}e^{-rT} \in [K_{r-1}(\mathcal{P}_2^*), K_r(\mathcal{P}_2^*)]$ for

$$K_r(\mathcal{P}_2^*) = \min\{K_{2,s} \in (K_{2,w}, K_u^A) \cap \mathbb{K}(\mathcal{P}_2^*) : A_{lb}^{rhs}(K_{2,s}, \mathcal{P}_2^*) > \mathbf{a}_{1,s}\}$$

and

$$K_w(\mathcal{P}_2^*) = \max\{K \in \mathbb{K}^A(\mathcal{P}_2^*) : K < K_u^A\},$$

then $(\overline{\mathbf{a}}_j, K_j^E e^{-rT})$ lies within the no-arbitrage bounds given by (3.12) and (3.23).

Proof. To see that this is the case we use induction on the number of auxiliary price constraints introduced in this iteration between $K_{r-1}(\mathcal{P}_2^*)$ and $K_r(\mathcal{P}_2^*)$. In the base step we consider the case where $(\overline{\mathbf{a}}_j, K_j^E e^{-rT})$ is the first auxiliary price constraint introduced between $K_{r-1}(\mathcal{P}_2^*)$ and $K_r(\mathcal{P}_2^*)$. According to Proposition 3.7.10 we can further rule out a violation of the upper bound on $[0, K_{r-1}(\mathcal{P}_2^*)]$. Since the price functions are given by (3.38) and (3.39) prior to a violation of the upper bound we can deduce that they must be convex up to $K_{2,s}$. We can therefore rule out a violation of convexity in any strike prior to $K_{2,s}$. It follows that $(\overline{\mathbf{a}}_j, K_j^E e^{-rT})$ has to be the first auxiliary price constraint added in this iteration.

Let us show now that the auxiliary price constraint $(\overline{\mathbf{a}}_j, K_j^E e^{-rT})$ satisfies $\overline{\mathbf{a}}_j < A_{ub}(K_j^E e^{-rT}, \mathcal{P}_{2,s})$. To do so, we argue first that $\overline{\mathbf{a}}_{j'} < A_{ub}(K_{j'}^E e^{-rT}, \mathcal{P}_{2,s})$ for $K_{j'}^E = \min\{K_l^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K_l^E \geq K_{2,s}\}$. According to Proposition 3.7.11 $K_{j'}^E \leq K_1^{aux}$ has to hold. Combined with the fact that $[K_u^A, K_1^{aux}) \cap \mathbb{K}^{aux}(\mathcal{P}_2^*) = \emptyset$ according to Proposition 3.10.30 and $K_{2,s} > K_{2,w}$ for $K_{2,w} = \max\{K \in \mathbb{K}^A(\mathcal{P}_2^*) : K < K_u^A\}$ we readily obtain that $[K_{2,s}, K_{j'}^E] \cap \mathbb{K}^{aux}(\mathcal{P}_2^*) = \emptyset$ has to hold. This in turn allows us to apply Proposition 3.10.14 guaranteeing that the prices $\overline{\mathbf{a}}_{2,s-1}$, $\overline{\mathbf{a}}_{2,s}$ and $\overline{\mathbf{a}}_{j'}$ are co-linear. We can further argue that $\mathbf{a}_{2,r-1} \leq \overline{A}(K_{2,r-1}, \mathcal{P}_{2,s})$ has to hold. Note first that the price for American options with strike $K_{2,r-1}$ remains unchanged between the two iterations of the algorithm according to Proposition 3.7.8. In addition, Proposition 3.7.8 shows that the European price function is not decreased on $[0, K_{2,s}]$ which in turn means that the upper bound is not decreased on $[0, K_{2,s}e^{-rT}]$. We then obtain $\mathbf{a}_{2,r-1} = \mathbf{a}_{1,r-1} \leq \overline{A}(K_{1,r-1}, \mathcal{P}_{1,p}) \leq \overline{A}(K_{2,r-1}, \mathcal{P}_{2,s})$. Recall also that we assumed that the first violation of the upper bound occurs at $K_{2,s}e^{-rT}$ and thus $\overline{\mathbf{a}}_{2,s-1} \geq A(K_{2,s-1}e^{-rT}, \mathcal{P}_{2,s})$ has to hold. We can thus conclude that $A(K, \mathcal{P}_{2,s}) \leq \overline{A}(K, \mathcal{P}_{2,s})$ for any strike $K \leq \max\{K_{2,r-1}, K_{2,s-1}e^{-rT}\}$. Combining $\overline{\mathbf{a}}_{2,s} < A(K_{2,s}e^{-rT}, \mathcal{P}_{2,s})$ with the fact that the prices $\overline{\mathbf{a}}_{2,s-1}$, $\overline{\mathbf{a}}_{2,s}$ and $\overline{\mathbf{a}}_{j'}$ are co-linear we readily obtain that $\overline{\mathbf{a}}_{j'} < A(K_{j'}^E e^{-rT}, \mathcal{P}_{2,s})$.

Since the price functions are obtained by interpolating linearly between the given option prices we must therefore have $\overline{\mathbf{a}}_{j'} < A_{ub}(K^E_{j'}e^{-rT}, \mathcal{P}_{2,s})$.

Let us assume now that $j > j'$, then this means that

$$\frac{\overline{\mathbf{a}}_j - \mathbf{a}_{2,r-1}}{K^E_j e^{-rT} - K_{2,r-1}} \leq \frac{\overline{\mathbf{a}}_{j'} - \mathbf{a}_{2,r-1}}{K^E_{j'} e^{-rT} - K_{2,s}}.$$

Having argued already that $\overline{\mathbf{a}}_{j'} < A_{ub}(K^E_{j'}e^{-rT}, \mathcal{P}_{2,s})$, we can now conclude that $\overline{\mathbf{a}}_j < A_{ub}(K^E_j e^{-rT}, \mathcal{P}_{2,s})$ to hold as well.

We are left to argue that the auxiliary price constraint $(\overline{\mathbf{a}}_j, K^E_j e^{-rT})$ satisfies

$$\overline{\mathbf{a}}_j \geq \max\{A_{lb}(K^E_j e^{-rT}, \mathcal{P}_{2,s}), A^{\overline{A}}_{lb}(K^E_j e^{-rT}, \mathcal{P}_{2,s}), A_{lf}(K^E_j e^{-rT}, \mathcal{P}_{2,s})\}.$$

Recall that the prices for American options with strikes up to $K_{2,r-1}$ remain unchanged. Hence, the left hand-side lower bound for American options with strikes in $(K_{2,r-1}, K_{2,r}]$ coincides with the left hand-side lower bound from the previous iteration. As no violation of the upper bound occurred on this interval in the previous iteration, we are guaranteed that $\overline{\mathbf{a}}_j \geq A^{lhs}_{lb}(K^E_j e^{-rT}, \mathcal{P}_{2,s})$. Further, we know that $\overline{\mathbf{a}}_j \geq A^{rhs}_{lb}(K^E_j e^{-rT}, \mathcal{P}_{2,s})$ holds as $\mathbf{a}^n_{1,q} \geq A^{\overline{A},l}_{lb}(K_{1,q}, \mathcal{P}^*_0)$.

Suppose now for the moment that the auxiliary price constraint violates $\overline{\mathbf{a}}_j \geq A^{\overline{A}}_{lb}(K^E_j e^{-rT}, \mathcal{P}_{2,s})$. According to (viii) of the Standing Assumptions $\hat{\mathbf{a}}_u \leq \overline{A}(K^A_u, \mathcal{P}^*_0)$ has to hold. Combined with the convexity of the upper bound $\overline{A}(\cdot, \mathcal{P}^*_0)$ we can therefore rule out $\overline{\mathbf{a}}_j < A^{\overline{A},r}_{lb}(K^E_j e^{-rT}, \mathcal{P}_{2,s})$. Hence, we must have $\overline{\mathbf{a}}_j < A^{\overline{A},l}_{lb}(K^E_j e^{-rT}, \mathcal{P}_{2,s})$. Suppose first that $K_{2,w}$ corresponds to K^A_{u-1}, then we can argue again using (viii) of the Standing Assumptions that $\overline{\mathbf{a}}_j \geq A^{\overline{A},l}_{lb}(K^E_j e^{-rT}, \mathcal{P}_{2,s})$ has to hold. If we assume that $K_{2,w} \in \mathbb{K}^{aux}_2(\mathcal{P}^*_2)$ where $K_{2,w}$ corresponds to $K^E_v e^{-rT}$ for $K^E_v \in \mathbb{K}^E(\mathcal{P}^*_0)$, then

$$A^{\overline{A},l}_{lb}(K^E_j e^{-rT}, \mathcal{P}_{2,s}) = \frac{\overline{\mathbf{a}}_v - \overline{\mathbf{a}}_{v-1}}{K^E_v e^{-rT} - K^E_{v-1} e^{-rT}}(K^E_j e^{-rT} - K^E_v e^{-rT}) + \overline{\mathbf{a}}_v.$$

In this case $\overline{\mathbf{a}}_j \geq A^{\overline{A},l}_{lb}(K^E_j e^{-rT}, \mathcal{P}_{2,s})$ follows immediately from the convexity of $\overline{A}(\cdot, \mathcal{P}^*_0)$.

Finally, we consider the case where a violation of $\overline{\mathbf{a}}_j \geq A_{lf}(K^E_j e^{-rT}, \mathcal{P}_{2,s})$ occurs. As we mentioned before, we know that the price for American options with strike $K_{r-1}(\mathcal{P}^*_2)$ remains unchanged between the two iterations of the algorithm according to Proposition 3.7.8. From $\mathcal{P}_{1,p-1}$ being a $K_{1,p-1}$-admissible $\mathcal{P}^*_0$-extension combined with the fact that $K_{2,s} \leq K^E_j \leq K^{aux}_1 \leq K_{1,p-1}$ we know that $\overline{\mathbf{a}}_j \geq A(K^E_j e^{-rT}, \mathcal{P}_{1,p-1})$ has to hold. Moreover, the price for American options with strike $K_{2,r}$ was computed such that the Legendre-Fenchel condition holds. Taking into account that the prices for European options in the interval $[K_{2,r-1}, K_{2,r}]$ remain unchanged it follows that $\overline{\mathbf{a}}_j \geq A_{lf}(K^E_j e^{-rT}, \mathcal{P}_{2,s})$ has to hold.

In the inductive step, we use the following induction hypothesis. We assume that any auxiliary price constraint already introduced to correct a violation of the upper bound on $[K_{r-1}(\mathcal{P}_2^*), K_r(\mathcal{P}_2^*)]$ lies within its respective no-arbitrage bounds and deduce that the new auxiliary price constraint will do so as well. Note further that the last constraint introduced has to be $(\overline{\mathbf{a}}_{j-1}, K_{j-1}^E e^{-rT})$ as we know that the upper bound $\overline{A}(\cdot, \mathcal{P}_0^*)$ is convex. In addition, we know that $A(K_{j-1}^E e^{-rT}, \mathcal{P}_{2,s}) = \overline{\mathbf{a}}_{j-1}$ has to hold.

We begin again by arguing that the new auxiliary constraint $(\overline{\mathbf{a}}_j, K_j^E e^{-rT})$ satisfies $\overline{\mathbf{a}}_j < A_{ub}(K_j^E e^{-rT}, \mathcal{P}_{2,s})$. To see that this is the case we apply the argument used in the base step to show that the prices $\overline{\mathbf{a}}_{j-1}$, $\overline{\mathbf{a}}_{2,s}$ and $\overline{\mathbf{a}}_j$ are co-linear. Combined with the fact that $A(K_{j-1}^E e^{-rT}, \mathcal{P}_{2,s}) = \overline{\mathbf{a}}_{j-1}$ we readily obtain $\overline{\mathbf{a}}_j < A_{ub}(K_j^E e^{-rT}, \mathcal{P}_{2,s})$.

Finally, we will argue that

$$\overline{\mathbf{a}}_j \geq \max\{A_{lb}(K_j^E e^{-rT}, \mathcal{P}_{2,s}), A_{lb}^{\overline{A}}(K_j^E e^{-rT}, \mathcal{P}_{2,s}), A_{lf}(K_j^E e^{-rT}, \mathcal{P}_{2,s})\}$$

has to hold. Note first that having introduced auxiliary constraints on the interval $[K_{r-1}(\mathcal{P}_2^*), K_j^E e^{-rT})$ has no effect on the lower bounds $A_{lb}^{rhs}(K_j^E e^{-rT}, \mathcal{P}_{2,s})$ or $A_{lb}^{\overline{A},r}(K_j^E e^{-rT}, \mathcal{P}_{2,s})$. Hence, the argument given in the base step applies here as well. Further it follows immediately from (3.28) that $\overline{\mathbf{a}}_j \geq A_{lb}^{lhs}(K_j^E e^{-rT}, \mathcal{P}_{2,s})$ holds. Taking into account that $K_{2,s-1}e^{-rT} \in \mathbb{K}_2^{aux}(\mathcal{P}_2^*)$, we readily obtain that $A_{lb}^{\overline{A},l}(K_j^E e^{-rT}, \mathcal{P}_{2,s}) \leq A_{lb}^{lhs}(K_j^E e^{-rT}, \mathcal{P}_{2,s})$ and thus $\overline{\mathbf{a}}_j \geq A_{lb}^{\overline{A},l}(K_j^E e^{-rT}, \mathcal{P}_{2,s})$ has to hold as well.

We are thus left to argue that $\overline{\mathbf{a}}_j \geq A_{lf}(K_j^E e^{-rT}, \mathcal{P}_{2,s})$ has to hold. To see that this is the case let us begin by pointing out that according to the induction hypothesis $K_{j-1}^E e^{-rT} \in (K_{r-1}(\mathcal{P}_2^*), K_r(\mathcal{P}_2^*))$ and $\overline{\mathbf{a}}_{j-1} \geq A_{lf}(K_{j-1}^E e^{-rT}, \mathcal{P}_{2,s})$. Moreover, we will set $\tilde{K} = \max\{K \in \mathbb{K}_2^{aux}(\mathcal{P}_2^*) : K < K_{j-1}^E e^{-rT}\}$. We can then define $K_{2,\tilde{s}} = \max\{K_{r-1}(\mathcal{P}_2^*), \tilde{K}\}$ to conclude that

$$\begin{aligned} cc(A_{lb}^{lhs}; K_{j-1}^E e^{-rT}, K_j^E e^{-rT}; \mathcal{P}_{2,s}) &= cc(A; K_{2,\tilde{s}}, K_{j-1}^E e^{-rT}; \mathcal{P}_{2,s}) \\ &\leq cc(A_{lf}; K_{2,\tilde{s}}, K_{j-1}^E e^{-rT}; \mathcal{P}_{2,\tilde{s}}) \\ &= cc(E; K_{2,\tilde{s}}, K_{j-1}^E e^{-rT}; \mathcal{P}_{2,\tilde{s}}) \\ &= cc(E; K_{r-1}(\mathcal{P}_2^*), K_r(\mathcal{P}_2^*); \mathcal{P}_{2,r-1}) \\ &= cc(A_{lf}; K_{j-1}^E e^{-rT}, K_j^E e^{-rT}; \mathcal{P}_{2,s}) \end{aligned} \tag{3.42}$$

Since the strikes at which auxiliary constraints are introduced are determined using (3.28), it follows that $\overline{\mathbf{a}}_j \geq A_{lb}^{lhs}(K_j^E e^{-rT}, \mathcal{P}_{2,s})$ has to hold. Combined with the inequality in (3.42) we can thus deduce that $\overline{\mathbf{a}}_j \geq A_{lf}(K_j^E e^{-rT}, \mathcal{P}_{2,s})$. We can therefore conclude that the new auxiliary price constraint $(\overline{\mathbf{a}}_j, K_j^E e^{-rT})$ lies within the no-arbitrage bounds given by (3.12) and (3.23). □

Next we will discuss how the price functions will look like for strikes in $[K_r(\mathcal{P}_2^*), K_1^{vc}]$ when the algorithm uses the initial price set $\mathcal{P}_2^*$ and only considers possible violations of the upper bound on $[0, K_r(\mathcal{P}_2^*)]$ while disregarding any such violations on $[K_r(\mathcal{P}_2^*), K_1^{vc}]$.

Proposition 3.7.13. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

Algorithm 2 stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of convexity. Algorithm 3 computes revised prices for non-traded options with strikes $K_{1,s} \in [K_{1,q}, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*)$. At the strike $K_{1,q} \in (K_u^A, K_{u+1}^A) \cap \mathbb{K}^E(\mathcal{P}_0^*)$ it stops and defines the new initial set of prices $\mathcal{P}_2^*$ by $\mathcal{P}_2^* = ((\mathcal{P}_1^*)^A \cup (\mathbf{a}_{1,q}^n, K_{1,q}); (\mathcal{P}_0^*)^E)$. Taking into account possible violations of the upper bound $\overline{A}$ on $[0, K_r(\mathcal{P}_2^*)]$, but disregarding any such violations on $[K_r(\mathcal{P}_2^*), K_1^{vc}]$ the American and European price functions, computed from the initial set $\mathcal{P}_2^*$, for strikes in $[K_{2,r}, K_1^{vc}]$ are given by

$$\mathbf{a}_{2,s} = \begin{cases} A_{lb}^{rhs}(K_{2,s}, \mathcal{P}_2^*), & \text{if } K_{2,s} \in [K_r(\mathcal{P}_2^*), K_u^A) \\ A_{ub}(K_{2,s}, \mathcal{P}_2^*), & \text{if } K_{2,s} \in [K_u^A, K_1^{aux}) \\ \mathbf{a}_{1,s}^n, & \text{if } K_{2,s} \in [K_1^{aux}, K_1^{vc}] \end{cases} \tag{3.43}$$

and

$$\mathbf{e}_{2,s} = \begin{cases} \mathbf{e}_{1,s}, & \text{if } K_{2,s} \in [K_{2,r}, K_1^{aux}] \\ \mathbf{e}_{1,s}^n, & \text{if } K_{2,s} \in [K_1^{aux}, K_1^{vc}]. \end{cases} \tag{3.44}$$

Proof. If we assume that the first violation of the upper bound occurred at the strike $K_{2,s_1}e^{-rT} \in [K_{r-1}(\mathcal{P}_2^*), K_r(\mathcal{P}_2^*)]$, then $K_{2,s_1} \geq K_r(\mathcal{P}_2^*)$ has to hold. Hence, the price for American options with strike $K_r(\mathcal{P}_2^*)$ was already computed and is thus given by $A_{lb}^{rhs}(K_r(\mathcal{P}_2^*), \mathcal{P}_2^*)$ according to Proposition 3.7.8. We note further that the introduction of auxiliary constraints on the prices of American put options has no effect on the prices of European options according to (3.30). It follows that the prices for European options with strikes less than or equal to $K_r(\mathcal{P}_2^*)$ remain unchanged between the two iterations.

We can now discuss the effect that the introduction of these constraints has on the prices of American options with strike $K > K_r(\mathcal{P}_2^*)$. Note first that the right hand-side lower bounds $A_{lb}^{rhs}(K, \mathcal{P}_{2,s})$ and $A_{lb}^{\overline{A},r}(K, \mathcal{P}_{2,s})$ remain unchanged. Similarly, the left hand-side lower bound $A_{lb}^{\overline{A},l}(K, \mathcal{P}_{2,s})$ is unaffected by the new prices.

Let us next investigate the possible effect the increased left hand-side lower bound $A_{lb}^{lhs}(K, \mathcal{P}_{2,s})$ has on the prices of American options with strike $K \in [K_r(\mathcal{P}_2^*), K_u^A]$. According to Proposition 3.7.12 we know that any of the auxiliary price constraints introduced in this iteration of the algorithm lies within its respective no-arbitrage bounds. We can therefore deduce that

$$\overline{\mathbf{a}}_j \geq \max\{A_{lb}^{rhs}(K_j^E e^{-rT}, \mathcal{P}_{2,s}), A_{lb}^{\overline{A},r}(K_j^E e^{-rT}, \mathcal{P}_{2,s})\}$$

for any such constraint $(\overline{\mathbf{a}}_j, K_j^E e^{-rT})$ has to hold. Combined with the fact that the price for American options with strike $K_r(\mathcal{P}_2^*)$ is given by $A_{lb}^{rhs}(K_r(\mathcal{P}_2^*), \mathcal{P}_{2,s})$, we obtain that $A_{lb}^{lhs}(K, \mathcal{P}_{2,s}) \leq A_{lb}^{rhs}(K, \mathcal{P}_{2,s})$ and thus the increased left hand-side lower bound does not effect the prices on $[K_r(\mathcal{P}_2^*), K_u^A]$.

We are left to argue that $A_{lb}^{rhs}(K, \mathcal{P}_{2,s}) \geq A_{lf}(K, \mathcal{P}_{2,s})$ holds for any strike $K \in [K_r(\mathcal{P}_2^*), K_u^A]$. Then again, we know that the Legendre-Fenchel condition has to hold with strict inequality on $[K_r(\mathcal{P}_2^*), K_u^A]$ and therefore we can rule out any change in the prices due to the Legendre-Fenchel condition.

As the prices for American options remain unchanged on $[K_r(\mathcal{P}_2^*), K_u^A]$ it follows that the prices for European options with strike K_u^A are unaffected by the introduction of the auxiliary price constraints on $[K_{r-1}(\mathcal{P}_2^*), K_r(\mathcal{P}_2^*)]$ as well. Moreover, none of the no-arbitrage bounds will contain an auxiliary price constraint and thus we have shown that the price functions must be given by (3.43) and (3.44) whenever we disregard any possible violation of the upper bound on $[K_r(\mathcal{P}_2^*), K_1^{vc}]$. ☐

This allows us to argue in the following result that a violation of the upper bound can be ruled out on $[K_r(\mathcal{P}_2^*), K_1^{vc}]$.

Proposition 3.7.14. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

The algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of convexity. Algorithm 3 computes revised prices for non-traded options with strikes $K_{1,s} \in [K_{1,q}, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*)$. At the strike $K_{1,q} \in (K_u^A, K_{u+1}^A) \cap \mathbb{K}^E(\mathcal{P}_0^*)$ it stops and defines the new initial set of prices $\mathcal{P}_2^*$ by $\mathcal{P}_2^* = ((\mathcal{P}_1^*)^A \cup (\mathbf{a}_{1,q}^n, K_{1,q}); (\mathcal{P}_0^*)^E)$.

If the algorithm is restarted using the initial set $\mathcal{P}_2^*$, then a violation of the upper bound $\overline{A}$ on $[K_r(\mathcal{P}_2^*), K_1^{vc}]$ can be ruled out.

Proof. To see this we consider the scenario where the algorithm constructed $\mathcal{P}_{2,s-1}$ a $K_{2,s-1}$-admissible $\mathcal{P}_0^*$-extension from the initial set $\mathcal{P}_2^*$ and stops at the strike $K_{2,s} \in [0, K_1^{vc}] \cap \mathbb{K}(\mathcal{P}_2^*)$ due to a violation of $\overline{\mathbf{a}}_{2,s} \geq A(K_{2,s}e^{-rT}, \mathcal{P}_{2,s})$, where $K_{2,s}e^{-rT} \in [K_r(\mathcal{P}_2^*), K_1^{vc}]$.

We then have to distinguish between the two cases where either $K_{2,s} \leq K_1^{aux}$ or $K_{2,s} > K_1^{aux}$. Suppose first that $K_{2,s} \leq K_1^{aux}$ and let us moreover assume for the moment that $K_{2,s}e^{-rT} \in [K_{2,\tilde{q}}, K_{2,\tilde{q}+1})$ for $K_{2,\tilde{q}}, K_{2,\tilde{q}+1} \in [K_r(\mathcal{P}_2^*), K_u^A]$. We would then like to apply Proposition 3.10.17 to argue that $K_{j'}^E e^{-rT} < K_u^A$ has to hold as well for

$$K_{j'}^E = \min\{K_l^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K_l^E \geq K_{2,s}\}.$$

To this end, we need to show that $K_{2,\tilde{q}} \notin \mathbb{K}_1^{aux}(\mathcal{P}_2^*)$ and that $[K_{2,s}, K_{j'}^E) \cap \mathbb{K}^{aux}(\mathcal{P}_2^*) = \emptyset$. Since $\mathbb{K}_1^{aux}(\mathcal{P}_2^*) = \{K_1^{vc}\}$ and $K_{2,\tilde{q}} < K_1^{aux} < K_1^{vc}$ we readily obtain that $K_{2,\tilde{q}} \notin$

$\mathbb{K}_1^{aux}(\mathcal{P}_2^*)$. To see that $[K_{2,s}, K_{j'}^E) \cap \mathbb{K}^{aux}(\mathcal{P}_2^*) = \emptyset$ holds we note that according to Proposition 3.10.30 $[K_u^A, K_1^{aux}) \cap \mathbb{K}^{aux}(\mathcal{P}_2^*) = \emptyset$. Combining the definition of $K_{2,w}$ with $K_{2,w} < K_{2,s}$ it follows that $[K_{2,s}, K_{j'}^E) \cap \mathbb{K}^{aux}(\mathcal{P}_2^*) = \emptyset$ and thus we can conclude that $K_{j'}^E e^{-rT} < K_u^A$.

We can further deduce from the fact that $\mathcal{P}_{2,s-1}$ is a $K_{2,s-1}$-admissible $\mathcal{P}_0^*$-extension that $\overline{\mathbf{a}}_{2,s-1} \geq A(K_{2,s-1}e^{-rT}, \mathcal{P}_{2,s})$ has to hold. Taking into account that the prices $\overline{\mathbf{a}}_{2,s-1}$, $\overline{\mathbf{a}}_{2,s}$ and $\overline{\mathbf{a}}_{j'}$ are co-linear according to Proposition 3.10.14 we can conclude from

$$\overline{\mathbf{a}}_{2,s} < \frac{\mathbf{a}_{1,q}^n - \hat{\mathbf{a}}_u}{K_{1,q} - K_u^A}(K_{2,s}e^{-rT} - K_u^A) + \hat{\mathbf{a}}_u$$

that

$$\overline{\mathbf{a}}_{j'} < \frac{\mathbf{a}_{1,q}^n - \hat{\mathbf{a}}_u}{K_{1,q} - K_u^A}(K_{j'}^E e^{-rT} - K_u^A) + \hat{\mathbf{a}}_u$$

has to hold as well. This, however, yields a contradiction to $\mathbf{a}_{1,q}^n \geq A_{lb}^{\overline{A},l}(K_{1,q}, \mathcal{P}_0^*)$ and it follows that we can rule out a violation of the upper bound for $K_{2,s}e^{-rT} \in [K_r(\mathcal{P}_2^*), K_u^A)$.

In the second case where $K_{2,s}e^{-rT} \in [K_u^A, K_1^{aux})$ we use the fact that the European price function is not decreased on $[0, K_1^{vc}]$ between the two iterations of the algorithm to deduce that $\overline{A}(K_{2,s}e^{-rT}, \mathcal{P}_{2,s}) \geq \overline{A}(K_{2,s}e^{-rT}, \mathcal{P}_{1,p})$. In addition, we know that the set of prices $\mathcal{P}_{1,p-1}$ is a $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension. Combined with the fact that $K_{2,s} \leq K_1^{aux} \leq K_{1,p-1}$ we obtain that $\overline{A}(K_{2,s}e^{-rT}, \mathcal{P}_{1,p}) \geq A(K_{2,s}e^{-rT}, \mathcal{P}_{1,p})$ holds. Taking into account that $\mathbf{a}_{1,q}^n < \mathbf{a}_{1,q}$, we further conclude that $A(K_{2,s}e^{-rT}, \mathcal{P}_{1,p}) \geq A(K_{2,s}e^{-rT}, \mathcal{P}_{2,s})$ has to hold and thus

$$\begin{aligned}
\overline{A}(K_{2,s}e^{-rT}, \mathcal{P}_{2,s}) &\geq \overline{A}(K_{2,s}e^{-rT}, \mathcal{P}_{1,p}) \\
&\geq A(K_{2,s}e^{-rT}, \mathcal{P}_{1,p}) \\
&\geq A(K_{2,s}e^{-rT}, \mathcal{P}_{2,s}).
\end{aligned}$$

This allows us to rule out a violation of the upper bound at strike $K_{2,s}e^{-rT}$ whenever $K_{2,s} \leq K_1^{aux}$.

We are left to rule out a violation of the upper bound for $K_{2,s} \in (K_1^{aux}, K_1^{vc}] \cap \mathbb{K}(\mathcal{P}_2^*)$. Since we assumed that the set $\mathcal{P}_{2,s-1}$ is a $K_{2,s-1}$-admissible $\mathcal{P}_0^*$-extension we must have

$$\overline{\mathbf{a}}_{2,s-1} \geq A(K_{2,s-1}e^{-rT}, \mathcal{P}_{2,s}).$$

According to Proposition 3.7.13 the Legendre-Fenchel condition has to hold with equality on $[K_{2,s-1}, K_{2,s}]$ and thus Proposition 3.10.7 yields a contradiction to the assumption that $\overline{\mathbf{a}}_{2,s} < A(K_{2,s}e^{-rT}, \mathcal{P}_{2,s})$. Hence, we have shown that a violation of the upper bound can be ruled out on $[K_r(\mathcal{P}_2^*), K_1^{vc}]$. $\square$

Since we now know how the price functions will look like on $[0, K_1^{vc}]$ after the algorithm restarts using the initial set of prices $\mathcal{P}_2^*$ we can show that the violation of convexity has been corrected successfully.

Proposition 3.7.15. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

The algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of convexity. Algorithm 3 computes revised prices for non-traded options with strikes $K_{1,s} \in [K_{1,q}, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*)$. At the strike $K_{1,q} \in (K_u^A, K_{u+1}^A) \cap \mathbb{K}^E(\mathcal{P}_0^*)$ it stops and defines the new initial set of prices $\mathcal{P}_2^*$ by $\mathcal{P}_2^* = ((\mathcal{P}_1^*)^A \cup (\mathbf{a}_{1,q}^n K_{1,q}); (\mathcal{P}_0^*)^E)$.

Then the algorithm computes a K_1^{vc}-admissible $\mathcal{P}_0^*$-extension from the new initial set $\mathcal{P}_2^*$.

Proof. When the algorithm computes price functions from an initial set two different types of violations may occur during the construction. On the one hand, the prices for American options may exceed the upper bound $\overline{A}$. On the other hand, the prices for either American or European options may violate convexity.

According to Proposition 3.7.10 and Proposition 3.7.14 a violation of the upper bound $\overline{A}$ is only possible on the interval $[K_{r-1}(\mathcal{P}_2^*), K_r(\mathcal{P}_2^*)]$. Proposition 3.7.12, however, guarantees that any such violation will be corrected successfully by Algorithm 4.

We are thus left to rule out a violation of convexity for both the American and European price functions on $[0, K_1^{vc}]$. Let us consider first the European price function. We know from Proposition 3.7.8 and Proposition 3.7.13 that the prices for European options in strikes $[0, K_1^{aux}] \cap \mathbb{K}(\mathcal{P}_2^*)$ coincide with the prices in the previous iteration of the algorithm. On $[K_1^{aux}, K_1^{vc}]$ the European price function will be given by $E^n(\cdot, \mathcal{P}_1^{rev})$ and thus the European price function has to be convex as argued in the proof of Proposition 3.7.8.

In Proposition 3.7.8 we argued that the American price function will be given by (3.38) if possible violations of the upper bound are ignored. It is furthermore shown that this price function is convex up to $K_p(\mathcal{P}_{1,p})$. When Algorithm 4 introduces an auxiliary price constraint to correct a violation of the upper bound on $[K_{r-1}(\mathcal{P}_2^*), K_r(\mathcal{P}_2^*)]$ the chosen price lies within its no-arbitrage bounds according to Proposition 3.7.12. As we have shown in Proposition 3.7.13 that the introduction of these auxiliary price constraints has no effect on the American price function outside of $[K_{r-1}(\mathcal{P}_2^*), K_r(\mathcal{P}_2^*)]$ it follows that the American price function is convex up to $K_p(\mathcal{P}_{1,p})$. Hence, we can conclude that the algorithm will construct a $K_p(\mathcal{P}_{1,p})$-admissible $\mathcal{P}_0^*$-extension from the initial set of prices $\mathcal{P}_2^*$. ☐

3.7.3 Violations under $\mathcal{P}_i^*$, $i \geq 2$

The major difference between an extended initial set of prices $\mathcal{P}_i^*$, $i \geq 2$, and $\mathcal{P}_1^*$ is the existence of auxiliary price constraints of type 1 that have to be accounted for during the construction of the price functions using $\mathcal{P}_i^*$. Subsequently, we will discuss the results required to argue that the algorithm can be applied to the extended initial set $\mathcal{P}_i^*$. Moreover, we provide reasons why they should hold and highlight situations they are required in. Note, however, that we are not able to give rigorous proofs here.

Before we start to discuss the situations in which the algorithm stops the construction of the price functions let us point out some structural properties we will be using

- $K_l^{aux} < K_k^{aux}$ for $l < k$ and $K_l^{aux}, K_k^{aux} \in \mathbb{K}_1^{aux}(\mathcal{P}_i^*)$.
- The Legendre-Fenchel condition holds with equality on $[K_{i-1}^{aux}, K_{i-1}^{vc}]$ under $\mathcal{P}_i^*$.
- If $K_{i-1}^{aux} \in (K_u^A, K_{u+1}^A)$, then $\max\{K \in \mathbb{K}_2^{aux}(\mathcal{P}_{i-1}^*)\} \leq K_u^A$.

Suppose now that the algorithm computed a K_{i-1}^{vc} admissible $\mathcal{P}_0^*$-extension using $\mathcal{P}_i^*$. We then have to discuss the different situations in which the algorithm is forced to stop the construction of the price functions in a strike strictly larger than K_{i-1}^{vc}.

Consider first the situation where a violation of the upper bound occurs. We could then argue as follows:

- A violation of the upper bound can be ruled out on $[K_{i-1}^{vc}, K_{i-1}^{vc}e^{rT}]$.

