Regularising Inverse Imaging Problems using Generative Machine Learning Models

Margaret Duff, Neill D F Campbell, Matthias J Ehrhardt

Society for Industrial and Applied Mathematics

Overview

- Inverse problem

$$y \approx Ax$$

where $x \in \mathcal{X}, y \in \mathcal{Y}$.

- Variational approach: solve

$$\arg\min_{x \in \mathcal{X}} \|y - Ax\|_2^2 + \lambda \mathcal{R}_G(x)$$

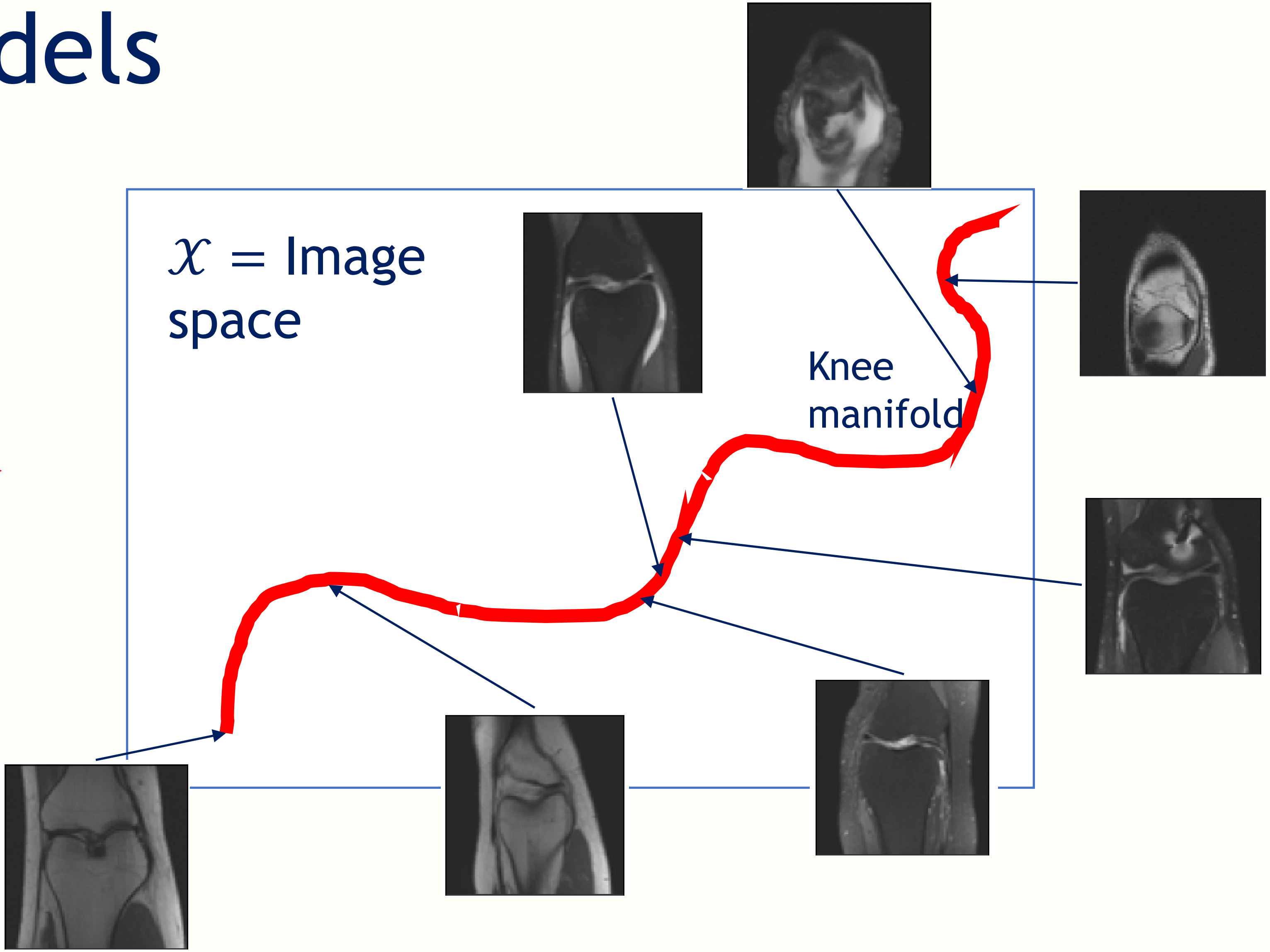
where $G: \mathcal{Z} \rightarrow \mathcal{X}$, a generative model.

- Penalise images far from the range of the generative model.

Generative Models

Latent Space

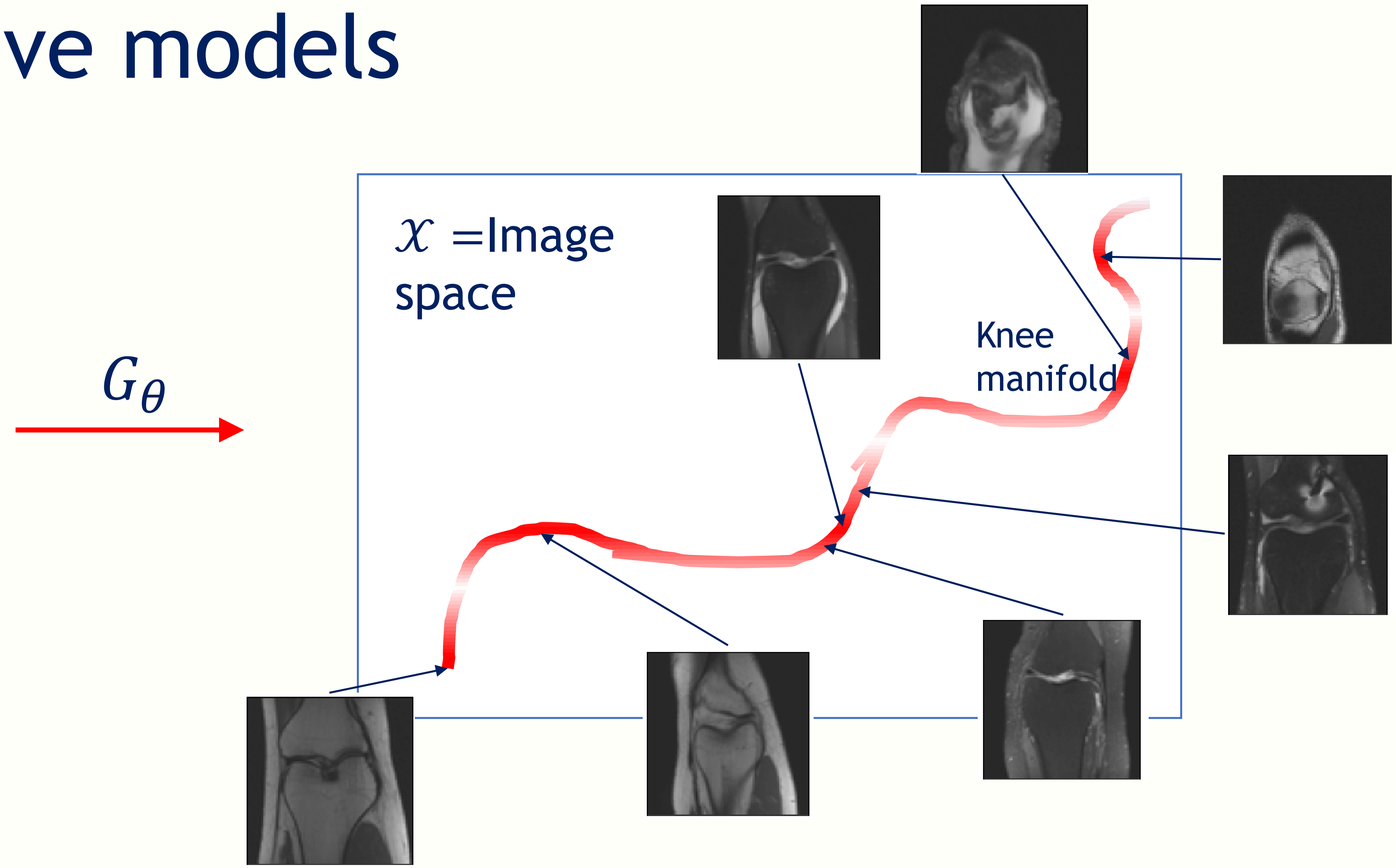
$$z \in \mathcal{Z} \xrightarrow{G_\theta}$$



Generative models

Latent Space

$z \sim N(0, I)$



Tomography example: MNIST

$A: X \to Y$

Original Problem: Find x s.t.

