Solution of a constraint generalised eigenvalue problem using the inexact Shift-and-Invert Lanczos method on a paper by V. Simoncini

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Introduction

The Lanzcos method

Motivation

The SI-Lanczos process on the constraint problem

Shift-and-Invert Lanczos

Inexact Shift-and-Invert Lanczos

Solution of the constraint inner system

Block definite preconditioning

Block indefinite preconditioning

The Augmented formulation and inexact SI-Lanczos

The modified formulation

Some numerics

Conclusions
Outline

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Problem

- Eigenproblem for $A \in \mathbb{C}^{n,n}$, $A = A^T$:
  \[ Ax = \lambda x. \]

- let the eigenvalues be
  \[ |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \]

- associated eigenvectors $x_1, x_2, \ldots, x_n$

- $A$ is large and sparse, need iterative methods.
The Lanzcos method

Idea behind Lanczos

- keep iterates from Power method $v, Av, \ldots, A^{k-1}v$ which form a Krylov subspace associated with $A$ and $v$

$$\mathcal{K}_j(A, v) = \text{span}\{v, Av, \ldots, A^{j-1}v\}.$$  

- $v, Av, \ldots, A^{k-1}v$ are usually ill-conditioned
- orthogonalise the vectors $v, Av, \ldots, A^{k-1}v$ in the Krylov space using a modified Gram-Schmidt process
The Lanzcos method

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  a modified Gram-Schmidt process
Lanczos algorithm

- choose initial vector $v$ and normalise $v_1 = \frac{v}{\|v\|_2}$
- On subsequent steps $k = 1, 2, \ldots$ take

$$\tilde{v}_{k+1} = Av_k - \sum_{j=1}^{k} v_j t_{jk}$$

where $t_{jk}$ is the Gram-Schmidt coefficient $t_{jk} = \langle Av_k, v_j \rangle$.
- normalise

$$v_{k+1} = \frac{\tilde{v}_{k+1}}{t_{k+1,k}} \quad \text{where} \quad t_{k+1,k} = \|\tilde{q}_{k+1}\|_2$$
Matrix formulation and calculation of eigenvalues

Lanczos in matrix form
The Lanczos process can be written in the form

$$AV_m = V_m T_m + v_{m+1} \beta_m e_m^T$$

where

$$T_m = \begin{bmatrix}
\alpha_1 & \beta_1 \\
\beta_1 & \alpha_2 & \cdot \\
& \cdot & \cdot & \cdot \\
& & \beta_{m-1} & \alpha_m \\
\end{bmatrix}$$

Theorem
Let $V_m$, $T_m$ and $\beta_m$ generated by the Lanczos process and

$$T_m s = \mu s, \quad \|s\|_2 = 1.$$ 

Let $y = V_m s \in \mathbb{C}^n$, then
The Lanzcos method

Matrix formulation and calculation of eigenvalues

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\[ AV_m = V_m T_m + v_{m+1} \beta_m e_m^T \]

where

\[ T_m = \begin{bmatrix}
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\beta_1 & \alpha_2 & \ddots \\
& \ddots & \ddots & \ddots \\
& & \beta_{m-1} & \alpha_m 
\end{bmatrix} \]

Theorem
Let \( V_m, T_m \) and \( \beta_m \) generated by the Lanczos process and

\[ T_m s = \mu s, \quad \|s\|_2 = 1. \]

Let \( y = V_m s \in \mathbb{C}^n \), then
The Lanzcos method

An example

first 10 Lanczos steps

10 steps of Lanczos (no reorthogonalization) applied to $A$

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Constraint eigenproblems and SI-Lanczos
An example

first 20 Lanczos steps

20 steps of Lanczos (no reorthogonalization) applied to A
The Lanczos method

An example

The first 30 Lanczos steps

30 steps of Lanczos (no reorthogonalization) applied to $A$
The constraint eigenvalue problem

- Computation of the smallest non-zero eigenvalues and corresponding eigenvectors of
  \[ Ax = \lambda Mx \]

  where \( M = M^T \) positive definite and \( A = A^T \) positive semidefinite.