  This would allow us to conclude that a violation of the upper bound to the right of K_{i-1}^{vc} has no effect on the already computed prices up to K_{i-1}^{vc}.
  Suppose that the algorithm stopped at the strike $K_{i,p} \in [K_{i-1}^{vc}, K_{i-1}^{vc}e^{rT}]$ and that $K_{i,p}e^{-rT} \in (K_{i,q}, K_{i,q+1})$. If we assume that the strike K_{i-1}^{vc} corresponds to $K_{i,s}$ and taking into account the convexity of the price functions we obtain

$$\begin{aligned} cc(A; K_{i,q}, K_{i,q+1}; \mathcal{P}_{i,p}) &\geq cc(A; K_{i,s-1}, K_{i,s,}; \mathcal{P}_{i,p}) \\ &= cc(E; K_{i,s-1}, K_{i,s}; \mathcal{P}_{i,p}) \\ &\geq cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) \\ &= cc(\overline{A}; K_{i,p-1}e^{-rT}, K_{i,p}e^{-rT}; \mathcal{P}_{i,p}). \end{aligned}$$

- If $[K_j^A, K_{j+1}^A] \cap \mathbb{K}_1^{aux}(\mathcal{P}_i^*) \neq \emptyset$ a violation of the upper bound on that interval can be ruled out.

This result could be used to generalise the propositions in Section 3.10.5.

We believe that this result holds as the auxiliary constraints of type 1 are increasing and the algorithm stopped to the right of $K^{vc}_{i-1}e^{rT}$. We then only need to consider the case where a violation of the upper bound occurs at a strike in $[K^{aux}_{i-1}, K^A_{j+1}]$. We further know that the Legendre-Fenchel condition has to hold with equality on $[K^{aux}_{i-1}, K^A_{j+1}]$ which contradicts a violation of the upper bound $\overline{A}$ on that interval.

We should then be able to apply the results in Section 3.10.5 to show that a violation of the upper bound can be corrected using Algorithm 4.

Suppose now that a violation of convexity occurs at the strike $K_{i,p} \in \mathbb{K}(\mathcal{P}^*_i)$. Moreover, we assume that $K^{aux}_{i-1} \in (K^A_u, K^A_{u+1})$ and $K^{vc}_{i-1} \in (K^A_v, K^A_{v+1})$ for $v \geq u$. We would then like to argue as follows:

- If Algorithm 3 introduces an auxiliary price constraint K^{aux}_i, then either $K^{aux}_i \in (K^A_u, K^{aux}_{i-1}]$ or $K^{aux}_i > K_{v+2}$.

  The Legendre-Fenchel condition has to hold with equality on $[K^{aux}_{i-1}, K^A_{v+1}]$. Hence Algorithm 3 will not stop revising option prices prior to K^{aux}_{i-1} if it did not stop prior to K^A_{v+1}. If we suppose that the algorithm stopped revising option prices at a strike $K_{i,q} \in (K^A_{v+1}, K^A_{v+2})$, then the price for American options with strike $K_{i,q}$ has to be given by $\mathbf{a}_{i,q} = A^{lhs}_{lb}(K_{i,q}, \mathcal{P}^A_{v+2})$. Since we know that the Legendre-Fenchel condition holds with equality on $[K^{aux}_{i-1}, K^A_{v+1}]$ it follows that it also has to hold with equality on $[K^A_{v+1}, K_{i,q}]$ as $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}^*_0$-extension. We can thus rule out that the algorithm would have stopped at the strike $K_{i,q} \in (K^A_{v+1}, K^A_{v+2})$.

- If $K^{aux}_{i-1} \in (K^A_u, K^A_{u+1})$, then $K^{aux}_l < K^A_u$ for any $K^{aux}_l \in \mathbb{K}^{aux}_1(\mathcal{P}^*_i)\backslash\{K^{aux}_{i-1}\}$,

- If $K^{aux}_i \in (K^A_u, K^{aux}_{i-1}]$, then $[K^A_u, \infty) \cap \mathbb{K}^{aux}_2(\mathcal{P}^*_i) = \emptyset$.

  We know that the Legendre-Fenchel condition holds with equality on $[K^{aux}_i, K^{vc}_i]$. Hence, a violation of the upper bound can be ruled out on that interval. In addition, we can exclude a violation of the upper bound for a strike larger than K^{vc}_i from consideration as the algorithm has never computed prices for non-traded options there before.

- The revised prices satisfy $\mathbf{a}^n_{i,s} < \mathbf{a}_{i,s}$ for $K_{i,s} \in \mathbb{K}^E(\mathcal{P}^*_i)\backslash\mathbb{K}^A(\mathcal{P}^*_i)$.

- If $K_i^{aux} \in (K_u^A, K_{i-1}^{aux}]$ the right hand-side lower bound on the prices of American options with strikes in $(K_{i,w}, K_u^A)$, where $K_{i,w} = \max\{K \in \mathbb{K}(\mathcal{P}_i^*) : K < K_u^A\}$, is increased between the iterations.

  This follows from $\mathbf{a}_{i,q}^n < \mathbf{a}_{i,q}$.

  We then reduced this situation to the situation in the previous iteration in which a violation of convexity was corrected successfully.

- If $K_{i,q} > K_{v+2}^A$ the situation is the same as under $\mathcal{P}_1^*$ as there exists at least one American interval between the new price constraint and any auxiliary price constraint of type 1.

Despite the fact that we are not able to give rigorous proof we hope that the arguments provided persuade the reader that these results are meaningful. In the absence of concrete proofs we are, however, only able to state the following results as conjectures.

Conjecture 3.7.16. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Assume further that the algorithm extended the initial set of prices from $\mathcal{P}_0^*$ to $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 0$. Starting with the initial set of prices $\mathcal{P}_i^*$ the algorithm computes the $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{i,p-1}$ for $p \geq 1$.*

Suppose further that the algorithm stops at the strike $K_{i,p} \in \mathbb{K}(\mathcal{P}_i^*)$ due to a violation of $\overline{\mathbf{a}}_{i,p} \geq A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p})$ where $K_{i,p}e^{-rT} \in (K_{i,q}, K_{i,q+1}]$ for $K_{i,q}, K_{i,q+1} \in \mathbb{K}(\mathcal{P}_i^*)$. If we assume that Algorithm 4 extended the price set $\mathcal{P}_{i,p}$ to

$$\mathcal{P}_{i,p+1} = ((\mathcal{P}_{i,p})^A \cup (\overline{\mathbf{a}}_j, K_j^E e^{-rT}); (\mathcal{P}_{i,p})^E \cup (E_{ub}(K_j^E e^{-rT}, \mathcal{P}_{i,p}), K_j^E e^{-rT}))$$

in order to correct the violation and that $K_{i,s} = K_j^E$, then $\mathcal{P}_{i,s}$ has to be a $K_{i,s}$-admissible $\mathcal{P}_0^*$-extension.

Conjecture 3.7.17. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Assume further that the algorithm extended the initial set of prices from $\mathcal{P}_0^*$ to $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 0$. Starting with the initial set of prices $\mathcal{P}_i^*$ the algorithm computes the $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{i,p-1}$ for $p \geq 1$.*

The algorithm stops at the strike $K_{i,p} \in \mathbb{K}(\mathcal{P}_i^*)$ due to a violation of convexity. Algorithm 3 computes revised prices for non-traded options with strikes $K_{i,s} \in [K_{i,q}, K_{i,p}] \cap \mathbb{K}(\mathcal{P}_i^*)$. At the strike $K_{i,q} \in (K_u^A, K_{u+1}^A) \cap \mathbb{K}^E(\mathcal{P}_0^*)$ it stops and defines the new initial set of prices $\mathcal{P}_{i+1}^*$ by $\mathcal{P}_{i+1}^* = ((\mathcal{P}_i^*)^A \cup (\mathbf{a}_{i,q}^n K_{i,q}); (\mathcal{P}_0^*)^E)$.

Then the algorithm computes a K_i^{vc}-admissible $\mathcal{P}_0^*$-extension from the new initial set $\mathcal{P}_{i+1}^*$.

3.8 Convergence of the algorithm

In this section we first argue that the algorithm given in Section 3.5 terminates in finitely many steps irrespective of its success in constructing admissible price functions. Subsequently, we will finally be able to show that given an initial set of prices $\mathcal{P}_0^*$ the algorithm either constructs American and European price functions satisfying the no-arbitrage conditions of Lemma 3.1.1 and Theorem 3.1.2 or provides an arbitrage portfolio.

Proposition 3.8.1. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are given by $\mathcal{P}_0^* \in \mathcal{M}$. Assuming that Conjecture 3.7.16 and Conjecture 3.7.17 hold the algorithm given in Section 3.5 will terminate in finitely many steps.*

Proof. Let us begin by assuming that the number of strikes in $\mathbb{K}^A(\mathcal{P}_0^*)$ and $\mathbb{K}^E(\mathcal{P}_0^*)$ are given by m_1 and m_2, respectively. It follows that the number of strikes in $\mathbb{K}(\mathcal{P}_0^*)$ can be at most $m_1 + m_2$.

To see that the algorithm terminates in finitely many steps we first argue that in each iteration i the number of strikes in $\mathbb{K}(\mathcal{P}_i^*)$ is bounded by $m_1 + 2m_2$. Subsequently, we show that the sequence of strikes $(K_j^{vc})_j$, at which the algorithm is restarted, is strictly increasing in j. Combining the two results then yields that the algorithm stops after finitely many steps.

In order to show that there are at most $m_1 + 2m_2$ strikes in $\mathbb{K}(\mathcal{P}_i^*)$ we need to discuss when and how the auxiliary price constraints are introduced. There are only two reasons for the algorithm to introduce an additional constraint. On the one hand either one of the price functions may violate convexity. On the other hand it is possible that the American price function violates its upper bound $\overline{A}$. Consider first the case where one of the price functions violates convexity. The algorithm then computes a constraint for the American price function at a strike in $\mathbb{K}^E(\mathcal{P}_0^*)$. We can therefore conclude that correcting a violation of convexity has no effect on the number of strikes in $\mathbb{K}(\mathcal{P}_i^*)$.

If the American price function violates the upper bound $\overline{A}$ the algorithm introduces an auxiliary price constraint at a strike of type $\mathbb{K}^E(\mathcal{P}_0^*)e^{-rT}$. Since the number of strikes in $\mathbb{K}^E(\mathcal{P}_0^*)$ is given by m_2, we readily obtain that the algorithm introduces at most m_2 constraints at strikes not included in $\mathbb{K}(\mathcal{P}_0^*)$. This implies that the number of strikes in $\mathbb{K}(\mathcal{P}_i^*)$ in each iteration i has to be bounded by $m_1 + 2m_2$.

Suppose now that the algorithm extended the initial set of prices from $\mathcal{P}_0^*$ to $\mathcal{P}_i^*$ and that a violation of convexity occurs at the strike K_i^{vc}. The algorithm then extends the initial set $\mathcal{P}_i^*$ by an auxiliary price constraint at K_i^{aux} and restarts. Assuming that Conjecture 3.7.17 holds, we know that the algorithm constructs a K_i^{vc} admissible $\mathcal{P}_0^*$-extension from the new initial set $\mathcal{P}_{i+1}^*$. It follows that the next violation of convexity has to occur at a strike $K_{i+1}^{vc} > K_i^{vc}$ and thus the sequence $(K_j^{vc})_j$ has to be strictly increasing in j. We can therefore conclude that the algorithm will terminate after finitely many steps. □

Finally, we are in a position to show that the algorithm in Section 3.5 can be used to determine whether or not a given set of American and co-terminal European put options allows for model-independent arbitrage.

Theorem 3.8.2. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are given by $\mathcal{P}_0^* \in \mathcal{M}$. Assuming that Conjecture 3.7.16 and Conjecture 3.7.17 hold the algorithm provided in Section 3.5 will either construct American and European price functions satisfying the no-arbitrage conditions of Lemma 3.1.1 and Theorem 3.1.2 or there exists arbitrage in the market.*

Proof. According to Proposition 3.6.9 there has to exist arbitrage in the market if $\mathcal{P}_0^* \in \mathcal{M}\backslash\overline{\mathcal{M}}$. Hence, it suffices to subsequently consider only price sets $\mathcal{P}_0^*$ with $\mathcal{P}_0^* \in \overline{\mathcal{M}}$.

We begin by pointing out that the only possible violations of the no-arbitrage conditions are either a violation of the upper bound $\overline{A}$ by the American price function or a violation of convexity by either one of the two price functions. This is due to the fact that the algorithm computes the prices for non-traded options using (3.22) and (3.23). Under the assumption that Conjecture 3.7.16 holds a violation of the upper bound can always be corrected by Algorithm 4. Assuming further that Conjecture 3.7.17 holds, a violation of convexity can either be corrected by introducing an auxiliary price constraint or Algorithm 3 stops revising option prices prematurely at a strike $K_{i,q} \in \mathbb{K}(\mathcal{P}_i^*)$ due to either

$$\mathbf{a}_{i,q}^n < \max\{A_{lb}(K_{i,q}, \mathcal{P}_0^*), A_{lb}^{\overline{A}}(K_{i,q}, \mathcal{P}_0^*), A_{lb}^{t_1}(K_{i,q}, \mathcal{P}_i^*)\}$$

or $\mathbf{e}_{i,q}^n > E_{ub}(K_{i,q}, \mathcal{P}_0^*)$. In the latter case there has to exist arbitrage in the market and depending on the violation of convexity the arbitrage portfolio can be found in either Section 3.6.3 or Section 3.6.4. Hence, we can conclude that the algorithm either computes a $K_{m_2}^E$-admissible $\mathcal{P}_0^*$-extension or there has to exist arbitrage in the market.

We are thus only left to argue that the price functions constructed by the algorithm satisfy the no-arbitrage conditions on the entire positive half-line and not only up to $K_{m_2}^E$. To this end, let us assume that the algorithm reached the strike $K_{m_2}^E$ in the i-th iteration and that the final price set is given by $\mathcal{P}_{i,n}$ for $i, n \geq 1$. Observe also that this readily implies that the strike $K_{m_2}^E$ corresponds to $K_{i,n}$ under $\mathcal{P}_{i,n}$.

Let us begin by arguing that the European price function $E(\cdot, \mathcal{P}_{i,n})$ is convex. According to the definition of E in (3.7) we know that $E(K, \mathcal{P}_{i,n}) = e^{-rT}K - S_0$ for any strike $K \geq K_{m_2}^E$. It follows that for any strike $K \geq K_{m_2}^E$ the right-hand side derivative of the European price function is given by $E'(K+, \mathcal{P}_{i,n}) = e^{-rT}$. Since we assumed that $\mathcal{P}_{i,n}$ is a $K_{i,n}$-admissible $\mathcal{P}_0^*$-extension we further know that the price for a European option with strike $K_{i,s} \in \mathbb{K}(\mathcal{P}_{i,n})$ has to satisfy $\mathbf{e}_{i,s} \geq e^{-rT} K_{i,s} - S_0$. Combined with the fact that $\hat{\mathbf{e}}_{m_2} = e^{-rT} K_{m_2}^E - S_0$ we can conclude that $E'(K_{m_2}^E -, \mathcal{P}_{i,n}) \leq e^{-rT}$ has to hold. Hence, the European price function $E(\cdot, \mathcal{P}_{i,n})$ has to be convex on the entire positive half-line.

Next we will show that the European price function $E(\cdot, \mathcal{P}_{i,n})$ also has to be increasing. Note first that the prices for European options with strikes in $\mathbb{K}^E(\mathcal{P}_{i,n})$ are non-negative. To see this observe that $E_{lb}^{lhs}(K, \mathcal{P}_{i,n}) \geq 0$ for any strike $K \geq 0$. In addition, Proposition 3.10.3 guarantees that the price for a European option with strike $K_{i,l} \in \mathbb{K}(\mathcal{P}_i^*)$, $\mathbf{e}_{i,l}$, exceeds $E_{lb}^{lhs}(K_{i,l}, \mathcal{P}_{i,n})$ and thus $\mathbf{e}_{i,l} \geq 0$ has to hold. Recall further that the price for a European option with strike 0 is given by 0. It thus follows that $E'(0+, \mathcal{P}_{i,n}) \geq 0$. Combined with the convexity of $E(\cdot, \mathcal{P}_{i,n})$ we obtain that $E'(K+, \mathcal{P}_{i,n}) \geq 0$ for any strike $K \geq 0$. This readily implies that the price function $E(\cdot, \mathcal{P}_{i,n})$ is increasing as well.

We proceed by showing that the American price function $A(\cdot, \mathcal{P}_{i,n})$ is convex. As $\mathcal{P}_{i,n}$ is a $K_{i,n}$-admissible $\mathcal{P}_0^*$-extension, we already know that $A(\cdot, \mathcal{P}_{i,n})$ is convex up to $K_{i,n}$ and that $\mathbf{a}_{i,s} \geq K_{i,s} - S_0$ for any strike $K_{i,s} \in \mathbb{K}(\mathcal{P}_{i,n})$. In addition, we argue in Proposition 3.10.45 that $\mathbf{a}_{i,s} = K_{i,s} - S_0$ for any strike $K_{i,s} \in (K_{m_2}^E e^{-rT}, \infty) \cap \mathbb{K}(\mathcal{P}_{i,n})$. It follows that $A'(K_{m_2}^E -, \mathcal{P}_{i,n}) \leq A'(K_{m_2}^E +, \mathcal{P}_{i,n})$ has to hold. The American price function $A(\cdot, \mathcal{P}_{i,n})$ thus has to be convex as $A'(K+, \mathcal{P}_{i,n}) = A'(K_{m_2}^E -, \mathcal{P}_{i,n})$ for any strike $K \geq K_{m_2}^E$.

Analogously to the European price function we can use non-negativity of the prices for American put options in combination with the convexity of $A(\cdot, \mathcal{P}_{i,n})$ to obtain that the American price function $A(\cdot, \mathcal{P}_{i,n})$ has to be increasing.

To see that the American price function $A(\cdot, \mathcal{P}_{i,n})$ remains below its upper bound $\overline{A}(\cdot, \mathcal{P}_{i,n})$ we will argue that $\overline{A}(K, \mathcal{P}_{i,n}) = A(K, \mathcal{P}_{i,n})$ has to hold for any strike $K \geq K_{m_2}^E e^{-rT}$. In particular, we will show that both price functions will coincide with the immediate exercise line. To this end, recall that we assumed that $\hat{\mathbf{e}}_{m_2} = e^{-rT} K_{m_2}^E - S_0$ and thus $\overline{\mathbf{a}}_{m_2} = e^{-rT} K_{m_2}^E - S_0$ has to hold as well. Since we assumed that $\mathcal{P}_{i,n}$ is a $K_{i,n}$-admissible $\mathcal{P}_0^*$-extension we further know that

$$K_{m_2}^E e^{-rT} - S_0 \leq A(K_{m_2}^E e^{-rT}, \mathcal{P}_{i,n}) \leq \overline{\mathbf{a}}_{m_2} = e^{-rT} K_{m_2}^E - S_0.$$

It then follows that $A(K_{m_2}^E e^{-rT}, \mathcal{P}_{i,n}) = K_{m_2}^E e^{-rT} - S_0$ has to hold. Combined with the result in Proposition 3.10.45 we see that $A(K, \mathcal{P}_{i,n}) = K - S_0$ for any strike $K \geq K_{m_2}^E e^{-rT} - S_0$. Taking into account that the European price function is extended beyond $K_{m_2}^E$ using $e^{-rT} K - S_0$ we obtain by the definition of the upper bound $\overline{A}$ that $\overline{A}(K, \mathcal{P}_{i,n}) = K - S_0$ for $K \geq K_{m_2}^E e^{-rT} - S_0$. Hence, the American price function A has to satisfy $\overline{A}(\cdot,\mathcal{P}_{i,n}) \geq A(\cdot,\mathcal{P}_{i,n})$. Moreover, we can conclude that $A(K,\mathcal{P}_{i,n}) \geq K - S_0$ has to hold for any strike $K \geq 0$.

Next we will demonstrate that the price functions $A(\cdot,\mathcal{P}_{i,n})$ and $E(\cdot,\mathcal{P}_{i,n})$ satisfy the Legendre-Fenchel condition. Since $A(K,\mathcal{P}_{i,n}) = \overline{A}(K,\mathcal{P}_{i,n})$ for $K \geq K_{i,n}e^{-rT}$ we must have $cc(A;K_{i,n}e^{-rT},\hat{K};\mathcal{P}_{i,n}) = cc(\overline{A};K_{i,n}e^{-rT},\hat{K};\mathcal{P}_{i,n})$ for any strike $\hat{K} > K_{i,n}e^{-rT}$. According to Proposition 3.10.4 we thus know that

$$cc(A;K_{i,n}e^{-rT},\hat{K};\mathcal{P}_{i,n}) = cc(E;K_{i,n},\hat{K}e^{rT};\mathcal{P}_{i,n})$$

holds. Taking into account that the European price function $E(\cdot;\mathcal{P}_{i,n})$ is convex we further know that $cc(E;K_{i,n}e^{-rT},\hat{K};\mathcal{P}_{i,n}) \geq cc(E;K_{i,n},\hat{K}e^{rT};\mathcal{P}_{i,n})$ has to hold. Combining these inequalities we obtain that $cc(A;K_{i,n}e^{-rT},\hat{K};\mathcal{P}_{i,n}) \leq cc(E;K_{i,n}e^{-rT},\hat{K};\mathcal{P}_{i,n})$ holds. Hence, the Legendre-Fenchel condition is satisfied on the entire positive half-line.

Finally, we still have to argue that the price for an American option with strike $K \geq 0$ exceeds the price for a co-terminal European option with the same strike. Note first that the Legendre-Fenchel condition implies that $\mathbf{a}_{i,s} \geq \mathbf{e}_{i,s}$ whenever $\mathbf{a}_{i,s-1} \geq \mathbf{e}_{i,s-1}$ and $s \geq 2$. In the case where $s = 1$ we can, moreover, see from the generalisation of the Legendre-Fenchel condition directly below (3.16) and (3.17), respectively, that $\mathbf{a}_{i,1} \geq \mathbf{e}_{i,1}$ has to hold. Hence the price functions have to satisfy $A(\cdot,\mathcal{P}_{i,n}) \geq E(\cdot,\mathcal{P}_{i,n})$.

We have therefore shown that given a set of prices $\mathcal{P}_0^* \in \mathcal{M}$ the algorithm either constructs American and European price functions satisfying the no-arbitrage conditions of Lemma 3.1.1 and Theorem 3.1.2 or there has to exist arbitrage in the market. □

3.9 Conclusion

Assuming that the conjectures in Section 3.7.3 hold we have shown that given a finite sets of American and European put option prices provided by $\mathcal{P}_0^* \in \mathcal{M}$ it is always possible to either construct American and European put price functions or there exists arbitrage.

We believe that this result should be of interest to market makers and speculators alike, as the arbitrage portfolios given in this paper will hold under any model. This is due to the fact that these portfolios are derived without making any assumptions on the underlying probability space generating the option prices. Furthermore, we would like to point out that the portfolios generating arbitrage are altogether semi-static — this means that the positions in the American and European options are fixed at the initial time and there are only finitely many trades in the underlying up to maturity. This is relevant, because semi-static portfolios generally exhibit smaller transaction costs than portfolios using delta hedging, where trading at infinitely many times is required.

3.10 Appendix

3.10.1 Properties of the Legendre-Fenchel condition

Lemma 3.10.1. *Suppose American and European options with maturity T are traded in the market at strikes $K_i^A, K_{i'}^A$ and $K_j^E, K_{j'}^E$, respectively, where $K_j^E \leq K_{j'}^E$, $K_i^A \leq K_{i'}^A$, $K_j^E \leq K_i^A$ and $K_{j'}^E \leq K_{i'}^A$. Then there exists model-independent arbitrage in the market if the extended Legendre-Fenchel condition*

$$\frac{\mathbf{a}_{i'} - \mathbf{a}_i}{K_{i'}^A - K_i^A} K_i^A - \mathbf{a}_i \geq \frac{\mathbf{e}_{j'} - \mathbf{e}_j}{K_{j'}^E - K_j^E} K_j^E - \mathbf{e}_j \tag{3.45}$$

is violated.

Proof. We will argue that there exists arbitrage whenever the extended Legendre-Fenchel condition is violated between the strikes $K_i^A, K_{i'}^A$, K_j^E and $K_{j'}^E$, in the case that $K_j^E \leq K_i^A \leq K_{j'}^E \leq K_{i'}^A$. Analogously, the situation where $K_j^E < K_{j'}^E \leq K_i^A < K_{i'}^A$ can be handled and is thus omitted.

To see that there exists arbitrage we have to find a portfolio that has negative initial value and only positive subsequent cashflows. The portfolio $P_{arb}^{LF}(K_j^E, K_i^A, K_{j'}^E, K_{i'}^A)$ that we want to consider consists of long positions of $K_i^A(K_{i'}^A - K_i^A)^{-1}$ units in an American option with strike $K_{i'}^A$ and $K_{j'}^E(K_{j'}^E - K_j^E)^{-1}$ units in a European option with strike K_j^E and short positions of $K_{i'}^A(K_{i'}^A - K_i^A)^{-1}$ units in an American option with strike K_i^A and $K_j^E(K_{j'}^E - K_j^E)^{-1}$ units in a European option with strike $K_{j'}^E$. If the condition (3.45) is violated this portfolio clearly has a strictly negative initial value and we are left to check that whatever happens to the price of the underlying up to T results only in positive cashflows. First we investigate what happens if the American options are not exercised before maturity T. In this case the payoff of the American options corresponds to the payoff of European options with the same strikes. Denoting $\Delta^A = K_{i'}^A - K_i^A$ and $\Delta^E = K_{j'}^E - K_j^E$ we obtain the following payoffs at maturity

$$\begin{cases} 0 & , S_T \geq K_{i'}^A \\ K_i^A(\Delta^A)^{-1}(K_{i'}^A - S_T) & , S_T \in [K_{j'}^E, K_{i'}^A] \\ K_i^A(\Delta^A)^{-1}(K_{i'}^A - S_T) + K_j^E(\Delta^E)^{-1}(S_T - K_{j'}^E) & , S_T \in [K_i^A, K_{j'}^E] \\ K_{j'}^E(\Delta^E)^{-1}(S_T - K_j^E) & , S_T \in [K_j^E, K_i^A] \\ 0 & , S_T \in [0, K_j^E] \end{cases}$$

which are all clearly positive.

The other possibility is that the shorted American is exercised strictly before maturity T. We then exercise the long American at the same time and hold the asset S

obtained this way until maturity to receive the following payoffs

$$\begin{cases} S_T & , S_T \geq K_{j'}^E \\ K_{j'}^E(\Delta^E)^{-1}(S_T - K_j^E) & , S_T \in [K_j^E, K_{j'}^E] \\ 0 & , S_T \in [0, K_j^E] \end{cases}$$

which are again all positive. We can therefore conclude that the condition in (3.45) is necessary for the absence of model-independent arbitrage. □

An important property of the Legendre-Fenchel condition is its transitivity over adjacent intervals.

Proposition 3.10.2. *Suppose the prices for American and European put options with strikes $K_1, ..., K_n$ are given by $\mathbf{a}_1, ..., \mathbf{a}_n$ and $\mathbf{e}_1, ..., \mathbf{e}_n$, respectively. Furthermore, the Legendre-Fenchel condition is satisfied with equality between any two adjacent strikes $K_i < K_{i+1}$ for $i \in \{1, ..., n-1\}$, i.e*

$$\frac{\mathbf{a}_{i+1} - \mathbf{a}_i}{K_{i+1} - K_i} K_i - \mathbf{a}_i = \frac{\mathbf{e}_{i+1} - \mathbf{e}_i}{K_{i+1} - K_i} K_i - \mathbf{e}_i. \tag{3.46}$$

Then it follows that the Legendre-Fenchel condition is also satisfied with equality between the prices $\mathbf{a}_q$, $\mathbf{a}_p$, $\mathbf{e}_q$ and $\mathbf{e}_p$ for $p, q \in \{1, ..., n\}$ with $q < p$.

Proof. To see that this result holds we will use induction on the number of strikes between K_q and K_p at which option prices are given. In the base step we assume that $p = q + 1$. The result then follows immediately from the assumption that the Legendre-Fenchel condition holds between any two adjacent strikes.

In the inductive step we suppose that $p > q + 1$ and that we know already that

$$\frac{\mathbf{a}_{p-1} - \mathbf{a}_q}{K_{p-1} - K_q} K_q - \mathbf{a}_q = \frac{\mathbf{e}_{p-1} - \mathbf{e}_q}{K_{p-1} - K_q} K_q - \mathbf{e}_q.$$

Note further that the condition in (3.46) is equivalent to writing

$$\mathbf{a}_{i+1} = \mathbf{e}_{i+1} + \frac{K_{i+1}}{K_i}[\mathbf{a}_i - \mathbf{e}_i].$$

It is then sufficient to show that $\mathbf{a}_p = \mathbf{e}_p + \frac{K_p}{K_q}[\mathbf{a}_q - \mathbf{e}_q]$. Since the Legendre-Fenchel condition holds with equality on $[K_{p-1}, K_p]$ we know that

$$\mathbf{a}_p = \mathbf{e}_p + \frac{K_p}{K_{p-1}}[\mathbf{a}_{p-1} - \mathbf{e}_{p-1}]. \tag{3.47}$$

In addition, we can use the induction hypothesis to write

$$\mathbf{a}_{p-1} = \mathbf{e}_{p-1} + \frac{K_{p-1}}{K_q}[\mathbf{a}_q - \mathbf{e}_q]. \tag{3.48}$$

Substituting $\mathbf{a}_{p-1}$ in (3.47) by (3.48), we obtain

$$\mathbf{a}_p = \mathbf{e}_p + \frac{K_p}{K_q}[\mathbf{a}_q - \mathbf{e}_q].$$

This readily implies that the Legendre-Fenchel condition has to hold with equality on the interval $[K_q, K_p]$. □

3.10.2 General properties of the price functions

Proposition 3.10.3. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Assume further that the algorithm extended the initial set of prices from $\mathcal{P}_0^*$ to $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 1$. Starting with the initial set of prices $\mathcal{P}_i^*$ the algorithm computes the $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{i,p-1}$ for $p \geq 1$.*

If the algorithm stops at the strike $K_{i,p} \in \mathbb{K}(\mathcal{P}_i^*)\backslash\mathbb{K}^E(\mathcal{P}_0^*)$ due to a violation of $\mathbf{e}_{i,p} \geq E_{lb}(K_{i,p}, \mathcal{P}_{i,p-1})$, then we must have $E_{lb}(K_{i,p}, \mathcal{P}_{i,p-1}) = E_{lb}^{rhs}(K_{i,p}, \mathcal{P}_0^*)$.

Proof. Let us assume for contradiction that $\mathbf{e}_{i,p} < E_{lb}^{lhs}(K_{i,p}, \mathcal{P}_{i,p-1})$. In the case where $p = 1$, the left hand-side lower bound is given by $E_{lb}^{lhs}(K_{i,p}, \mathcal{P}_{i,p-1}) = 0$. For $\mathbf{e}_{i,p} < 0$ we must then have that either $E_{lf}(K_{i,p}, \mathcal{P}_{i,p-1})$ or $E_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})$ is negative. Note further that $\mathbf{e}_{i,p} = E_{lf}(K_{i,p}, \mathcal{P}_{i,p-1})$ implies that $\mathbf{e}_{i,p} = \mathbf{a}_{i,p}$ and thus that American options with strike $K_{i,p}$ are traded at a negative price in the market. This, however, can be ruled out as $\mathcal{P}_0^* \in \overline{\mathcal{M}}$.

Let us now consider the case that $p \geq 2$. We will then show that the situation $\mathbf{e}_{i,p} < E_{lb}^{lhs}(K_{i,p}, \mathcal{P}_{i,p-1})$ cannot occur as the algorithm would have stopped prior to $K_{i,p}$ already. To this end we distinguish between the cases where $\mathbf{e}_{i,p}$ is either given by $E_{lf}(K_{i,p}, \mathcal{P}_{i,p-1})$ or $E_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})$. We start with the situation where $\mathbf{e}_{i,p} = E_{lf}(K_{i,p}, \mathcal{P}_{i,p-1})$. Since $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension we can conclude that $cc(A; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) \leq cc(A; K_{i,p-2}, K_{i,p-1}; \mathcal{P}_{i,p})$ has to hold. Also $\mathbf{e}_{i,p} = E_{lf}(K_{i,p}, \mathcal{P}_{i,p-1})$ means that the Legendre-Fenchel conditions holds with equality between $K_{i,p-1}$ and $K_{i,p}$ and therefore we must have $cc(A; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) = cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p})$. The assumption that $\mathbf{e}_{i,p} < E_{lb}^{lhs}(K_{i,p}, \mathcal{P}_{i,p-1})$, furthermore, implies that $cc(E; K_{i,p-2}, K_{i,p-1}; \mathcal{P}_{i,p}) < cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p})$. Combined we obtain

$$cc(A; K_{i,p-2}, K_{i,p-1}; \mathcal{P}_{i,p}) > cc(E; K_{i,p-2}, K_{i,p-1}; \mathcal{P}_{i,p}),$$

a contradiction to the fact that $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension.

Suppose now that we are in the situation where $\mathbf{e}_{i,p} = E_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})$, then $E_{ub}(K_{i,p}, \mathcal{P}_{i,p-1}) < E_{lb}^{lhs}(K_{i,p}, \mathcal{P}_{i,p-1})$ immediately implies that

$$\hat{\mathbf{e}}_j < \frac{\mathbf{e}_{i,p-1} - \mathbf{e}_{i,p-2}}{K_{i,p-1} - K_{i,p-2}}(K_j^E - K_{i,p-1}) + \mathbf{e}_{p-1} \tag{3.49}$$

for $j = \arg\min\{K_s \in \mathbb{K}^E(\mathcal{P}_0^*) : K_s > K_{i,p}\}$. We then have to distinguish between the cases when European options with strike $K_{i,p-1}$ are traded in the market or not. If $K_{i,p-1} \in \mathbb{K}^E(\mathcal{P}_0^*)$, then this implies that $\mathbf{e}_{i,p-2} < E_{lb}^{rhs}(K_{i,p-2}, \mathcal{P}_0^*)$ yielding a contradiction to the fact that $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension. In the case where $K_{i,p-1} \notin \mathbb{K}^E(\mathcal{P}_0^*)$ the price $\mathbf{e}_{i,p-1}$ would have been determined by $\mathbf{e}_{i,p-1} = \min\{E_{lf}(K_{i,p-1}, \mathcal{P}_{i,p-2}), E_{ub}(K_{i,p-1}, \mathcal{P}_{i,p-2})\}$. Then again, the inequality in (3.49) implies that $\mathbf{e}_{i,p-1} > E_{ub}(K_{i,p-1}, \mathcal{P}_{i,p-2})$ which is not possible. Thus we have shown that the price for European options with strike $K_{i,p}$ cannot be below $E_{lb}^{lhs}(K_{i,p}, \mathcal{P}_{i,p-1})$. $\square$

Proposition 3.10.4. *Suppose we are given a finite set of European put option prices*

$$\mathcal{P}^E = \{(\mathbf{e}_0, 0), (\mathbf{e}_1, K_1), ..., (\mathbf{e}_n, K_n)\}$$

and that the functions E and $\overline{A}$ are defined by (3.7) and $\overline{A}(K, \mathcal{P}) = E(Ke^{rT}, \mathcal{P}^E)$ for $K \geq 0$, then

$$cc(\overline{A}; K_{p-1}e^{-rT}, K_p e^{-rT}; \mathcal{P}^E) = cc(E; K_{p-1}, K_p; \mathcal{P}^E) \tag{3.50}$$

has to hold for any $p \in \{1, ..., n\}$.