$$y = Ax + \epsilon$$

Generative model $G: Z \to X$

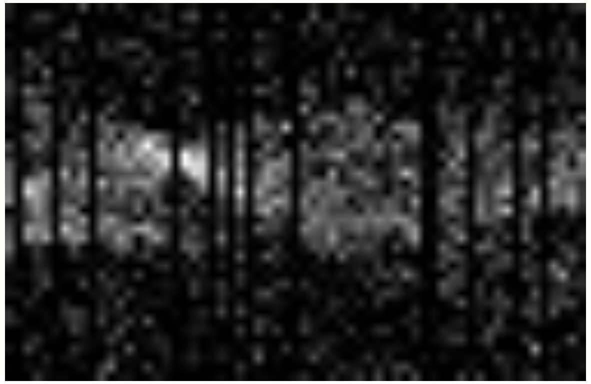
New Problem: Find z s.t.

$$y = A(G(z)) + \epsilon$$

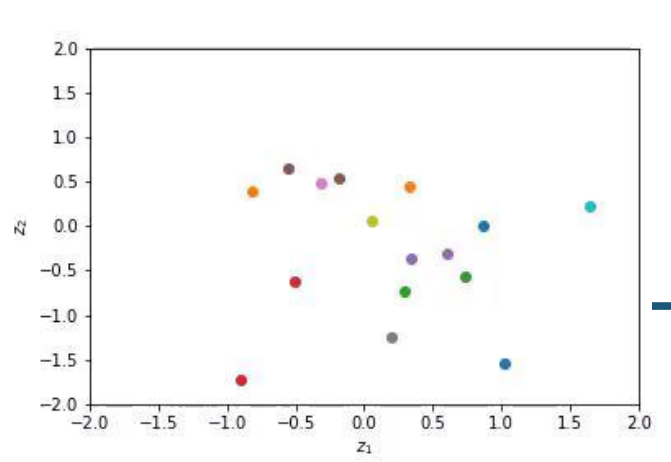
$$x = G(z)$$

x

A

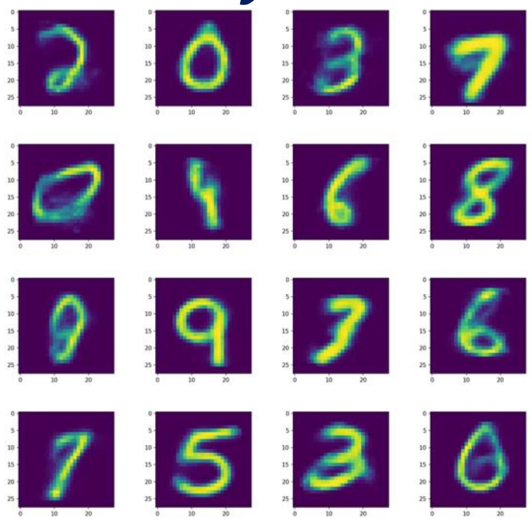


y



Z

G

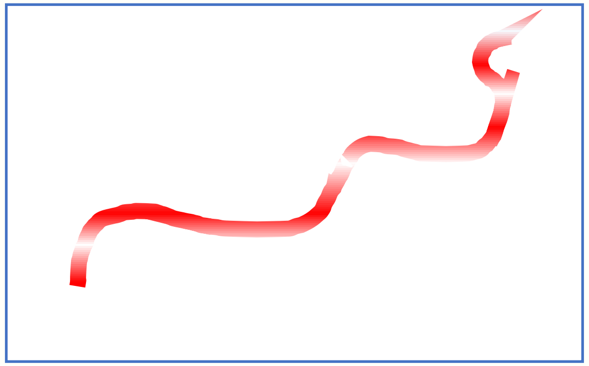


X

Incorporating the generator

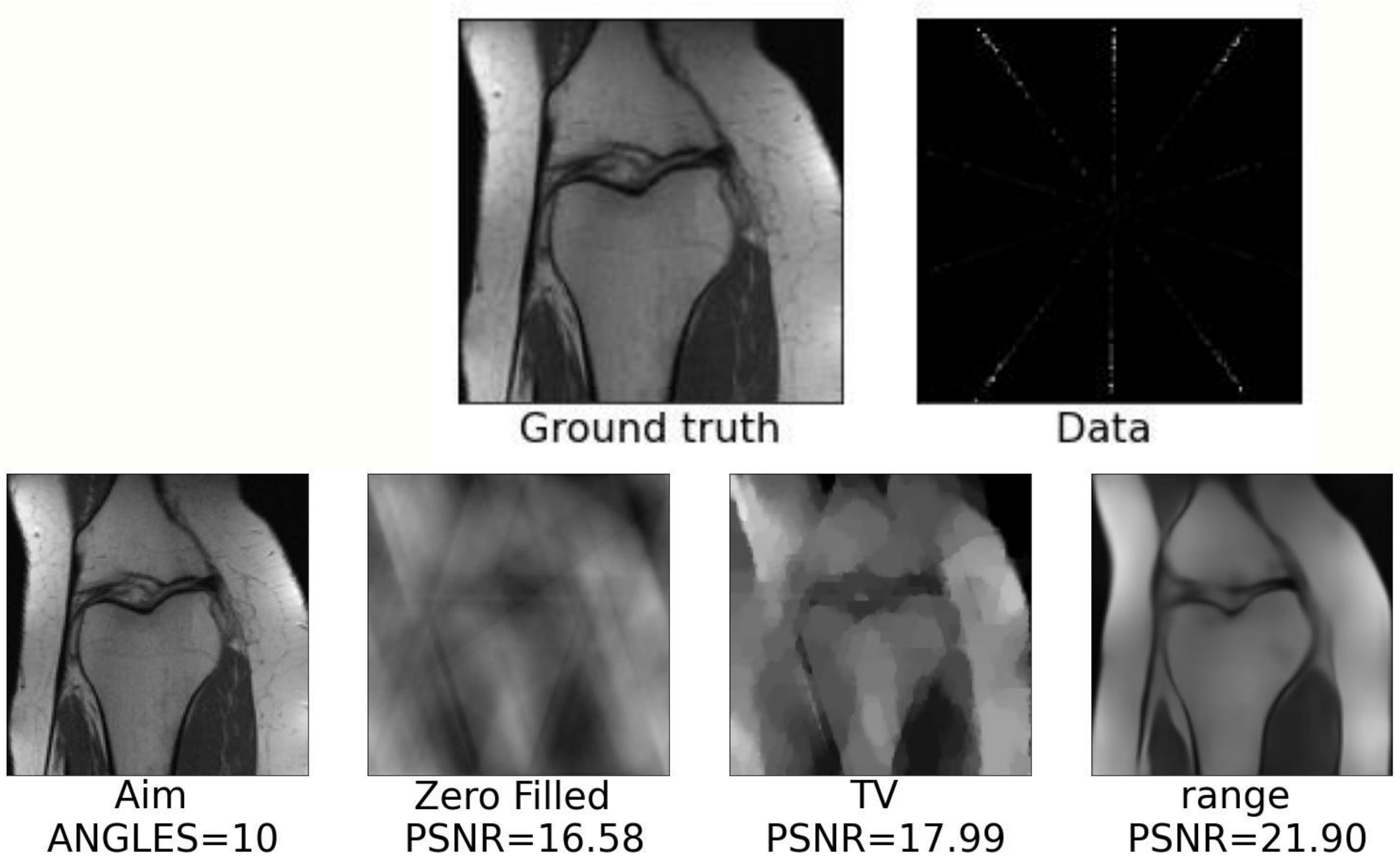
Image in the range of the generator

$$\mathcal{R}_G(x) = \min_{z \in \mathcal{Z}} \iota_{\{0\}}(G(z) - x) + \|z\|_2^2$$



Bora et al. "Compressed sensing using generative models". ICML 2017.