- Assume sparse basis \( C \) for null-space of \( A \) is available.

- Dimension of the null-space is high compared with the problem dimension.

- Constraint in terms of the null-space orthogonality, for smallest non-zero eigenvalue:

  \[
  \min_{\substack{C^T M x = 0 \\ 0 \neq x \in \mathbb{R}}} \frac{x^T A x}{x^T M x}
  \]
The constraint eigenvalue problem

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- Dimension of the null-space is high compared with the problem dimension
- Constraint in terms of the null-space orthogonality, for smallest non-zero eigenvalue:
  \[
  \min_{C^TMx=0, 0 \neq x \in \mathbb{R}} \frac{x^TAx}{x^TMx}
  \]
Motivation

Application areas

**Electromagnetic cavity resonator**

\[ \text{curl}(\mu^{-1}\text{curl}u) = \omega^2 u \quad \text{in} \quad \Omega \]

\[ \text{div}(\varepsilon u) = 0 \quad \text{in} \quad \Omega \]

\[ u \times n = 0 \quad \text{on} \quad \partial\Omega \]

where \( u \) is the electric field, \( n \) denotes the outward normal vector, \( \mu \) the magnetic permeability, \( \varepsilon \) the electric permittivity.

**Network problems**

\[ Ax = \lambda x, \quad \text{with} \quad Ac = 0 \]

where \( A = A^T \) SPD, \( M = I \) and the eigenpair \((0, c)\) is known, looking for second smallest eigenvalue \( \lambda_2 \) with the constraint \( c^T x = 0 \).
Motivation

Application areas

Electromagnetic cavity resonator

\[
\text{curl}(\mu^{-1}\text{curl}\mathbf{u}) = \omega^2 \mathbf{u} \quad \text{in} \quad \Omega \\
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\mathbf{u} \times \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega
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Network problems

\[
Ax = \lambda x, \quad \text{with} \quad Ac = 0
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where \( A = A^T \) SPD, \( M = I \) and the eigenpair \((0, c)\) is known, looking for second smallest eigenvalue \( \lambda_2 \) with the constraint \( c^T x = 0 \).
Simplify the problem

- Consider smallest non-zero eigenvalues and corresponding eigenvectors of

\[ Ax = \lambda x \]

where \( M = I \) positive definite and \( A = A^T \) positive semidefinite.

- The null-space is one-dimensional

\[ Ac = 0 \]

- Constraint in terms of the null-space orthogonality, for smallest non-zero eigenvalue:

\[
\min_{c^T x = 0, \, 0 \neq x \in \mathbb{R}} \frac{x^T Ax}{x^T x}
\]
**Motivation**

**Different formulations of the problem**

- \[ Ax = \lambda x, \quad c^T x = 0 \]
- Shifting the null eigenvalue
  \[
  (A + cH^{-1}c^T)x = \eta x,
  \]
  where \( H = \frac{1}{\gamma}c^Tc \) shifts zero eigenvalue to \( \gamma \).
  Smallest eigenvalues coincide.
- Enforce constraint with augmented system
  \[
  \begin{bmatrix}
  A & c \\
  c^T & 0
  \end{bmatrix}
  \begin{bmatrix}
  x \\
  y
  \end{bmatrix}
  = \lambda
  \begin{bmatrix}
  I & 0 \\
  0 & 0
  \end{bmatrix}
  \begin{bmatrix}
  x \\
  y
  \end{bmatrix}.
  \]
  Smallest eigenvalues coincide.
Different formulations of the problem

- $Ax = \lambda x, \quad c^T x = 0$
- Shifting the null eigenvalue

$$ (A + cH^{-1}c^T)x = \eta x, $$

$H = \frac{1}{\gamma}c^T c$ shifts zero eigenvalue to $\gamma$.  

Smallest eigenvalues coincide.