Proof. To see that this is the case we apply the definition of the function cc to both the European price function and the upper bound $\overline{A}$. We then readily obtain that

$$\begin{aligned}
&cc(\overline{A}; K_{i,p-1}e^{-rT}, K_{i,p}e^{-rT}; \mathcal{P}^E) = \\
&= \overline{A}(K_{i,p-1}e^{-rT}, \mathcal{P}^E) - \overline{A}'(K_{i,p-1}e^{-rT}+, \mathcal{P}^E)K_{i,p-1}e^{-rT} \\
&= \overline{A}(K_{i,p-1}e^{-rT}, \mathcal{P}^E) - \frac{\overline{A}(K_{i,p}e^{-rT}, \mathcal{P}^E) - \overline{A}(K_{i,p-1}e^{-rT}, \mathcal{P}^E)}{K_{i,p}e^{-rT} - K_{i,p-1}e^{-rT}} K_{i,p-1}e^{-rT} \\
&= E(K_{i,p-1}, \mathcal{P}^E) - \frac{E(K_{i,p}, \mathcal{P}^E) - E(K_{i,p-1}, \mathcal{P}^E)}{K_{i,p} - K_{i,p-1}} K_{i,p-1} \\
&= E(K_{i,p-1}, \mathcal{P}^E) - E'(K_{i,p-1}+, \mathcal{P}^E)K_{i,p-1} \\
&= cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}^E).
\end{aligned}$$

$\square$

Proposition 3.10.5. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Assume* *further that the algorithm extended the initial set of prices from* $\mathcal{P}_0^*$ *to* $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 1$. *Starting with the initial set of prices* $\mathcal{P}_i^*$ *the algorithm computes the* $K_{i,p}$*-admissible* $\mathcal{P}_0^*$*-extension* $\mathcal{P}_{i,p}$ *for* $p \geq 1$.

If we assume that the price for an American option with strike $K_{i,p} \in (K_u^A, K_{u+1}^A) \cap (\mathbb{K}(\mathcal{P}_i^*) \backslash \mathbb{K}^A(\mathcal{P}_i^*))$ *is given by*

$$\mathbf{a}_{i,p} = \max\{A_{lb}^{rhs}(K_{i,p}, \mathcal{P}_i^*), A_{lb}^{\overline{A},r}(K_{i,p}, \mathcal{P}_0^*)\}, \tag{3.51}$$

and that $[K_{i,p}, K_{u+1}^A) \cap \mathbb{K}^{aux}(\mathcal{P}_i^*) = \emptyset$ *then the algorithm will compute* $\mathcal{P}_{i,l}$ *successfully for* $K_{i,l} \in (K_{i,p}, K_{u+1}^A)$ *with*

$$\mathbf{a}_{i,\tilde{l}} = \max\{A_{lb}^{rhs}(K_{i,\tilde{l}}, \mathcal{P}_i^*), A_{lb}^{\overline{A},r}(K_{i,\tilde{l}}, \mathcal{P}_0^*)\} \tag{3.52}$$

for any strike $K_{i,\tilde{l}} \in [K_{i,p}, K_{i,l}]$.

Alternatively, we can assume that the price for European options with strike $K_{i,p} \in (K_v^E, K_{v+1}^E) \cap (\mathbb{K}(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_i^*))$ *is given by* $\mathbf{e}_{i,p} = E_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})$ *and that* $[K_{i,p}, K_{v+1}^E) \cap \mathbb{K}^{aux}(\mathcal{P}_i^*) = \emptyset$. *The algorithm will then successfully compute* $\mathcal{P}_{i,l}$ *for* $K_{i,l} \in (K_{i,p}, K_{v+1}^E)$ *and* $\mathbf{e}_{i,\tilde{l}} = E_{ub}(K_{i,\tilde{l}}, \mathcal{P}_{i,\tilde{l}-1})$ *for any strike* $K_{i,\tilde{l}} \in [K_{i,p}, K_{i,l})$.

Proof. Consider first the situation where $K_{i,p} \in (K_u^A, K_{u+1}^A) \cap (\mathbb{K}^E(\mathcal{P}_i^*) \backslash \mathbb{K}^A(\mathcal{P}_i^*))$ and the price for American options with strike $K_{i,p}$ is given by (3.51). Let us assume for contradiction that there exists a strike $K_{i,s}$ with

$$\begin{aligned} K_{i,s} = \min\{K_{i,\tilde{s}} \in (K_{i,p}, K_{i,l}] \cap (\mathbb{K}^E(\mathcal{P}_i^*) \backslash \mathbb{K}^A(\mathcal{P}_i^*)) : \\ \mathbf{a}_{i,\tilde{s}} > \max\{A_{lb}^{rhs}(K_{i,\tilde{s}}, \mathcal{P}_i^*), A_{lb}^{\overline{A},r}(K_{i,\tilde{s}}, \mathcal{P}_0^*)\}\}. \end{aligned}$$

We will begin by showing that we can exclude $\mathbf{a}_{i,s} = A_{lb}^{lhs}(K_{i,s}, \mathcal{P}_{i,s-1})$ from consideration. To this end, we assume for contradiction that $\mathbf{a}_{i,s} = A_{lb}^{lhs}(K_{i,s}, \mathcal{P}_{i,s-1})$. Taking into account the definition of $K_{i,s}$, the price for American options with strike $K_{i,s-1}$ has to be given by (3.52). We thus obtain that $\mathbf{a}_{i,s-2}$ has to satisfy

$$\mathbf{a}_{i,s-2} < \max\{A_{lb}^{rhs}(K_{i,s-2}, \mathcal{P}_i^*), A_{lb}^{\overline{A},r}(K_{i,s-2}, \mathcal{P}_0^*)\}.$$

This can be ruled out, however, as either $\mathbf{a}_{i,s-2}$ is given by (3.52) for $s-2 \geq p$ or because $\mathcal{P}_{i,p}$ is a $K_{i,p}$-admissible $\mathcal{P}_0^*$-extension for $s-1 = p$ and thus convex.

Suppose now for contradiction that $\mathbf{a}_{i,s} = A_{lf}(K_{i,s}, \mathcal{P}_{i,s-1})$, then we have to distinguish between the two cases where the right hand-side lower bound is either given by $A_{lb}^{\overline{A},r}(K_{i,s}, \mathcal{P}_0^*)$ or $A_{lb}^{rhs}(K_{i,s}, \mathcal{P}_i^*)$. In the first case we assume that the right hand-side lower bound $A_{lb}^{\overline{A},r}$ is given by the prices $\hat{\mathbf{a}}_{u+1}$ and $\overline{\mathbf{a}}_j$. We can then argue that

$$cc(A, \overline{A}; K_{u+1}^A, K_j^E e^{-rT}; \mathcal{P}_0^*) \leq cc(\overline{A}; K_{j-1}^E e^{-rT}, K_j^E e^{-rT}; \mathcal{P}_0^*). \tag{3.53}$$

To do this, we need to distinguish between the two cases where $K_{j-1}^E e^{-rT} \leq K_{u+1}^A$ or not. If $K_{j-1}^E e^{-rT} \leq K_{u+1}^A$ then the inequality in (3.53) follows immediately from $(viii)$ of the Standing Assumption. In the case where $K_{j-1}^E e^{-rT} > K_{u+1}^A$ we can deduce from the definition of $A_{lb}^{\overline{A},r}$ in (3.18) that

$$\overline{\mathbf{a}}_{j-1} \geq \frac{\overline{\mathbf{a}}_j - \hat{\mathbf{a}}_{u+1}}{K_j^E e^{-rT} - K_{u+1}^A}(K_{j-1}^E e^{-rT} - K_{u+1}^A) + \hat{\mathbf{a}}_{u+1}.$$

Hence we must have

$$\begin{aligned} cc(A,\overline{A};K_{u+1}^A,K_j^E e^{-rT};\mathcal{P}_0^*) &\leq cc(\overline{A};K_{j-1}^E e^{-rT},K_j^E e^{-rT};\mathcal{P}_0^*) \\ &= cc(E;K_{j-1}^E,K_j^E;\mathcal{P}_0^*) \\ &\leq cc(E;K_{i,s-1},K_{i,s};\mathcal{P}_0^*). \end{aligned}$$

This, however, contradicts the assumption that $A_{lf}(K_{i,\tilde{l}},\mathcal{P}_{i,s-1}) > A_{lb}^{\overline{A},r}(K_{i,\tilde{l}},\mathcal{P}_0^*)$ and thus we must have $\mathbf{a}_{i,\tilde{l}} = A_{lb}^{\overline{A},r}(K_{i,\tilde{l}},\mathcal{P}_{i,\tilde{l}-1})$.

In the second case there exist ordered strikes

$$(K_{i,s-1},K_{i,s},K_{u+1}^A,K_{i,r}) \in (\mathbb{K}^E(\mathcal{P}_0^*),\mathbb{K}^E(\mathcal{P}_0^*),\mathbb{K}^A(\mathcal{P}_0^*),\mathbb{K}^A(\mathcal{P}_i^*))$$

such that

$$\begin{aligned} cc(E;K_{i,s-1},K_{i,s};\mathcal{P}_0^*) &= cc(A;K_{i,s-1},K_{i,s};\mathcal{P}_{i,s}) \\ &< cc(A;K_{i,p},K_{u+1}^A;\mathcal{P}_{i,s}) \\ &= cc(A;K_{u+1}^A,K_{i,r};\mathcal{P}_i^*) \end{aligned} \tag{3.54}$$

has to hold. We can thus immediately rule out that the strike $K_{i,r} \in \mathbb{K}^A(\mathcal{P}_0^*)$, as this would imply a violation of (iv) of the Standing Assumptions. Note, moreover, that for $K_{i,r} \in \mathbb{K}_2^{aux}(\mathcal{P}_i^*)$ the two right hand-side lower bounds coincide and thus this case can be ruled out using the argument from the first case. We are therefore left with the situation where $K_{i,r} \in \mathbb{K}_1^{aux}(\mathcal{P}_i^*)$. In this situation we only have to distinguish between the two cases where $K_{i,r-1} \in \mathbb{K}^E(\mathcal{P}_0^*)$ or $K_{i,r-1} = K_{u+1}^A$ as the definition of the right hand-side lower bound in (3.14) implies that $K_{i,r} = \min\{K \in \mathbb{K}^A(\mathcal{P}_i^*) : K > K_{u+1}^A\}$. If $K_{i,r-1} \in \mathbb{K}^E(\mathcal{P}_0^*)$, then we know from the stopping condition in Algorithm 3 that

$$cc(A;K_{u+1}^A,K_{i,r};\mathcal{P}_i^*) \leq cc(E;K_{i,r-1},K_{i,r};\mathcal{P}_0^*).$$

Combined with the inequality in (3.54) this yields

$$cc(E;K_{i,s-1},K_{i,s};\mathcal{P}_0^*) < cc(E;K_{i,r-1},K_{i,r};\mathcal{P}_0^*)$$

which is a contradiction to the convexity assumption for $E(\cdot, \mathcal{P}_0^*)$ in the Standing Assumptions.

Suppose now that $K_{i,r-1} = K_{u+1}^A$ and define $K_j^E = \max\{K_{j'} \in \mathbb{K}^E(\mathcal{P}_0^*) : K_{j'} \leq K_{u+1}^A\}$, then we can conclude that

$$cc(A; K_{u+1}^A, K_{i,r}; \mathcal{P}_i^*) \leq cc(E; K_j^E, K_{i,r}; \mathcal{P}_0^*)$$

as there would exist arbitrage in the market otherwise according to the stopping condition of Algorithm 3 which in turn would have prompted the algorithm to stop instead of restarting under $\mathcal{P}_i^*$. Combined with (3.54) we obtain that

$$cc(E; K_{i,s-1}, K_{i,s}; \mathcal{P}_0^*) < cc(E; K_{i,r-1}, K_{i,r}; \mathcal{P}_0^*)$$

which again yields a contradiction to the convexity assumption for $E(\cdot, \mathcal{P}_0^*)$ in the Standing Assumptions. We can thus conclude that

$$\mathbf{a}_{i,l} = \max\{A_{lb}^{rhs}(K_{i,l}, \mathcal{P}_i^*), A_{lb}^{\overline{A},r}(K_{i,l}, \mathcal{P}_0^*)\}$$

for any strike $K_{i,l} \in [K_{i,p}, K_{u+1}^A) \cap (\mathbb{K}^E(\mathcal{P}_i^*) \backslash \mathbb{K}^A(\mathcal{P}_i^*))$ whenever (3.52) holds.

Suppose now that $\mathbf{e}_{i,p} = E_{ub}(K_{i,p}; \mathcal{P}_{i,p-1})$ for the strike $K_{i,p} \in (K_v^E, K_{v+1}^E) \cap \mathbb{K}^A(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_i^*)$. Let us furthermore assume for contradiction that there exists a strike $K_{i,s}$ with

$$\begin{aligned} K_{i,s} = \min\{K_{i,\tilde{s}} \in (K_{i,p}, K_{v+1}^E) \cap (\mathbb{K}^A(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_i^*)) : \\ \mathbf{e}_{i,\tilde{s}} = E_{lf}(K_{i,\tilde{s}}, \mathcal{P}_{i,\tilde{s}-1}) \text{ and } \mathbf{e}_{i,\tilde{s}} < E_{ub}(K_{i,\tilde{s}}, \mathcal{P}_{i,\tilde{s}-1})\} \end{aligned}$$

This means that $cc(E; K_{i,s-1}, K_{i,s}; \mathcal{P}_{i,s}) > cc(E; K_{i,p-1}, K_{i,s-1}; \mathcal{P}_{i,s})$. Additionally, the Legendre-Fenchel condition has to hold with strict inequality on $[K_{i,p-1}, K_{i,s-1}]$ and with equality on $[K_{i,s-1}, K_{i,s}]$ and thus we obtain that

$$\begin{aligned} cc(A; K_{i,p-1}, K_{i,s-1}; \mathcal{P}_{i,s}) &< cc(E; K_{i,p-1}, K_{i,s-1}; \mathcal{P}_{i,s}) \\ &< cc(E; K_{i,s-1}, K_{i,s}; \mathcal{P}_{i,s}) \\ &= cc(A; K_{i,s-1}, K_{i,s}; \mathcal{P}_{i,s}). \end{aligned}$$

Hence, the convexity of the American price function A has to be violated on $[K_{i,p-1}, K_{i,s}]$. Then again, we assumed that $[K_{i,p}, K_{v+1}^E) \cap \mathbb{K}^{aux}(\mathcal{P}_i^*) = \emptyset$ and thus $(K_{i,p}, K_{i,s}) \in (\mathbb{K}^A(\mathcal{P}_0^*), \mathbb{K}^A(\mathcal{P}_0^*))$ has to hold. We now distinguish between the two cases where either $s > p+1$ or $s = p+1$.

In the first case there exists $K_{i,s-1} \in (K_{i,p}, K_{i,s}) \cap \mathbb{K}^A(\mathcal{P}_0^*)$ such that the American price function between the three prices $\mathbf{a}_{i,p}$, $\mathbf{a}_{i,s-1}$ and $\mathbf{a}_{i,s}$ cannot be convex. Since

American options are traded at these prices in the market, the price set $\mathcal{P}_0^*$ would have to violate the Standing Assumptions.

In the second case $K_{i,s} = K_{i,p+1}$ would imply that $\mathbf{a}_{i,p-1} < A_{lb}^{rhs}(K_{i,p-1}, \mathcal{P}_0^*)$ which cannot be the case as we assumed that $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension and we can thus conclude that $\mathbf{e}_{i,s} = E_{ub}(K_{i,s}, \mathcal{P}_i^*)$. □

Proposition 3.10.6. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Assume further that the algorithm extended the initial set of prices from $\mathcal{P}_0^*$ to $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 1$. Starting with the initial set of prices $\mathcal{P}_i^*$ the algorithm computes the $K_{i,p}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{i,p}$ for $p \geq 1$.*

If we assume that the price for an American option with strike $K_{i,p} \in (K_u^A, K_{u+1}^A) \cap (\mathbb{K}(\mathcal{P}_i^*) \backslash \mathbb{K}^A(\mathcal{P}_i^*))$ is given by $\mathbf{a}_{i,p} = A_{lb}^{\overline{A},l}(K_{i,p}, \mathcal{P}_0^*)$, then $\mathbf{a}_{i,l} = A_{lb}^{\overline{A},l}(K_{i,l}, \mathcal{P}_0^*)$ for any strike $K_{i,l} \in (K_u^A, K_{i,p}]$.

Proof. Suppose for contradiction that there exists a strike $K_{i,l}$ such that

$$K_{i,l} = \max\{K_{i,s} \in (K_u^A, K_{i,p}) \cap \mathbb{K}(\mathcal{P}_i^*) : \mathbf{a}_{i,l} > A_{lb}^{\overline{A},l}(K_{i,l}, \mathcal{P}_0^*)\}.$$

We can deduce from the fact that $\mathcal{P}_{i,p}$ is a $K_{i,p}$-admissible $\mathcal{P}_0^*$-extension that the American price function has to be convex up to $K_{i,p}$ and thus

$$cc(A; K_{i,l-1}, K_{i,l}; \mathcal{P}_{i,p}) \leq cc(A; K_u^A, K_{i,l}; \mathcal{P}_{i,p})$$

has to hold. Moreover, we must have

$$cc(A; K_u^A, K_{i,l}; \mathcal{P}_{i,p}) < cc(A_{lb}^{\overline{A},l}; K_u^A, K_{i,l}; \mathcal{P}_0^*),$$

as we assumed that $\mathbf{a}_{i,l} > A_{lb}^{\overline{A},l}(K_{i,l}, \mathcal{P}_0^*)$. We can therefore conclude that

$$\begin{aligned} cc(A; K_{i,l-1}, K_{i,l}; \mathcal{P}_{i,p}) &\leq cc(A; K_u^A, K_{i,l}; \mathcal{P}_{i,p}) \\ &< cc(A_{lb}^{\overline{A},l}; K_u^A, K_{i,l}; \mathcal{P}_0^*) \\ &= cc(A_{lb}^{\overline{A},l}; K_{i,p-1}, K_{i,p}; \mathcal{P}_0^*). \end{aligned}$$

Then again, this yields a contradiction to the assumption that the American price function is convex and thus we can conclude that the price for American options with strike $K_{i,l}$ has to be given by $\mathbf{a}_{i,l} = A_{lb}^{\overline{A},l}(K_{i,l}, \mathcal{P}_0^*)$. □

Proposition 3.10.7. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Assume further that the algorithm extended the initial set of prices from $\mathcal{P}_0^*$ to $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 1$.*

Starting with the initial set of prices $\mathcal{P}_i^*$ the algorithm computes the $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{i,p-1}$ for $p \geq 1$.

If the Legendre-Fenchel condition holds with equality on $[K_{i,p-1}, K_{i,p}]$ a violation of $\overline{\mathbf{a}}_{i,p} \geq A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p})$ can be ruled out.

Proof. Since we assumed that $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension we know that $\overline{\mathbf{a}}_{i,p-1} \geq A(K_{i,p-1}, \mathcal{P}_{i,p})$ as to hold. According to the assumptions we also know that the Legendre-Fenchel condition holds with equality on $[K_{i,p-1}, K_{i,p}]$. That is, $cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) = cc(A; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p})$. Taking into account that

$$cc(\overline{A}; K_{i,p-1}e^{-rT}, K_{i,p}e^{-rT}; \mathcal{P}_{i,p}) = cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p})$$

according to Proposition 3.10.4 we obtain

$$\begin{aligned} cc(\overline{A}; K_{i,p-1}e^{-rT}, K_{i,p}e^{-rT}; \mathcal{P}_{i,p}) &= cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) \\ &= cc(A; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) \\ &\leq cc(A; K_{i,q}, K_{i,q+1}; \mathcal{P}_{i,p}). \end{aligned} \tag{3.55}$$

We then have to distinguish between the two cases where either

$$cc(A; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) < 0$$

or not. In the case where $cc(A; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) < 0$ holds we readily obtain from $\overline{\mathbf{a}}_{i,p-1} \geq A(K_{i,p-1}e^{-rT}, \mathcal{P}_{i,p})$ and (3.55) that $\overline{\mathbf{a}}_{i,p} \geq A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p})$.

Let us consider now the situation where $cc(A; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) = 0$. The non-positivity of the convex conjugate of the price functions A and E then yields that both sides of the inequality in (3.55) are zero. If $K_{i,p-1} > 0$ we thus obtain that $A(K, \mathcal{P}_{i,p}) = E(K, \mathcal{P}_{i,p})$ for any strike $K \in [0, K_{i,p}]$ and hence $\overline{A}(K, \mathcal{P}_{i,p}) = 0$ has to hold as well. We can therefore rule out a violation of the upper bound in $K_{i,p}e^{-rT}$ in this case as well.

In the case where $K_{i,p-1} = 0$ we note that either $\mathbf{a}_{i,1} \geq A_{lf}(K_{i,1}, \mathcal{P}_{i,p})$ holds for $K_{i,1} \in (\mathbb{K}(\mathcal{P}_i^*) \backslash \mathbb{K}^A(\mathcal{P}_i^*))$ or $\mathbf{e}_{i,1} \leq E_{lf}(K_{i,1}, \mathcal{P}_{i,p})$ for $K_{i,1} \in (\mathbb{K}(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_i^*))$. This not only implies that $\mathbf{a}_{i,1} \geq \mathbf{e}_{i,p}$, but also $A(K, \mathcal{P}_{i,p}) \geq E(K, \mathcal{P}_{i,p})$ for any strike $K \in [0, K_{i,1}]$. We can thus conclude that $\overline{\mathbf{a}}_{i,p} \geq A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p})$ whenever the Legendre-Fenchel condition holds with equality on $[K_{i,p-1}, K_{i,p}]$. □

Proposition 3.10.8. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Assume further that the algorithm extended the initial set of prices from $\mathcal{P}_0^*$ to $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 1$. Starting with the initial set of prices $\mathcal{P}_i^*$ the algorithm computes $\mathcal{P}_{i,p}$ a $K_{i,p}$-admissible $\mathcal{P}_0^*$-extension.*

If we suppose now that the price for American options with strike $K_{i,p}$ is given by $\mathbf{a}_{i,p} = K_{i,p} - S_0$ under $\mathcal{P}_{i,p}$, then $\hat{\mathbf{a}}_j = K_j^A - S_0$ for any strike $K_j^A \in [K_{i,p}, \infty) \cap \mathbb{K}^A(\mathcal{P}_0^*)$.

Proof. We begin by showing that $\mathbf{a}_{j'} = K_{j'}^A - S_0$ for $K_{j'}^A = \min\{K_s^A \in \mathbb{K}^A(\mathcal{P}_0^*) : K_s^A > K_{i,p}\}$. Suppose for contradiction that the price of an American option with strike $K_{j'}^A \in \mathbb{K}^A(\mathcal{P}_0^*)$ is given by $\mathbf{a}_{j'} \neq K_{j'}^A - S_0$. Observe first that the option price has to satisfy $\hat{\mathbf{a}}_{j'} \geq K_{j'}^A - S_0$ as $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. It follows that $\hat{\mathbf{a}}_{j'} > K_{j'}^A - S_0$ has to hold. We then have to distinguish between the two cases where $K_{i,p} \in \mathbb{K}^A(\mathcal{P}_0^*)$ or $K_{i,p} \in \mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_0^*)$. The case where $K_{i,p} \in \mathbb{K}^A(\mathcal{P}_0^*)$ can be ruled out immediately as $A'(K_{i,p}+, \mathcal{P}_0^*) \leq 1$ has to hold according to (iii) of the Standing Assumptions which contradicts $\hat{\mathbf{a}}_{j'} > K_{j'}^A - S_0$ for $\mathbf{a}_{i,p} = K_{i,p} - S_0$.

In the second case where $K_{i,p} \in \mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_0^*)$ we can deduce from $A'(K+, \mathcal{P}_0^*) \leq 1$ for $K \geq 0$ together with $\hat{\mathbf{a}}_{j'} > K_{j'}^A - S_0$ that $\hat{\mathbf{a}}_s > K_s^A - S_0$ for any strike $K_s^A \in [0, K_{j'}^A] \cap \mathbb{K}^A(\mathcal{P}_0^*)$. It thus follows that $K_{j'}^A \leq K_{l_1(\mathcal{P}_0^*)}$ has to hold. The Standing Assumptions then guarantee in (iii) that $A'(K_{j'}^A+, \mathcal{P}_0^*) < 1$ has to hold. We can therefore conclude that $A_{lb}^{rhs}(K_{i,p}, \mathcal{P}_0^*) > K_{i,p} - S_0$, thereby contradicting the assumption that $\mathbf{a}_{i,p} = K_{i,p} - S_0$. Hence, the price for an American option with strike $K_{j'}^A \in \mathbb{K}^A(\mathcal{P}_0^*)$ has to be given by $K_{j'}^A - S_0$.

Finally, we can use (iii) of the Standing Assumptions to deduce from $\hat{\mathbf{a}}_{j'} = K_{j'}^A - S_0$ that $\hat{\mathbf{a}}_j = K_j^A - S_0$ for any strike $K_j^A \in [K_{i,p}, \infty) \cap \mathbb{K}^A(\mathcal{P}_0^*)$. □

Proposition 3.10.9. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Assume further that the algorithm extended the initial set of prices from $\mathcal{P}_0^*$ to $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 1$. Starting with the initial set of prices $\mathcal{P}_i^*$ the algorithm computes the $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{i,p-1}$ for $p \geq 1$. If we further assume that $K_{i,p} \in \mathbb{K}^A(\mathcal{P}_i^*)$ with $\mathbf{a}_{i,p} \geq K_{i,p} - S_0$, then $\mathbf{e}_{i,p} \geq e^{-rT} K_{i,p} - S_0$ has to hold.*

Remark 3.10.10. *Note that $\mathbf{a}_{i,p} \geq K_{i,p} - S_0$ has to hold for $K_{i,p} \in \mathbb{K}^A(\mathcal{P}_0^*)$ whenever $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Moreover, this result remains valid after introducing auxiliary price constraints that lie within their respective no-arbitrage bounds.*

Proof. We consider first the case where $K_{i,p} \in [0, K_{m_2}^E) \cap (\mathbb{K}^A(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*))$. The algorithm then computes the price for a European option with strike $K_{i,p}$ using

$$\mathbf{e}_{i,p} = \min\{E_{ub}(K_{i,p}, \mathcal{P}_{i,p-1}), E_{lf}(K_{i,p}, \mathcal{P}_{i,p-1})\}.$$

Since $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension we also know that the price for European options with strike $K_{i,p-1}$ satisfies $\mathbf{e}_{i,p-1} \geq e^{-rT} K - S_0$. Combined with the fact that $\hat{\mathbf{e}}_j \geq e^{-rT} K_j^E - S_0$ for any strike $K_j^E \in \mathbb{K}^E(\mathcal{P}_0^*)$ we can then deduce that $E_{ub}(K_{i,p}, \mathcal{P}_{i,p-1}) \geq e^{-rT} K_{i,p} - S_0$ has to hold as well. Hence, it follows that

$\mathbf{e}_{i,p} = E_{lf}(K_{i,p}, \mathcal{P}_{i,p-1})$. Note further that this implies that the Legendre-Fenchel condition holds with equality between the strikes $K_{i,p-1}$ and $K_{i,p}$. Combined with the convexity of the American put price function $A(\cdot, \mathcal{P}_{i,p})$ we get that

$$\begin{aligned} cc(\overline{A}, K_{i,p-1}e^{-rT}, K_{i,p}e^{-rT}; \mathcal{P}_{i,p}) &= cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) \\ &= cc(A; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) \\ &\leq cc(A; K_{i,p-1}e^{-rT}, K_{i,p}e^{-rT}; \mathcal{P}_{i,p}). \end{aligned} \tag{3.56}$$

Since $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension, we must have

$$\overline{\mathbf{a}}_{i,p-1} \geq A(K_{i,p-1}e^{-rT}; \mathcal{P}_{i,p}).$$

Using the inequality in (3.56) we can further conclude that $\overline{\mathbf{a}}_{i,p} \geq A(K_{i,p}e^{-rT}; \mathcal{P}_{i,p})$ must hold as well. Then again, we assumed that the European price function violates its lower bound at $K_{i,p}$, which means that

$$e^{-rT}K_{i,p} - S_0 > \mathbf{e}_{i,p} = \overline{\mathbf{a}}_{i,p} \geq A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p}). \tag{3.57}$$

Since $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension we can deduce that $A(K, \mathcal{P}_{i,p-1}) \geq K - S_0$ has to hold for any strike $K \leq K_{i,p-1}$. Combined with the assumption that $\mathbf{a}_{i,p} \geq K_{i,p} - S_0$, we readily obtain that $A(K, \mathcal{P}_{i,p}) \geq K - S_0$ for any strike $K \leq K_{i,p}$, thereby yielding a contradiction to (3.57). It follows that $\mathbf{e}_{i,p} \geq e^{-rT}K_{i,p} - S_0$ has to hold for $K_{i,p} \in [0, K_{m_2}^E) \cap (\mathbb{K}^A(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*))$.

We are thus only left to argue that $\mathbf{e}_{i,p} \geq e^{-rT}K_{i,p} - S_0$ for $K_{i,p} \in (K_{m_2}^E, \infty) \cap (\mathbb{K}^A(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*))$. Then again, the algorithm stops computing option prices at $K_{m_2}^E$ and extends the European price function to $e^{-rT}K - S_0$ for any strike $K \geq K_{m_2}^E$. We can thus conclude that $\mathbf{e}_{i,p} \geq K_{i,p}e^{-rT} - S_0$ for $K_{i,p} \in \mathbb{K}^A(\mathcal{P}_i^*)$. □

3.10.3 General properties of the revised price functions

Proposition 3.10.11. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Assume further that the algorithm extended the initial set of prices from $\mathcal{P}_0^*$ to $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 1$. Starting with the initial set of prices $\mathcal{P}_i^*$ the algorithm computes the $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{i,p-1}$ for $p \geq 1$.*

Suppose further that the algorithm stops at the strike $K_{i,p}$ either due to a violation of $\mathbf{e}_{i,p} \geq E_{lb}(K_{i,p}, \mathcal{P}_{i,p-1})$ for $K_{i,p} \in (\mathbb{K}^A(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_i^*))$ or due to a violation of $\mathbf{a}_{i,p} \leq A_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})$ for $K_{i,p} \in (\mathbb{K}^E(\mathcal{P}_i^*) \backslash \mathbb{K}^A(\mathcal{P}_i^*))$ and Algorithm 3 revises the already computed prices on $[K_{i,\tilde{q}}, K_{i,p}]$.

If the previously computed price functions $A(\cdot, \mathcal{P}_i)$ and $E(\cdot, \mathcal{P}_i)$ satisfy the Legendre-Fenchel condition with equality on $[K_{i,\tilde{q}}, K_{i,p}]$, then we must have $\mathbf{e}_{i,s}^n > \mathbf{e}_{i,s}$ for any *strike* $K_{i,s} \in [K_{i,\tilde{q}}, K_{i,p}) \cap (\mathbb{K}^A(\mathcal{P}_i^*)\backslash\mathbb{K}^E(\mathcal{P}_i^*))$ *and* $\mathbf{a}_{i,s}^n < \mathbf{a}_{i,s}$ *for any strike* $K_{i,s} \in [K_{i,\tilde{q}}, K_{i,p}) \cap \mathbb{K}^E(\mathcal{P}_i^*)$.

Proof. Note that both the price functions constructed using the initial set of prices $\mathcal{P}_i^*$ and the revised price functions satisfy the Legendre-Fenchel condition with equality on $[K_{i,\tilde{q}}, K_{i,p}]$. This means that the following equations have to hold for $K_{i,s} \in (\mathbb{K}^A(\mathcal{P}_i^*)\backslash\mathbb{K}^E(\mathcal{P}_i^*))$

$$\begin{aligned}\mathbf{e}_{i,s} &= \mathbf{a}_{i,s} - \frac{K_{i,s}}{K_{i,p}}[\mathbf{a}_{i,p} - \mathbf{e}_{i,p}]\\ \mathbf{e}_{i,s}^n &= \mathbf{a}_{i,s} - \frac{K_{i,s}}{K_{i,p}}[\mathbf{a}_{i,p}^n - \mathbf{e}_{i,p}^n]\end{aligned}$$

and for $K_{i,s} \in \mathbb{K}^E(\mathcal{P}_i^*)$

$$\begin{aligned}\mathbf{a}_{i,s} &= \mathbf{e}_{i,s} + \frac{K_{i,s}}{K_{i,p}}[\mathbf{a}_{i,p} - \mathbf{e}_{i,p}]\\ \mathbf{a}_{i,s}^n &= \mathbf{e}_{i,s} + \frac{K_{i,s}}{K_{i,p}}[\mathbf{a}_{i,p}^n - \mathbf{e}_{i,p}^n].\end{aligned}$$

Looking at the difference between $\mathbf{e}_{i,s}^n$ and $\mathbf{e}_{i,s}$ we see that

$$\mathbf{e}_{i,s}^n - \mathbf{e}_{i,s} = \frac{K_{i,s}}{K_{i,p}}\left[(\mathbf{a}_{i,p} - \mathbf{a}_{i,p}^n) + (\mathbf{e}_{i,p}^n - \mathbf{e}_{i,p})\right] \tag{3.58}$$

is strictly positive and thus $\mathbf{e}_{i,s}^n > \mathbf{e}_{i,s}$. For the price difference of an American option with strike $K_{i,s} \in \mathbb{K}^E(\mathcal{P}_i^*)$ we see that

$$\mathbf{a}_{i,s} - \mathbf{a}_{i,s}^n = \frac{K_{i,s}}{K_{i,p}}\left[(\mathbf{a}_{i,p} - \mathbf{a}_{i,p}^n) + (\mathbf{e}_{i,p}^n - \mathbf{e}_{i,p})\right] \tag{3.59}$$

which is again strictly positive and thus $\mathbf{a}_{i,s} > \mathbf{a}_{i,s}^n$. ☐

Proposition 3.10.12. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by* $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. *Assume further that the algorithm extended the initial set of prices from* $\mathcal{P}_0^*$ *to* $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 1$. *Starting with the initial set of prices* $\mathcal{P}_i^*$ *the algorithm computes the* $K_{i,p-1}$-*admissible* $\mathcal{P}_0^*$-*extension* $\mathcal{P}_{i,p-1}$ *for* $p \geq 1$.