NYU FastMRI dataset



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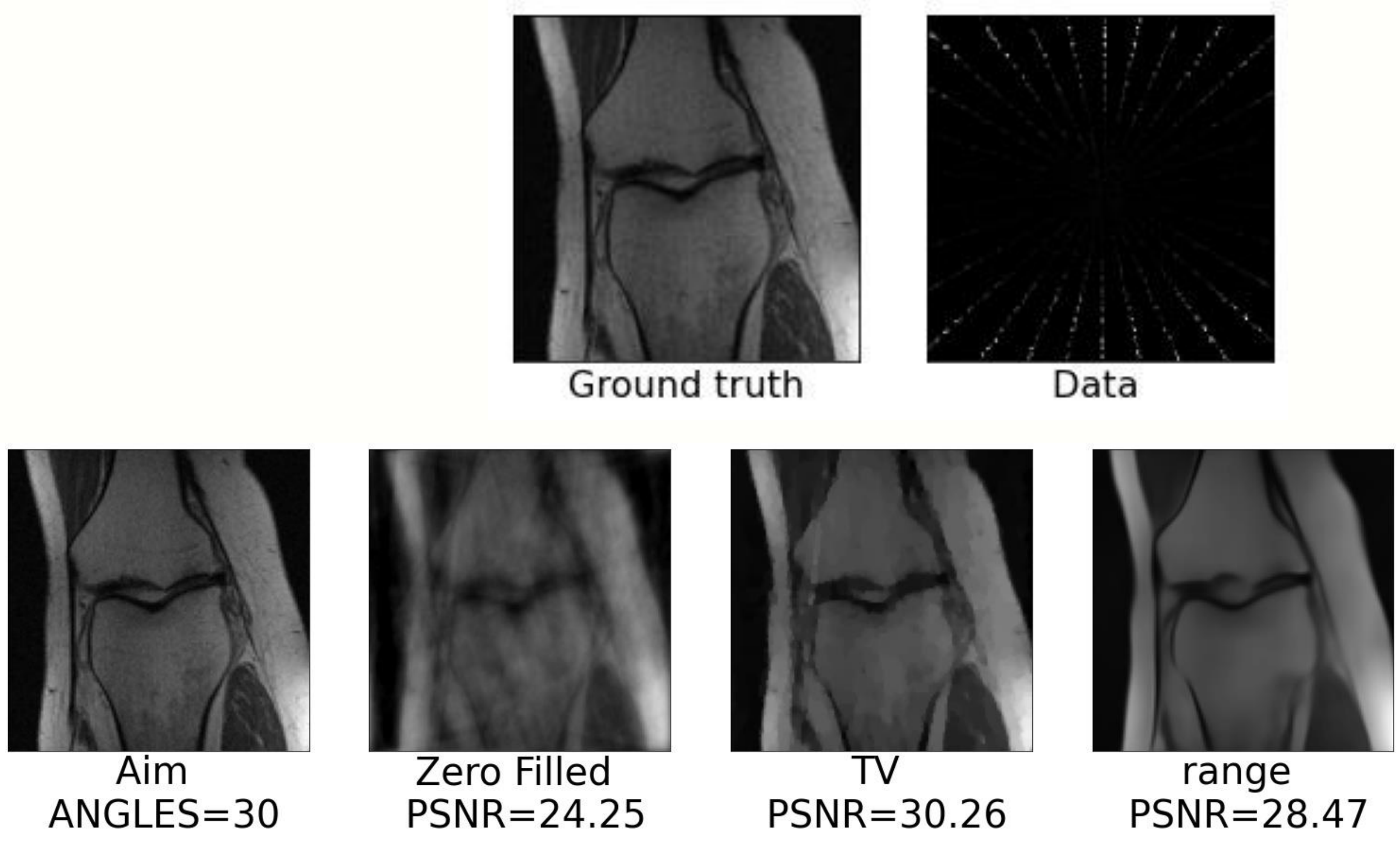

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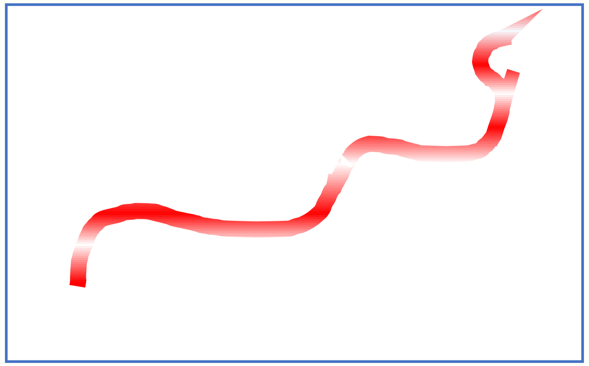
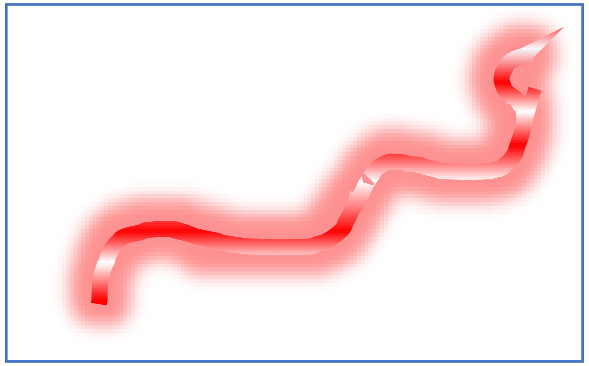
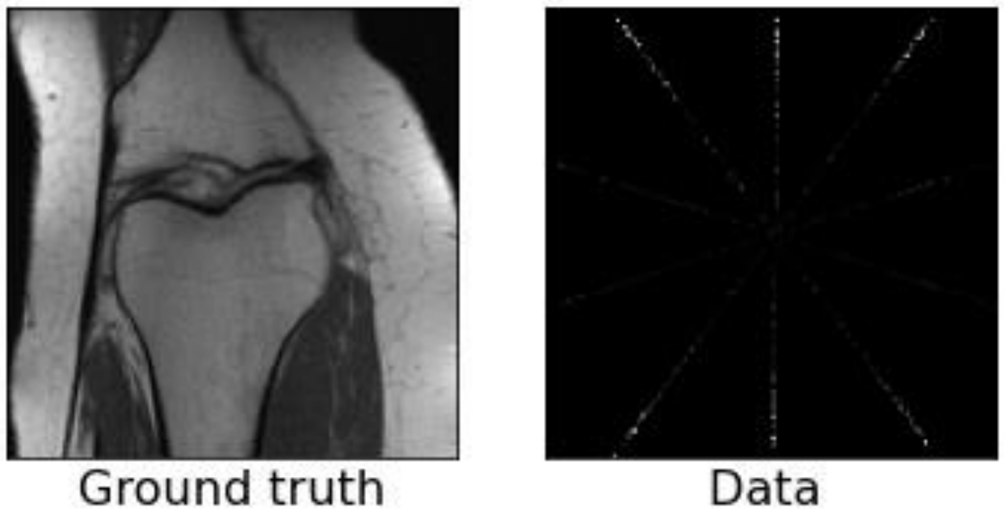


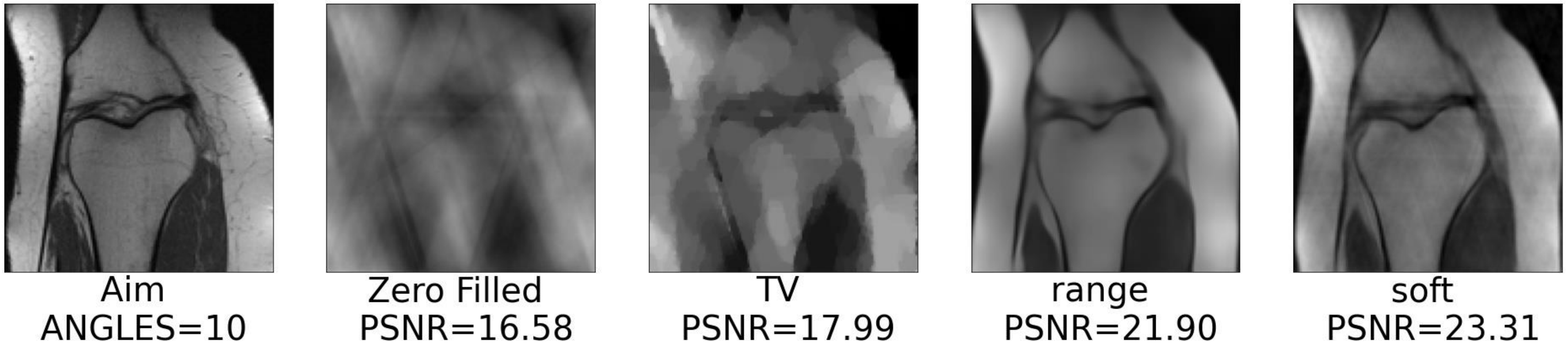
Image close to the range of the generator