- Enforce constraint with augmented system

$$ \begin{bmatrix} A & c \\ c^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. $$

Smallest eigenvalues coincide.
Different formulations of the problem

- $Ax = \lambda x$, \quad $c^T x = 0$

- Shifting the null eigenvalue

  \[(A + cH^{-1}c^T)x = \eta x,\]

  \[H = \frac{1}{\gamma}c^T c \text{ shifts zero eigenvalue to } \gamma.\]

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- Enforce constraint with augmented system

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  \end{bmatrix}.
  \]

  Smallest eigenvalues coincide.
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Shift-and-Invert Lanczos for generalised eigenproblem

- Consider
  \[ Ax = \lambda I x, \quad A = A^T, \]

- apply Lanczos to spectrally transformed problem
  \[ (A - \sigma I)^{-1} I x = \eta x, \quad \eta = (\lambda - \sigma)^{-1} \]

- basic recursion for SI-Lanczos
  \[ (A - \sigma I)^{-1} V_j = V_j T_j + v_{j+1} t_{j+1,j} e_j^T, \]
  where \( V_j = [v_1, \ldots, v_j] \) is an orthogonal basis, \( T_j \) tridiagonal with
  \[ T_j = V_j^T (A - \sigma I)^{-1} V_j \]

- If \( T_j s_j^{(i)} = \eta_j^{(i)} s_j^{(i)} \) we get eigenpairs for \( A \) by \( (1/\eta_j^{(i)} + \sigma, V_j s_j^{(i)}) \).
Consider

\[ Ax = \lambda I x, \quad A = A^T, \]

apply Lanczos to spectrally transformed problem

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If \( T_j s_j^{(i)} = \eta_j^{(i)} s_j^{(i)} \) we get eigenpairs for \( A \) by \( (1/\eta_j^{(i)} + \sigma, V_j s_j^{(i)}) \)
Shift-and-Invert Lanczos and constraints

- \((A - \sigma I)^{-1} x = \eta x, \quad c^T x = 0\).
- start iteration with \(v_1\) such that \(c^T v_1 = 0\) and \(v_1^T v_1 = 1\)
- \(c^T x\) is automatically satisfied by exact eigenvectors
- finite precision arithmetic orthogonality constraint not satisfied
- let \(\pi = c(c^T c)^{-1} c^T\), then \(I - \pi\) projects onto \(\mathbb{R}^n\) orthogonal to the null-space of \(A\)
- modify Lanczos algorithm to enforce orthogonality constraint \(c^T v_j = 0\):

\[
\tilde{v} = (I - \pi)(A - \sigma I)^{-1} v_j \\
v_{j+1} t_{j+1,j} = \tilde{v} - V_j T_{:,j}, \quad T_{:,j} = V_j^T \tilde{v}
\]
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Inexact Shift-and-Invert Lanczos and constraints

- let $z_j$ be approximate solution to the system

$$ (A - \sigma I)z = v_j $$

- Set $Z_j = [z_1, \ldots, z_j]$

$$ (A - \sigma I)^{-1}V_j = V_j T_j + v_{j+1} t_{j+1,j} e_j^T, $$

becomes

$$ Z_j = V_j T_j + v_{j+1} t_{j+1,j} e_j^T, \quad \bar{T}_j = V_j^T Z_j $$

- Problem: $c^T z_j = 0$ depends on the iterative solver and preconditioning strategy

- enforce the constraint in the outer Lanczos iteration:

$$ (I - \pi)Z_j = V_j T_j + v_{j+1} t_{j+1,j} e_j^T, \quad V_j = (I - \pi)V_j $$

- enforce the constraint during the solution of the inner system
Inexact Shift-and-Invert Lanczos and constraints

- Let $z_j$ be approximate solution to the system
  
  $$(A - \sigma I)z = v_j$$

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  $$(A - \sigma I)^{-1}V_j = V_jT_j + v_{j+1}t_{j+1,j}e_j^T,$$

  becomes

  $$Z_j = V_j\bar{T}_j + v_{j+1}t_{j+1,j}e_j^T, \quad \bar{T}_j = V_j^T Z_j$$

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- Enforce the constraint during the solution of the inner system.
Inexact Shift-and-Invert Lanczos and constraints