The algorithm stops at the strike $K_{i,p} \in (K_u^A, K_{u+1}^A] \cap \mathbb{K}(\mathcal{P}_i^*)$ *either due to a violation of* $\mathbf{e}_{i,p} \geq E_{lb}(K_{i,p}, \mathcal{P}_{i,p-1})$ *for the strike* $K_{i,p} \in \mathbb{K}^A(\mathcal{P}_0^*)$ *or due to a violation of* $\mathbf{a}_{i,p} \leq A_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})$ *for the strike* $K_{i,p} \in (\mathbb{K}^E(\mathcal{P}_i^*)\backslash\mathbb{K}^A(\mathcal{P}_i^*))$. *If Algorithm 3 revises the already computed prices on* $[K_{i,q}, K_{i,p}]$, *then the prices* $\mathbf{a}_{i,p-1}^n$, $\mathbf{a}_{i,p}^n$ *and* $\hat{\mathbf{a}}_{u+1}$ *have to be co-linear.*

Proof. Suppose first that $K_{i,p} \in \mathbb{K}^A(\mathcal{P}_0^*)$, then this readily implies that $K_{i,p} = K_{u+1}^A$.

Since we are only considering the two different prices $\mathbf{a}_{i,p-1}$ and $\mathbf{a}_{i,p}$ in this case, we trivially must have that they are co-linear.

Let us consider now the second case where $K_{i,p} \in (\mathbb{K}^E(\mathcal{P}_0^*)\backslash\mathbb{K}^A(\mathcal{P}_0^*))$. Here the price for the American option with strike $K_{i,p}$ depends on whether $K_{i,p-1} \in \mathbb{K}^A(\mathcal{P}_0^*)$ or $K_{i,p-1} \in \mathbb{K}^E(\mathcal{P}_0^*)\backslash\mathbb{K}^A(\mathcal{P}_0^*)$.

In the first case Algorithm 3 determines the price for American options with strike $K_{i,p}$ to be $\mathbf{a}_{i,p}^n = A_{ub}(K_{i,p}, \mathcal{P}_0^*)$ and it follows from the definition of $A_{ub}(\cdot, \mathcal{P}_0^*)$ that the prices $\mathbf{a}_{i,p-1}^n$, $\mathbf{a}_{i,p}^n$ and $\hat{\mathbf{a}}_{u+1}$ have to be co-linear.

In the second case the price for American options with strike $K_{i,p}$ is determined to be

$$\mathbf{a}_{i,p}^n = \frac{\hat{\mathbf{a}}_{u+1} - cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p})}{K_{u+1}^A}(K_{i,p} - K_{u+1}^A) + \hat{\mathbf{a}}_{u+1}.$$

Note further that the price for American options at the strike $K_{i,p-1}$ is computed so as to satisfy the Legendre-Fenchel condition with equality on $[K_{i,p-1}, K_{i,p}]$. It follows that

$$\begin{aligned} cc(A; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) &= cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) \\ &= cc(A; K_{i,p}, K_{u+1}^A; \mathcal{P}_{i,p}). \end{aligned}$$

and thus we obtain again that the prices $\mathbf{a}_{i,p-1}^n$, $\mathbf{a}_{i,p}^n$ and $\hat{\mathbf{a}}_{u+1}$ have to be co-linear. □

3.10.4 Properties of the price functions when a violation of $\overline{A} \geq A$ occurs

Proposition 3.10.13. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Assume further that the algorithm extended the initial set of prices from $\mathcal{P}_0^*$ to $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 0$. Starting with the initial set of prices $\mathcal{P}_i^*$ the algorithm computes the $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{i,p-1}$ for $p \geq 1$.*

Suppose further that the algorithm stops at the strike $K_{i,p} \in \mathbb{K}(\mathcal{P}_i^*)$ due to a violation of $\overline{\mathbf{a}}_{i,p} \geq A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p})$ where $K_{i,p}e^{-rT} \in (K_{i,q}, K_{i,q+1}]$ for $K_{i,q}, K_{i,q+1} \in \mathbb{K}(\mathcal{P}_i^*)$. If we set $j' = \arg\min\{K_v^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K_v^E \geq K_{i,p}\}$ and assume that both $[K_{i,p}, K_{j'}^E) \cap \mathbb{K}^{aux}(\mathcal{P}_i^*) = \emptyset$ and $cc(A; K_{i,q}, K_{i,q+1}; \mathcal{P}_{i,p}) < 0$ hold, then the Legendre-Fenchel condition has to hold with strict inequality on every subinterval of $[K_{i,q}, K_{j'}^E]$ for any $K_{j'}^E$-admissible $\mathcal{P}_{i,p}$-extension.

Proof. We start by noting that $\overline{\mathbf{a}}_{i,p-1} \geq A(K_{i,p-1}e^{-rT}, \mathcal{P}_{i,p-1})$ must hold, as $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension. Moreover, we can conclude that the price functions are increasing and convex up to the strike $K_{i,p}$ as these properties are checked by the algorithm prior to a possible violation of the upper bound. It then follows from

$cc(A; K_{i,q}, K_{i,q+1}; \mathcal{P}_{i,p}) < 0$ that there has to exist $r \in \{1, ..., q\}$ such that

$$cc(A; K_{i,r}, K_{i,r+1}; \mathcal{P}_{i,p}) < cc(\overline{A}; K_{i,p-1}e^{-rT}, K_{i,p}e^{-rT}; \mathcal{P}_{i,p}),$$

since we would have $\overline{\mathbf{a}}_{i,p} \geq A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p})$ otherwise. According to Proposition 3.10.4 we further know that

$$cc(\overline{A}; K_{i,p-1}e^{-rT}, K_{i,p}e^{-rT}; \mathcal{P}_{i,p}) = cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}).$$

Since both the American and the European price functions are convex up to $K_{i,p}$, we can conclude that $cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) \leq cc(E; K_{i,q}, K_{i,q+1}; \mathcal{P}_{i,p})$ and

$$cc(A; K_{i,q}, K_{i,q+1}; \mathcal{P}_{i,p}) \leq cc(A; K_{i,r}, K_{i,r+1}; \mathcal{P}_{i,p}).$$

Combined we obtain for $s \in \{q, ..., p-1\}$ that

$$\begin{aligned} cc(A; K_{i,s}, K_{i,s+1}; \mathcal{P}_{i,p}) &\leq cc(A; K_{i,r}, K_{i,r+1}; \mathcal{P}_{i,p}) \\ &< cc(\overline{A}; K_{i,p-1}e^{-rT}, K_{i,p}e^{-rT}; \mathcal{P}_{i,p}) \\ &= cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) \\ &\leq cc(E; K_{i,s}, K_{i,s+1}; \mathcal{P}_{i,p}) \end{aligned}$$

which shows that the Legendre-Fenchel condition holds with strict inequality on the interval $[K_{i,q}, K_{i,p}]$. In the case that $K_{i,p} \in \mathbb{K}^E(\mathcal{P}_0^*)$, this readily implies that the Legendre-Fenchel conditions holds with strict inequality on $[K_{i,q}, K_{j'}^E]$. We are thus left to consider the case where $K_{i,p} \in \mathbb{K}^A(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*)$. It then follows from the strict inequality in the Legendre-Fenchel condition on $[K_{i,q}, K_{i,p}]$ that the price for European options with strike $K_{i,p}$ is computed to be $\mathbf{e}_{i,p} = E_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})$. According to Proposition 3.10.5, we can thus conclude that $\mathbf{e}_{i,l} = E_{ub}(K_{i,l}, \mathcal{P}_{i,l-1})$ for any strike $K_{i,l} \in [K_{i,p}, K_{j'}^E) \cap (\mathbb{K}(\mathcal{P}_i^*) \backslash K^E(\mathcal{P}_i^*))$. Hence, the convex conjugate for the European price function remains unchanged on the interval $[K_{i,p-1}, K_{j'}^E]$ and thus the Legendre-Fenchel condition has to hold with strict inequality on $[K_{i,q}, K_{j'}^E]$. □

Proposition 3.10.14. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Assume further that the algorithm extended the initial set of prices from $\mathcal{P}_0^*$ to $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 0$. Starting with the initial set of prices $\mathcal{P}_i^*$ the algorithm computes the $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{i,p-1}$ for $p \geq 1$.*

Suppose further that the algorithm stops at the strike $K_{i,p} \in \mathbb{K}(\mathcal{P}_i^*)$ due to a violation of $\overline{\mathbf{a}}_{i,p} \geq A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p})$ where $K_{i,p}e^{-rT} \in (K_{i,q}, K_{i,q+1}]$ for $K_{i,q}, K_{i,q+1} \in \mathbb{K}(\mathcal{P}_i^*)$. If we set $j' = \arg\min\{K_v^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K_v^E \geq K_{i,p}\}$ and assume that $[K_{i,p}, K_{j'}^E) \cap \mathbb{K}^{aux}(\mathcal{P}_i^*) = \emptyset$, then the prices $\mathbf{e}_{i,p-1}$, $\mathbf{e}_{i,p}$ and $\hat{\mathbf{e}}_{j'}$ are co-linear.

Remark 3.10.15. *Note that for $K_{i,p} \in \mathbb{K}^E(\mathcal{P}_0^*)$ we have $K_{j'}^E = K_{i,p}$ and thus the prices $\mathbf{e}_{i,p-1}$, $\mathbf{e}_{i,p}$ and $\hat{\mathbf{e}}_{j'}$ are trivially co-linear.*

Remark 3.10.16. *Note, moreover, that Proposition 3.10.14 combined with Proposition 3.10.4 readily implies that $\overline{\mathbf{a}}_{i,p-1}$, $\overline{\mathbf{a}}_{i,p}$ and $\overline{\mathbf{a}}_{j'}$ have to be co-linear as well.*

Proof. We need to distinguish between the two cases where either

$$cc(A; K_{i,q}, K_{i,q+1}; \mathcal{P}_{i,p}) < 0$$

or not. In the first case we know from Proposition 3.10.13 that the Legendre-Fenchel condition holds with strict inequality on $[K_{i,q}, K_{j'}^E]$ and that $\mathbf{e}_{i,s} = E_{ub}(K_{i,s}, \mathcal{P}_{i,s-1})$ for any strike $K_{i,s} \in [K_{i,q}, K_{j'}^E] \cap (\mathbb{K}(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_i^*))$. Hence, the prices $\mathbf{e}_{i,p-1}$, $\mathbf{e}_{i,p}$ and $\hat{\mathbf{e}}_{j'}$ have to be co-linear.

In the case where $cc(A; K_{i,q}, K_{i,q+1}; \mathcal{P}_{i,p}) = 0$, we cannot apply Proposition 3.10.13 and thus have to examine the situation separately. We begin by showing that $K_{i,p} = K_{i,1}$ whenever $\overline{\mathbf{a}}_{i,p} < A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p})$. To see this, we assume for contradiction that $K_{i,p} > K_{i,1}$. Since we know that $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension, we can conclude that $\overline{\mathbf{a}}_{i,s} \geq A(K_{i,s}e^{-rT}, \mathcal{P}_i^*)$ has to hold for any $s \in \{0, ..., p-1\}$. Moreover, we know that the European price function is convex up to the strike $K_{i,p}$ and thus

$$\overline{\mathbf{a}}_{i,p} \geq \frac{\overline{\mathbf{a}}_{i,1} - \overline{\mathbf{a}}_{i,0}}{K_{i,1}}(K_{i,p} - K_{i,1}) + \overline{\mathbf{a}}_{i,1}.$$

Taking into account that $cc(A; K_{i,q}, K_{i,q+1}; \mathcal{P}_{i,p}) = 0$ and $\overline{\mathbf{a}}_{i,0} = 0$, we thus obtain that

$$\begin{aligned}
\overline{\mathbf{a}}_{i,p} &\geq \frac{\overline{\mathbf{a}}_{i,1} - \overline{\mathbf{a}}_{i,0}}{K_{i,1}}(K_{i,p} - K_{i,1}) + \overline{\mathbf{a}}_{i,1}. \\
&= \frac{\overline{\mathbf{a}}_{i,1} - cc(A; K_{i,q}, K_{i,q+1}; \mathcal{P}_i^*)}{K_{i,1}}(K_{i,p} - K_{i,1}) + \overline{\mathbf{a}}_{i,1} \\
&\geq \frac{A(K_{i,1}e^{-rT}, \mathcal{P}_{i,p}) - cc(A; K_{i,q}, K_{i,q+1}; \mathcal{P}_i^*)}{K_{i,1}}(K_{i,p} - K_{i,1}) + \overline{\mathbf{a}}_{i,1} \\
&\geq A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p}).
\end{aligned}$$

Hence, we can rule out $K_{i,p} > K_{i,1}$ and are left to consider the situation where $K_{i,p} = K_{i,1}$. For $K_{i,1} \in \mathbb{K}^E(\mathcal{P}_0^*)$ we have $\mathbf{e}_{i,p} = \hat{\mathbf{e}}_{j'}$ which readily implies that the prices $\mathbf{e}_{i,p-1}$, $\mathbf{e}_{i,p}$ and $\hat{\mathbf{e}}_{j'}$ are co-linear. In the case where $K_{i,1} \in \mathbb{K}^A(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_i^*)$, the price for European options with strike $K_{i,1}$ has to be given by $\mathbf{e}_{i,1} = E_{ub}(K_{i,1}, \mathcal{P}_i^*)$ as $\mathbf{e}_{i,1} = E_{lf}(K_{i,1}, \mathcal{P}_i^*) = \hat{\mathbf{a}}_1$ contradicts $\overline{\mathbf{a}}_{i,p} < A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p})$. It thus follows that $\mathbf{e}_{i,p-1}$, $\mathbf{e}_{i,p}$ and $\hat{\mathbf{e}}_j$ are co-linear. ☐

Proposition 3.10.17. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$.*

Assume also that the algorithm extended the initial set of prices from $\mathcal{P}_0^*$ to $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 0$. Starting with the initial set of prices $\mathcal{P}_i^*$ the algorithm computes the $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{i,p-1}$ for $p \geq 1$.

Suppose further that the algorithm stops at the strike $K_{i,p} \in \mathbb{K}(\mathcal{P}_i^*)$ due to a violation of $\overline{\mathbf{a}}_{i,p} \geq A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p})$ where $K_{i,p}e^{-rT} \in (K_{i,q}, K_{i,q+1}]$ for $K_{i,q}, K_{i,q+1} \in \mathbb{K}(\mathcal{P}_i^*)$. We further set

$$j' = \arg\min\{K_v^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K_v^E \geq K_{i,p}\}$$

and

$$u = \arg\max\{K_s^A \in \mathbb{K}^A(\mathcal{P}_0^*) : K_s^A < K_{i,p}e^{-rT}\}.$$

If we assume that $K_{i,q} \notin \mathbb{K}_1^{aux}(\mathcal{P}_i^*)$ and $[K_{i,p}, K_{j'}^E] \cap \mathbb{K}^{aux}(\mathcal{P}_i^*) = \emptyset$, then $K_{j'}^E e^{-rT} < K_{u+1}^A$.

Proof. Suppose first for contradiction that $K_{j'}^E e^{-rT} > K_{u+1}^A$. According to Proposition 3.10.14 the European price function is linear on $[K_{i,p-1}, K_{j'}^E]$. Let us thus denote the smallest price co-linear with the prices $\mathbf{e}_{i,p-1}$ and $\hat{\mathbf{e}}_{j'}$ by $\mathbf{e}_{i,l}$, where we must have $0 \leq l \leq p-1$. It then follows that the upper bound $\overline{A}$ is linear on $[K_{i,l}e^{-rT}, K_{j'}^E e^{-rT}]$ as well.

Suppose now that there exists a strike $K_{i,s}$, $s \in \{l, ..., p-1\}$, such that $K_{i,s} \in \mathbb{K}^E(\mathcal{P}_0^*)$. As we assumed that $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension, we must have $\overline{\mathbf{a}}_{i,s} \geq A(K_{i,s}e^{-rT}, \mathcal{P}_{i,p})$. Taking into account that $\overline{\mathbf{a}}_{i,p} < A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p})$ we obtain that $\hat{\mathbf{a}}_{u+1} > \overline{A}(K_{u+1}^A, \mathcal{P}_0^*)$ thereby violating the Standing Assumptions. We can thus conclude that

$$[K_{i,l}, K_{i,p-1}] \cap \mathbb{K}^E(\mathcal{P}_0^*) \neq \emptyset. \tag{3.60}$$

Moreover, we can deduce for $l \geq 1$ that $\mathbf{e}_{i,l} = E_{lf}(K_{i,l}, \mathcal{P}_{i,l-1})$, as we assumed that $\mathbf{e}_{i,l}$ is the smallest price co-linear with $\mathbf{e}_{i,p-1}$ and $\hat{\mathbf{e}}_{j'}$. This readily implies that the Legendre-Fenchel condition holds with equality on $[K_{i,l-1}, K_{i,l}]$ or $l = 0$.

In the case where $l > 0$, we can use the argument in the proof of Proposition 3.10.14 to deduce that $cc(A; K_{i,q}, K_{i,q+1}; \mathcal{P}_{i,p}) < 0$. Proposition 3.10.13 then states that the Legendre-Fenchel condition has to hold with strict inequality on the interval $[K_{i,q}, K_{j'}^E]$. Combined with the fact that $K_{i,l} \leq K_{i,p-1} < K_{j'}^E$ we obtain that $K_{i,l} \leq K_{i,q}$. In the second case where $l = 0$, it follows directly from $q \geq 0$ that $K_{i,l} \leq K_{i,q}$.

Consider now the strike $K_{i,q}$, where the above implies that $K_{i,q} \in \mathbb{K}^A(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*)$. Note first that this allows us to exclude the case where $K_{i,q} = 0$ from consideration as $0 \in \mathbb{K}^E(\mathcal{P}_0^*)$. We can further deduce from $\mathcal{P}_{i,p-1}$ being a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension that $\overline{\mathbf{a}}_{i,p-1} \geq A(K_{i,p-1}, \mathcal{P}_{i,p})$ has to hold. Note further that $K_{u+1}^A \leq K_{i,p}$ as we either have that $K_{i,p} \in \mathbb{K}^A(\mathcal{P}_0^*)$ or $K_{i,p} \in (\mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_0^*))$ with $K_{i,p}e^{-rT} = K_{j'}^E e^{-rT} > K_{u+1}^A$. We can thus conclude that the American price function has to be convex up to K^A_{u+1}. Combined with the assumption that $\overline{\mathbf{a}}_{i,p} < A(K_{i,p}e^{-rT}, \mathcal{P}_{i,p})$ we obtain that

$$\begin{aligned} cc(A; K_{i,q}, K^A_{u+1}, \mathcal{P}_{i,p}) &\leq cc(A; K_{i,q}, K_{i,q+1}; \mathcal{P}_{i,p}) \\ &< cc(\overline{A}; K_{i,p-1}e^{-rT}, K_{i,p}e^{-rT}; \mathcal{P}_{i,p}) \\ &= cc(\overline{A}; K_{i,p-1}e^{-rT}, K^E_{j'}e^{-rT}; \mathcal{P}_{i,p}) \end{aligned} \tag{3.61}$$

where the last equality holds because the upper bound $\overline{A}$ is linear on the interval $[K_{i,l}e^{-rT}, K^E_{j'}e^{-rT}]$. If we suppose now that $K_{i,q} \in (0,\infty) \cap \mathbb{K}^A(\mathcal{P}^*_0)$, it follows that $A^{lhs}_{lb}(K^E_j e^{-rT}, \mathcal{P}^*_0) > \overline{\mathbf{a}}_j$ has to hold which contradicts the Standing Assumptions.

We will now argue that $K_{i,q} \notin \mathbb{K}^{aux}_2(\mathcal{P}^*_i)$. Suppose thus for contradiction that $K_{i,q} \in \mathbb{K}^{aux}_2(\mathcal{P}^*_i)$, then there has to exist a strike $K_{i,\tilde{s}} \in \mathbb{K}^E(\mathcal{P}^*_0)$ with $K_{i,\tilde{s}}e^{-rT} = K_{i,q}$. Since we assumed that $K_{i,p}e^{-rT} \in (K_{i,q}, K_{i,q+1}]$, we must have $K_{i,\tilde{s}} < K_{i,p}$. This, however, would imply that $[K_{i,l}, K_{i,p-1}] \cap \mathbb{K}^E(\mathcal{P}^*_0) \neq \emptyset$ which yields a contradiction to (3.60).

We are now only left to argue that $K^E_{j'}e^{-rT} \neq K^A_{u+1}$. Then again, this follows immediately from (*viii*) of the Standing Assumptions and we can therefore conclude that $K^E_{j'}e^{-rT} < K^A_{u+1}$. □

3.10.5 Properties of the price functions under $\mathcal{P}^*_1$ when a violation of $\overline{A} \geq A$ occurs

Proposition 3.10.18. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by $\mathcal{P}^*_0 \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}^*_1$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}^*_0$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

If the algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}^*_1)$ due to a violation of $\overline{\mathbf{a}}_{1,p} \geq A(K_{1,p}e^{-rT}, \mathcal{P}_{1,p})$ where $K_{1,p}e^{-rT} \in (K_{1,q}, K_{1,q+1}]$ for $K_{1,q}, K_{1,q+1} \in \mathbb{K}(\mathcal{P}^*_1)$, then $[K_{1,p}e^{-rT}, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}^*_1) = \emptyset$.

Proof. Before we start we would like to point out that during the whole argument we will use the enumeration of the strikes with respect to the price set $\mathcal{P}_{1,p}$. Let us assume for contradiction that there exists a strike

$$K_{1,s} \in [K_{1,p}e^{-rT}, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}^*_1).$$

We can then immediately rule out that $K_{1,s} \in \mathbb{K}^{aux}_1(\mathcal{P}^*_1)$ as the initial set is given by $\mathcal{P}^*_1$ and thus no violation of convexity has occurred so far. Hence, the constraint at the strike $K_{1,s}$ must have been introduced to correct a violation of the upper bound. Suppose now that this violation occurred at the strike $K_{1,\tilde{p}}e^{-rT} \in [K_{1,\tilde{q}}, K_{1,\tilde{q}+1}]$ for $K_{1,\tilde{p}}, K_{1,\tilde{q}}, K_{1,\tilde{q}+1} \in \mathbb{K}(\mathcal{P}^*_1)$. We can then conclude from the way the strike $K_{1,s}$ is chosen that $K_{1,s} = K_{1,\tilde{q}+1}$. Note also that $K_{1,\tilde{p}} < K_{1,p}$ has to hold, as the algorithm did not compute option prices for non-traded strikes to the right of $K_{1,p}$ yet. We can thus conclude that $K_{1,p}e^{-rT} \in (K_{1,\tilde{p}}e^{-rT}, K_{1,s}]$, which in turn implies that $q = \tilde{q}$. Combining $K_{1,\tilde{p}} < K_{1,p}$ with the fact that $\mathcal{P}_{1,p-1}$ is a $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension, we can thus deduce that $\mathbf{a}_{1,\tilde{q}} \leq \overline{A}(K_{1,\tilde{q}}, \mathcal{P}_{1,p})$. Note further that Proposition 3.10.14 guarantees that for $j' = \arg\min\{K_r \in \mathbb{K}^E(\mathcal{P}_0^*) : K_r \geq K_{1,p}\}$ the prices $\mathbf{e}_{1,p-1}$, $\mathbf{e}_{1,p}$ and $\mathbf{e}_{j'}$ have to be co-linear. It follows further from $K_{1,s}e^{rT} \in \mathbb{K}^E(\mathcal{P}_0^*)$ that $K^E_{j'}e^{-rT} \leq K_{1,s}$. We can thus conclude from $A(K_{1,p-1}e^{-rT}, \mathcal{P}_{1,p}) \leq \overline{\mathbf{a}}_{1,p-1}$ and $A(K_{1,p}e^{-rT}, \mathcal{P}_{1,p}) > \overline{\mathbf{a}}_{1,p}$ that $A(K^E_{j'}e^{-rT}, \mathcal{P}_{1,p}) > \overline{\mathbf{a}}_{j'}$ has to hold. Then again, this contradicts the fact that the strike $K_{1,s}$ was chosen such that

$$\frac{\overline{A}(K_{1,s}, \mathcal{P}_0^*) - \mathbf{a}_{1,q}}{K_{1,s} - K_{1,q}} \leq \frac{\overline{\mathbf{a}}_v - \mathbf{a}_{1,q}}{K^E_v e^{-rT} - K_{1,q}}$$

for any strike $K^E_v e^{-rT} \in [K_{1,\tilde{q}}, K_{1,\tilde{q}+1}]$ and $K^E_v \in \mathbb{K}^E(\mathcal{P}_0^*)$. We can thus rule out that there exists a strike $K_{1,s} \in [K_{1,p}, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*)$ when $\overline{\mathbf{a}}_{1,p} < A(K_{1,p}e^{-rT}, \mathcal{P}_{1,p})$ occurs. □

Proposition 3.10.19. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

Suppose further that the algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of $\overline{\mathbf{a}}_{1,p} \geq A(K_{1,p}e^{-rT}, \mathcal{P}_{1,p})$ where $K_{1,p}e^{-rT} \in (K_{1,q}, K_{1,q+1}]$ for $K_{1,q}, K_{1,q+1} \in \mathbb{K}(\mathcal{P}_1^*)$. If $[K_{1,p}e^{-rT}, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$, then we must either have

$$\mathbf{a}_{1,q} = \max\{A^{lhs}_{lb}(K_{1,q}, \mathcal{P}_{1,q-1}), A^{\overline{A},l}_{lb}(K_{1,q}, \mathcal{P}_{1,q-1}), A_{lf}(K_{1,q}, \mathcal{P}_{1,q-1})\}$$

for $K_{1,q} \in \mathbb{K}^E(\mathcal{P}_1^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*)$ or $K_{1,q} \in \mathbb{K}^A(\mathcal{P}_1^*)$.

Proof. Suppose for contradiction that the price for American options with strike $K_{1,q} \in \mathbb{K}^E(\mathcal{P}_1^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*)$ is given by

$$\mathbf{a}_{1,q} = \max\{A^{rhs}_{lb}(K_{1,q}, \mathcal{P}_{1,q-1}), A^{\overline{A},r}_{lb}(K_{1,q}, \mathcal{P}_{1,q-1})\}.$$

We then start by pointing out that the assumption that $[K_{1,p}e^{-rT}, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ combined with $K_{1,q} \in \mathbb{K}^E(\mathcal{P}_1^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*)$ readily implies that $[K_{1,q}, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$. Hence, we can apply Proposition 3.10.5 and see that

$$\mathbf{a}_{1,s} = \max\{A^{rhs}_{lb}(K_{1,s}, \mathcal{P}_{1,s-1}), A^{\overline{A},r}_{lb}(K_{1,s}, \mathcal{P}_{1,s-1})\}$$

for any $K_{1,s} \in [K_{1,q}, K^A_{u+1})$, where $u = \arg\max\{K_s \in \mathbb{K}^A(\mathcal{P}_0^*) : K_s < K_{1,p}e^{-rT}\}$. In addition, we have seen in Proposition 3.10.17 that $K_{j'}^E e^{-rT} < K_{u+1}^A$. Then again, this would imply that

$$\overline{\mathbf{a}}_{j'} < \max\{A_{lb}^{rhs}(K_{j'}^E e^{-rT}, \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{j'}^E e^{-rT}, \mathcal{P}_0^*)\}$$

which contradicts either (v) or $(viii)$ of the Standing Assumptions. Hence, we can conclude that

$$\mathbf{a}_{1,q} = \max\{A_{lb}^{lhs}(K_{1,q}, \mathcal{P}_{1,q-1}), A_{lb}^{\overline{A},l}(K_{1,q}, \mathcal{P}_{1,q-1}), A_{lf}(K_{1,q}, \mathcal{P}_{1,q-1})\}$$

for $K_{1,q} \in \mathbb{K}^E(\mathcal{P}_1^*)\backslash\mathbb{K}^A(\mathcal{P}_1^*)$. $\square$

Let us write $w = \arg\max\{K_{1,s} \in \mathbb{K}^A(\mathcal{P}_1^*) : K_{1,s} < K_{1,p}e^{-rT}\}$, then the result above can be readily extended to the price of any American option with strike $K_{i,s} \in (K_{i,w}, K_{i,q}]$ using the argument above.

Corollary 3.10.20. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

The algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of $\overline{\mathbf{a}}_{1,p} \geq A(K_{1,p}e^{-rT}, \mathcal{P}_{1,p})$ where $K_{1,p}e^{-rT} \in (K_{1,q}, K_{1,q+1}]$ for $K_{1,q}, K_{1,q+1} \in \mathbb{K}(\mathcal{P}_1^*)$. If we assume that $[K_{1,p}e^{-rT}, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$, then we must have

$$\mathbf{a}_{1,s} = \max\{A_{lb}^{lhs}(K_{1,s}, \mathcal{P}_{1,s-1}), A_{lb}^{\overline{A},l}(K_{1,s}, \mathcal{P}_{1,s-1}), A_{lf}(K_{1,s}, \mathcal{P}_{1,s-1})\}$$

for any strike $K_{1,s} \in (K_{1,w}, K_{1,q}] \cap \mathbb{K}^E(\mathcal{P}_1^*)$.

Proposition 3.10.21. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

The algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of $\overline{\mathbf{a}}_{1,p} \geq A(K_{1,p}e^{-rT}, \mathcal{P}_{1,p})$ where $K_{1,p}e^{-rT} \in (K_{1,q}, K_{1,q+1}]$ for $K_{1,q}, K_{1,q+1} \in \mathbb{K}(\mathcal{P}_1^*)$. Setting

$$u = \arg\max\{K_s^A \in \mathbb{K}^A(\mathcal{P}_0^*) : K_s^A < K_{1,p}e^{-rT}\},$$

we must have $\hat{\mathbf{a}}_u \leq \overline{A}(K_u^A, \mathcal{P}_{i,p})$.

Proof. In the case where $K_u^A \leq K_{1,p-1}e^{-rT}$, we can deduce from $\mathcal{P}_{1,p-1}$ being a $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension that $\hat{\mathbf{a}}_u \leq \overline{A}(K_u^A, \mathcal{P}_{1,p})$.

Thus we are left to consider the case $K_u^A > K_{1,p-1}e^{-rT}$, where we assume for contradiction that $\hat{\mathbf{a}}_u > \overline{A}(K_u^A, \mathcal{P}_{1,p})$. According to Proposition 3.10.18 we can apply

Proposition 3.10.14 in this setting and are thus guaranteed that the European price function is linear on $[K_{1,p-1}, K_{j'}^E]$. Moreover, this implies that the upper bound $\overline{A}$ is linear on $[K_{1,p-1}e^{-rT}, K_{j'}^E e^{-rT}]$ as well. We then have to distinguish between the two situations where the European price function is either linear on $[K_{j'-1}^E, K_{j'}^E]$ or not. If it were we would have $\hat{\mathbf{a}}_u > \overline{A}(K_u^A, \mathcal{P}_0^*)$, as $K_{j'-1}^E \leq K_{1,p-1}$. This, however, can be ruled out due to $(viii)$ of the Standing Assumptions.

Suppose now that the European price function is not linear on $[K_{j'-1}^E, K_{j'}^E]$. In this case there has to exist a strike $K_{1,\tilde{s}} \in (K_{j'-1}^E, K_{1,p-1}] \cap (\mathbb{K}(\mathcal{P}_i^*) \backslash \mathbb{K}^E(\mathcal{P}_1^*))$ such that

$$\begin{aligned} K_{1,\tilde{s}} = \max\{K_{1,l} \in (K_{j'-1}^E, K_{1,p-1}] \cap (\mathbb{K}(\mathcal{P}_1^*) \backslash \mathbb{K}^E(\mathcal{P}_1^*)) :& \\ \mathbf{e}_{1,l} = E_{lf}(K_{1,l}, \mathcal{P}_{1,l-1}) \text{ and } \mathbf{e}_{1,l} < E_{ub}(K_{1,l}, \mathcal{P}_{1,l-1})\}.& \end{aligned}$$

In addition, we know that $\overline{\mathbf{a}}_{1,p-1} \geq A(K_{1,p-1}e^{-rT}, \mathcal{P}_{1,p-1})$ as the price set $\mathcal{P}_{1,p-1}$ is a $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension. It then follows from $\hat{\mathbf{a}}_u > \overline{A}(K_u^A, \mathcal{P}_{1,p})$ and $K_u^A > K_{1,p-1}e^{-rT}$ that there has to exist a strike $K_{1,s} \in [0, K_u^A) \cap \mathbb{K}(\mathcal{P}_1^*)$ such that

$$\begin{aligned} K_{1,s} = \min\{K_{1,l} \in [0, K_u^A) \cap \mathbb{K}(\mathcal{P}_1^*) :& \\ cc(A; K_{1,l}, K_{1,l+1}; \mathcal{P}_{1,p}) < cc(\overline{A}; K_{1,p-1}e^{-rT}, K_{1,p}e^{-rT}; \mathcal{P}_{1,p})\}.& \end{aligned}$$

Note first that the definition of $K_{1,s}$ readily implies that $\mathbf{a}_{1,s} \leq \overline{A}(K_{1,s}, \mathcal{P}_{1,p})$. The linearity of the European price function on $[K_{1,p-1}, K_{j'}^E]$, furthermore, allows us to conclude that the Legendre-Fenchel condition has to hold with strict inequality on every subinterval of $[K_{1,s}, K_{j'}^E]$. Then again, this implies that $K_{1,\tilde{s}} \leq K_{1,s}$, as we assumed that the price for European options with strike $K_{1,\tilde{s}}$ is given by $\mathbf{e}_{1,\tilde{s}} = E_{lf}(K_{1,\tilde{s}}, \mathcal{P}_{1,p})$ and $K_{1,\tilde{s}} \leq K_{1,p-1}$. We are thus given strikes

$$K_{j'-1}^E < K_{1,\tilde{s}} \leq K_{1,s} < K_u^A \leq K_{1,q} < K_{1,q+1} \leq K_{1,p} \leq K_{j'}^E.$$

If the strike $K_{1,s} \in \mathbb{K}^A(\mathcal{P}_0^*)$, then we can use $\mathbf{a}_{1,s} \leq \overline{A}(K_{1,s}, \mathcal{P}_{1,p})$ and $\hat{\mathbf{a}}_u > \overline{A}(K_u^A, \mathcal{P}_{1,p})$ together with the convexity of the American price function $A(\cdot, \mathcal{P}_0^*)$ to argue that

$$cc(A; K_{u-1}^A, K_u^A; \mathcal{P}_0^*) < cc(\overline{A}; K_{1,p-1}e^{-rT}, K_{j'}^E e^{-rT}; \mathcal{P}_{1,p}).$$

Since $\hat{\mathbf{a}}_u > \overline{A}(K_u^A, \mathcal{P}_{1,p})$ we thus obtain $A_{lb}^{lhs}(K_{j'}^E, \mathcal{P}_0^*) > \overline{\mathbf{a}}_{j'}$, a contradiction to (v) of the Standing Assumptions.