$$\mathcal{R}_G(x) = \min_{z \in \mathcal{Z}} \|G(z) - x\|_2^2 + \mu\|z\|_2^2$$



NYU FastMRI dataset





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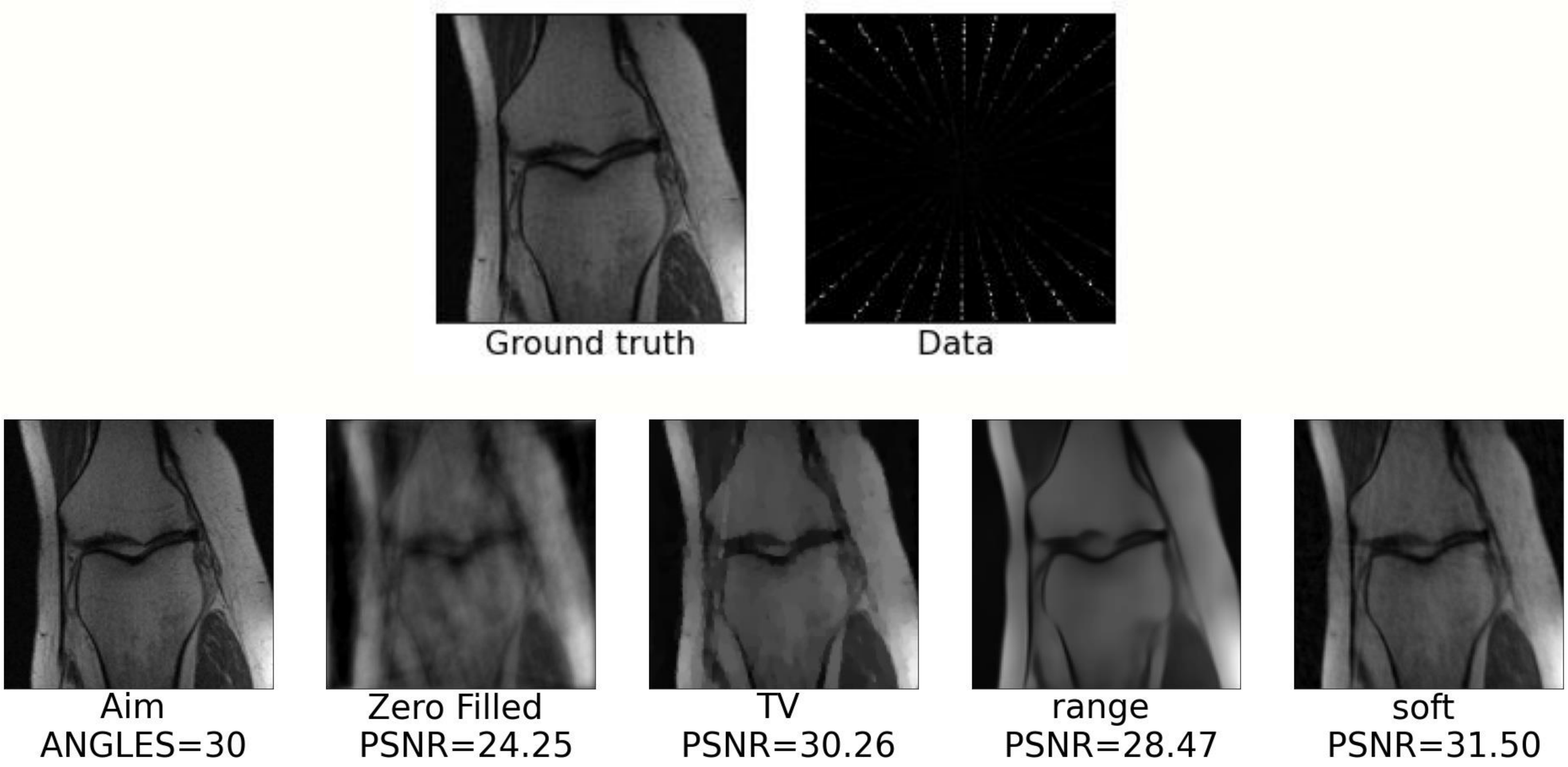
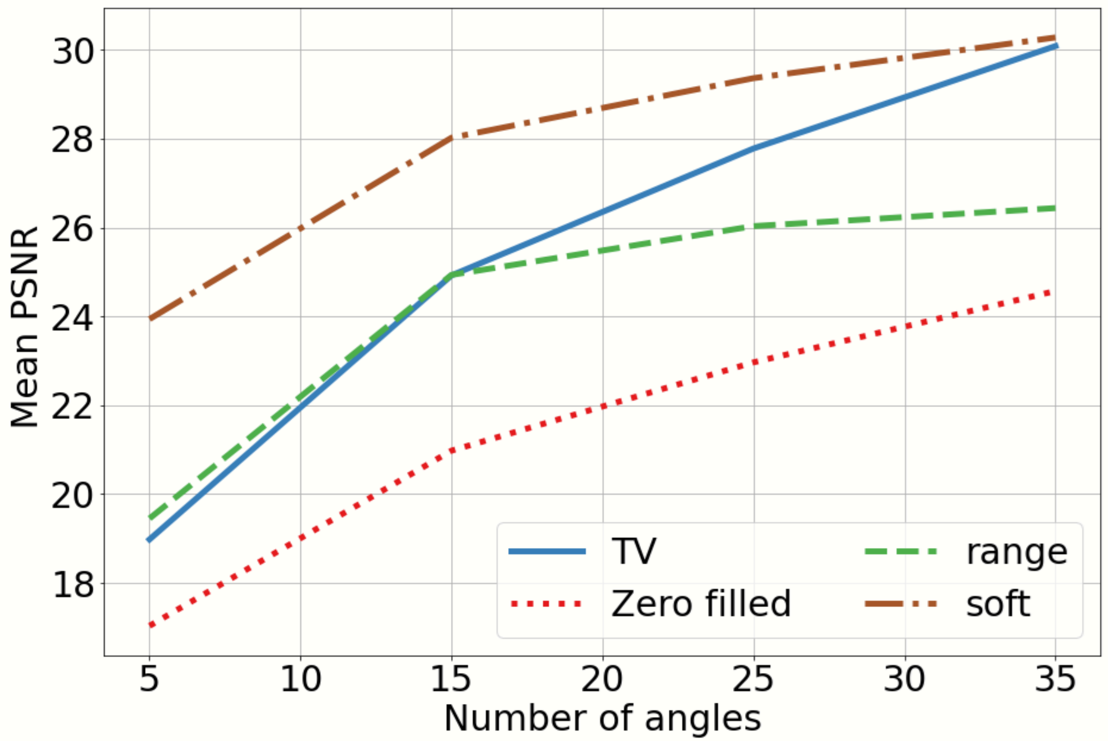
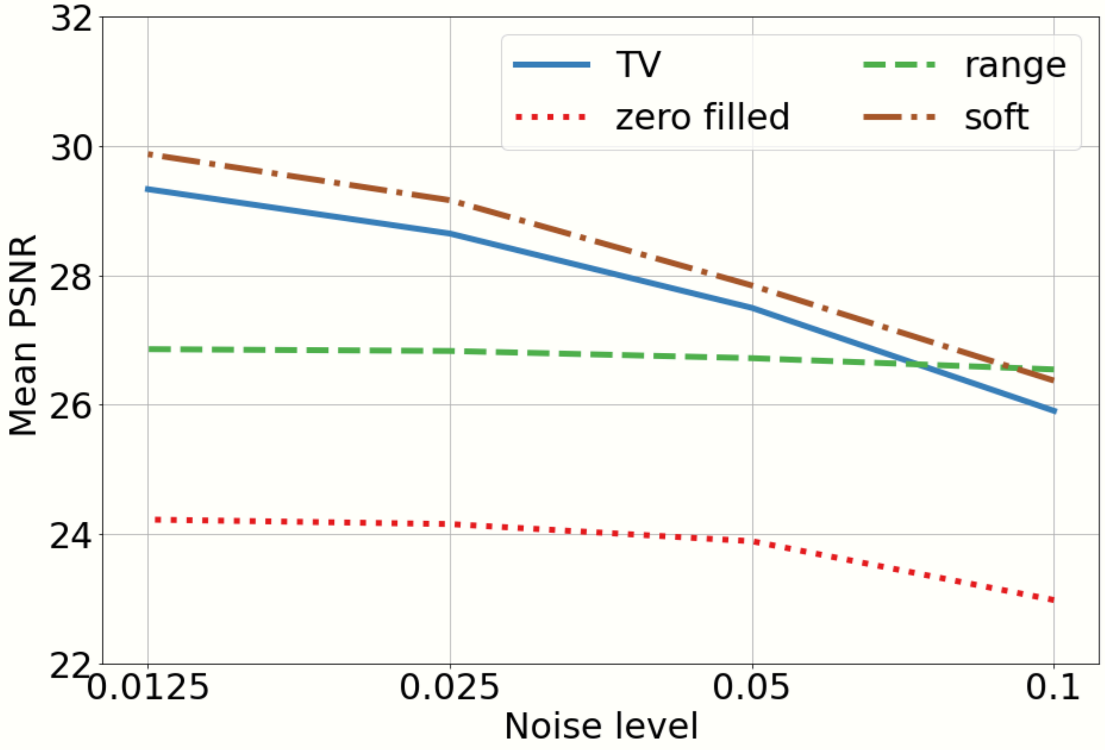