- let $z_j$ be approximate solution to the system
  $$(A - \sigma I)z = v_j$$

- Set $Z_j = [z_1, \ldots, z_j]$
  $$(A - \sigma I)^{-1}V_j = V_j T_j + v_{j+1} t_{j+1,j} e_j^T,$$ becomes
  $$Z_j = V_j \tilde{T}_j + v_{j+1} t_{j+1,j} e_j^T, \quad \tilde{T}_j = V_j^T Z_j$$

- Problem: $c^T z_j = 0$ depending on the iterative solver and preconditioning strategy
  - enforce the constraint in the outer Lanczos iteration:
    $$(I - \pi) Z_j = V_j T_j + v_{j+1} t_{j+1,j} e_j^T, \quad V_j = (I - \pi) V_j$$
  - enforce the constraint during the solution of the inner system
let \( z_j \) be approximate solution to the system

\[
(A - \sigma I)z = v_j
\]

Set \( Z_j = [z_1, \ldots, z_j] \)

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becomes

\[
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Problem: \( c^T z_j = 0 \) depending on the iterative solver and preconditioning strategy

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enforce the constraint during the solution of the inner system
Inexact Shift-and-Invert Lanczos and constraints

- Let \( z_j \) be approximate solution to the system
  \[
  (A - \sigma I)z = v_j
  \]

- Set \( Z_j = [z_1, \ldots, z_j] \)
  \[
  (A - \sigma I)^{-1}V_j = V_jT_j + v_{j+1}t_{j+1,j}e_j^T,
  \]
  becomes
  \[
  Z_j = V_j\bar{T}_j + v_{j+1}t_{j+1,j}e_j^T, \quad \bar{T}_j = V_j^TZ_j
  \]
- Problem: \( c^Tz_j = 0 \) depending on the iterative solver and preconditioning strategy
- Enforce the constraint in the **outer** Lanczos iteration:
  \[
  (I - \pi)Z_j = V_jT_j + v_{j+1}t_{j+1,j}e_j^T, \quad V_j = (I - \pi)V_j
  \]
- Enforce the constraint during the solution of the **inner** system
On Krylov subspace methods (for solving linear systems)

- want to solve

\[(A - \sigma I)z = v\]

- using right preconditioner \(P\) we obtain

\[(A - \sigma I)P^{-1}\hat{z} = v\]

- minimise the residual \(v - (A - \sigma I)P^{-1}\hat{z}\) with zero starting guess

\[z^{(m)} = P^{-1}\hat{z}^{(m)} \quad \text{with} \quad \hat{z}^{(m)} \in \mathcal{K}_m((A - \sigma I)P^{-1}, v)\]

- examples: CG, MINRES, GMRES
On Krylov subspace methods (for solving linear systems)

- want to solve

\[(A - \sigma I)z = v\]

- using right preconditioner \(P\) we obtain

\[(A - \sigma I)P^{-1}\hat{z} = v\]

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- examples: CG, MINRES, GMRES
Original system - Augmented system

Augmented System

\[
\begin{bmatrix}
A - \sigma I & c \\
c^T & 0
\end{bmatrix}
\begin{bmatrix}
P^{-1} \hat{z} \\
v
\end{bmatrix}
\Leftrightarrow
(A - \sigma I)P^{-1} \hat{z} = Ib
\]

Vectors generating the subspace \( \mathcal{K}_m((A - \sigma I)P^{-1}, Ib) \)

\[
((A - \sigma I)^k P^{-1})^k Ib = \begin{bmatrix} G^k v \\ 0 \end{bmatrix}
\]

Minimisation procedure

\[
\hat{z}^{(m)} \in \mathcal{K}_m((A - \sigma I)P^{-1}, Ib) \quad \hat{z}^{(m)} = [\hat{x}^{(m)}; 0]
\]

optimal approximate solution of \( G \hat{x} = v \) in
Original system - Augmented system