We are now only left to rule out $K_{1,s} \in \mathbb{K}_2^{aux}(\mathcal{P}_1^*)$. This, however, would mean that $K_{1,s}e^{rT} \in \mathbb{K}^E(\mathcal{P}_0^*)$. Moreover, we know that the upper bound $\overline{A}$ is linear on $[K_{1,s}, K_{j'}^E e^{-rT}]$ as $K_{1,s} \geq K_{1,\tilde{s}}$. Hence, we must have $\hat{\mathbf{a}}_u > \overline{A}(K_u^A, \mathcal{P}_0^*)$, violating $(viii)$ of the Standing Assumptions. □

First violation of the upper bound $\overline{A}$ on $[K_u^A, K_{u+1}^A]$

We will first discuss the situation where the current violation of the upper bound is the first violation of this type in $[K_u^A, K_{u+1}^A]$.

Proposition 3.10.22. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

The algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of $\overline{\mathbf{a}}_{1,p} \geq A(K_{1,p}e^{-rT}, \mathcal{P}_{1,p})$ where $K_{1,p}e^{-rT} \in (K_{1,q}, K_{1,q+1}]$ for $K_{1,q}, K_{1,q+1} \in \mathbb{K}(\mathcal{P}_1^*)$. If we assume that $[K_u^A, K_{1,p}e^{-rT}) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ for

$$u = \arg\max\{K_s^A \in \mathbb{K}^A(\mathcal{P}_0^*) : K_s^A < K_{1,p}e^{-rT}\},$$

then $\overline{A}(K_{1,s}, \mathcal{P}_{1,p}) \geq \mathbf{a}_{1,s}$ for any strike $K_{1,s} \in [K_u^A, K_{1,q}] \cap \mathbb{K}(\mathcal{P}_1^*)$.

Proof. We will use induction on s to show that this result holds. In the base step we can use Proposition 3.10.21 to argue that $\hat{\mathbf{a}}_u \leq \overline{A}(K_u^A, \mathcal{P}_{1,p})$.

In the inductive step we will assume that $\overline{A}(K_{1,s-1}, \mathcal{P}_{1,p}) \geq \mathbf{a}_{1,s-1}$ holds for $K_{1,s-1} \in [K_u^A, K_{1,q}) \cap \mathbb{K}(\mathcal{P}_1^*)$ and show that this implies $\overline{A}(K_{1,s}, \mathcal{P}_{1,p}) \geq \mathbf{a}_{1,s}$. Analogously to Proposition 3.10.21 we have to distinguish between the two cases where either $K_{1,s} \leq K_{1,p-1}e^{-rT}$ or not. In the first case $\overline{A}(K_{1,s}, \mathcal{P}_{1,p}) \geq \mathbf{a}_{1,s}$ follows immediately from $\mathcal{P}_{1,p-1}$ being a $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension.

Suppose now that $K_{1,s} > K_{1,p-1}e^{-rT}$. We begin by pointing out that according to Corollary 3.10.20

$$\mathbf{a}_{1,s} = \max\{A_{lb}^{lhs}(K_{1,s}, \mathcal{P}_{1,s-1}), A_{lb}^{\overline{A},l}(K_{1,s}, \mathcal{P}_{1,s-1}), A_{lf}(K_{1,s}, \mathcal{P}_{1,s-1})\} \tag{3.62}$$

for any strike $K_{1,s} \in (K_u^A, K_{1,q}] \cap (\mathbb{K}^E(\mathcal{P}_1^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*))$ as $K_{1,w} = K_u^A$.

Suppose for the moment that $\mathbf{a}_{1,s} = A_{lf}(K_{1,s}, \mathcal{P}_{1,s-1})$. Since the price functions $A(\cdot, \mathcal{P}_{i,p})$ and $E(\cdot, \mathcal{P}_{i,p})$ have to be convex up to $K_{1,q}$ if the algorithm stops in $K_{i,p}$ due to a violation of the upper bound in $K_{i,p}e^{-rT}$, we must have

$$\begin{aligned} cc(A; K_{i,s-1}, K_{i,s}; \mathcal{P}_{i,p}) &= cc(E; K_{i,s-1}, K_{i,s}; \mathcal{P}_{i,p}) \\ &= cc(\overline{A}; K_{i,s-1}e^{-rT}, K_{i,s}e^{-rT}; \mathcal{P}_{i,p}) \\ &\geq cc(\overline{A}; K_{i,\tilde{s}-1}, K_{i,\tilde{s}}; \mathcal{P}_{i,p}), \end{aligned}$$

for $\tilde{s} \in \{s, ..., p\}$, where the equality in the second line follows from Proposition 3.10.4. Combined with the induction hypothesis that $\mathbf{a}_{i,s-1} \leq \overline{A}(K_{i,s-1}, \mathcal{P}_{i,p})$ we readily obtain that $\mathbf{a}_{i,s} \leq \overline{A}(K_{i,s-1}, \mathcal{P}_{i,p})$ has to hold.

We assume next that the price for American options with strike $K_{1,s}$ is given by

$\mathbf{a}_{1,s} = A_{lb}^{lhs}(K_{1,s}, \mathcal{P}_{1,s-1})$. We then start by considering the case where the lower bound $A_{lb}^{lhs}(K_{1,s}, \mathcal{P}_{1,s-1}) = A_{lb}^{lhs}(K_{1,s}, \mathcal{P}_{1,w})$ for

$$A_{lb}^{lhs}(K_{1,s}, \mathcal{P}_{1,w}) = \frac{\hat{\mathbf{a}}_u - \mathbf{a}_{1,w-1}}{K_u^A - K_{1,w-1}}(K_{1,s} - K_u^A) + \hat{\mathbf{a}}_u,$$

with $w = \arg\max\{K_{1,s} \in \mathbb{K}^A(\mathcal{P}_1^*) : K_{1,s} < K_{1,p}e^{-rT}\}$. To see that $\mathbf{a}_{1,s} \le \overline{A}(K_{1,s}, \mathcal{P}_{1,p})$ holds, we have to distinguish between the cases where $K_{1,w-1} \in \mathbb{K}^E(\mathcal{P}_0^*)\backslash\mathbb{K}^A(\mathcal{P}_1^*)$, $K_{1,w-1} \in \mathbb{K}^A(\mathcal{P}_0^*)$ or $K_{1,w-1} \in \mathbb{K}_2^{aux}(\mathcal{P}_1^*)$.

In the first case the price for an American option with strike $K_{1,w-1}$ satisfies $\mathbf{a}_{1,w-1} \ge A_{lb}^{\overline{A},r}(K_{1,w-1}, \mathcal{P}_1^*)$ according to (3.23) and thus $\overline{\mathbf{a}}_{j'} \ge A_{lb}^{lhs}(K_{j'}^E e^{-rT}, \mathcal{P}_{1,w})$ has to hold for any strike $K_{j'}^E e^{-rT} \in [K_u^A, K_{u+1}^A]$, where $K_{j'}^E \in \mathbb{K}^E(\mathcal{P}_0^*)$. Note further that according to Proposition 3.10.17 $K_{j'}^E e^{-rT} < K_{u+1}^A$ has to hold for

$$j' = \arg\min\{K_s^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K_s^E \ge K_{i,p}\}.$$

It then follows that $\overline{\mathbf{a}}_{j'} \ge A_{lb}^{lhs}(K_{j'}^E e^{-rT}, \mathcal{P}_{i,w})$. Moreover, we can deduce from the fact that $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension that $\overline{\mathbf{a}}_{i,p-1} \ge A(K_{i,p-1}e^{-rT}, \mathcal{P}_{i,p})$. According to Remark 3.10.16 we further know that the prices $\overline{\mathbf{a}}_{i,p-1}$, $\overline{\mathbf{a}}_{i,p}$ and $\overline{\mathbf{a}}_{j'}$ are co-linear. Recall also that we assumed that $K_{i,p-1}e^{-rT} < K_{i,s} \le K_{i,q} \le K_{i,p}e^{-rT}$ holds. Taking into account that the American price function $A(\cdot, \mathcal{P}_{i,p})$ is linear on $[K_{i,w-1}, K_{i,s}]$ we can then conclude that $\mathbf{a}_{i,s} \le \overline{A}(K_{i,s}, \mathcal{P}_{i,p})$ has to hold.

In the second case American options with strike $K_{1,w-1}$ are traded in the market for $\mathbf{a}_{1,w-1}$. According to the Standing Assumptions we must then have that $\overline{\mathbf{a}}_{j'} \ge A_{lb}^{lhs}(K_{j'}^E e^{-rT}, \mathcal{P}_0^*)$ for any strike $K_{j'}^E e^{-rT} \in (K_u^A, K_{u+1}^A)$ with $K_{j'}^E \in \mathbb{K}^E(\mathcal{P}_0^*)$. Analogously to the first case we obtain $\mathbf{a}_{1,s} \le \overline{A}(K_{1,s}, \mathcal{P}_{1,p})$.

We are thus left to consider the case where $K_{1,w-1} \in \mathbb{K}_2^{aux}(\mathcal{P}_1^*)$. Note first that this implies that $\mathbf{a}_{1,w-1} = \overline{A}(K_{1,w-1}; \mathcal{P}_0^*)$ has to hold. According to the induction hypothesis we further know that $\mathbf{a}_{1,s-1} \le \overline{A}(K_{1,s-1}, \mathcal{P}_{1,p})$. It follows that we must have

$$cc(\overline{A}; K_{1,w-1}, K_{1,s-1}; \mathcal{P}_{1,p}) \le cc(A; K_{1,w-1}, K_{1,s-1}; \mathcal{P}_{1,p}).$$

We, moreover, know that the price functions $A(\cdot, \mathcal{P}_{i,p})$ and $E(\cdot, \mathcal{P}_{i,p})$ are convex as the algorithm stops at strike $K_{i,p}$ due to a violation of the upper bound and not due to a violation of convexity which is checked first. Hence, the upper bound $\overline{A}(\cdot, \mathcal{P}_{1,p})$ has to be convex as well. In addition, we know from the assumption that $\mathbf{a}_{1,s} = A_{lb}^{lhs}(K_{1,s}, \mathcal{P}_{1,w})$ that

$$cc(A; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}) = cc(A; K_{1,w-1}, K_u^A; \mathcal{P}_{1,p})$$

has to hold. The convexity of the American price function then implies that

$$cc(A; K_{1,w-1}, K_{1,\tilde{s}}; \mathcal{P}_{1,p}) = cc(A; K_{1,w-1}, K_u^A; \mathcal{P}_{1,p})$$

for any strike $K_{1,\tilde{s}} \in [K_u^A, K_{1,s}]$. From the convexity of the upper bound we can further deduce that

$$cc(\overline{A}; K_{1,w-1}, K_{1,s}; \mathcal{P}_{1,p}) \leq cc(\overline{A}; K_{1,w-1}, K_{1,s-1}; \mathcal{P}_{1,p}).$$

It follows that

$$cc(\overline{A}; K_{1,w-1}, K_{1,s}; \mathcal{P}_{1,p}) \leq cc(A; K_{1,w-1}, K_{1,s}; \mathcal{P}_{1,p})$$

has to hold. Combined with $\mathbf{a}_{1,w-1} = \overline{A}(K_{1,w-1}; \mathcal{P}_{1,p})$, we then obtain that $\mathbf{a}_{1,s} \leq \overline{A}(K_{1,s}; \mathcal{P}_{1,p})$ is satisfied.

Before we continue with the situation where $\mathbf{a}_{1,s} = A_{lb}^{lhs}(K_{1,s}, \mathcal{P}_{1,s-1})$, we will show that the case where $\mathbf{a}_{1,s} = A_{lb}^{\overline{A},l}(K_{1,s}, \mathcal{P}_{1,p})$ can be reduced to the situation in the previous paragraph. To this end, we assume that the price for an American option with strike $K_{1,s}$ is given by $\mathbf{a}_{1,s} = A_{lb}^{\overline{A},l}(K_{1,s}, \mathcal{P}_{1,s-1})$, where

$$A_{lb}^{\overline{A},l}(K_{1,s}, \mathcal{P}_{1,s-1}) = \frac{\hat{\mathbf{a}}_u - \overline{\mathbf{a}}_j}{K_u^A - K_j^E e^{-rT}}(K_{1,s} - K_u^A) + \hat{\mathbf{a}}_u$$

for $K_j^E e^{-rT} \in (K_{u-1}^A, K_u^A)$ and $K_j^E \in \mathbb{K}^E(\mathcal{P}_0^*)$. We can then immediately observe from $K_j^E e^{-rT} < K_u^A < K_{1,p} e^{-rT}$ that $K_j^E \leq K_{1,p-1}$ has to hold. Since we assumed that $\mathcal{P}_{1,p-1}$ is a $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension we can also conclude that $\overline{\mathbf{a}}_j \geq A(K_j^E e^{-rT}, \mathcal{P}_{1,p})$ has to hold. Note further that the prices $\overline{\mathbf{a}}_j$, $\hat{\mathbf{a}}_u$ and $\mathbf{a}_{1,s}$ have to be co-linear as we assumed that the price for an American option with strike $K_{1,s}$ is given by $\mathbf{a}_{1,s} = A_{lb}^{\overline{A},l}(K_{1,s}, \mathcal{P}_{1,p})$. The convexity of the American price function $A(\cdot, \mathcal{P}_{1,p})$ then allows us to conclude that $A(K_j^E e^{-rT}, \mathcal{P}_{1,p}) = \overline{\mathbf{a}}_j$ has to hold. Hence, we have successfully reduced this case to the one in the previous paragraph and can therefore deduce that $\mathbf{a}_{1,s} \leq \overline{A}(K_{1,s}, \mathcal{P}_{1,p})$.

This leaves us with the situation where $\mathbf{a}_{1,s} = A_{lb}^{lhs}(K_{1,s}, \mathcal{P}_{1,s-1})$ and

$$A_{lb}^{lhs}(K_{1,s}, \mathcal{P}_{1,s-1}) > A_{lb}^{lhs}(K_{1,s}, \mathcal{P}_{1,w}). \tag{3.63}$$

In this case there has to exist a strike $K_{1,\tilde{s}} \in (K_u^A, K_{1,s}] \cap (\mathbb{K}^E(\mathcal{P}_1^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*))$ with

$$\begin{aligned} K_{1,\tilde{s}} = \max\{K_{1,l} \in (K_u^A, K_{1,s}] \cap \mathbb{K}^E(\mathcal{P}_1^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*) :& \\ \mathbf{a}_{1,l} = \max\{A_{lb}^{\overline{A},l}(K_{1,l}, \mathcal{P}_{1,l-1}), A_{lf}(K_{1,l}, \mathcal{P}_{1,l-1})\}\}&, \end{aligned}$$

since we ruled out a price given by $\mathbf{a}_{1,\tilde{s}} = \max\{A_{lb}^{rhs}(K_{1,\tilde{s}}, \mathcal{P}_1^*), A_{lb}^{\overline{A},r}(K_{1,\tilde{s}}, \mathcal{P}_1^*)\}$ in (3.62).

Suppose first that $\mathbf{a}_{1,\tilde{s}} = A_{lb}^{\overline{A},l}(K_{1,\tilde{s}}, \mathcal{P}_{1,\tilde{s}-1})$, then $\mathbf{a}_{1,s} = A_{lb}^{\overline{A},l}(K_{1,s}, \mathcal{P}_{1,s-1})$ has to hold as well. We can therefore apply the argument above to conclude that $\mathbf{a}_{1,s} \leq \overline{A}(K_{1,s}, \mathcal{P}_{1,p})$.

Let us now assume that $\mathbf{a}_{1,\tilde{s}} = A_{lf}(K_{1,\tilde{s}}, \mathcal{P}_{1,\tilde{s}-1})$ then we must have

$$cc(A; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}) = cc(A; K_{1,\tilde{s}-1}, K_{1,\tilde{s}}; \mathcal{P}_{1,p})$$

if $\mathbf{a}_{1,s}$ is given by (3.62). As the Legendre-Fenchel condition holds with equality on $[K_{1,\tilde{s}-1}, K_{1,\tilde{s}}]$ we also have $cc(E; K_{1,\tilde{s}-1}, K_{1,\tilde{s}}; \mathcal{P}_{1,p}) = cc(A; K_{1,\tilde{s}-1}, K_{1,\tilde{s}}; \mathcal{P}_{1,p})$. Combined with the convexity of the European price function $E(\cdot, \mathcal{P}_{1,p})$ we obtain

$$\begin{aligned} cc(E; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}) &\leq cc(E; K_{1,\tilde{s}-1}, K_{1,\tilde{s}}; \mathcal{P}_{1,p}) \\ &= cc(A; K_{1,\tilde{s}-1}, K_{1,\tilde{s}}; \mathcal{P}_{1,p}) \\ &= cc(A; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}). \end{aligned}$$

Then again, we know that the Legendre-Fenchel condition holds on $[K_{1,s-1}, K_{1,s}]$ as $\mathcal{P}_{1,p-1}$ is a $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension and thus

$$cc(E; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}) = cc(A; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}).$$

However, this means that the price $\mathbf{a}_{1,s}$ is given by $\mathbf{a}_{1,s} = A_{lf}(K_{1,s}, \mathcal{P}_{1,s-1})$ and we argued already at the beginning that $\mathbf{a}_{1,s} \leq \overline{A}(K_{1,s}, \mathcal{P}_{1,p})$ in this case. $\square$

Proposition 3.10.23. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

The algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of $\overline{\mathbf{a}}_{1,p} \geq A(K_{1,p}e^{-rT}, \mathcal{P}_{1,p})$ where $K_{1,p}e^{-rT} \in (K_{1,q}, K_{1,q+1}]$ for $K_{1,q}, K_{1,q+1} \in \mathbb{K}(\mathcal{P}_1^*)$. If we assume that $[K_u^A, K_{1,p}e^{-rT}] \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ for

$$u = \arg\max\{K_s^A \in \mathbb{K}^A(\mathcal{P}_0^*) : K_s^A < K_{1,p}e^{-rT}\},$$

then

$$\overline{\mathbf{a}}_j \geq \frac{\mathbf{a}_{1,q} - \mathbf{a}_{1,q-1}}{K_{1,q} - K_{1,q-1}}(K_j^E e^{-rT} - K_{1,q}) + \mathbf{a}_{1,q} \tag{3.64}$$

for any strike $K_j^E e^{-rT} \in [K_{1,q}, K_{1,q+1}]$ with $K_j^E \in \mathbb{K}^E(\mathcal{P}_0^*)$.

Proof. We are required to distinguish between the two situations where either $K_{1,q} \in \mathbb{K}^E(\mathcal{P}_0^*)\backslash\mathbb{K}^A(\mathcal{P}_1^*)$ or $K_{1,q} \in \mathbb{K}^A(\mathcal{P}_1^*)$. Since we assumed that $[K_u^A, K_{1,p}e^{-rT}] \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ the second case can further be reduced to $K_{1,q} \in \mathbb{K}^A(\mathcal{P}_0^*)$.

We begin by discussing the case where $K_{1,q} \in \mathbb{K}^E(\mathcal{P}_0^*)\backslash\mathbb{K}^A(\mathcal{P}_1^*)$. According to

Corollary 3.10.20 the price for an American option with strike $K_{1,q}$ is given by

$$\mathbf{a}_{1,s} = \max\{A_{lb}^{lhs}(K_{1,s}, \mathcal{P}_{1,s-1}), A_{lb}^{\overline{A},l}(K_{1,s}, \mathcal{P}_{1,s-1}), A_{lf}(K_{1,s}, \mathcal{P}_{1,s-1})\}.$$

To see that the inequality in (3.64) holds we will consider the different price possibilities separately. We begin by assuming that $\mathbf{a}_{1,q} = A_{lf}(K_{1,q}, \mathcal{P}_{1,q-1})$. Let us further assume for contradiction that inequality (3.64) is violated. Combined with the fact that $\mathbf{a}_{1,q} \leq \overline{A}(K_{1,q}, \mathcal{P}_{1,p})$ holds according to Proposition 3.10.22, there has to exist a strike $K_{1,s}$ such that $K_{1,s}e^{-rT} \in (K_{1,q}, K_j^E]$ such that

$$cc(\overline{A}; K_{1,s-1}e^{-rT}, K_{1,s}e^{-rT}; \mathcal{P}_{1,p}) > cc(A; K_{1,q-1}, K_{1,q}; \mathcal{P}_{1,p}).$$

This however, implies that

$$\begin{aligned} cc(\overline{A}; K_{1,s-1}e^{-rT}, K_{1,s}e^{-rT}; \mathcal{P}_{1,p}) &> cc(A; K_{1,q-1}, K_{1,q}; \mathcal{P}_{1,p}) \\ &= cc(E; K_{1,q-1}, K_{1,q}; \mathcal{P}_{1,p}) \\ &= cc(\overline{A}; K_{1,q-1}e^{-rT}, K_{1,q}e^{-rT}; \mathcal{P}_{1,p}) \end{aligned}$$

which contradicts the fact that the upper bound $\overline{A}(\cdot, \mathcal{P}_{1,p})$ is convex. Hence, we can rule out $\mathbf{a}_{1,q} = A_{lf}(K_{1,q}, \mathcal{P}_{1,q-1})$.

Next we will consider the case where $\mathbf{a}_{1,q} = A_{lb}^{\overline{A},l}(K_{1,q}, \mathcal{P}_{1,q-1})$. If we assume that the inequality in (3.64) does not hold, it follows that $\hat{\mathbf{a}}_u > \overline{A}(K_u^A, \mathcal{P}_0^*)$ which can be ruled out according to (v) of the Standing Assumptions.

We are now left to discuss the situation where $\mathbf{a}_{1,q} = A_{lb}^{lhs}(K_{1,q}, \mathcal{P}_{1,q-1})$. To do so, we have to distinguish between the two cases where the left hand-side lower bound is either given by $A_{lb}^{lhs}(K_{1,q}, \mathcal{P}_{1,w})$ or $A_{lb}^{lhs}(K_{1,q}, \mathcal{P}_{1,q-1})$ with $A_{lb}^{lhs}(K_{1,q}, \mathcal{P}_{1,q-1}) > A_{lb}^{lhs}(K_{1,q}, \mathcal{P}_{1,w})$.

Let us consider first the situation where $\mathbf{a}_{1,q} = A_{lb}^{lhs}(K_{1,q}, \mathcal{P}_{1,w})$ and $K_{1,w-1} \in \mathbb{K}^A(\mathcal{P}_0^*)$, then the left hand-side lower bound is given by the prices of two traded American options as $[K_u^A, K_{1,p}e^{-rT}] \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$. In this case we can deduce from (v) of the Standing Assumptions that $\overline{\mathbf{a}}_j \geq A_{lb}^{lhs}(K_{1,q}, \mathcal{P}_{1,w})$ has to hold. If $K_{1,w-1} \in \mathbb{K}_2^{aux}(\mathcal{P}_1^*)$, then we must have $\mathbf{a}_{1,w-1} = \overline{A}(K_{1,w-1}, \mathcal{P}_{1,p})$ for $K_{1,w-1}e^{rT} \in \mathbb{K}^E(\mathcal{P}_0^*)$ and thus (3.64) has to hold according to $(viii)$ of the Standing Assumptions. Alternatively, we may have $K_{1,w-1} \in \mathbb{K}^E(\mathcal{P}_0^*)$ which means that the algorithm determined $\mathbf{a}_{1,w-1}$ using (3.23) and thus $\mathbf{a}_{1,w-1} \geq A_{lb}^{\overline{A},r}(K_{1,w-1}, \mathcal{P}_0^*)$ has to hold, readily implying (3.64).

Suppose now that $A_{lb}^{lhs}(K_{1,q}, \mathcal{P}_{1,q-1})$ with $A_{lb}^{lhs}(K_{1,q}, \mathcal{P}_{1,q-1}) > A_{lb}^{lhs}(K_{1,q}, \mathcal{P}_{1,w})$. In this situation there has to exist a strike

$$K_{1,\tilde{s}} = \max\{K_{1,l} \in (K_u^A, K_{1,q}] : \mathbf{a}_{1,l} = \max\{A_{lb}^{\overline{A},l}(K_{1,l}, \mathcal{P}_{1,l-1}), A_{lf}(K_{1,l}, \mathcal{P}_{1,l-1})\}\}$$

as neither $\mathbf{a}_{1,\tilde{s}} = A_{lb}^{\overline{A},r}(K_{1,l}, \mathcal{P}_{1,l-1})$ nor $\mathbf{a}_{1,\tilde{s}} = A_{lb}^{rhs}(K_{1,l}, \mathcal{P}_{1,l-1})$ are possible according to Corollary 3.10.20. Let us assume for the moment that $\mathbf{a}_{1,\tilde{s}} = A_{lb}^{\overline{A},l}(K_{1,\tilde{s}}, \mathcal{P}_{1,\tilde{s}-1})$, then we must also have $\mathbf{a}_{1,q} = A_{lb}^{\overline{A},l}(K_{1,q}, \mathcal{P}_{1,q-1})$ and we can thus argue as above to see that the inequality in (3.64) has to hold. Analogously, we can deduce from $\mathbf{a}_{1,\tilde{s}} = A_{lf}(K_{1,\tilde{s}}, \mathcal{P}_{1,\tilde{s}-1})$ that $\mathbf{a}_{1,q} = A_{lf}(K_{1,q}, \mathcal{P}_{1,q-1})$ and thus (3.64) has to hold.

We are thus left to consider the case where $K_{1,q} \in \mathbb{K}^A(\mathcal{P}_0^*)$. Note, however, that the arguments used to show that (3.64) holds in case that $\mathbf{a}_{1,q} = A_{lb}^{lhs}(K_{1,q}, \mathcal{P}_{1,w})$ apply here as well, since the argument uses the co-linearity of the prices $\mathbf{a}_{1,w-1}$, $\mathbf{a}_{1,w}$ and $\mathbf{a}_{1,q}$ to draw conclusions about the prices $\mathbf{a}_{1,w-1}$, $\mathbf{a}_{1,w}$ and $\overline{\mathbf{a}}_j$. We have therefore shown that the inequality in (3.64) indeed has to hold. □

Proposition 3.10.24. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

The algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of $\overline{\mathbf{a}}_{1,p} \geq A(K_{1,p}e^{-rT}, \mathcal{P}_{1,p})$ where $K_{1,p}e^{-rT} \in (K_{1,q}, K_{1,q+1}]$ for $K_{1,q}, K_{1,q+1} \in \mathbb{K}(\mathcal{P}_1^*)$. If we assume that $[K_u^A, K_{1,p}e^{-rT}] \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ for

$$u = \arg\max\{K_s^A \in \mathbb{K}^A(\mathcal{P}_0^*) : K_s^A < K_{1,p}e^{-rT}\},$$

then we must either have

$$\mathbf{a}_{1,q+1} = \max\{A_{lb}^{rhs}(K_{1,q+1}, \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{1,q+1}, \mathcal{P}_0^*)\}$$

for $K_{1,q+1} \in \mathbb{K}^E(\mathcal{P}_1^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*)$ or $K_{1,q+1} \in \mathbb{K}^A(\mathcal{P}_0^*)$.

Proof. Note first that we only need to consider the case where the strike $K_{1,q+1} \in \mathbb{K}^E(\mathcal{P}_1^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*)$, as $K_{1,q+1} \notin \mathbb{K}^{aux}(\mathcal{P}_1^*)$. Suppose for contradiction that the price for an American option with strike $K_{1,q+1}$ was computed to be given by

$$\mathbf{a}_{1,q+1} = \max\{A_{lb}^{lhs}(K_{1,q+1}, \mathcal{P}_{1,q}), A_{lb}^{\overline{A},l}(K_{1,q+1}, \mathcal{P}_{1,q}), A_{lf}(K_{1,q+1}, \mathcal{P}_{1,q})\}.$$

We thus have to consider the situations where $\mathbf{a}_{1,q+1} = A_{lb}^{lhs}(K_{1,q+1}, \mathcal{P}_{1,q})$, $\mathbf{a}_{1,q+1} = A_{lb}^{\overline{A},l}(K_{1,q+1}, \mathcal{P}_{1,q})$ or $\mathbf{a}_{1,q+1} = A_{lf}(K_{1,q+1}, \mathcal{P}_{1,q})$ separately. The first case where the price for American options with strike $K_{1,q+1}$ is given by $\mathbf{a}_{1,q+1} = A_{lb}^{lhs}(K_{1,q+1}, \mathcal{P}_{1,q})$ can be ruled out according to Proposition 3.10.23 as the right hand-side in (3.64) corresponds to the left hand-side lower bound.

In the second case we assume that $\mathbf{a}_{1,q+1} = A_{lb}^{\overline{A},l}(K_{i,q+1}, \mathcal{P}_0^*)$. Proposition 3.10.22 then states that $\mathbf{a}_{1,q} \leq \overline{A}(K_{1,q}, \mathcal{P}_{1,p})$. From Proposition 3.10.14 we further know that the prices $\mathbf{e}_{1,p-1}$, $\mathbf{e}_{1,p}$ and $\hat{\mathbf{e}}_{j'}$ have to be co-linear. Moreover, we argued in Proposition 3.10.17 that $K_{j'}^E e^{-rT} < K_{u+1}^A$ has to hold. Combined this yields $\overline{\mathbf{a}}_{j'} <$

$A_{lb}^{\overline{A},l}(K_{j'}e^{-rT}, \mathcal{P}_0^*)$. Then again, this would imply that $\hat{\mathbf{a}}_u > \overline{A}(K_u^A, \mathcal{P}_0^*)$ and therefore violate the Standing assumptions. We can thus rule out $\mathbf{a}_{1,q+1} = A_{lb}^{\overline{A},l}(K_{i,q+1}, \mathcal{P}_0^*)$ as well.

In the third case where $\mathbf{a}_{1,q+1} = A_{lf}(K_{1,q+1}, \mathcal{P}_{1,q})$, we have to distinguish between the two cases where $cc(A; K_{1,q}, K_{1,q+1}; \mathcal{P}_{1,p}) < 0$ or not. Suppose first that $cc(A; K_{1,q}, K_{1,q+1}; \mathcal{P}_{1,p}) < 0$, then we can combine $\overline{\mathbf{a}}_{1,p} < A(K_{1,p}e^{-rT}, \mathcal{P}_{1,p})$ with the fact from Proposition 3.10.22 that $\mathbf{a}_{1,q} \leq \overline{A}(K_{1,q}, \mathcal{P}_{1,p})$ to obtain

$$\begin{aligned} cc(A; K_{1,q}, K_{1,q+1}; \mathcal{P}_{1,p}) &< cc(\overline{A}; K_{1,p-1}e^{-rT}, K_{1,p}e^{-rT}; \mathcal{P}_{1,p}) \\ &= cc(E; K_{1,p-1}, K_{1,p}; \mathcal{P}_{1,p}) \\ &\leq cc(E; K_{1,q}, K_{1,q+1}; \mathcal{P}_{1,p}) \end{aligned}$$

This, however, contradicts the assumption that the price for American options with strike $K_{1,q+1}$ was computed to be $\mathbf{a}_{1,q+1} = A_{lf}(K_{1,q+1}, \mathcal{P}_{1,q})$.

In the case where $cc(A; K_{1,q}, K_{1,q+1}; \mathcal{P}_{1,p}) = 0$, we argued in the proof of Proposition 3.10.14 that $p = 1$. This, however, means that $\mathbf{a}_{1,p} = \mathbf{e}_{1,p}$ and thus $\overline{\mathbf{a}}_{1,p} < A(K_{1,p}, \mathcal{P}_{1,p})$ can be ruled out.

We can therefore conclude that the price for an American option with strike $K_{1,q+1}$ has to be given by

$$\mathbf{a}_{1,q+1} = \max\{A_{lb}^{rhs}(K_{1,q+1}, \mathcal{P}_{1,q}), A_{lb}^{\overline{A},r}(K_{1,q+1}, \mathcal{P}_{1,q})\}$$

for $K_{1,q+1} \in \mathbb{K}^E(\mathcal{P}_1^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*)$ or $K_{1,q+1} \in \mathbb{K}^A(\mathcal{P}_0^*)$. □

3.10.6 Properties of the price functions under $\mathcal{P}_1^*$ when a violation of convexity occurs

Proposition 3.10.25. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

If we assume that the algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of convexity, then $K_{1,s} < K_{1,p}$ where $K_{1,s}e^{-rT} = \max\{K \in \mathbb{K}_2^{aux}(\mathcal{P}_1^*)\}$.

Remark 3.10.26. *This readily implies that $\mathcal{P}_{1,s}$ is a $K_{1,s}$-admissible $\mathcal{P}_0^*$-extension.*

Remark 3.10.27. *Note that it is theoretically possible that $K_{1,s} > K_{1,p}$ although the strike at which the violation of the upper bound is detected has to be strictly to the left of $K_{1,p}$. The reason being that the auxiliary price constraint is not necessarily introduced at the strike where the violation occurs but at a strike chosen by (3.28).*

Proof. Using the convention that $\max\{\emptyset\} = -\infty$, we readily obtain for $\mathbb{K}_2^{aux}(\mathcal{P}_1^*) = \emptyset$ that $K_{1,s} < K_{1,p}$. We therefore only need to consider the case where $\mathbb{K}_2^{aux}(\mathcal{P}_1^*) \neq \emptyset$ in the sequel.