Method comparison

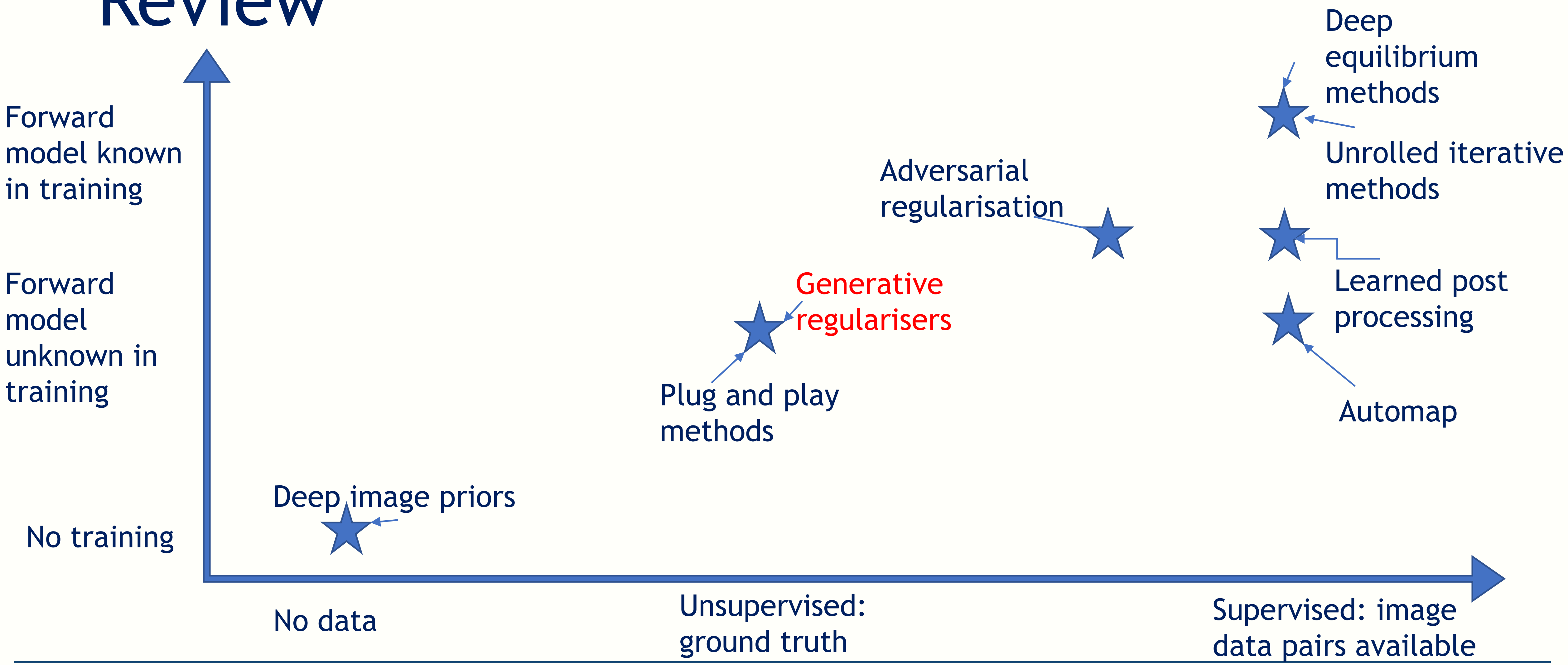




The Benefits of Generative Regularisers

- Don’t require supervised (paired) training data
- Flexible to changes in the forward problem
- Some degree of mathematical insight and control.

Deep Learning and Inverse Problems: Review



What properties do we need for the generator?

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Generator properties

- Generator produces all ‘feasible’ images
- Generator produces no ‘unfeasible’ images
- The generated probability distribution matches the training data distribution

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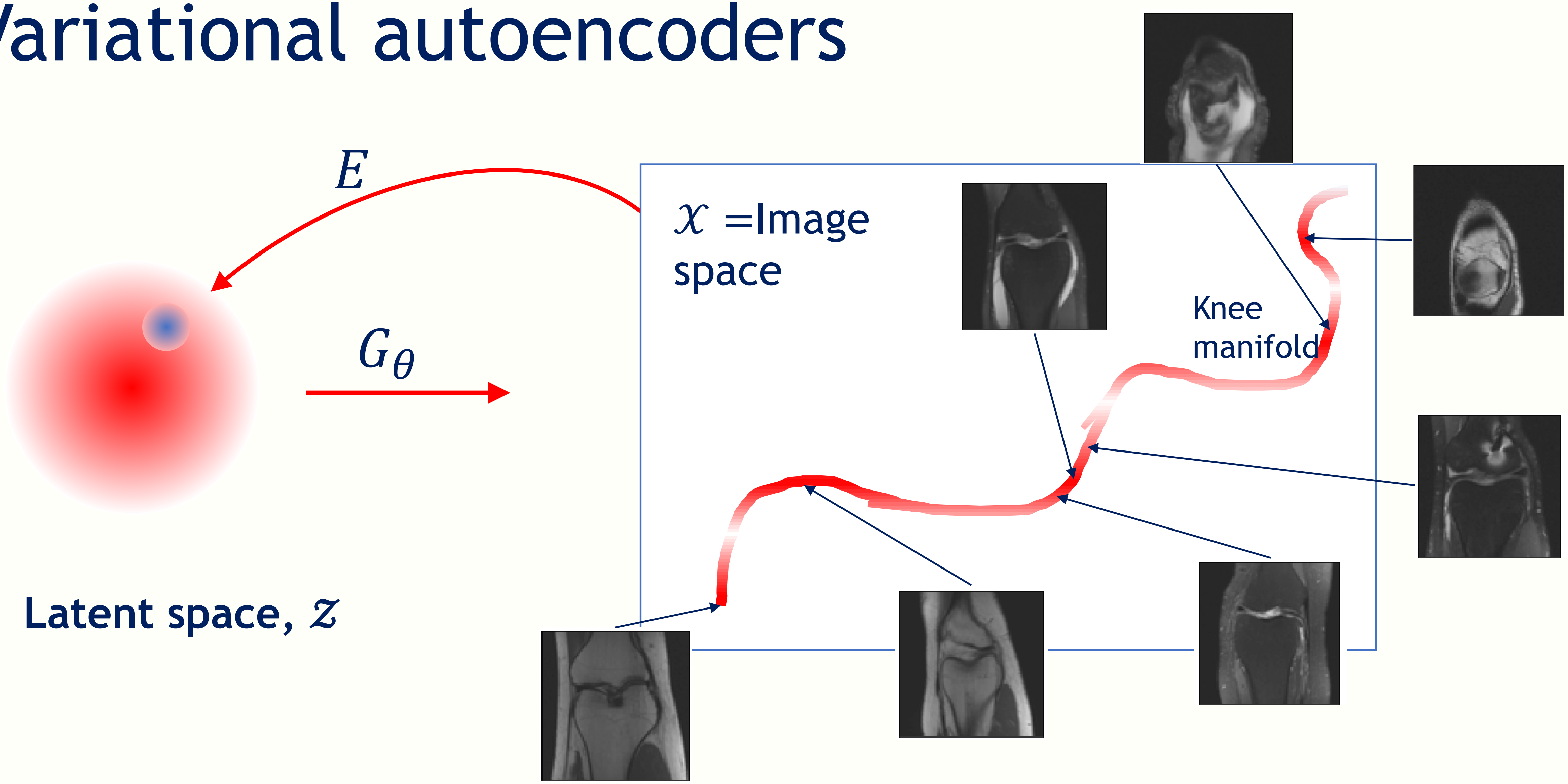
Latent space properties

- Smoothness of the generator with respect to z
- The area of the latent space that maps to feasible images is known