Augmented System

\[
\begin{bmatrix}
A - \sigma I & c \\
c^T & 0
\end{bmatrix}
\begin{bmatrix}
P^{-1}\hat{z} \\
0
\end{bmatrix}
\Leftrightarrow
\begin{bmatrix}
(A - \sigma I)P^{-1}\hat{z} \\
0
\end{bmatrix} = Ib
\]

Vectors generating the subspace \( \mathcal{K}_m((A - \sigma I)P^{-1}, Ib) \)

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((A - \sigma I)^k P^{-1})^k Ib = \begin{bmatrix} G^k v \\ 0 \end{bmatrix}
\]

Minimisation procedure

\[
\hat{z}^{(m)} \in \mathcal{K}_m((A - \sigma I)P^{-1}, Ib) \quad \hat{z}^{(m)} = [\hat{x}^{(m)}; 0]
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optimal approximate solution of \( G\hat{x} = v \) in
Original system - Augmented system

Augmented System

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\begin{bmatrix}
A - \sigma I & c \\
c^T & 0
\end{bmatrix}
\begin{bmatrix}
P^{-1} \hat{z} \\
0
\end{bmatrix} \Leftrightarrow (A - \sigma I)P^{-1} \hat{z} = I b
\]

Vectors generating the subspace \( \mathcal{K}_m((A - \sigma I)P^{-1}, Ib) \)

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((A - \sigma I)^k P^{-1})^k Ib = \begin{bmatrix}
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optimal approximate solution of \( G \hat{x} = v \) in
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Conclusions
Original system - Augmented system

- **Original system:**
  \[(A - \sigma I)x = v\] with \[c^T x = 0\]

- **Augmented system**
  \[
  \begin{bmatrix}
  A - \sigma I & c \\
  c^T & 0
  \end{bmatrix}
  \begin{bmatrix}
  x \\
  y
  \end{bmatrix}
  =
  \begin{bmatrix}
  v \\
  0
  \end{bmatrix}
  \iff
  (A - \sigma I)z = Ib
  \]

- show that augmented system is not better than original system
- analyse 2 preconditioning techniques
Original system - Augmented system

- Original system:
  \[(A - \sigma I)x = v \quad \text{with} \quad c^T x = 0\]

- Augmented system
  \[
  \begin{bmatrix}
  A - \sigma I & c \\
  c^T & 0
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  \begin{bmatrix}
  x \\
  y
  \end{bmatrix}
  =
  \begin{bmatrix}
  v \\
  0
  \end{bmatrix}
  \Leftrightarrow (A - \sigma I)z = Ib
  \]

- show that augmented system is not better than original system
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Original system - Augmented system

- Original system:
  \[(A - \sigma I)x = v \quad \text{with} \quad c^T x = 0\]

- Augmented system
  \[
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  \begin{bmatrix}
  x \\
  y \\
  \end{bmatrix}
  =
  \begin{bmatrix}
  v \\
  0 \\
  \end{bmatrix}
  \iff
  (A - \sigma I)z = Ib
  \]

- show that augmented system is not better than original system
- analyse 2 preconditioning techniques
The preconditioner and its properties

- structured symmetric definite preconditioner

\[ P_D = \begin{bmatrix} K_1 & 0 \\ 0 & c^T K_1^{-1} c \end{bmatrix}, \quad K_1 = A_1 - \tau I, \quad \tau \in \mathbb{R} \]

where \( K_1 = K_1^T \) nonsingular and \( A_1 c = 0 \) so \( \tau \neq 0 \) \((A_1 = 0)\)

- We have

\[ c^T K_1^{-1} = -\frac{1}{\tau} c^T \]

\[ c^T K_1^{-1} c = -\frac{1}{\tau} c^T c \quad \text{symplifies} \quad P_D \]

and with \( K = A - \sigma I \)

\[ c^T (KK_1^{-1})^k = \frac{\sigma}{\tau} c^T \quad \text{for} \quad k \geq 0 \]
The preconditioner and its properties

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Equivalence of optimal solutions

**Theorem**

Let \( v \) satisfy \( c^T v = 0 \). The optimal Krylov subspace solution of the augmented system \( z^{(m)} \) with right preconditioner \( P_D \) can be written as

\[
z^{(m)} = [x^{(m)}; 0],
\]

where \( x^{(m)} \) is the optimal Krylov subspace solution of the original (non-augmented) system with preconditioner \( K_1 \).