Suppose for contradiction that we have $K_{1,s} \geq K_{1,p}$ but note that $K_{1,s}e^{-rT} < K_{1,p}$ has to hold as the auxiliary constraint is introduced between the same two strikes of $\mathbb{K}(\mathcal{P}_i^*)$ where the violation occurs. Let us further define

$$K_{1,r} = \min\{K_{1,\tilde{r}} \in (K_{1,s}e^{-rT}, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*) : \\ cc(A; K_{1,\tilde{r}-1}, K_{1,\tilde{r}}; \mathcal{P}_{1,p}) = cc(E; K_{1,\tilde{r}-1}, K_{1,\tilde{r}}; \mathcal{P}_{1,p})\}. \quad (3.65)$$

Since a violation of convexity at strike $K_{1,p}$ can only occur if the Legendre-Fenchel condition holds with equality on $[K_{1,p-1}, K_{1,p}]$, we can conclude that $K_{1,r}$ has to exist. If we, moreover, assume that $K_{1,s}e^{-rT} = K_{1,q+1}$ under $\mathcal{P}_{1,p}$, then we can conclude from the previous violation of the upper bound at that strike that

$$cc(A; K_{1,q+1}, K_{1,q+2}; \mathcal{P}_{1,p}) < cc(A; K_{1,q}, K_{1,q+2}; \mathcal{P}_{1,p}) \\ < cc(A; K_{1,q}, K_{1,q+1}; \mathcal{P}_{1,p}). \quad (3.66)$$

Note also that the European price function remains unchanged on $[K_{1,q}, K_{1,q+2}]$. Combined with the fact that the price functions satisfied the Legendre-Fenchel condition prior to the introduction of the auxiliary price constraint at $K_{1,q+1}$, we can conclude that it now has to hold with strict inequality on $[K_{1,q+1}, K_{1,q+2}]$. It then follows by the definition of $K_{1,r}$ that $\mathbf{e}_{1,l} = E_{ub}(K_{1,l}, \mathcal{P}_{1,l-1})$ for any strike $K_{1,l} \in (K_{1,q+1}, K_{1,r-1}] \cap (\mathbb{K}^A(\mathcal{P}_1^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*))$.

Suppose that $K_{1,l} \in (K_v^E, K_{v+1}^E) \cap \mathbb{K}^A(\mathcal{P}_1^*)$, then we can apply Proposition 3.10.5 as $K_{1,q+1} = \max\{K \in \mathbb{K}_2^{aux}(\mathcal{P}_1^*)\}$ implies that $[K_{1,l}, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$. It follows that $\mathbf{e}_{1,\tilde{l}} = E_{ub}(K_{1,\tilde{l}}, \mathcal{P}_{1,\tilde{l}-1})$ for any strike $K_{1,\tilde{l}} \in [K_{1,l}, K_{v+1}^E)$. Hence the European price function has to be linear on $[K_{1,l}, K_{v+1}^E]$. This readily implies that $cc(E; K_{1,l}, K_{v+1}^E; \mathcal{P}_{1,p}) = cc(E; K_{1,\tilde{l}}, K_{1,\tilde{l}+1}; \mathcal{P}_{1,p})$.

Suppose for contradiction that $K_{1,r-1} \in \mathbb{K}^A(\mathcal{P}_0^*)$, then we know that the price for European options with strike $K_{1,r-1}$ has to be given by $\mathbf{e}_{1,r-1} = E_{ub}(K_{1,r-1}, \mathcal{P}_{1,r-2})$ as the Legendre-Fenchel condition holds with strict inequality according to the definition of $K_{1,r}$. Then again, we know that

$$cc(A; K_{1,r-1}, K_{1,r}; \mathcal{P}_{1,p}) \leq cc(A; K_{1,r-2}, K_{1,r-1}; \mathcal{P}_{1p})$$

which implies that

$$cc(A; K_{1,r-1}, K_{1,r}; \mathcal{P}_{1,p}) \leq cc(A; K_{1,r-2}, K_{1,r-1}; \mathcal{P}_{1p})$$

$$
\begin{aligned}
&< cc(E; K_{1,r-2}, K_{1,r-1}; \mathcal{P}_{1p}) \\
&= cc(E; K_{1,r-1}, K_{1,r}; \mathcal{P}_{1p}).
\end{aligned}
$$

Hence, the Legendre-Fenchel condition holds with strict inequality on the interval $[K_{1,r-1}, K_{1,r}]$, thereby contradicting the definition of $K_{1,r}$. We can therefore conclude that $K_{1,r-1} \in \mathbb{K}^E(\mathcal{P}_0^*)$.

We can furthermore deduce from $K_{1,s} \in \mathbb{K}^E(\mathcal{P}_0^*)$ that there has to exist a strike K_j^E for $j \in \{1, ..., m_2\}$ with $K_j^E = K_{1,s}$. According to (3.28) we must then have that

$$
cc(A; K_{1,q}, K_{1,q+1}; \mathcal{P}_{1,p}) \leq cc(\overline{A}; K_{j-1}^E e^{-rT}, K_j^E e^{-rT}; \mathcal{P}_0^*), \tag{3.67}
$$

as $\overline{\mathbf{a}}_j = A(K_{1,q+1}, \mathcal{P}_1^*)$ and $\mathbf{a}_{1,q} \leq \overline{A}(K_{1,q}, \mathcal{P}_{1,p})$.

According to (3.22) the algorithm computes the price for a European option with strike $K_{1,l} \in \mathbb{K}^A(\mathcal{P}_1^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*)$ to be

$$
\mathbf{e}_{1,l} = \min\{E_{lf}(K_{1,l}, \mathcal{P}_{1,l-1}), E_{ub}(K_{1,l}, \mathcal{P}_{1,l-1})\}
$$

which readily implies that $cc(E; K_{1,l-1}, K_{1,l}; \mathcal{P}_{1,l}) \geq cc(E, E_{ub}; K_{1,l-1}, K_{1,l}; \mathcal{P}_{1,l})$. If $K_{1,l-1} \in \mathbb{K}^E(\mathcal{P}_0^*)$ we must thus have that

$$
cc(E; K_{1,l-1}, K_{1,l}; \mathcal{P}_{1,l}) \geq cc(E; K_{1,l-1}, K_{1,l}; \mathcal{P}_0^*).
$$

Combined with the result in Proposition 3.10.4 we obtain that

$$
cc(\overline{A}; K_{1,l-1}e^{-rT}, K_{1,l}e^{-rT}; \mathcal{P}_{1,l}) \geq cc(\overline{A}; K_{1,l-1}e^{-rT}, K_{1,l}e^{-rT}; \mathcal{P}_0^*). \tag{3.68}
$$

Taking into account that the Legendre-Fenchel condition holds with equality on $[K_{1,r-1}, K_{1,r}]$, we can finally combine the inequalities in (3.66),(3.67) and (3.68) to obtain

$$
\begin{aligned}
cc(\overline{A}; K_{1,r-1}e^{-rT}, K_{1,r}e^{-rT}; \mathcal{P}_{1,p}) &= cc(E; K_{1,r-1}, K_{1,r}; \mathcal{P}_{1,p}) \\
&= cc(A; K_{1,r-1}, K_{1,r}; \mathcal{P}_{1,p}) \\
&\leq cc(A; K_{1,q}, K_{1,q+2}; \mathcal{P}_{1,p}) \\
&< cc(A; K_{1,q}, K_{1,q+1}; \mathcal{P}_{1,p}) \\
&\leq cc(\overline{A}; K_{j-1}^E e^{-rT}, K_j^E e^{-rT}; \mathcal{P}_0^*) \\
&\leq cc(\overline{A}; K_{1,r-1}e^{-rT}, K_{1,r}e^{-rT}; \mathcal{P}_{1,p}).
\end{aligned}
$$

Since this is impossible, we can conclude that the Legendre-Fenchel condition has to hold with strict inequality up to $K_{1,s}$. This, however, readily implies that $K_{1,s} < K_{1,p}$. □

Proposition 3.10.28. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

The algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of convexity and starts revising the prices for non-traded options using Algorithm 3. If we write

$$K_{1,\tilde{q}} = \max\{K_{1,s} \in [0, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*) : \\ cc(A; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}) < cc(E; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p})\},$$

then $[K_{1,\tilde{q}}, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ has to hold.

Proof. As the algorithm uses the initial set $\mathcal{P}_1^*$ to compute the American and European price functions a violation of convexity prior to $K_{1,p}$ can be ruled out and thus $\mathbb{K}^{aux}(\mathcal{P}_1^*) = \mathbb{K}_2^{aux}(\mathcal{P}_1^*)$ has to hold. Having never restarted the algorithm before we can further conclude that $[K_{1,p}, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$. Hence, we are left to argue that $[K_{1,\tilde{q}}, K_{1,p}) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ holds as well.

Suppose now for contradiction that

$$K_{1,r} = \max\{K_{1,\tilde{r}} \in [K_{1,\tilde{q}}, K_{1,p}) \cap \mathbb{K}_2^{aux}(\mathcal{P}_1^*)\}$$

exists. To detect a violation of the upper bound at $K_{1,r}$ the algorithm has to have computed option prices up to the strike $K_{1,r}e^{rT} \in \mathbb{K}(\mathcal{P}_1^*)$. Since $K_{1,r+1} = \min\{K \in \mathbb{K}(\mathcal{P}_1^*) : K > K_{1,r}\}$ it readily follows that $K_{1,r}e^{rT} \geq K_{1,r+1}$ has to hold. Taking into account that the algorithm prices non-traded options so that the Legendre-Fenchel condition holds, we can deduce that the Legendre-Fenchel condition has to hold on $[K_{1,r-1}, K_{1,r+1}]$ prior to the introduction of the auxiliary price constraint at $K_{1,r}$. Since the new constraint reduces the price for American options with strike $K_{1,r}$ it follows that the Legendre-Fenchel condition now has to hold with strict inequality on $[K_{1,r}, K_{1,r+1}]$. According to the definition of $K_{1,\tilde{q}}$ we must then have $K_{1,\tilde{q}} \geq K_{1,r+1}$. It then follows that $K_{1,r} < K_{1,\tilde{q}}$ yielding a contradiction. We can therefore rule out this situation as well and thus obtain that $[K_{1,\tilde{q}}, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$. □

Proposition 3.10.29. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

The algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of convexity and starts revising the prices for non-traded options using Algorithm 3. If the algorithm *reaches the strike* $K_{1,\tilde{q}}$*, where*

$$K_{1,\tilde{q}} = \max\{K_{1,s} \in [0, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*) : \\ cc(A; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}) < cc(E; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p})\},$$

without finding an arbitrage, then $K_{1,\tilde{q}} \in (\mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*))$ *has to hold.*

Proof. Observe that a violation of convexity at the strike $K_{1,p}$ can only occur if the price functions $A(\cdot, \mathcal{P}_{1,p})$ and $E(\cdot, \mathcal{P}_{1,p})$ satisfy the Legendre-Fenchel condition with equality. According to the definition of $K_{1,\tilde{q}}$ we can then conclude that the Legendre-Fenchel condition has to hold with equality on $[K_{1,\tilde{q}}, K_{1,p}]$. Hence, Proposition 3.10.11 can be applied to see that $\mathbf{a}^n_{1,\tilde{q}} < \mathbf{a}_{1,\tilde{q}}$.

We proceed by excluding $K_{1,\tilde{q}} \in \mathbb{K}^E(\mathcal{P}_0^*) \cap \mathbb{K}^A(\mathcal{P}_1^*)$ and $K_{1,\tilde{q}} \in \mathbb{K}^A(\mathcal{P}_1^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*)$ from consideration. Suppose first that $K_{1,\tilde{q}} \in \mathbb{K}^E(\mathcal{P}_0^*) \cap \mathbb{K}^A(\mathcal{P}_1^*)$. Since this implies that American options are traded in the market for $\mathbf{a}_{1,\tilde{q}}$ it follows that $A_{lb}(K_{1,\tilde{q}}, \mathcal{P}_0^*) = \mathbf{a}_{1,\tilde{q}}$ holds. Depending on whether $K_{1,p} \in \mathbb{K}^A(\mathcal{P}_0^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*)$ or $K_{1,p} \in \mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_0^*)$ we can then either use Proposition 3.6.14 or Proposition 3.6.18 to conclude that there has to exist arbitrage in the market.

Consider now the situation where $K_{1,\tilde{q}} \in (\mathbb{K}^A(\mathcal{P}_1^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*))$. According to the definition of $K_{1,\tilde{q}}$ we know that the Legendre-Fenchel condition has to hold with strict inequality on $[K_{1,\tilde{q}-1}, K_{1,\tilde{q}}]$. Hence, the price for a European option with strike $K_{1,\tilde{q}}$ has to have been determined by Algorithm 2 to be $\mathbf{e}_{1,\tilde{q}} = E_{ub}(K_{1,\tilde{q}}, \mathcal{P}_{1,\tilde{q}-1})$. Taking into account that $[K_{1,\tilde{q}}, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ according to Proposition 3.10.28 we can apply Proposition 3.10.5 to see that $\mathbf{e}_{1,s} = E_{ub}(K_{1,s}, \mathcal{P}_{1,s-1})$ has to hold for any strike $K_{1,s} \in [K_{1,\tilde{q}}, K_j^E)$ where

$$K_j^E = \min\{K_r^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K > K_{1,\tilde{q}}\}.$$

Then again, this would imply that the price functions $A(\cdot, \mathcal{P}_{1,p})$ and $E(\cdot, \mathcal{P}_{1,p})$ satisfy the Legendre-Fenchel condition with strict inequality on $[K_{1,\tilde{q}}, K_j^E]$ which yields a contradiction to the definition of the strike $K_{1,\tilde{q}}$. We can therefore conclude that $K_{1,\tilde{q}} \in (\mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*))$. □

Proposition 3.10.30. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by* $\mathcal{P}_0^* \in \overline{\mathcal{M}}$*. Starting with the initial set of prices* $\mathcal{P}_1^*$ *the algorithm computes the* $K_{1,p-1}$*-admissible* $\mathcal{P}_0^*$*-extension* $\mathcal{P}_{1,p-1}$ *for* $p \geq 1$*.*

Algorithm 2 stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ *due to a violation of convexity. Suppose further that Algorithm 3 revises option prices for non-traded options on* $[K_{1,\tilde{q}}, K_{1,p}]$*,* *where*

$$K_{1,\tilde{q}} = \max\{K_{1,s} \in [0, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*) : \\ cc(A; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}) < cc(E; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p})\}$$

without finding an arbitrage. If we write

$$K_u^A = \max\{K_s^A \in \mathbb{K}^A(\mathcal{P}_0^*) : K_s^A < K_{1,\tilde{q}}\}$$

then $[K_u^A, K_{1,\tilde{q}}) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ *has to hold.*

Remark 3.10.31. *Note that combining Proposition 3.10.28 with Proposition 3.10.30 we readily obtain that* $[K_u^A, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ *has to hold.*

Proof. As the algorithm uses the initial set $\mathcal{P}_1^*$ to compute the American and European price functions a violation of convexity prior to $K_{1,p}$ can be ruled out and thus $\mathbb{K}^{aux}(\mathcal{P}_1^*) = \mathbb{K}_2^{aux}(\mathcal{P}_1^*)$ has to hold.

We are thus left to show that $[K_u^A, K_{1,\tilde{q}}] \cap \mathbb{K}_2^{aux}(\mathcal{P}_1^*) = \emptyset$. For that purpose let us assume for contradiction that there exists $K_{\tilde{r}}(\mathcal{P}_{1,p}) = \min\{K \in (K_u^A, K_{1,\tilde{q}}) \cap \mathbb{K}_2^{aux}(\mathcal{P}_1^*)\}$. Suppose further that the auxiliary price constraint at $K_{\tilde{r}}(\mathcal{P}_{1,p})$ was introduced to correct a violation of the upper bound at $K_{\tilde{p}}(\mathcal{P}_{1,p})e^{-rT}$, where $K_{\tilde{p}}(\mathcal{P}_{1,p}) < K_p(\mathcal{P}_{1,p})$ has to hold. We proceed by analysing the situation under $\mathcal{P}_{1,\tilde{p}}$ which will then allow us to draw conclusions about the prices under $\mathcal{P}_{1,p}$. Suppose first that $K_{\tilde{p}}(\mathcal{P}_{1,p})e^{-rT} \in [K_l(\mathcal{P}_{1,\tilde{p}}), K_{l+1}(\mathcal{P}_{1,\tilde{p}})]$ for $K_l(\mathcal{P}_{1,\tilde{p}}), K_{l+1}(\mathcal{P}_{1,\tilde{p}}) \in \mathbb{K}(\mathcal{P}_0^*)$. Note, moreover, that the definition of $K_{\tilde{r}}(\mathcal{P}_{1,p})$ allows us to deduce that $[K_u^A, K_{\tilde{p}}(\mathcal{P}_{1,p})e^{-rT}] \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ has to hold. We can therefore apply Proposition 3.10.24 to see that

$$\mathbf{a}_{1,l+1} = \max\{A_{lb}^{rhs}(K_{l+1}(\mathcal{P}_{1,\tilde{p}}), \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{l+1}(\mathcal{P}_{1,\tilde{p}}), \mathcal{P}_0^*)\}$$

for $K_{l+1}(\mathcal{P}_{1,\tilde{p}}) \in (\mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_0^*))$ or $K_{l+1}(\mathcal{P}_{1,\tilde{p}}) \in \mathbb{K}^A(\mathcal{P}_0^*)$.

If we suppose that the strike $K_{l+1}(\mathcal{P}_{1,\tilde{p}})$ corresponds to $K_{r+1}(\mathcal{P}_{1,p})$, then we can conclude from Remark 3.7.5 that $K_r(\mathcal{P}_{1,p}) = \max\{K \in \mathbb{K}_2^{aux}(\mathcal{P}_1^*)\}$. In addition, we know that

$$\mathbf{a}_{1,r+1} = \max\{A_{lb}^{rhs}(K_{r+1}(\mathcal{P}_{1,p}), \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{r+1}(\mathcal{P}_{1,p}), \mathcal{P}_0^*)\}$$

for $K_{r+1}(\mathcal{P}_{1,p}) \in (\mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_0^*))$ or $K_{r+1}(\mathcal{P}_{1,p}) \in \mathbb{K}^A(\mathcal{P}_0^*)$, as correcting a violation of the upper bound has no effect on the prices already determined by the algorithm.

Let us first assume that $K_{r+1}(\mathcal{P}_{1,p}) \in (\mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_0^*))$. According to the definition of $K_r(\mathcal{P}_{1,p})$ we know that $[K_{r+1}(\mathcal{P}_{1,p}), K_{\tilde{q}}(\mathcal{P}_{1,p})) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ has to hold. Combined with the result in Proposition 3.10.28 we can then conclude that $[K_{r+1}(\mathcal{P}_{1,p}), \infty) \cap$

$\mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ has to hold as well. We can thus apply Proposition 3.10.5 to obtain that

$$\mathbf{a}_{1,\tilde{q}} = \max\{A_{lb}^{rhs}(K_{\tilde{q}}(\mathcal{P}_{1,p}), \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{\tilde{q}}(\mathcal{P}_{1,p}), \mathcal{P}_0^*)\}.$$

Then again, we argued already in Proposition 3.10.11 that $\mathbf{a}_{1,\tilde{q}}^n < \mathbf{a}_{1,\tilde{q}}$ has to hold. This, however, would imply that there exists arbitrage in the market yielding a contradiction to the assumptions.

Moreover, we can rule out that $K_{r+1}(\mathcal{P}_{1,p}) \in \mathbb{K}^A(\mathcal{P}_0^*)$ according to the definition of K_u^A. We can thus conclude that $[K_u^A, K_{\tilde{q}}(\mathcal{P}_{1,p}) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ has to hold. □

Proposition 3.10.32. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

Algorithm 2 stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of convexity. Suppose further that Algorithm 3 revises option prices for non-traded options on $[K_{1,\tilde{q}}, K_{1,p}]$, where

$$K_{1,\tilde{q}} = \max\{K_{1,s} \in [0, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*) : \\ cc(A; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}) < cc(E; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p})\},$$

without finding an arbitrage, then $\mathbf{a}_{1,\tilde{q}} = A_{lb}^{lhs}(K_{1,\tilde{q}}, \mathcal{P}_{1,\tilde{q}-1})$, where

$$A_{lb}^{lhs}(K_{1,\tilde{q}}, \mathcal{P}_{1,\tilde{q}-1}) > \max\{A_{lf}(K_{1,\tilde{q}}, \mathcal{P}_{1,\tilde{q}-1}), A_{lb}(K_{1,\tilde{q}}, \mathcal{P}_0^*), A_{lb}^{\overline{A}}(K_{1,\tilde{q}}, \mathcal{P}_0^*)\}.$$

Proof. We begin by pointing out that Proposition 3.10.29 guarantees that $K_{1,\tilde{q}} \in (\mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_0^*))$. Suppose for contradiction that the price for an American option with strike $K_{1,\tilde{q}}$ is given by

$$\mathbf{a}_{1,\tilde{q}} = \max\{A_{lf}(K_{1,\tilde{q}}, \mathcal{P}_{1,\tilde{q}-1}), A_{lb}(K_{1,\tilde{q}}, \mathcal{P}_0^*), A_{lb}^{\overline{A}}(K_{1,\tilde{q}}, \mathcal{P}_0^*)\}.$$

According to the definition of the strike $K_{1,\tilde{q}}$ we can immediately rule out that $\mathbf{a}_{1,\tilde{q}} = A_{lf}(K_{1,\tilde{q}}, \mathcal{P}_{1,\tilde{q}-1})$, as the Legendre-Fenchel condition holds with strict inequality on $[K_{1,\tilde{q}-1}, K_{1,\tilde{q}}]$.

Recall further that we know from Proposition 3.10.11 that $\mathbf{a}_{1,\tilde{q}}^n < \mathbf{a}_{1,\tilde{q}}$ has to hold. If the price for an American option with strike $K_{1,\tilde{q}}$ was given by

$$\mathbf{a}_{1,\tilde{q}} = \max\{A_{lb}(K_{1,\tilde{q}}, \mathcal{P}_0^*), A_{lb}^{\overline{A}}(K_{1,\tilde{q}}, \mathcal{P}_0^*)\}$$

Algorithm 3 would have stopped revising option prices at the strike $K_{1,\tilde{q}}$ due to the existence of an arbitrage. The respective arbitrage portfolio can then be found either in Section 3.6.3 or Section 3.6.4. Hence, we can deduce that $\mathbf{a}_{1,\tilde{q}} = A_{lb}^{lhs}(K_{1,\tilde{q}}, \mathcal{P}_{1,\tilde{q}-1})$ has to hold. $\square$

Proposition 3.10.33. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

Algorithm 2 stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of convexity. Suppose further that Algorithm 3 revises option prices for non-traded options on $[K_{1,\tilde{q}}, K_{1,p}]$, where

$$\begin{aligned} K_{1,\tilde{q}} = \max\{K_{1,s} \in [0, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*) :& \\ & cc(A; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}) < cc(E; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p})\}, \end{aligned}$$

without finding an arbitrage. If we further write

$$K_u^A = \max\{K_s^A \in \mathbb{K}^A(\mathcal{P}_0^*) : K_s^A < K_{1,\tilde{q}}\}$$

and assume that the strike K_u^A corresponds to $K_{1,w}$ under $\mathcal{P}_{1,p}$, then $\mathbf{a}_{1,w-1}$, $\mathbf{a}_{1,w}$ and $\mathbf{a}_{1,\tilde{q}}$ are co-linear.

Proof. According to Proposition 3.10.32 the price for an American option with strike $K_{1,\tilde{q}}$ was computed by Algorithm 2 to be $\mathbf{a}_{1,\tilde{q}} = A_{lb}^{lhs}(K_{1,\tilde{q}}, \mathcal{P}_{1,\tilde{q}-1})$. It then follows that the prices $\mathbf{a}_{1,\tilde{q}-2}$, $\mathbf{a}_{1,\tilde{q}-1}$ and $\mathbf{a}_{1,\tilde{q}}$ are co-linear which readily implies that the result holds for $K_u^A = K_{1,\tilde{q}-1}$. We are thus only left to consider the case where $K_u^A < K_{1,\tilde{q}-1}$. Suppose now for contradiction that $\mathbf{a}_{1,\tilde{q}} > A_{lb}^{lhs}(K_{1,\tilde{q}}, \mathcal{P}_{1,w})$, then there has to exist a strike $K_{1,s} \in (K_u^A, K_{1,\tilde{q}})$ such that

$$\begin{aligned} K_{1,s} = \max\{K_{1,\tilde{s}} \in (K_u^A, K_{1,\tilde{q}}) :& \\ & \mathbf{a}_{1,s} = \max\{A_{lf}(K_{1,s}, \mathcal{P}_{1,s-1}), A_{lb}^{rhs}(K_{1,s}, \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{1,s}, \mathcal{P}_0^*)\}\}. \end{aligned}$$

Let us first rule out that $\mathbf{a}_{1,s} = \max\{A_{lb}^{rhs}(K_{1,s}, \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{1,s}, \mathcal{P}_0^*)\}$. According to Remark 3.10.31 we know that $[K_u^A, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$. If the price was indeed given by $\mathbf{a}_{1,s} = \max\{A_{lb}^{rhs}(K_{1,s}, \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{1,s}, \mathcal{P}_0^*)\}$, we could apply Proposition 3.10.5 to argue that $\mathbf{a}_{1,\tilde{q}} = \max\{A_{lb}^{rhs}(K_{1,\tilde{q}}, \mathcal{P}_0^*), A_{lb}^{\overline{A},r}(K_{1,\tilde{q}}, \mathcal{P}_0^*)\}$ has to hold as well which we ruled out in Proposition 3.10.32.

Suppose now that the price for an American option with strike $K_{1,s}$ is given by $\mathbf{a}_{1,s} = A_{lf}(K_{1,s}, \mathcal{P}_{1,s-1})$. By the definition of the strike $K_{1,s}$ we can then conclude that the price for an American option with strike $K_{1,s+1}$ has to be given by $\mathbf{a}_{1,s+1} = A_{lb}^{lhs}(K_{1,s+1}, \mathcal{P}_{1,s})$. Observe that this implies that the Legendre-Fenchel condition has to hold with strict inequality on $[K_{1,s}, K_{1,s+1}]$. In addition,

$$cc(A; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}) = cc(A; K_{1,s}, K_{1,s+1}; \mathcal{P}_{1,p})$$

has to hold. Taking into account that $\mathcal{P}_{1,p-1}$ is a $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension and that $K_{1,\tilde{q}} < K_{1,p}$, we can further deduce that the European price function $E(\cdot, \mathcal{P}_{1,p})$ has to be convex on $[K_{1,s-1}, K_{1,s+1}]$. Hence,

$$cc(E; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}) \geq cc(E; K_{1,s}, K_{1,s+1}; \mathcal{P}_{1,p})$$

has to hold. It thus follows that

$$\begin{aligned} cc(A; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}) &= cc(A; K_{1,s}, K_{1,s+1}; \mathcal{P}_{1,p}) \\ &< cc(E; K_{1,s}, K_{1,s+1}; \mathcal{P}_{1,p}) \\ &\leq cc(E; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}) \end{aligned}$$

yielding a contradiction to the assumption that $\mathbf{a}_{1,s} = A_{lf}(K_{1,s}, \mathcal{P}_{1,s-1})$. Hence, we can rule out the existence of the strike $K_{1,s}$ and thus the prices $\mathbf{a}_{1,w-1}$, $\mathbf{a}_{1,w}$ and $\mathbf{a}_{1,\tilde{q}}$ have to be co-linear. $\square$

Remark 3.10.34. *Note further that $K_{1,w-1} \in \mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*)$ has to hold, as there exists arbitrage in the market otherwise due to*

$$\mathbf{a}_{1,\tilde{q}}^n < \mathbf{a}_{1,\tilde{q}} = \frac{\hat{\mathbf{a}}_u - \mathbf{a}_{1,w-1}}{K_u^A - K_{1,w-1}}(K_{1,\tilde{q}} - K_u^A) + \hat{\mathbf{a}}_u.$$

Proposition 3.10.35. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

The algorithm then stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ due to a violation of convexity. Assume further that Algorithm 3 revised the prices for non-traded options on $[K_{1,s}, K_{1,p}]$, where $K_{1,s} \in [K_u^A, K_{1,p})$ for

$$K_u^A = \max\{K_{\tilde{s}}^A \in \mathbb{K}^A(\mathcal{P}_0^*) : K_{\tilde{s}}^A < K_{1,\tilde{q}}\}$$

and

$$\begin{aligned} K_{1,\tilde{q}} = \max\{K_{1,\tilde{s}} \in [0, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*) :& \\ & cc(A; K_{1,\tilde{s}-1}, K_{1,\tilde{s}}; \mathcal{P}_{1,p}) < cc(E; K_{1,\tilde{s}-1}, K_{1,\tilde{s}}; \mathcal{P}_{1,p})\}, \end{aligned}$$

without finding an arbitrage. Then we must have $\mathbf{e}_{1,\tilde{s}}^n > \mathbf{e}_{1,\tilde{s}}$ for any strike $K_{1,\tilde{s}} \in$

$[K_{1,s}, K_{1,p}) \cap (\mathbb{K}^A(\mathcal{P}_1^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*))$ and $\mathbf{a}^n_{1,\tilde{s}} < \mathbf{a}_{1,\tilde{s}}$ for any strike $K_{1,\tilde{s}} \in [K_{1,s}, K_{1,p}) \cap \mathbb{K}^E(\mathcal{P}_0^*)$.

Proof. According to Proposition 3.10.11 we know already that $\mathbf{e}^n_{1,\tilde{s}} > \mathbf{e}_{1,\tilde{s}}$ for any strike $K_{1,\tilde{s}} \in [K_{1,\tilde{q}}, K_{1,p}) \cap (\mathbb{K}^A(\mathcal{P}_1^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*))$ and $\mathbf{a}^n_{1,\tilde{s}} < \mathbf{a}_{1,\tilde{s}}$ for any strike $K_{1,\tilde{s}} \in [K_{1,\tilde{q}}, K_{1,p}) \cap \mathbb{K}^E(\mathcal{P}_0^*)$.

Next we consider the case where $K_{1,\tilde{s}} \in (K_u^A, K_{1,\tilde{q}})$. Observe first that $[K_u^A, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}_1^*) = \emptyset$ holds according to Remark 3.10.31. Moreover, we know from Proposition 3.10.29 that $K_{1,\tilde{q}} \in (\mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*))$. Taking into account the definition of K_u^A, we readily obtain that $(K_u^A, K_{1,\tilde{q}}] \cap \mathbb{K}^A(\mathcal{P}_1^*) = \emptyset$. We are thus required to show that $\mathbf{a}^n_{1,\tilde{s}} < \mathbf{a}_{1,\tilde{s}}$ holds. To this end, let us assume for contradiction that there exists a strike $K_{1,r} \in [K_{1,\tilde{s}}, K_{1,\tilde{q}}) \cap \mathbb{K}(\mathcal{P}_1^*)$ with

$$K_{1,r} = \max\{K_{1,\tilde{r}} \in [K_{1,\tilde{s}}, K_{1,\tilde{q}}) \cap \mathbb{K}(\mathcal{P}_1^*) : \mathbf{a}^n_{1,\tilde{r}} \geq \mathbf{a}_{1,\tilde{r}}\}.$$

It then follows that $\mathbf{a}^n_{1,\tilde{r}+1} < \mathbf{a}_{1,\tilde{r}+1}$ has to hold as $K_{1,\tilde{r}+1} \in (\mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_1^*))$. Since the algorithm did not stop revising option prices at $K_{1,\tilde{r}+1}$ we must have

$$cc(A, A^n; K_u^A, K_{1,\tilde{r}+1}; \mathcal{P}_{1,p}) > cc(E; K_{1,\tilde{r}}, K_{1,\tilde{r}+1}; \mathcal{P}_{1,p})$$

according to line 34 of Algorithm 3. Recall further that we used equality in the Legendre-Fenchel condition to compute the revised prices. Hence, the revised price for American options with strike $K_{1,\tilde{r}}$ corresponds to

$$\mathbf{a}^n_{1,\tilde{r}} = \mathbf{e}_{1,\tilde{r}} + \frac{K_{1,\tilde{r}}}{K_{1,\tilde{r}+1}}[\mathbf{a}^n_{1,\tilde{r}+1} - \mathbf{e}_{1,\tilde{r}+1}].$$

Taking into account that the prices for American options with strikes in $[K_{1,w-1}, K_{1,\tilde{q}}]$, computed by Algorithm 2, are co-linear, we readily obtain that $\mathbf{a}^n_{1,\tilde{r}} < \mathbf{a}_{1,\tilde{r}}$ has to hold. Since this yields a contradiction we can conclude that $\mathbf{a}^n_{1,\tilde{s}} < \mathbf{a}_{1,\tilde{s}}$ for any strike $K_{1,\tilde{s}} \in (K_u^A, K_{1,\tilde{q}}]$.

We are thus left to show that $\mathbf{e}^n_{1,\tilde{s}} > \mathbf{e}_{1,\tilde{s}}$ for $K_{1,\tilde{s}} = K_u^A$. According to the stopping condition in Algorithm 3 the revised price for European options with strike K_u^A is computed if and only if

$$cc(A, A^n; K_{1,\tilde{s}}, K_{1,\tilde{s}+1}; \mathcal{P}_{1,p}) > cc(E; K_{1,\tilde{s}}, K_{1,\tilde{s}+1}; \mathcal{P}_{1,p})$$

holds. Using equality in the Legendre-Fenchel condition to recompute the price for a European option with strike K_u^A we now readily obtain that $\mathbf{e}^n_{1,\tilde{s}} > \mathbf{e}_{1,\tilde{s}}$. We have therefore shown that either $\mathbf{e}^n_{1,\tilde{s}} > \mathbf{e}_{1,\tilde{s}}$ for any strike $K_{1,\tilde{s}} \in [K_{1,s}, K_{1,p}) \cap (\mathbb{K}^A(\mathcal{P}_1^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*))$ or $\mathbf{a}^n_{1,\tilde{s}} < \mathbf{a}_{1,\tilde{s}}$ for any strike $K_{1,\tilde{s}} \in [K_{1,s}, K_{1,p}) \cap \mathbb{K}^E(\mathcal{P}_0^*)$. □

Corollary 3.10.36. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}_1^*$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$. The algorithm stops at the strike $K_{1,p}$ due to a violation of convexity and Algorithm 3 is used to revise the already computed prices on $[K_{1,q}, K_{1,p}]$.*

If neither $\mathbf{a}_{1,s}^n < \max\{A_{lb}(K_{1,s}, \mathcal{P}_0^*), A_{lb}^{\overline{A}}(K_{1,s}, \mathcal{P}_0^*)\}$ nor $\mathbf{e}_{1,s}^n > E_{ub}(K_{1,s}, \mathcal{P}_0^*)$ occurred for any strike $K_{1,s} \in [K_{1,q}, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*)$, then the revised price functions $A^n(\cdot, \mathcal{P}_1^{rev})$ and $E^n(\cdot, \mathcal{P}_1^{rev})$ are convex on $[K_{1,q}, K_{1,p}]$.