Variational autoencoders



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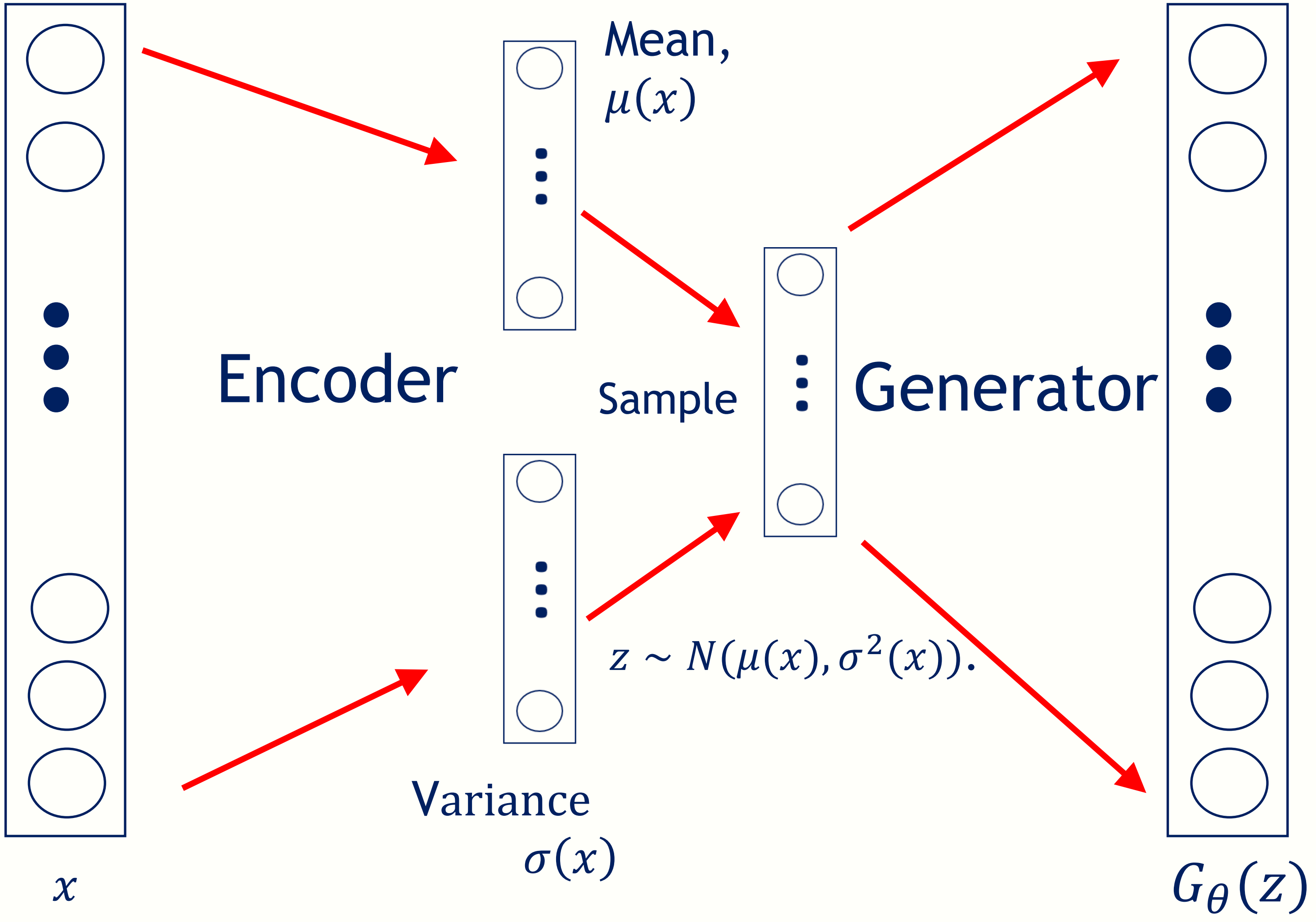
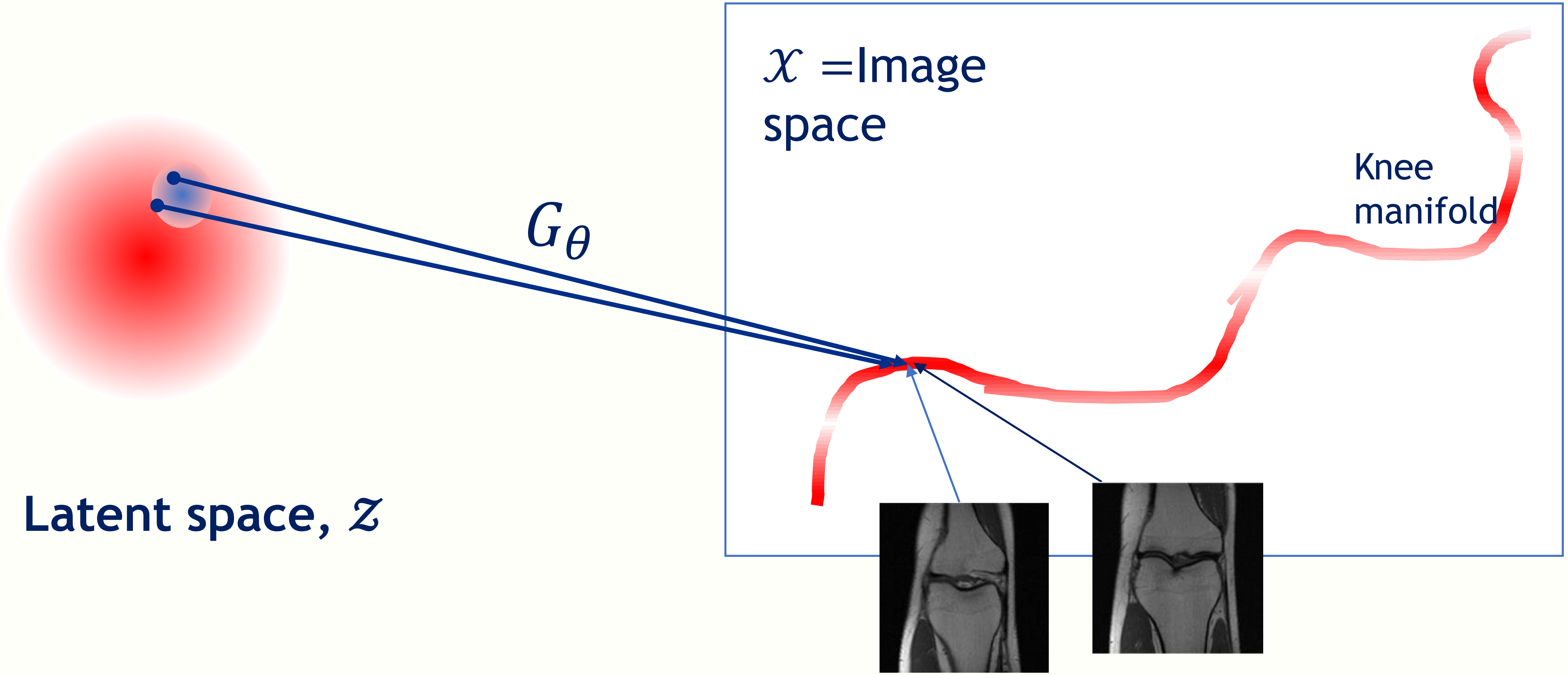


Objective function:

$$\min_{\mu,\sigma,\theta} \mathbb{E}_x[\mathbb{E}_{z|x}\|x - G_\theta\,(z)\|^2 + KL(p_z|p_{N(0,1)})]$$

Where $z|x \sim N(\mu(x), \sigma^2(x))$.

Variational autoencoders



Generative Adversarial Networks (GANs)