**Proof Idea**

\[
((A - \sigma I)P_D^{-1})^k \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} (KK_1^{-1})^k v \\ 0 \end{bmatrix}
\]

If \( \hat{z}^{(m)} \in \mathcal{K}_m((A - \sigma I)P_D^{-1}, Ib) \) then \( \hat{x}^{(m)} \in \mathcal{K}_m((KK_1^{-1}, v) \) both optimal approximate solutions.
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Remark
The solution $z^{(m)} = [x^{(m)}; 0]$, satisfies $c^T x^{(m)} = 0$.

Proof

- Since $\hat{x}^{(m)}$ is optimal approximate solution in $K_m((K K_1^{-1}, v)$

$$\hat{x}^{(m)} = \phi_{m-1}(K K_1^{-1}) v$$

and $c^T (K K_1^{-1})^k = \frac{\sigma}{\tau} c^T$ we have $c^T \hat{x}^{(m)} = 0$.

- Then

$$c^T x^{(m)} = -\frac{1}{\tau} c^T K_1 x^{(m)} = -\frac{1}{\tau} c^T \hat{x}^{(m)} = 0$$
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The preconditioner and its properties

- structured symmetric indefinite preconditioner

\[ P_I = \begin{bmatrix} K_1 & c \\ c^T & 0 \end{bmatrix}, \quad K_1 = A_1 - \tau I, \quad \tau \in \mathbb{R} \]

where $K_1 = K_1^T$ nonsingular and $A_1 c = 0$ so $\tau \neq 0$ ($A_1 = 0$ possible)

- also possible $K_1 = A_1 - \sigma M_1$ for general $M_1 = M_1^T$
Block indefinite preconditioning

The preconditioner and its properties

- structured symmetric indefinite preconditioner

\[ P_I = \begin{bmatrix} \kappa_1 & c \\ c^T & 0 \end{bmatrix}, \quad K_1 = A_1 - \tau I, \quad \tau \in \mathbb{R} \]

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**Theorem**

Let \( v \) satisfy \( c^T v = 0 \). The optimal Krylov subspace solution of the augmented system \( z^{(m)} \) with right preconditioner \( P_I \) can be written as

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**Remark**

The solution $z^{(m)} = [x^{(m)}; 0]$, satisfies $c^Tx^{(m)} = 0$. 
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- augmented formulation of the problem

\[
\begin{bmatrix}
A & c \\
c^T & 0 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
\end{bmatrix}
= \lambda
\begin{bmatrix}
I & 0 \\
0^T & 0 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
\end{bmatrix}
\]

- $n_c = 1$ zero eigenvalue of the original problem become infinite
- $n_c = 1$ more eigenvalues arise (corresponding) to the singular part of $I$; infinite
- non-zero eigenvalues remain unchanged; find smallest eigenvalues of the augmented system; eigenvectors are of the form $[x; 0]$
- exact SI-Lanczos - inexact SI Lanczos
Back to the solve of the outer system

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Theorem
Let $u_1$ satisfy $c^T u_1 = 0$. Inexact SI-Lanczos with shift $\sigma$ applied to the augmented formulation with staring vector $v_1 = [u_1; 0]$ and inner right preconditioner

$$P_D = \begin{bmatrix} K_1 & 0 \\ 0 & c^T K_1^{-1} c \end{bmatrix}, \quad K_1 = A_1 - \tau I, \quad A_1 c = 0, \quad A_1 \in \mathbb{R}^{n \times n}$$

with $K_1 = K_1^T$ nonsingular generates the same approximation iterates as inexact SI-Lanczos with shift $\sigma$ applied to the original problem with starting vector $u_1$ and inner right preconditioner $K_1$.