Proof. To see that the revised price functions $A^n(\cdot, \mathcal{P}_i^{rev})$ and $E^n(\cdot, \mathcal{P}_i^{rev})$ are convex we check whether or not the convex conjugate is decreasing as a function of the strike. Note further that $A^n(\cdot, \mathcal{P}_1^{rev})$ is convex if and only if $E^n(\cdot, \mathcal{P}_1^{rev})$ is convex, as the convex conjugates of $A^n(\cdot, \mathcal{P}_1^{rev})$ and $E^n(\cdot, \mathcal{P}_1^{rev})$ coincide. This allows us to choose in each situation individually for which function we show convexity. All possible situations can be characterised via the type of the three adjacent strikes between which the two linear pieces in question are given. Hence, there are eight different cases in which a violation of the convexity of the functions $A^n(\cdot, \mathcal{P}_1^{rev})$ and $E^n(\cdot, \mathcal{P}_1^{rev})$ could occur. Let us assume that we want to check convexity between the three strikes $K_{1,l}$, $K_{1,l+1}$ and $K_{1,l+2} \in \mathbb{K}(\mathcal{P}_1^*)$. According to the Standing Assumptions the price functions $A(\cdot, \mathcal{P}_0^*)$ and $E(\cdot, \mathcal{P}_0^*)$ are convex. We can therefore immediately exclude from consideration the cases where $(K_{1,l}, K_{1,l+1}, K_{1,l+2}) \in (\mathbb{K}^A(\mathcal{P}_0^*), \mathbb{K}^A(\mathcal{P}_0^*), \mathbb{K}^A(\mathcal{P}_0^*))$ or $(K_{1,l}, K_{1,l+1}, K_{1,l+2}) \in (\mathbb{K}^E(\mathcal{P}_0^*), \mathbb{K}^E(\mathcal{P}_0^*), \mathbb{K}^E(\mathcal{P}_0^*))$.

We further know that neither $\mathbf{a}_{1,s}^n < \max\{A_{lb}(K_{1,s}, \mathcal{P}_0^*), A_{lb}^{\overline{A}}(K_{1,s}, \mathcal{P}_0^*)\}$ nor $\mathbf{e}_{1,s}^n > E_{ub}(K_{1,s}, \mathcal{P}_0^*)$ occurred for any $K_{1,s} \in [K_{1,q}, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*)$. The corresponding strike triplets are then given by

$$(K_{1,l}, K_{1,l+1}, K_{1,l+2}) \in (\mathbb{K}^A(\mathcal{P}_1^*), \mathbb{K}^A(\mathcal{P}_1^*), \mathbb{K}^E(\mathcal{P}_1^*)),$$

$$(K_{1,l}, K_{1,l+1}, K_{1,l+2}) \in (\mathbb{K}^E(\mathcal{P}_1^*), \mathbb{K}^A(\mathcal{P}_1^*), \mathbb{K}^A(\mathcal{P}_1^*))$$

and

$$(K_{1,l}, K_{1,l+1}, K_{1,l+2}) \in (\mathbb{K}^E(\mathcal{P}_1^*), \mathbb{K}^A(\mathcal{P}_1^*), \mathbb{K}^E(\mathcal{P}_1^*))$$

and can thus be ruled out as well.

We are therefore left to discuss the situations where

$$(K_{1,l}, K_{1,l+1}, K_{1,l+2}) \in (\mathbb{K}^A(\mathcal{P}_1^*), \mathbb{K}^E(\mathcal{P}_1^*), \mathbb{K}^A(\mathcal{P}_1^*)),$$

$$(K_{1,l}, K_{1,l+1}, K_{1,l+2}) \in (\mathbb{K}^A(\mathcal{P}_1^*), \mathbb{K}^E(\mathcal{P}_1^*), \mathbb{K}^E(\mathcal{P}_1^*))$$

or

$$(K_{1,l}, K_{1,l+1}, K_{1,l+2}) \in (\mathbb{K}^E(\mathcal{P}_1^*), \mathbb{K}^E(\mathcal{P}_1^*), \mathbb{K}^A(\mathcal{P}_1^*)).$$

In the first case a violation of convexity implies that $\mathbf{a}^n_{1,l+1} > A_{ub}(K_{1,l+1}, \mathcal{P}^*_0)$. Combined with Proposition 3.10.35 we could then deduce that $\mathbf{a}_{1,l+1} > A_{ub}(K_{1,l+1}, \mathcal{P}^*_0)$ holds as well, thereby contradicting the assumption that $\mathcal{P}_{1,p-1}$ is a $K_{1,p-1}$-admissible $\mathcal{P}^*_0$-extension.

In the last two cases a violation of convexity corresponds either to the violation $\mathbf{e}^n_{1,l} < E^{rhs}_{lb}(K_{1,l}, \mathcal{P}^*_0)$ or $\mathbf{e}^n_{1,l+2} < E^{lhs}_{lb}(K_{1,l+2}, \mathcal{P}^*_0)$. Using Proposition 3.10.35 we see that $\mathbf{e}^n_{1,s} > \mathbf{e}_{1,s}$ for any $K_{1,s} \in [K_{1,q}, K_{1,p}] \cap \mathbb{K}^A(\mathcal{P}^*_1)$. Hence, $\mathbf{e}_{1,\tilde{s}} < E_{lb}(K_{1,\tilde{s}}, \mathcal{P}^*_0)$ would have to hold as well for $\tilde{s} = l$ in the first case and $\tilde{s} = l + 2$ in the second case. As we assumed that $\mathcal{P}_{1,p-1}$ is a $K_{1,p-1}$-admissible $\mathcal{P}^*_0$-extension this cannot be the case and we can therefore rule out the last two situations as well. It follows that the revised price functions $A^n(\cdot, \mathcal{P}^{rev}_1)$ and $E^n(\cdot, \mathcal{P}^{rev}_1)$ are convex on $[K_{1,q}, K_{1,p}]$. $\square$

Proposition 3.10.37. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by $\mathcal{P}^*_0 \in \overline{\mathcal{M}}$. Starting with the initial set of prices $\mathcal{P}^*_1$ the algorithm computes the $K_{1,p-1}$-admissible $\mathcal{P}^*_0$-extension $\mathcal{P}_{1,p-1}$ for $p \geq 1$.*

Algorithm 2 stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}^*_1)$ due to a violation of convexity. Suppose further that Algorithm 3 revises option prices down to K^A_u, where

$$K^A_u = \max\{K^A_s \in \mathbb{K}^A(\mathcal{P}^*_0) : K^A_s < K_{1,\tilde{q}}\},$$

and

$$\begin{aligned} K_{1,\tilde{q}} = \max\{K_{1,s} \in [0, K_{1,p}] \cap \mathbb{K}(\mathcal{P}^*_1) : \\ cc(A; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}) < cc(E; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p})\}, \end{aligned}$$

then there exists arbitrage in the market.

Proof. We begin by assuming that the strike K^A_u corresponds to $K_{1,w}$ under $\mathcal{P}_{1,p}$. Recall further that Proposition 3.10.33 guarantees that the prices $\mathbf{a}_{1,w-1}$, $\mathbf{a}_{1,w}$ and $\mathbf{a}_{1,\tilde{q}}$ are co-linear. The definition of $K_{1,\tilde{q}}$ moreover implies that the Legendre-Fenchel condition holds with strict inequality on $[K_{1,\tilde{q}-1}, K_{1,\tilde{q}}]$. Taking into account the convexity of the European price function $E(\cdot, \mathcal{P}_{1,p})$ on $[0, K_{1,\tilde{q}}]$ we obtain that

$$\begin{aligned} cc(A; K_{1,w-1}, K_{1,w}; \mathcal{P}_{1,p}) &= cc(A; K_{1,\tilde{q}-1}, K_{1,\tilde{q}}; \mathcal{P}_{1,p}) \\ &< cc(E; K_{1,\tilde{q}-1}, K_{1,\tilde{q}}; \mathcal{P}_{1,p}) \\ &\leq cc(E; K_{1,w-1}, K_{1,w}; \mathcal{P}_{1,p}). \end{aligned}$$

We can thus conclude that the Legendre-Fenchel condition has to hold with strict inequality on $[K_{1,w-1}, K_{1,\tilde{q}}]$. Moreover, we can use Remark 3.10.34 to argue that $K_{1,w-1} \in \mathbb{K}^E(\mathcal{P}^*_0) \backslash \mathbb{K}^A(\mathcal{P}^*_1)$ has to hold. It then follows that $\mathbf{e}_{1,w} = E_{ub}(K_{1,w}, \mathcal{P}^*_0)$.

Depending on whether $K_{1,p} \in (\mathbb{K}^A(\mathcal{P}_0^*) \backslash \mathbb{K}^E(\mathcal{P}_0^*))$ or $K_{1,p} \in (\mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_0^*))$ we can apply either Proposition 3.6.13 or Proposition 3.6.17 to see that there has to exist arbitrage in the market. $\square$

Remark 3.10.38. *If Algorithm 3 introduces the auxiliary price constraint* $(\mathbf{a}_{1,q}^n, K_{1,q})$ *to correct a violation of convexity at the strike* $K_{1,p}$*, then* $K_{1,q} \in (K_u^A, K_{1,\tilde{q}}]$ *has to hold. The reason being that the algorithm will not stop to the right of* $K_{1,\tilde{q}}$ *nor will it introduce an auxiliary price constraint once the strike* K_u^A *is reached.*

Proposition 3.10.39. *Consider a market trading finitely many American and co-terminal European put options and suppose that their prices are given by* $\mathcal{P}_0^* \in \overline{\mathcal{M}}$*. Starting with the initial set of prices* $\mathcal{P}_1^*$ *the algorithm computes the* $K_{1,p-1}$*-admissible* $\mathcal{P}_0^*$*-extension* $\mathcal{P}_{1,p-1}$ *for* $p \geq 1$*.*

The algorithm stops at the strike $K_{1,p} \in \mathbb{K}(\mathcal{P}_1^*)$ *due to a violation of convexity and computes revised prices for non-traded options with strikes* $K_{1,s} \in [K_{1,q}, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*)$*. At the strike* $K_{1,q} \in (K_u^A, K_{u+1}^A) \cap \mathbb{K}^E(\mathcal{P}_0^*)$ *the algorithm stops and defines the new initial set of prices* $\mathcal{P}_2^*$ *by* $\mathcal{P}_2^* = ((\mathcal{P}_1)^A \cup (\mathbf{a}_{1,q}^n, K_{1,q}); (\mathcal{P}_0^*)^E)$*. The algorithm is then restarted with the initial set* $\mathcal{P}_2^*$*. If possible violations of the upper bound* $\overline{A}$ *are disregarded there has to exist a strike* $K_{2,r} \in (K_{2,w}, K_u^A) \cap \mathbb{K}^E(\mathcal{P}_0^*)$ *for* $K_{2,w} = \max\{K \in \mathbb{K}^A(\mathcal{P}_2^*) : K < K_u^A\}$ *with* $\mathbf{a}_{2,r} = A_{lb}^{rhs}(K_{2,r}, \mathcal{P}_2^*)$*.*

Proof. Suppose that the strike K_u^A corresponds to $K_{1,s}$ under $\mathcal{P}_{1,p}$ and let us write

$$K_{1,\tilde{q}} = \max\{K_{1,s} \in [0, K_{1,p}] \cap \mathbb{K}(\mathcal{P}_1^*) : \\ cc(A; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p}) < cc(E; K_{1,s-1}, K_{1,s}; \mathcal{P}_{1,p})\}.$$

According to Proposition 3.10.33 we then know that the prices for American options with strikes in $[K_{1,s-1}, K_{1,\tilde{q}}]$, computed by Algorithm 2 using the initial set of prices $\mathcal{P}_1^*$, are co-linear. Additionally, Remark 3.10.38 ensures that $K_{1,q} \in (K_u^A, K_{1,\tilde{q}}]$ and thus $\mathbf{a}_{1,s-1}$, $\mathbf{a}_{1,s}$ and $\mathbf{a}_{1,q}$ have to be co-linear. Moreover, we argue in Proposition 3.10.35 that the revised price $\mathbf{a}_{1,q}^n$ for American options with strike $K_{1,q}$ satisfies $\mathbf{a}_{1,q}^n < \mathbf{a}_{1,q}$. Hence, the price for American options at strike $K_{1,s-1}$ will be increased to $\mathbf{a}_{2,l} = A_{lb}^{rhs}(K_{2,l}, \mathcal{P}_2^*)$ after restarting. Since we disregard any possible violation of the upper bound we must have $K_{2,w} = K_{1,w}$ and thus we can conclude that $K_{2,r} = K_{1,s-1} \in (K_{2,w}, K_u^A) \cap \mathbb{K}^E(\mathcal{P}_0^*)$ with $\mathbf{a}_{2,r} = A_{lb}^{rhs}(K_{2,r}, \mathcal{P}_2^*)$. $\square$

The following result shows that the algorithm will compute the price for a European option with strike $K_{2,p} \in \mathbb{K}_2^{aux}(\mathcal{P}_2^*)$ to be $\mathbf{e}_{2,p} = E_{ub}(K_{2,p}, \mathcal{P}_{2,p-1})$.

Proposition 3.10.40. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by* $\mathcal{P}_0^* \in \overline{\mathcal{M}}$*. Assume further that the algorithm extended the initial set of prices from* $\mathcal{P}_0^*$ *to* $\mathcal{P}_2^* \supseteq \mathcal{P}_0^*$*.*

Starting with the initial set of prices $\mathcal{P}_2^*$ *the algorithm computes the* $K_{2,p-1}$*-admissible* $\mathcal{P}_0^*$*-extension* $\mathcal{P}_{2,p-1}$ *for* $p \geq 1$. *If we assume that* $K_{2,p} \in \mathbb{K}_2^{aux}(\mathcal{P}_2^*)$, *then* $\mathbf{e}_{2,p} = E_{ub}(K_{2,p}, \mathcal{P}_{2,p-1})$.

Proof. According to Proposition 3.10.30 we know that $[K_u^A, \infty) \cap \mathbb{K}_2^{aux}(\mathcal{P}_2^*) = \emptyset$ for $K_u^A = \max\{K \in \mathbb{K}^A(\mathcal{P}_0^*) : K < K_1^{aux}\}$ and thus $K_{2,p} < K_u^A$ has to hold. From the definition of the set $\mathbb{K}_2^{aux}(\mathcal{P}_2^*)$, we can further deduce that $K_{2,p}e^{rT} \in \mathbb{K}^E(\mathcal{P}_0^*)$. Hence, there has to exist a $j \in \{1, ..., m_2\}$ with $K_{2,p}e^{rT} = K_j^E$. In addition, we can assume without loss of generality that $K_{2,s} = K_{2,p}e^{rT}$ for some $s > p$ under $\mathcal{P}_{2,p-1}$.

To see that the algorithm determines $\mathbf{e}_{2,p}$ to be $E_{ub}(K_{2,p}, \mathcal{P}_{2,p-1})$, we first consider the case where $K_{2,p} = \min\{K \in \mathbb{K}_2^{aux}(\mathcal{P}_2^*)\}$. We start by showing that $\mathbf{a}_{2,p-1} \leq \overline{A}(K_{2,p-1}, \mathcal{P}_{2,p-1})$. To do so, we note that the European price function $E(\cdot, \mathcal{P}_{2,p-1})$ can only have kinks in strikes of type $\mathbb{K}^E(\mathcal{P}_0^*)$ or $[0, K_{2,p-1}] \cap \mathbb{K}^A(\mathcal{P}_2^*)$. Similarly, the upper bound $\overline{A}(\cdot, \mathcal{P}_{2,p-1})$ can only have kinks in strikes K with $Ke^{rT} \in \mathbb{K}^E(\mathcal{P}_0^*)$ or $Ke^{rT} \in [0, K_{2,p-1}] \cap \mathbb{K}^A(\mathcal{P}_2^*)$ and thus we have to distinguish between the two cases where either $K_{2,p-1} \geq K_{j-1}^E$ or $K_{2,p-1} < K_{j-1}^E$.

In the first case we can use the fact that $\mathcal{P}_{2,p-1}$ is a $K_{2,p-1}$-admissible $\mathcal{P}_0^*$-extension to infer that $\overline{\mathbf{a}}_{2,p-1} \geq A(K_{2,p-1}e^{-rT}, \mathcal{P}_{2,p-1})$ has to hold. Taking into account that $A(\cdot, \mathcal{P}_{2,p-1})$ is convex, we can deduce from $\mathbf{a}_{2,p} = \overline{\mathbf{a}}_{2,s}$ that

$$
\begin{aligned}
\overline{A}(K_{2,p-1}, \mathcal{P}_{2,p-1}) &= \frac{\overline{\mathbf{a}}_{2,s} - \overline{\mathbf{a}}_{2,p-1}}{K_{2,p} - K_{2,p-1}e^{-rT}}(K_{2,p-1} - K_{2,p}) + \overline{\mathbf{a}}_{2,s} \\
&\geq \frac{\mathbf{a}_{2,p} - A(K_{2,p-1}e^{-rT}, \mathcal{P}_{2,p-1})}{K_{2,p} - K_{2,p-1}e^{-rT}}(K_{2,p-1} - K_{2,p}) + \mathbf{a}_{2,p} \\
&\geq \mathbf{a}_{2,p-1}
\end{aligned}
$$

For $K_{2,p-1} < K_{j-1}^E$ we have to further distinguish whether $K_{2,p-1} < K_{j-1}e^{-rT}$ or not. Let us first assume that $K_{2,p-1} < K_{j-1}e^{-rT}$ holds, then we can immediately conclude from (3.28) that $\overline{\mathbf{a}}_{j-1} \geq A(K_{j-1}e^{-rT}, \mathcal{P}_{2,p-1})$. The convexity of $E(\cdot, \mathcal{P}_{2,p-1})$ then readily implies that $\mathbf{a}_{2,p-1} \leq \overline{A}(K_{2,p-1}, \mathcal{P}_{2,p-1})$ has to hold.

The other situation occurs when $K_{j-1}^E e^{-rT} \in (K_{2,p-1}e^{-rT}, K_{2,p-1}]$. Let us now assume for contradiction that $\mathbf{a}_{2,p-1} > \overline{A}(K_{2,p-1}, \mathcal{P}_{2,p-1})$. We can then conclude from $K_{2,p-1}e^{-rT} < K_{j-1}^E e^{-rT}$ that $\mathbf{a}_{2,p-1} > \overline{A}(K_{2,p-1}, \mathcal{P}_0^*)$. Note, moreover, that the price sets $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ differ only by the auxiliary price constraint $(A(K_1^{aux}, \mathcal{P}_2^*), K_1^{aux})$ as any auxiliary price constraint introduced to correct a violation of the upper bound during the first iteration is added to $\mathcal{P}_1^*$. Hence, a change in the price functions has to be caused either by this new constraint or by the algorithm pricing European options with strikes in $\mathbb{K}_2^{aux}(\mathcal{P}_2^*)$ differently to (3.29). If we set

$$K_{2,w} = \max\{K \in \mathbb{K}^A(\mathcal{P}_2^*) : K < K_u^A\},$$

it follows that the new constraint first appears in the pricing formula for American options with strikes $K > K_{2,w}$. The definition of $K_{2,w}$, however, guarantees that $K_{2,p} \leq K_{2,w}$ and thus we are guaranteed that the option prices are unchanged up to $K_{2,p-1}$. If we further assume that the strike K_j^E corresponds to $K_{1,\tilde{s}}$ and $K_{2,p-1}$ corresponds to $K_{1,\tilde{p}-1}$ under $\mathcal{P}_{1,\tilde{s}}$, then

$$\begin{aligned}\mathbf{a}_{1,\tilde{p}-1} &= \mathbf{a}_{2,p-1} \\ &> \overline{A}(K_{2,p-1}, \mathcal{P}_0^*) \\ &\geq \overline{A}(K_{1,\tilde{p}-1}, \mathcal{P}_{1,\tilde{s}})\end{aligned}$$

where the inequality in the last line holds due to the fact that $\mathcal{P}_{1,\tilde{s}}$ is a $K_{1,\tilde{s}}$-admissible $\mathcal{P}_0^*$-extension according to Remark 3.10.26. Then again, we can rule out $\mathbf{a}_{1,\tilde{p}-1} > \overline{A}(K_{1,\tilde{p}-1}, \mathcal{P}_{1,\tilde{s}})$ for $\mathcal{P}_{1,\tilde{s}}$ a $K_{1,\tilde{s}}$-admissible $\mathcal{P}_0^*$-extension, thereby contradicting the assumption that $\mathbf{a}_{2,p-1} > \overline{A}(K_{2,p-1}, \mathcal{P}_{2,p-1})$. We have thus shown that regardless of whether $K_{2,p-1} < K_{j-1}^E$ or not $\mathbf{a}_{2,p-1} \leq \overline{A}(K_{2,p-1}, \mathcal{P}_{2,p-1})$ has to hold.

We are only left to show that $\mathbf{e}_{2,p} < E_{ub}(K_{2,p}, \mathcal{P}_{2,p-1})$ cannot hold. To this end, we assume for contradiction that $\mathbf{e}_{2,p} = E_{lf}(K_{2,p}, \mathcal{P}_{2,p-1})$ with

$$E_{lf}(K_{2,p}, \mathcal{P}_{2,p-1}) < E_{ub}(K_{2,p}, \mathcal{P}_{2,p-1}). \tag{3.69}$$

We can then argue that

$$\begin{aligned}cc(E; K_{2,p-1}, K_{2,p}; \mathcal{P}_{2,p-1}) &= cc(A; K_{2,p-1}, K_{2,p}; \mathcal{P}_{2,p-1}) \\ &\leq cc(\overline{A}; K_{2,s-1}e^{-rT}, K_{2,s}e^{-rT}; \mathcal{P}_{2,p-1}) \\ &= cc(E; K_{2,s-1}, K_{2,s}; \mathcal{P}_{2,p-1})\end{aligned}$$

where the inequality in the second line follows from $\mathbf{a}_{2,p-1} \leq \overline{A}(K_{2,p-1}, \mathcal{P}_{2,p-1})$ and $\mathbf{a}_{2,p} = \overline{\mathbf{a}}_{2,s}$. This, however, implies that

$$cc(E; K_{2,p-1}, K_{2,p}; \mathcal{P}_{2,p-1}) = cc(E; K_{2,s-1}, K_{2,s}; \mathcal{P}_{2,p-1}).$$

Taking into account that $K_{2,s} \in \mathbb{K}^E(\mathcal{P}_0^*)$ and $K_{2,s} > K_{2,p}$, we must therefore have

$$cc(E; K_{2,p-1}, K_{2,p}; \mathcal{P}_{2,p-1}) = cc(E_{ub}; K_{2,p-1}, K_{2,p}; \mathcal{P}_{2,p-1}).$$

As this is a contradiction to the assumption in (3.69), we can conclude that $\mathbf{e}_{2,p} = E_{ub}(K_{2,p}, \mathcal{P}_{2,p-1})$ has to hold.

This result can now readily be extended to any strike $K \in \mathbb{K}_2^{aux}(\mathcal{P}_2^*)$, as the European price function remains unchanged and $K \leq K_{2,w}$. We have therefore shown that $\mathbf{e}_{2,p} = E_{ub}(K_{2,p}, \mathcal{P}_{2,p-1})$ for any strike $K_{2,p} \in \mathbb{K}_2^{aux}(\mathcal{P}_2^*)$. □

3.10.7 Miscellaneous

Proposition 3.10.41. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Assume further that the algorithm extended the initial set of prices from $\mathcal{P}_0^*$ to $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 1$, then we can super-replicate an American option with strike $K_{i,r} \in \mathbb{K}^{aux}(\mathcal{P}_i^*)$ for $\mathbf{a}_{i,r}$ in the market.*

Proof. Let us first consider the case where $K_{i,r} \in \mathbb{K}_1^{aux}(\mathcal{P}_i^*)$. To this end, we assume that the violation of the no-arbitrage bounds that lead to the introduction of the auxiliary price constraint at the strike $K_{i,r}$ occurred under the initial price set $\mathcal{P}_j^*$, where $1 \leq j \leq i$. If we suppose further that the strike $K_r(\mathcal{P}_i^*)$ corresponds to the strike $K_s(\mathcal{P}_j^*)$, then there has to exist a strike $K_{j,p} > K_{j,s}$ where the actual violation of convexity occurred. We then have to distinguish between the different types of violations.

Suppose first that $K_{j,p} \in \mathbb{K}^A(\mathcal{P}_0^*)\backslash\mathbb{K}^E(\mathcal{P}_0^*)$, then the price for European options with strike $K_{j,p}$ computed by the algorithm using the initial price set $\mathcal{P}_j^*$ violated the right-hand side lower bound $E_{lb}^{rhs}(K_{j,p}, \mathcal{P}_0^*)$. The auxiliary price constraint at $K_{j,s}$, then satisfies

$$\mathbf{a}_{j,s} = \mathbf{e}_{j,s} + \frac{K_{j,s}}{K_{j,p}}[\mathbf{a}_{j,p} - E_{lb}^{rhs}(K_{j,p}, \mathcal{P}_0^*)], \tag{3.70}$$

as the prices are revised such that the Legendre-Fenchel condition holds between $K_{j,s}$ and $K_{j,p}$. Observe further that (3.70) corresponds to the price for the super-replicating portfolio $P_1^{LF}(K_{j,s}, K_{j,p})$ given in Proposition 3.6.7. Then again, European options with strike $K_{j,p}$ are not traded in the market and thus we need to find a (sub)-replicating portfolio for the position in the European option with strike $K_{j,p}$ with cost of at least $E_{lb}^{rhs}(K_{j,p}, \mathcal{P}_0^*)$ to replace it. From Proposition 3.10.9 we furthermore know that $K_{j,p} < K_{m_2-1}^E$. Hence, there exist at least two strikes larger than $K_{j,p}$ at which European options are traded in the market. We can thus use the portfolio P_3^E from Proposition 3.6.4 to do exactly that. We have therefore shown that an American option with strike $K_{j,s}$ can be super-replicated in the market for $\mathbf{a}_{j,s}$. Since we assumed that the price remains unchanged between the j-th and the i-th iteration of the algorithm we can conclude that an American option with strike $K_{j,s}$ can be super-replicated in the market for $\mathbf{a}_{i,r}$.

Next we will consider the situation where a violation of $\mathbf{a}_{j,p} \leq A_{ub}(K_{j,p}, \mathcal{P}_{j,p-1})$ occurred at the strike $K_{j,p} \in (\mathbb{K}^E(\mathcal{P}_0^*)\backslash\mathbb{K}^A(\mathcal{P}_0^*))$. Recall first that the algorithm revised the option prices between $K_{j,s}$ and $K_{j,p}$ using the starting prices $A_{ub}(K_{j,p}, \mathcal{P}_{j,p-1})$ and $\mathbf{e}_{j,p}$ together with equality in the Legendre-Fenchel condition. Hence, we must have

$$\mathbf{a}_{j,s} = \mathbf{e}_{j,s} + \frac{K_{j,s}}{K_{j,p}}[A_{ub}(K_{j,p}, \mathcal{P}_{j,p-1}) - \mathbf{e}_{j,p}].$$

According to Proposition 3.6.7 this corresponds to the price of the super-replicating portfolio $P_1^{LF}(K_{j,s}, K_{j,p})$. Then again, we know that American options are not traded in the market at the strike $K_{j,p}$ and thus we have to find a super-replicating portfolio for that position in P_1^{LF} that costs no more than $A_{ub}(K_{j,p}, \mathcal{P}_{j,p-1})$. Depending on whether $K_{j,p-1} \in \mathbb{K}^A(\mathcal{P}_0^*)$ or not the super-replicating portfolio will be given either by P_1^A from Proposition 3.6.5 or P_3^{LF} from Proposition 3.6.8. In any case the cost of the super-replicating portfolio will be given by $A_{ub}(K_{j,p}, \mathcal{P}_{j,p-1})$ and thus we found a super-replicating portfolio for the American option with strike $K_{j,s}$ that costs $\mathbf{a}_{j,s}$. Having assumed that the price remains unchanged between the j-th and the i-th iteration of the algorithm we can conclude that an American option with strike $K_{j,s}$ can be super-replicated in the market for $\mathbf{a}_{i,r}$.

We are then only left to consider the case where $K_{i,r} \in \mathbb{K}_2^{aux}(\mathcal{P}_i^*)$. From Theorem 2.2.3 of Chapter 2 we already know that an American option with strike $K_{i,r}$ can be super-replicated by a European option with strike $K_{i,r}e^{rT}$. According to (3.28) the algorithm only ever introduces an auxiliary constraint at a strike $K_{i,r}$ if $K_{i,r}e^{rT} \in \mathbb{K}^E(\mathcal{P}_0^*)$ holds. Hence, we can use the traded European option with strike $K_{i,r}e^{rT}$ to super-replicate the American option with strike $K_{i,r}$ in the market. Note further that the definition of the upper bound $\overline{A}$ guarantees that the cost of the super-replicating portfolio is given by $\mathbf{a}_{i,r}$. We have therefore shown that an American option with strike $K_{i,r} \in \mathbb{K}^{aux}(\mathcal{P}_i^*)$ can be super-replicated in the market for $\mathbf{a}_{i,r}$. □

Proposition 3.10.42. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Assume further that the algorithm extended the initial set of prices from $\mathcal{P}_0^*$ to $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 1$. Starting with the initial set of prices $\mathcal{P}_i^*$ the algorithm computes $\mathcal{P}_{i,p-1}$ a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension.*

If we suppose now that the price for American options with strike $K_{i,p-1}$ is given by $\mathbf{a}_{i,p-1} = K_{i,p-1} - S_0$ under $\mathcal{P}_{i,p-1}$ and that $K_{i-1}^{vc} \leq K_{i,p-1}$ holds, then the price for American options with strike $K_{i,p}$ has to be $\mathbf{a}_{i,p} = K_{i,p} - S_0$ and the price for a European option with strike $K_{i,p}$ is $\mathbf{e}_{i,p} = E_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})$ under $\mathcal{P}_{i,p}$.

Remark 3.10.43. *It follows that a violation of convexity can be ruled out at the strike $K_{i,p}$. Note further that a possible violation of the upper bound in $K_{i,p}e^{-rT}$ has no effect on the option prices with strike $K \in [K_{i,p}, \infty)$. We can therefore conclude that the algorithm will compute a $K_{i,p}$-admissible $\mathcal{P}_0^*$-extension without having to restart. Proposition 3.10.42 can thus be applied repeatedly thereby showing that a violation of convexity can be ruled out on $[K_{i,p}, \infty)$.*

Proof. Note that $[K_{i,p}, \infty) \cap \mathbb{K}^{aux}(\mathcal{P}_i^*) = \emptyset$ has to hold as we assumed that $K_{i-1}^{vc} \leq K_{i,p-1}$. Hence, we only have to discuss the cases where either $K_{i,p} \in \mathbb{K}^A(\mathcal{P}_0^*)$ or $K_{i,p} \in \mathbb{K}^E(\mathcal{P}_0^*)$. In the first case we can apply Proposition 3.10.8 to see that $\mathbf{a}_{i,p} = K_{i,p} - S_0$ has to hold. Let us thus argue now that the price for a European option with strike $K_{i,p} \in \mathbb{K}^A(\mathcal{P}_0^*)$ will be computed to be $E_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})$. Note first that for $K_{i,p} \in \mathbb{K}^E(\mathcal{P}_0^*)$ $\mathbf{e}_{i,p} = E_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})$ trivially has to be satisfied according to the definition of E_{ub} in (3.8). Hence, we will only consider the situation where $K_{i,p} \in \mathbb{K}^A(\mathcal{P}_0^*)\backslash\mathbb{K}^E(\mathcal{P}_0^*)$. According to (3.22) the algorithm determines the price using

$$\mathbf{e}_{i,p} = \min\{E_{lf}(K_{i,p}, \mathcal{P}_{i,p-1}), E_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})\}.$$

It thus suffices to show that $E_{ub}(K_{i,p}, \mathcal{P}_{i,p-1}) \leq E_{lf}(K_{i,p}, \mathcal{P}_{i,p-1})$ holds which is the case if

$$cc(A; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p-1}) \leq cc(E; K_{i,p-1}, K_j^E; \mathcal{P}_{i,p-1})$$

for $j = \arg\min_{1 \leq j' \leq m_2}\{K_{j'}^E \in \mathbb{K}^E(\mathcal{P}_0^*) : K_{j'}^E \geq K_{i,p}\}$. Since we assumed that $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension it follows that the European price function $E(\cdot, \mathcal{P}_{i,p-1})$ is convex. This readily implies that

$$cc(E; K_{i,p-1}, K_j^E; \mathcal{P}_{i,p-1}) \geq cc(E; K_{i,p-1}, K_{m_2}^E; \mathcal{P}_{i,p-1})$$

has to hold. We can further deduce from $\mathcal{P}_{i,p-1}$ being a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension that $\mathbf{e}_{i,s} \geq e^{-rT}K_{i,s} - S_0$ for any strike $K_{i,s} \in \mathbb{K}^E(\mathcal{P}_{i,p-1})$. Taking into account that $\hat{\mathbf{e}}_{m_2} = e^{-rT}K_{m_2}^E - S_0$ it follows that $cc(E; K_{i,p-1}, K_{m_2}^E; \mathcal{P}_{i,p-1}) \geq -S_0$ has to hold. Then again, we know that $cc(A; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p-1}) = -S_0$ as $A(K, \mathcal{P}_{i,p-1}) = K - S_0$ for any strike $K \in [K_{i,p-1}, K_{i,p}]$. We can therefore conclude that the price for European options with strike $K_{i,p}$ is computed to be $\mathbf{e}_{i,p} = E_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})$.

Suppose now that $K_{i,p} \in \mathbb{K}^E(\mathcal{P}_0^*)\backslash\mathbb{K}^A(\mathcal{P}_i^*)$, then we know already that the price for European options with strike $K_{i,p}$ is given by $E_{ub}(K_{i,p}, \mathcal{P}_{i,p-1})$. We are therefore only left to argue that the price for American options with strike $K_{i,p}$ will be computed to be $\mathbf{a}_{i,p} = K_{i,p} - S_0$ by the algorithm. To see this we will first show that $A_{lb}^{rhs}(K_{i,p}, \mathcal{P}_{i,p-1}) = K_{i,p} - S_0$ has to hold, thereby guaranteeing that $\mathbf{a}_{i,p} \geq K_{i,p} - S_0$.