$z \sim p_z$ $\quad z \in Z$

Generator

Fake data

$G(z) \in X$

Discriminator

$D(x) \in (0,1)$

Probability the image is real

$x \sim p_{data}$

Real data

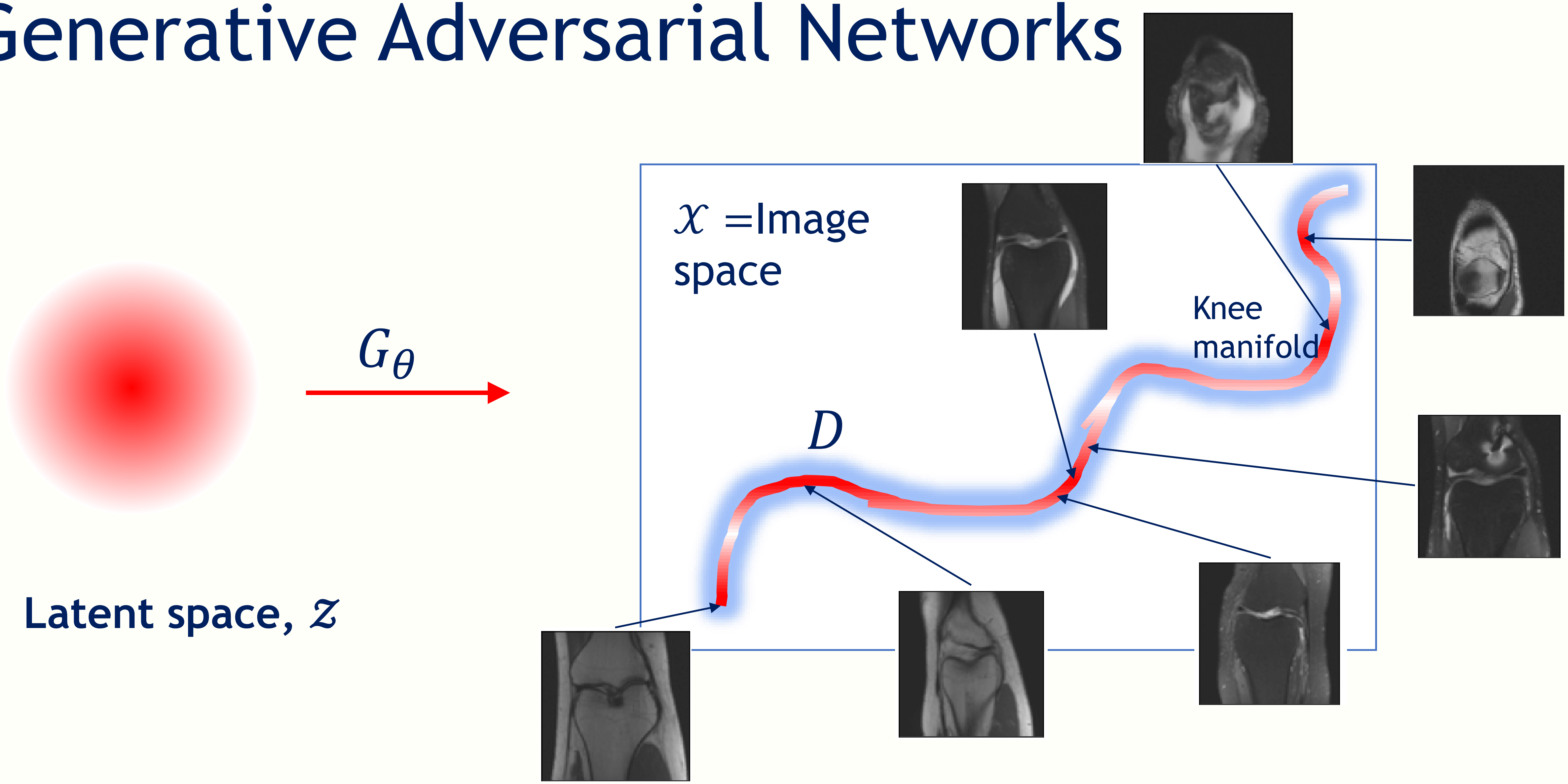
$x \in X$

Objective function:

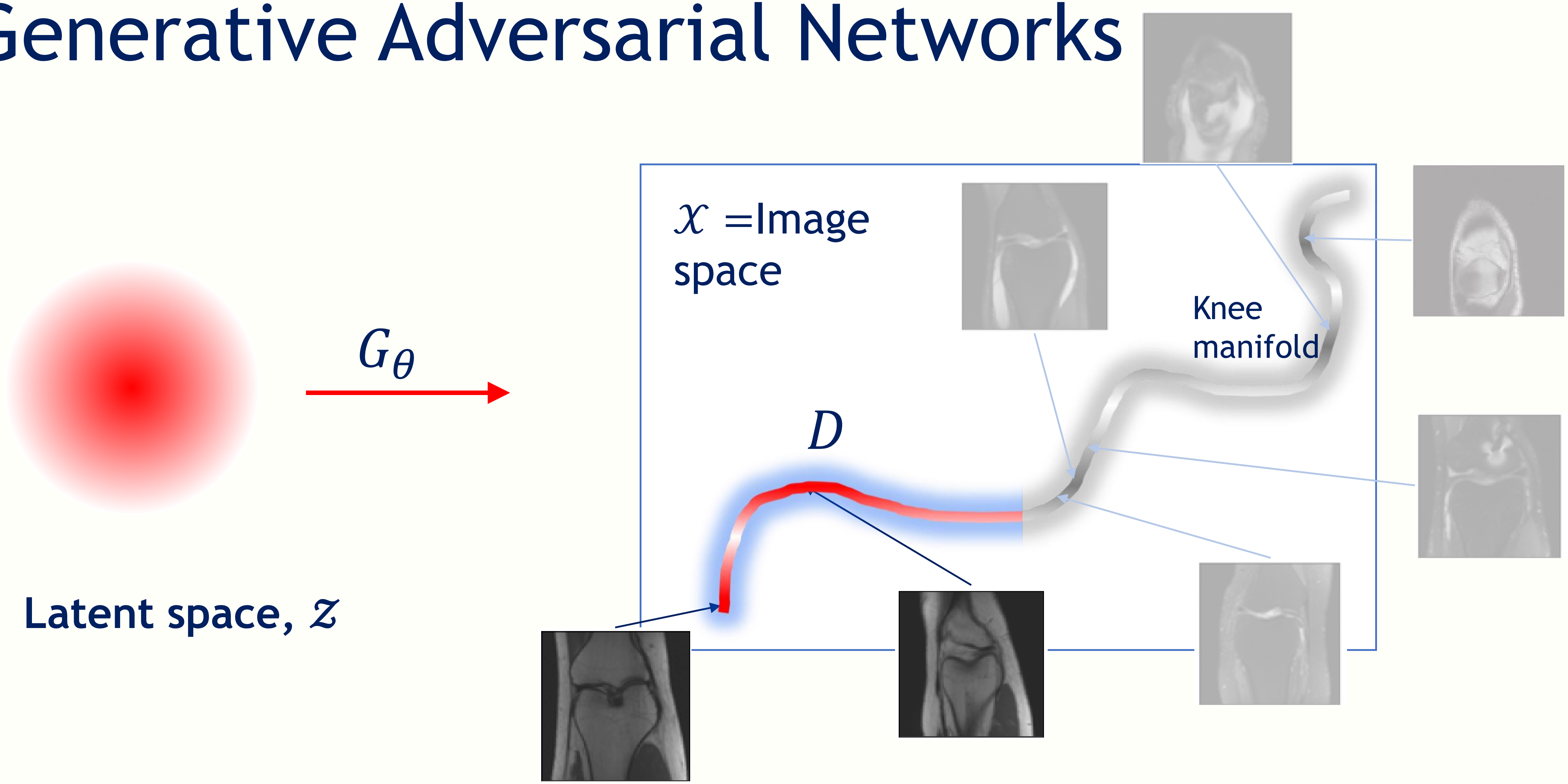
$$\min_G \max_D V(G,D)$$

$$V(G,D) = \mathbb{E}_{x \sim p_{data}} \log D(x) + \mathbb{E}_{z \sim p_z} \log(1 - D(G(z)))$$