Proof Idea
Uses results that optimal Krylov subspace approximate solution of inner systems are essentially the same. Induction.
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Preconditioning with $P_I$

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Proof Idea
Uses results that optimal Krylov subspace approximate solution of inner systems are essentially the same. Induction.
Remarks

- key condition is $c^T K_1 = \beta c^T$ for $\beta \neq 0$
- here: $K_1 = A_1 - \tau I$
- could use $K_1 = \alpha A_1 + cH^{-1}c^T$ with $H = c^T c$, $\alpha \in \mathbb{R}$, $A_1 c = 0$. 
Remarks

- also possible: higher dimensional null-spaces of $A$, where $C$ is a basis of the null-space such that $AC' = 0$

- also possible: generalised eigenproblem $Ax = \lambda Mx$
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Regularisation of the problem

- other than augmented formulation, a so-called regularised formulation is available
- move zero eigenvalues away from the origin and also (hopefully) far away from the sought after eigenvalues
- let \( H \in \mathbb{R}^{n_c \times n_c} \) (here \( H \) is just a scalar) be symmetric and nonsingular, then the transformed generalised eigenvalue problem is given by

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(A + cH^{-1}c^T)x = \eta x
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Let

\[ Ax = \lambda x \quad \text{and} \quad (A + cH^{-1}c^T)x = \eta x \]

and \( \lambda_i, \eta_i \) be eigenvalues.

- If \( \lambda_i \neq 0 \) there exists \( j \) such that \( \lambda_i = \eta_j \).
- If \( \lambda_i = 0 \) there corresponds an eigenvalue \( \eta_j \) with \( \eta_j \in \Lambda(c^T c, H) \).

**Remarks**

- no practical advantage
- inner solver \( (A + cH^{-1}c^T)z = v \) produces the same Krylov subspace as \( Az = v \)
Shifting of the zero eigenvalue

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2D computational model of an electromagnetic cavity resonator

- variational formulation: Find $\omega_h \in \mathbb{R}$ s.t. $\exists 0 \neq u_h \in \Sigma_h \subset \Sigma$

\[
(\text{rot} u_h, \text{rot} v_h) = \omega_h^2 (u_h, v_h) \quad \forall v_h \in \Sigma_h,
\]

where $\text{rot}(v_1, v_2) = (v_2)_x - (v_1)_y$,

$\Sigma = \{ v \in L^2(\Omega)^2 : \text{rot} v \in L^2(\Omega), v \cdot t = 0 \text{ on } \partial \Omega \}$ and $t$ is the counterclockwise oriented tangent unit vector to the boundary

- FEM discretisation
- size $n = 3229$, null-space dimension $n_c = 1036$
- solver: right preconditioned GMRES
- preconditioner $K_1 = A_1 - \sigma M$ with $A_1 = 0$ and $\sigma = 0.8$
- inner tolerance $10^{-8}$ for the solve of the inner system
Results

<table>
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<th>j</th>
<th>$\frac{(A - \sigma M)^{-1}Mx = \eta x}{K_1}$</th>
<th>$\frac{(A - \sigma M)^{-1}Mz = \eta z}{P_D}$</th>
<th>$P_I$</th>
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<td>0.000000000158827</td>
<td>0.000000000158882</td>
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</tbody>
</table>

Table: Relative eigenvalue residual norm $\frac{Ax_j - \lambda_j Mx_j}{\lambda_j}$ of approximate smallest eigenpair in the inexact SI-Lanczos method applied to different formulations and different preconditioners.
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- dependent on the fact that the constraint matrix $C$ is a basis for the null-space of the problem
- approximation space is maintained $M$-orthogonal to the null-space without explicit orthogonalisation (constraint $C^T M x = 0$ automatically satisfied)
- inner accuracy influences the performance of the method
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