If there is at most one American option traded in the market to the right of $K_{i,p}$ this follows immediately from the definition of the right hand-side lower bound in (3.14). In the other case where two or more traded American options exist to the right of $K_{i,p}$ Proposition 3.10.8 yields the result. If we can now also rule out that $\mathbf{a}_{i,p} > K_{i,p} - S_0$ we have shown that $\mathbf{a}_{i,p} = K_{i,p} - S_0$. Recall that the algorithm uses

$$\mathbf{a}_{i,p} = \max\{A_{lb}(K_{i,p}, \mathcal{P}_{i,p-1}), A_{lb}^{\overline{A}}(K_{i,p}, \mathcal{P}_{i,p-1}), A_{lf}(K_{i,p}, \mathcal{P}_{i,p-1})\}$$

to determine the price for an American option with strike $K_{i,p} \in \mathbb{K}^E(\mathcal{P}_0^*)\backslash\mathbb{K}^A(\mathcal{P}_i^*)$. We will thus discuss each of the possible prices separately. Recall that the right-hand side lower bound in $K_{i,p}$ has to be given by $A_{lb}^{rhs}(K_{i,p}, \mathcal{P}_{i,p-1}) = K_{i,p} - S_0$. It follows that $\mathbf{a}_{i,p} > K_{i,p} - S_0$ cannot be caused by the right-hand side lower bound.

Suppose now that $\mathbf{a}_{i,p} = A_{lb}^{lhs}(K_{i,p}, \mathcal{P}_{i,p-1})$. Since we assumed that $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension, we can conclude that $\mathbf{a}_{i,s} \geq K_{i,s} - S_0$ has to hold for any strike $K_{i,s} \in [0, K_{i,p-1}] \cap \mathbb{K}(\mathcal{P}_{i,p-1})$. Taking into account that $\mathbf{a}_{i,p-1} = K_{i,p-1} - S_0$, we readily obtain that $A_{lb}^{lhs}(K_{i,p}, \mathcal{P}_{i,p-1}) \leq K_{i,p} - S_0$. We can therefore conclude that the price for the American options with strike $K_{i,p}$ was not determined by $\mathbf{a}_{i,p} = A_{lb}^{lhs}(K_{i,p}, \mathcal{P}_{i,p-1})$ if $\mathbf{a}_{i,p} > K_{i,p} - S_0$.

Next we will consider the case where $\mathbf{a}_{i,p} = A_{lb}^{\overline{A},r}(K_{i,p}, \mathcal{P}_{i,p})$. We then have to distinguish between the two cases where there either exists a traded American option to the right of $K_{i,p}$ or not. If it does exist its price has to lie on the immediate exercise line $K - S_0$ according to Proposition 3.10.8. Moreover, it follows from (i) of the Standing Assumptions that $\hat{\mathbf{e}}_j \geq e^{-rT}K_j^E - S_0$ has to hold. From this we can then readily deduce that $\overline{\mathbf{a}}_j \geq e^{-rT}K_j^E - S_0$. Hence, the right-hand side lower bound $A_{lb}^{\overline{A},r}$ has to satisfy $A_{lb}^{\overline{A},r}(K_{i,p}, \mathcal{P}_{i,p-1})$ and can thus be excluded from consideration as well.

In the case where $K_{m_1}^A < K_{i,p}$ the definition in (3.18) implies that the right hand-side lower bound $A_{lb}^{\overline{A},r}$ has to be given by $A_{lb}^{\overline{A},r}(K_{i,p}, \mathcal{P}_{i,p-1}) = -\infty$. This, however, allows us to rule out $\mathbf{a}_{i,p} = A_{lb}^{\overline{A},r}(K_{i,p}, \mathcal{P}_{i,p-1})$ in this situation as well.

Let us assume now for the moment that $\mathbf{a}_{i,p} = A_{lb}^{\overline{A},l}(K_{i,p}, \mathcal{P}_{i,p-1})$. Since $\mathcal{P}_0^* \in \overline{\mathcal{M}}$ we know that the price for an American option with strike $K_j^A \in \mathbb{K}^A(\mathcal{P}_0^*)$ has to satisfy $\hat{\mathbf{a}}_j \geq K_j^A - S_0$. In particular, we must have $\hat{\mathbf{a}}_u \geq K_u^A - S_0$ for $K_u^A = \max\{K \in \mathbb{K}^A(\mathcal{P}_0^*) : K \leq K_{i,p-1}\}$. Moreover, we know that the price for an American option with strike $K_{i,p-1}$ has to satisfy $\mathbf{a}_{i,p-1} \geq A_{lb}^{\overline{A},l}(K_{i,p-1}, \mathcal{P}_{i,p-2})$ as we assumed that $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension. Combined we obtain that $A_{lb}^{\overline{A},l}(K_{i,p}, \mathcal{P}_{i,p-1}) \leq K_{i,p} - S_0$ has to hold. It thus follows that we cannot have $\mathbf{a}_{i,p} = A_{lb}^{\overline{A}}(K_{i,p}, \mathcal{P}_{i,p-1})$.

We consider now the last case where $\mathbf{a}_{i,p} = A_{lf}(K_{i,p}, \mathcal{P}_{i,p-1})$. We then know that

$$cc(A; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) = cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}).$$

Taking further into account that $\mathbf{a}_{i,p-1} = K_{i,p-1} - S_0$ and $\mathbf{a}_{i,p} > K_{i,p} - S_0$, we obtain that $cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) < -S_0$ has to hold. We then proceed by showing that this is not possible. Since $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension we can deduce that the European price function $E(\cdot, \mathcal{P}_{i,p-1})$ is convex. It then follows from $K_{i,p} \in \mathbb{K}^E(\mathcal{P}_0^*)$ that

$$cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p-1}) \geq cc(E; K_{i,p-1}, K_{m_2}^E; \mathcal{P}_{i,p-1}) \tag{3.71}$$

and

$$cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) = cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p-1}) \tag{3.72}$$

have to hold. We can further observe that the price for European options with strike

$K_{i,p-1}$ has to satisfy $\mathbf{e}_{i,p-1} \geq e^{-rT}K_{i,p-1} - S_0$ as $\mathcal{P}_{i,p-1}$ is a $K_{i,p-1}$-admissible $\mathcal{P}_0^*$-extension. Combined with $\hat{\mathbf{e}}_{m_2} = e^{-rT}K^E_{m_2} - S_0$ we readily obtain that

$$cc(E; K_{i,p-1}, K^E_{m_2}; \mathcal{P}_{i,p-1}) \geq -S_0.$$

Taking into account the inequalities in (3.71) and (3.72) we see that

$$cc(E; K_{i,p-1}, K_{i,p}; \mathcal{P}_{i,p}) \geq -S_0$$

which contradicts $A_{lf}(K_{i,p}, \mathcal{P}_{i,p-1}) > K_{i,p} - S_0$. We can therefore rule out that the price for an American options with strike $K_{i,p}$ exceeds $K_{i,p} - S_0$. □

Proposition 3.10.44. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Assume further that the algorithm extended the initial set of prices from $\mathcal{P}_0^*$ to $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 1$. Starting with the initial set $\mathcal{P}_i^*$ the algorithm computes the $K_{i,q}$-admissible $\mathcal{P}_0^*$-extension $\mathcal{P}_{i,q}$ where $K_{i,q} = \max\{K \in \mathbb{K}(\mathcal{P}_i^*) : K < K^E_{m_2}e^{-rT}\}$. If $K^{vc}_{i-1} \leq K_{i,q}$ the price for an American option with strike $K_{i,q+1}$ is given by $\mathbf{a}_{i,q+1} = K_{i,q+1} - S_0$ under $\mathcal{P}_{i,q+1}$.*

Proof. From $K^{vc}_{i-1} \leq K_{i,q}$ we can immediately conclude that $K_{i,q+1} \notin \mathbb{K}^{aux}(\mathcal{P}_i^*)$. We thus only have to distinguish between the case where either $K_{i,q+1} \in \mathbb{K}^A(\mathcal{P}_0^*)$ or $K_{i,q+1} \in \mathbb{K}^E(\mathcal{P}_0^*)$. If we assume first that $K_{i,q+1} \in \mathbb{K}^A(\mathcal{P}_0^*)$, then we can apply Corollary 3.6.12 to see that $\mathbf{a}_{i,s} = K_{i,s} - S_0$ has to hold for any strike $K_{i,s} \in [K^E_{m_2}e^{-rT}, \infty) \cap \mathbb{K}^A(\mathcal{P}_0^*)$ including $K_{i,q+1}$.

Suppose now that $K_{i,q+1} \in \mathbb{K}^E(\mathcal{P}_0^*) \backslash \mathbb{K}^A(\mathcal{P}_0^*)$ we then need to distinguish between the situations where either $\mathbf{a}_{i,q} > K_{i,q} - S_0$ or not. In the first case we must have $A(K^E_{m_2}e^{-rT}, \mathcal{P}_{i,q+1}) > \overline{\mathbf{a}}_{m_2}$ as $\mathbf{a}_{i,q+1} \geq K_{i,q+1} - S_0$ has to hold according to (3.23). It follows that a violation of the upper bound occurs at the strike $K^E_{m_2}e^{-rT}$. If a generalised version of Proposition 3.10.24 holds we could then argue that the price for an American option with strike $K_{i,q+1}$ has to be given by

$$\mathbf{a}_{i,q+1} = \max\{A^{rhs}_{lb}(K_{i,q+1}, \mathcal{P}_{i,q}), A^{\overline{A},r}_{lb}(K_{i,q+1}, \mathcal{P}_{i,q})\}.$$

Since $\mathcal{P}_0^* \in \overline{\mathcal{M}}$ we are guaranteed that the prices for traded European options exceed the lower bound $e^{-rT}K - S_0$. The definition of the upper bound $\overline{A}$ then readily implies that $\overline{\mathbf{a}}_j \geq e^{-rT}K^E_j - S_0$ for $K^E_j \in \mathbb{K}^E(\mathcal{P}_0^*)$. If $K_{i,q+1} < K^A_{m_1}$ this yields a right hand-side lower bound $A^{\overline{A},r}_{lb}$ at $K_{i,q+1}$ given by $A^{\overline{A},r}_{lb}(K_{i,q+1}, \mathcal{P}_{i,q}) \leq K_{i,q+1} - S_0$. In the case that $K^A_{m_1} < K_{i,q+1}$ the right-hand side lower bound is given by $A^{\overline{A},r}_{lb}(K_{i,q+1}, \mathcal{P}_{i,q}) = -\infty$ according to the definition in (3.18). The lower bound A^{rhs}_{lb} is furthermore given by $K_{i,q+1} - S_0$ at the strike $K_{i,q+1}$. The reason begin that either all the prices for traded American options with strikes larger than $K^E_{m_2}e^{-rT}$ lie on the immediate exercise line, as argued above, or because the right-hand side lower bound is defined to be $K_{i,q+1} - S_0$ in the case where $K_{i,q+1} > K^A_{m_1-1}$ holds.

If we consider now the situation where the price for American options with strike $K_{i,q}$ is given by $\mathbf{a}_{i,q} = K_{i,q} - S_0$ we can apply Proposition 3.10.42 to see that $\mathbf{a}_{i,q+1} = K_{i,q+1} - S_0$ has to hold. We can therefore conclude that the price for an American option with strike $K_{i,q+1}$ has to be given by $\mathbf{a}_{i,q+1} = K_{i,q+1} - S_0$ under $\mathcal{P}_{i,q+1}$. □

Proposition 3.10.45. *Suppose finitely many American and co-terminal European put options are traded in the market and that their prices are provided by $\mathcal{P}_0^* \in \overline{\mathcal{M}}$. Assume further that the algorithm extended the initial set of prices from $\mathcal{P}_0^*$ to $\mathcal{P}_i^* \supseteq \mathcal{P}_0^*$, $i \geq 1$.*

Using the initial set of prices $\mathcal{P}_i^*$ the algorithm terminates at the strike $K^E_{m_2}$ having constructed a $K^E_{m_2}$-admissible $\mathcal{P}_0^*$-extension. Suppose that the final set of prices is given by $\mathcal{P}_{i,n}$, then $\mathbf{a}_{i,s} = K_{i,s} - S_0$ for any strike $K_{i,s} \in [K_{i,q+1}, \infty) \cap \mathbb{K}(\mathcal{P}_{i,n})$ where $K_{i,q+1} = \min\{K \in \mathbb{K}(\mathcal{P}_{i,n}) : K > K^E_{m_2} e^{-rT}\}$.

Proof. According to Proposition 3.10.44 the algorithm will initially compute the price for an American option with strike $K_{i,q+1}$ to be $\mathbf{a}_{i,q+1} = K_{i,q+1} - S_0$. We then argue in Proposition 3.10.42 that the price for any American option with a strike $K_{i,s} \in [K_{i,q+1}, \infty) \cap \mathbb{K}(\mathcal{P}_i^*)$ has to be given by $\mathbf{a}_{i,s} = K_{i,s} - S_0$ as well. Moreover, Remark 3.10.43 points out that a violation of convexity can be ruled out at any such strike. Hence, the algorithm will not have to be restarted and thus the prices for American options with strike $K \geq K_{i,q+1}$ in $\mathcal{P}_{i,n}$ will be given by the prices calculated initially. □

3.11 Flowchart

In this section we will provide a flowchart for the algorithm given in Section 3.5. The first flowchart contains the entire Algorithm 2. Algorithm 3 is then presented on the following two pages whereas Algorithm 4 concludes this section.

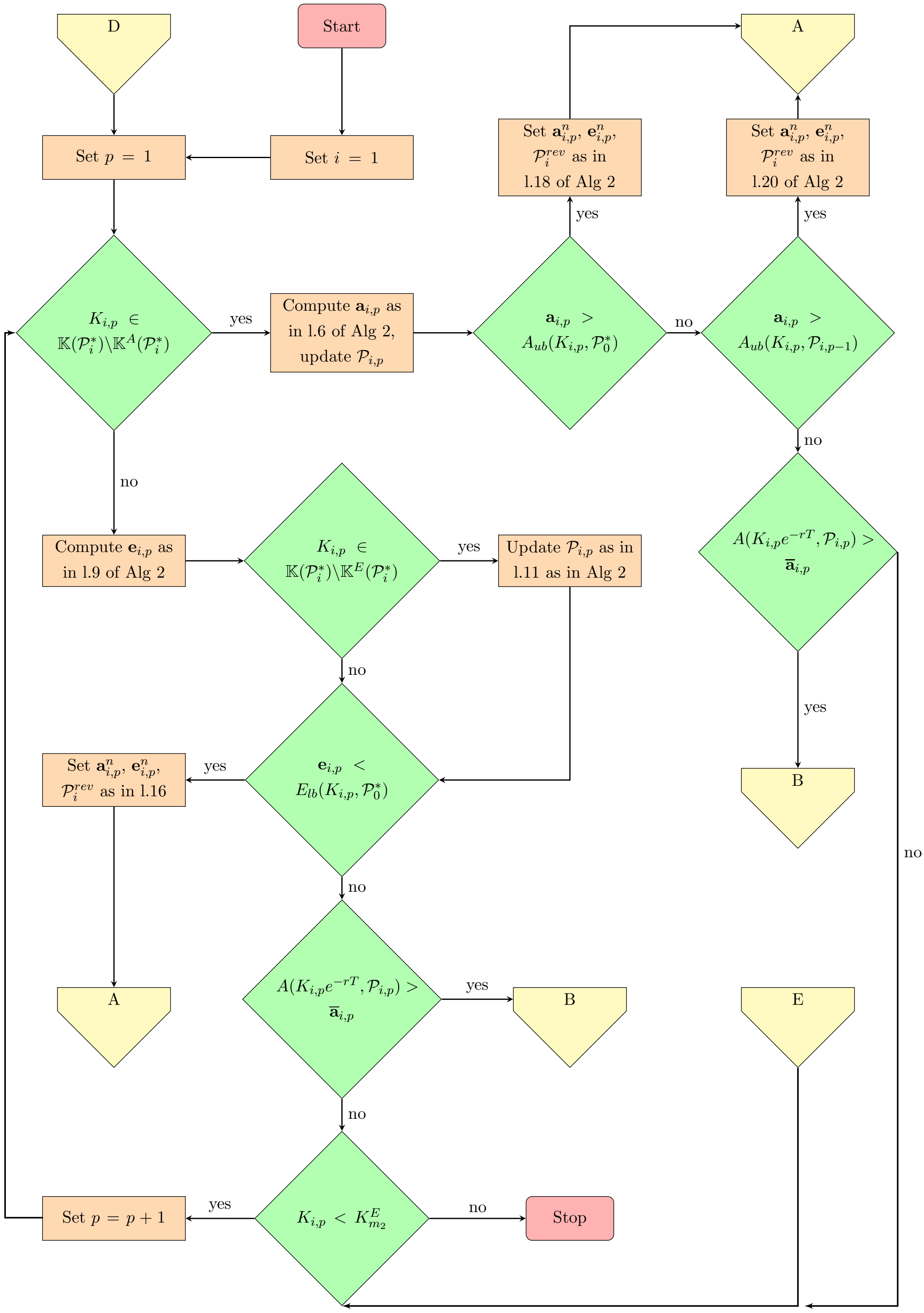

Figure 3-3: Flowchart for Algorithm 2

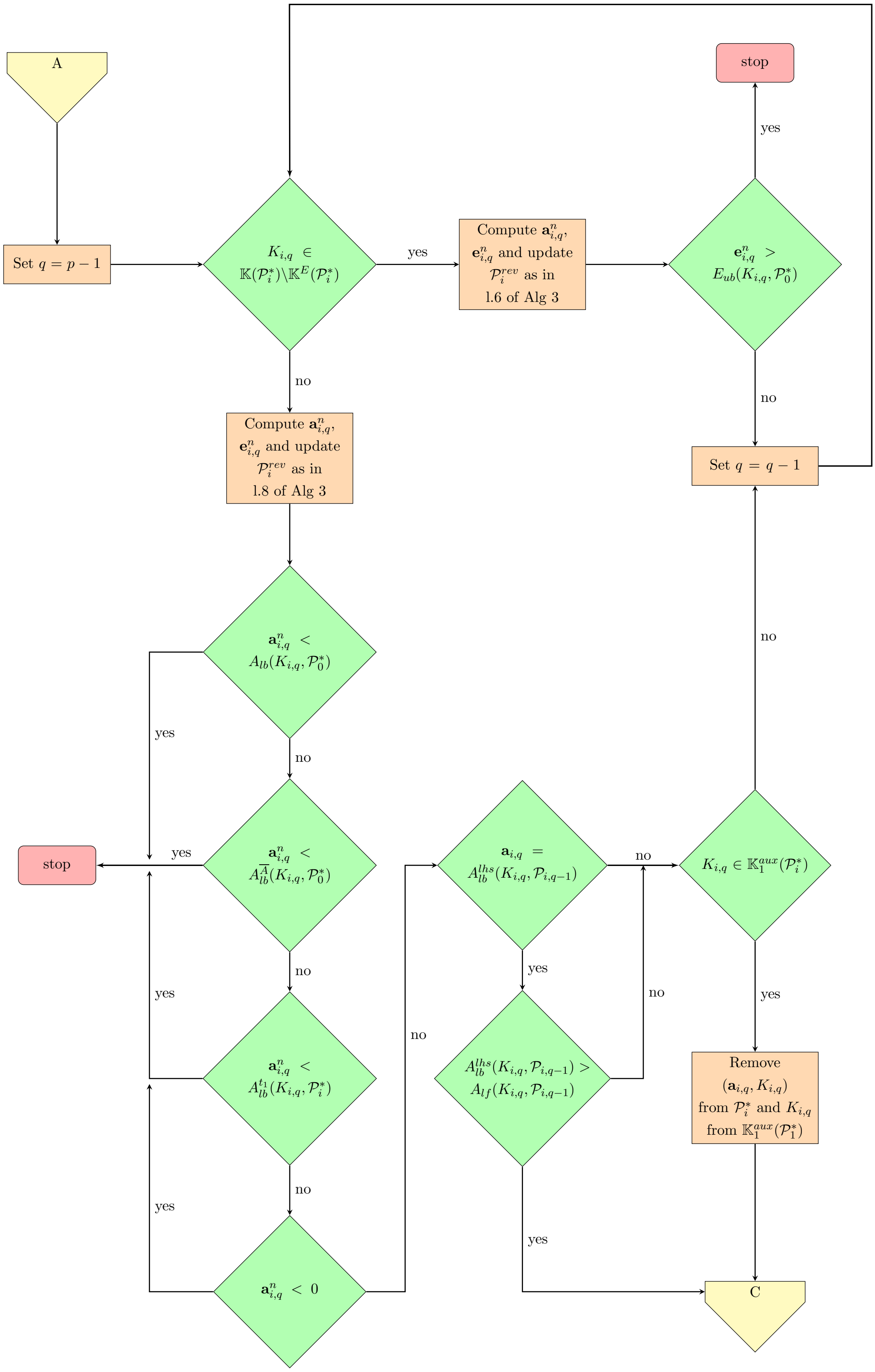

Figure 3-4: Part 1 of the flowchart for Algorithm 3

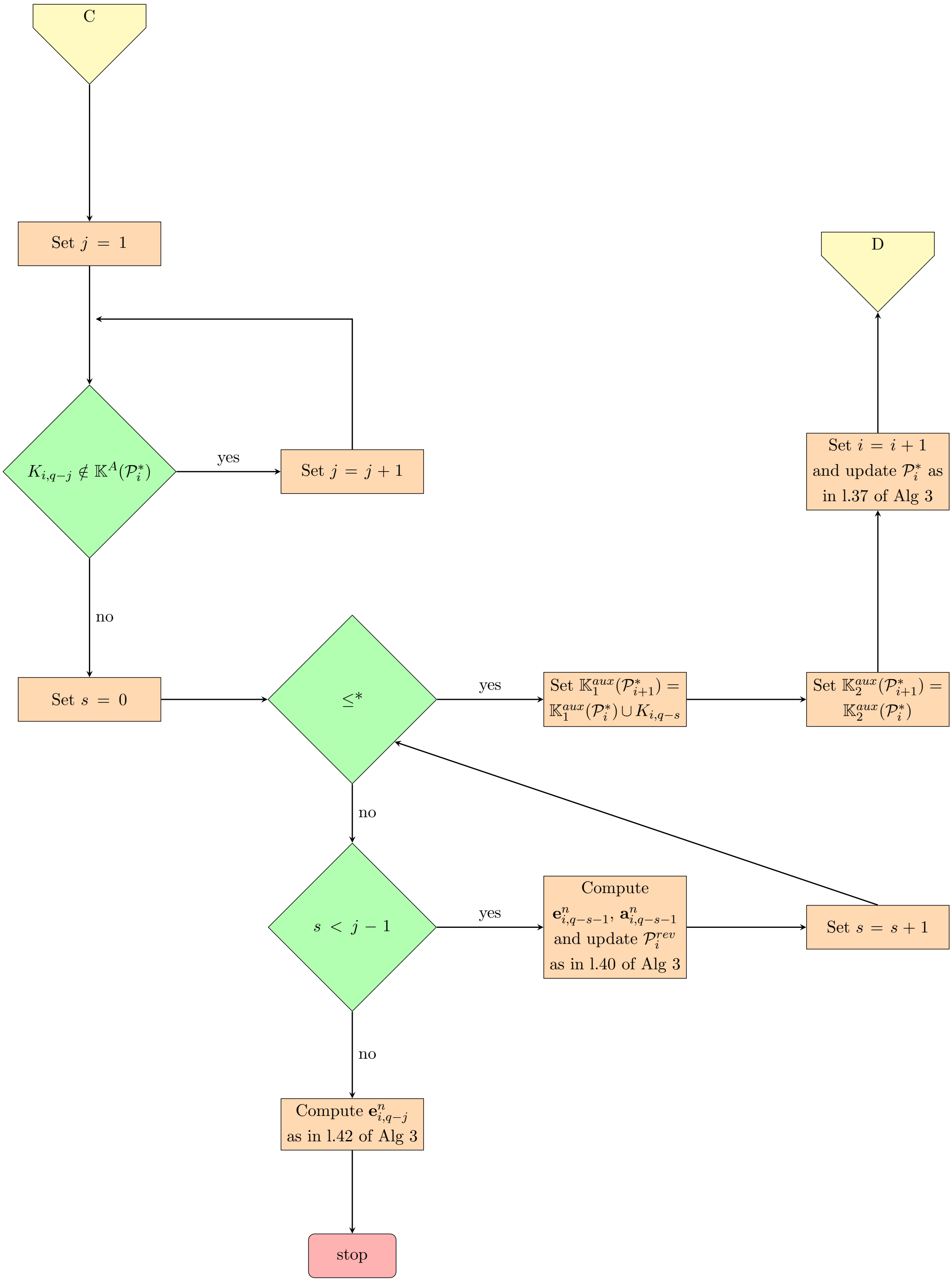

Figure 3-5: Part 2 of the flowchart for Algorithm 3

$^*cc(A, A^n; K_{i,q-j}, K_{i,q-s}; \mathcal{P}_{i,p}) \leq cc(E; K_{i,q-s-1}, K_{i,q-s}; \mathcal{P}_{i,p})$

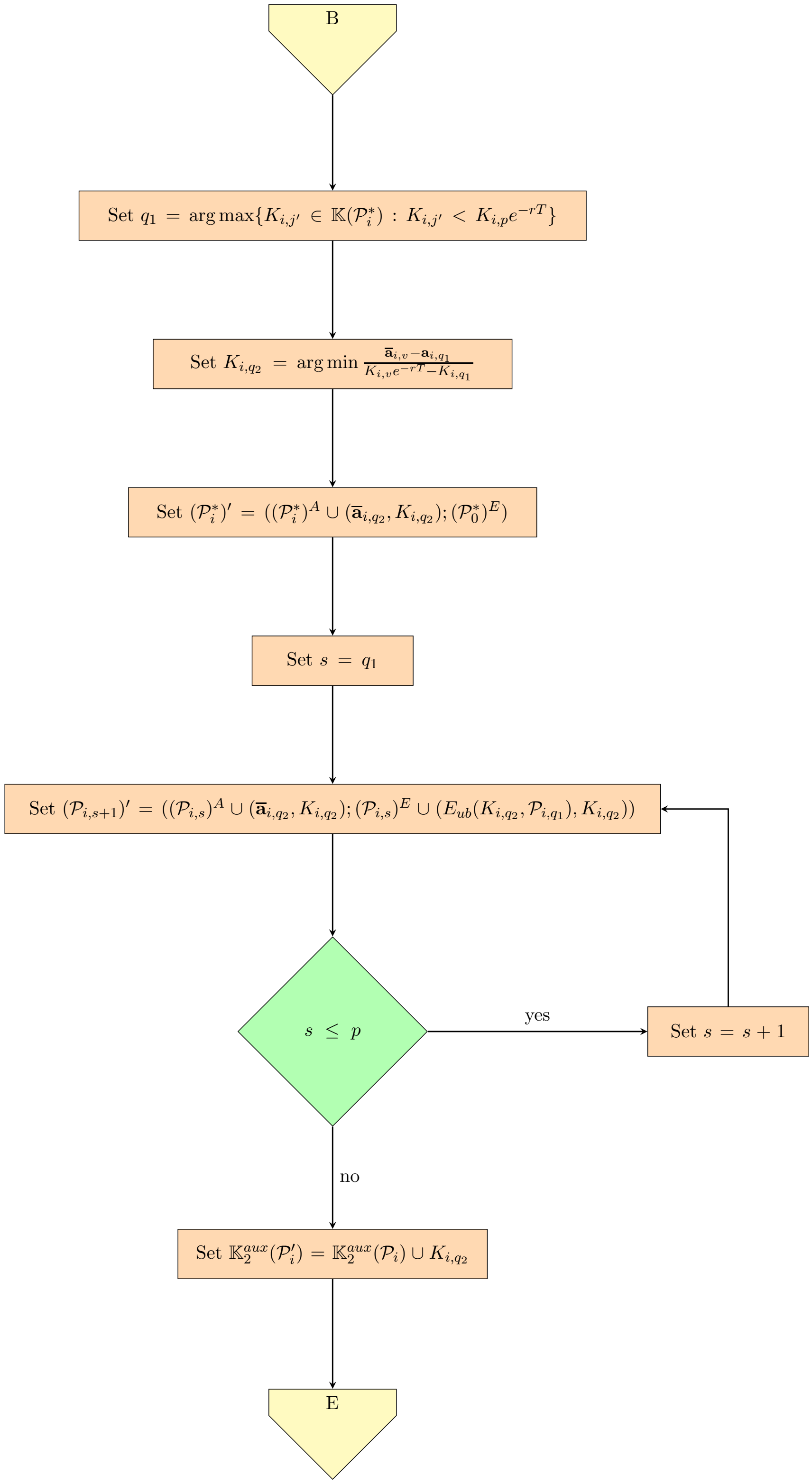

Figure 3-6: Flowchart for Algorithm 4

Bibliography

L. Bachelier. Théorie de la spéculation. *Annales de l'Ecole Supérieure*, 17:21–86, 1900.

T. Björk. *Arbitrage Theory in Continuous Time*. Oxford University Press, 3rd edition, 2009.

F. Black and M. Scholes. The pricing of options and corporate liabilities. *The Journal of Political Economy*, 81(3):637–654, 1973.

D.T. Breeden and R.H. Litzenberger. Prices of state-contingent claims implicit in option prices. *Journal of Business*, 51(4):621–51, 1978.

H. Brown, D.G. Hobson, and L.C.G. Rogers. Robust hedging of barrier options. *Mathematical Finance*, 11:285–314, 2001.

H. Buehler. Expensive martingales. *Quantitative Finance*, 6:207–218, 2006.

P. Carr and R. Lee. Hedging variance options on continuous semimartingales. *Finance and Stochastics*, 14:179–207, 2010.

P. Carr and D.B. Madan. A note on sufficient conditions for no arbitrage. *Finance Research Letters*, 2:125–130, 2005.

R.V. Chacon. Potential processes. *Transactions of the American Mathematical Society*, 226:39–58, 1977.

R.V. Chacon and J.B. Walsh. One-dimensional potential embedding. pages 19–23. Lecture Notes in Math., Vol. 511, 1976.

L. Cousot. Conditions on option prices for absence of arbitrage and exact calibration. *Journal of Banking & Finance*, 31(11):3377–3397, 2007.

A.M.G. Cox. Robust Hedging with given option prices: The Skorokhod embedding approach. Preprint, 2014.

A.M.G. Cox and C. Hoeggerl. Model-independent no-arbitrage conditions on american put options. *Mathematical Finance, To Appear*, 2013. doi: 10.1111/mafi.12058.

A.M.G. Cox and J. Obłój. Robust pricing and hedging of double no-touch options. *Finance and Stochastics*, 15:573–605, 2011a.

A.M.G. Cox and J. Obłój. Robust hedging of double touch barrier options. *SIAM Journal of Financial Mathematics*, 2:141–182, 2011b.

A.M.G. Cox and J. Wang. Root's barrier: Construction, optimality and applications to variance options. *The Annals of Applied Probability*, 2012. To Appear.

A.M.G. Cox and J. Wang. Optimal robust bounds for variance options. *http://arxiv.org/abs/1308.4363*, 2013.

M.H.A. Davis and D.G. Hobson. The range of traded option prices. *Math. Finance*, 17 (1):1–14, 2007.

B. Dupire. Arbitrage bounds for volatility derivatives as free boundary problem. *Presentation at PDE and Mathematical Finance, KTH, Stockholm*, 2005.

E. Ekström and D.G. Hobson. Recovering a time-homogeneous stock price process from perpetual option prices. *Annals of Applied Probability*, 21:1102–1135, 2009.

A. Galichon, P. Henry-Labordère, and N. Touzi. *A stochastic control approach to no-arbitrage bounds given marginals, with an application to Lookback options.* http://ssrn.com/abstract=1912477. 2011.

J.M. Harrison and D.M. Kreps. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20:381–408, 1979.

J.M. Harrison and S.R. Pliska. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and Applications*, 11:251–260, 1981.

V. Henderson, J. Sun, and A.E. Whalley. *Portfolios of American Options Under General Preferences: Results and Counterexamples. Mathematical Finance*, To Appear. 2013.

D.G. Hobson. Robust hedging of the lookback option. *Finance and Stochastics*, 2: 329–347, 1998.

D.G. Hobson. The Skorokhod Embedding Problem and Model-Independent Bounds for Option Prices. In R.A. Carmona, E. Çinlar, I. Ekeland, E. Jouini, J.A. Scheinkman, and N. Touzi, editors, *Paris-Princeton Lectures on Mathematical Finance 2010*, volume 2003 of *Lecture Notes in Math.*, pages 267–318. Springer, 2011.

D.G. Hobson and M. Klimmek. Constructing time-homogeneous generalized diffusions consistent with optimal stopping values. *Stochastics An International Journal of Probability and Stochastic Processes*, 83(4-6):477–503, 2011.

I. Karatzas. On the pricing of american options. *Applied Mathematics & Optimization*, 17:37–60, 1988.

I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer, 2nd edition, 1998. ISBN 978-0387-97655-6.

J.-P. Laurent and D. Leisen. Building a consistent pricing model from observed option prices. In *Quantitative Analysis in Financial Markets: Collected Papers of the New York University Mathematical Finance Seminar*, volume 2, pages 216–238, 2000.

W. Margrabe. The value of an option to exchange one asset for another. *The Journal of Finance*, 33(1):177–186, 1978.

A. Miller and R. Vyborny. Some remarks on functions with one-sided derivatives. *The American Mathematical Monthly*, 93(6):471–475, 1986.

A. Neuberger. Bounds and robust hedging of the american option. *http://www2.warwick.ac.uk/fac/soc/wbs/subjects/finance/faculty1 /anthony_neuberger/ bounds.pdf*, 2009.

J. Obłój. The Skorokhod embedding problem and its offspring. *Probab. Surv.*, 1: 321–390, 2004.

G. Peskir and A. Shiryaev. *Optimal Stopping and Free-Boundary Problems*. Birkhäuser Basel, 2006. ISBN 978-3-7643-2419-3.

P.A. Samuelson. Rational theory of warrant pricing. *Industrial Management Review*, 6:13–31, 1965.

P. Shah. No-arbitrage bounds on american put options with a single maturity. *http://hdl.handle.net/1721.1/36232*, 2006.

A.V. Skorokhod. *Studies in the theory of random processes.* Translated from the Russian by Scripta Technica, Inc. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965.

D. Williams. *Probability with martingales.* Cambridge Mathematical Textbooks. Cambridge University Press, 12th edition, 2010. ISBN 978-0-521-40605-5.