Generative Adversarial Networks

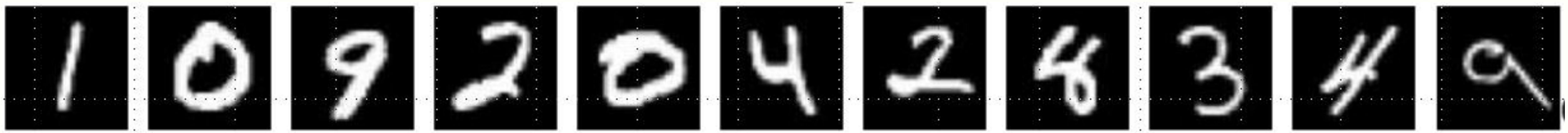


Generative Adversarial Networks

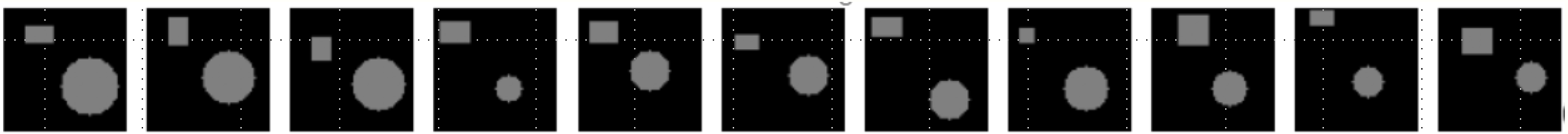


Generative model comparisons

- Datasets:
  - MNIST

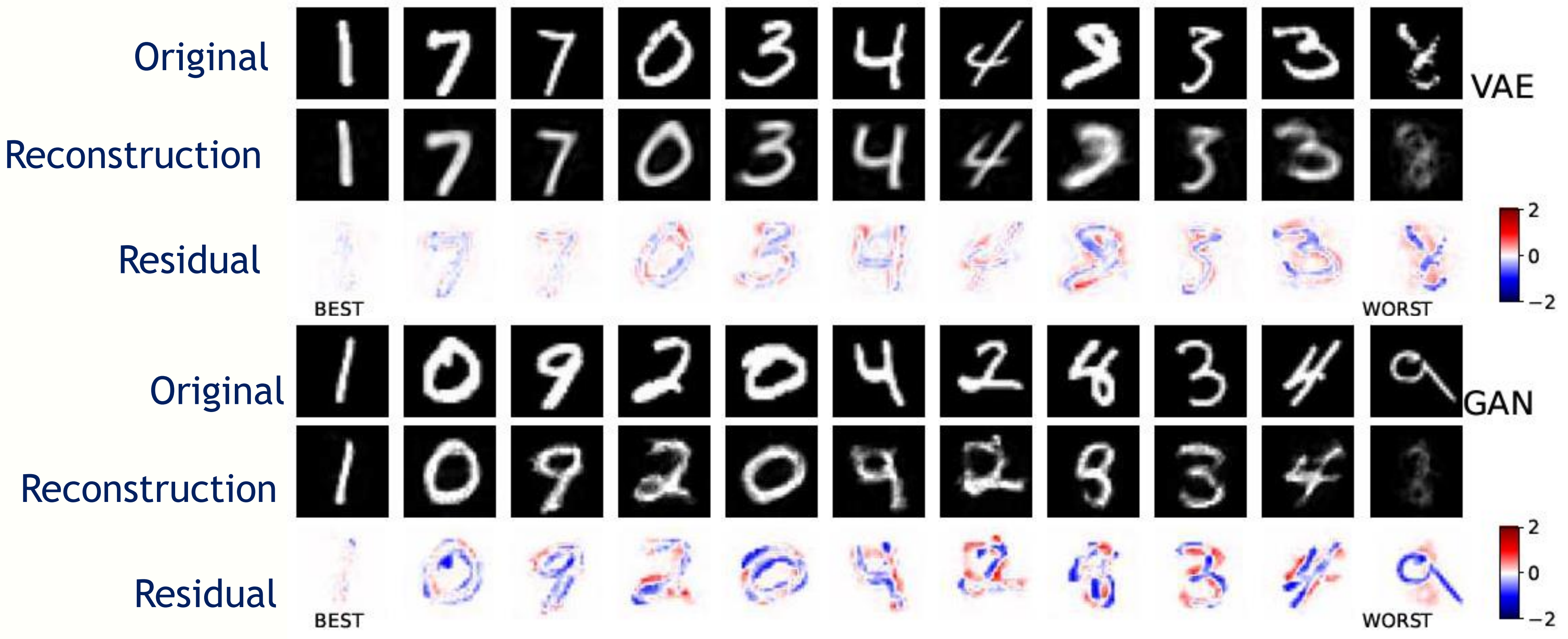


  - Squares and circles



Generative model comparisons

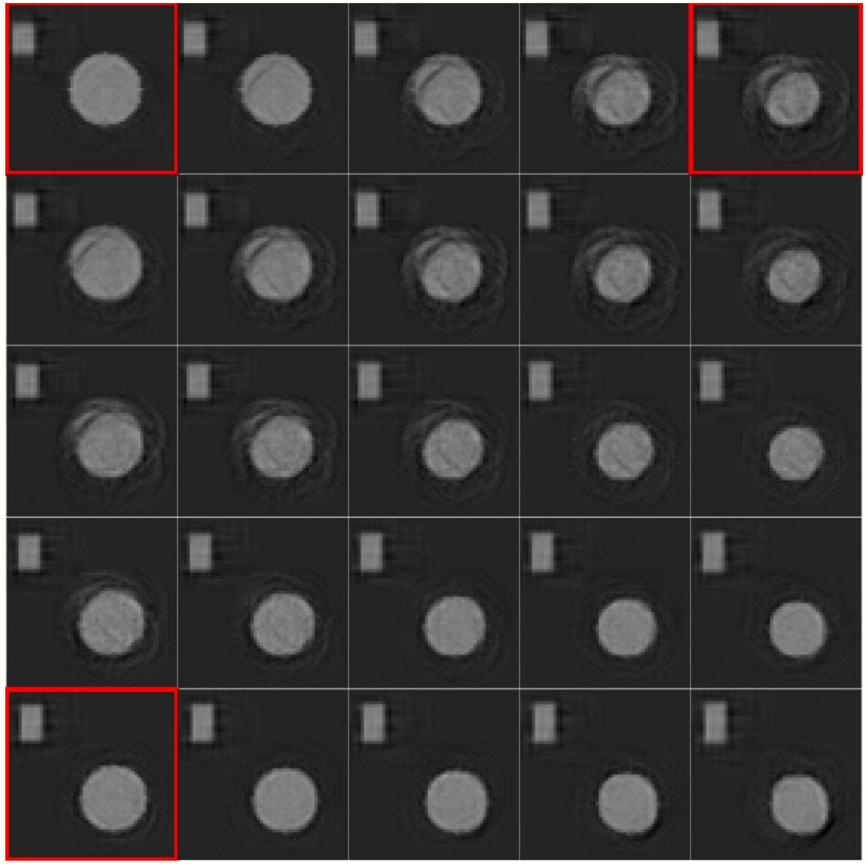
- Generator produces all ‘feasible’ images
- Generator produces no ‘unfeasible’ images

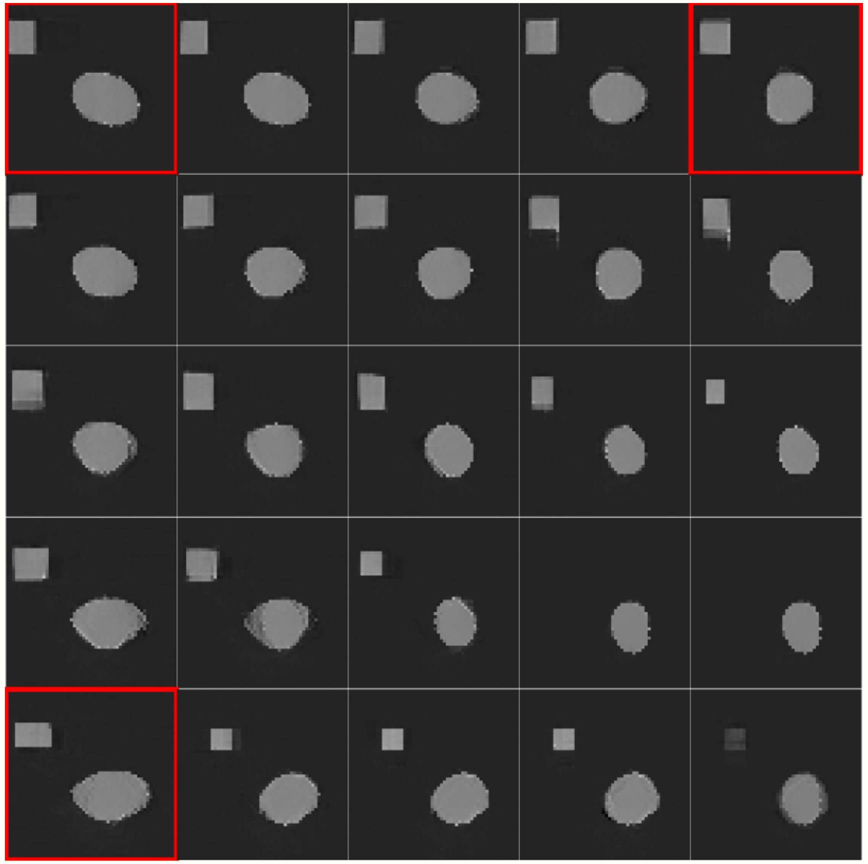


Generative model comparisons

- Smoothness of the generator with respect to z

VAE





GAN

VAE and GAN Comparison

	Variational Autoencoder	Generative Adversarial Network
Generate all 'feasible' images	✓	Susceptible to mode collapse
Generate no 'unfeasible' images	Can produce blurry images	✓
Smoothness with respect to z	Depends on the network Encoder distribution	Depends on the network
Known latent space distribution	Only the prior is known	Only the prior is known

Takeaway points

$A: X \to Y$

Original Problem: Find x s.t.

$$y \approx Ax$$

Generative model $G: Z \to X$

New Problem:

$$\arg\min_{x \in \mathcal{X}} \|y - Ax\|_2^2 + \lambda \mathcal{R}_G(x)$$

- Generative models can be used as priors for inverse problems
  - Penalise images far from the range of a generative model

- Requires generative models that produce more than a few good images.

https://arxiv.org/abs/2107.11191