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THE SCALING LIMIT OF THE INTERFACE OF THE CONTINUOUS-SPACE SYMBIOTIC BRANCHING MODEL[1]

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The continuous-space symbiotic branching model describes the evolution of two interacting populations that can reproduce locally only in the simultaneous presence of each other. If started with complementary Heaviside initial conditions, the interface where both populations coexist remains compact. Together with a diffusive scaling property, this suggests the presence of an interesting scaling limit. Indeed, in the present paper, we show weak convergence of the diffusively rescaled populations as measure-valued processes in the Skorokhod, respectively the Meyer–Zheng, topology (for suitable parameter ranges). The limit can be characterized as the unique solution to a martingale problem and satisfies a "separation of types" property. This provides an important step toward an understanding of the scaling limit for the interface. As a corollary, we obtain an estimate on the moments of the width of an approximate interface.

1. Introduction.

1.1. *The symbiotic branching model and its interface.* The symbiotic branching model of Etheridge and Fleischmann [11] is a spatial stochastic model of two interacting populations described by the nonnegative solutions of the stochastic partial differential equations

$$\text{(1)}\qquad \mathrm{cSBM}(\varrho,\gamma)_{u_0,v_0}: \begin{cases} \dfrac{\partial}{\partial t}u_t(x) = \dfrac{\Delta}{2}u_t(x) + \sqrt{\gamma u_t(x)v_t(x)}\,\dot{W}_t^{(1)}(x), \\ \dfrac{\partial}{\partial t}v_t(x) = \dfrac{\Delta}{2}v_t(x) + \sqrt{\gamma u_t(x)v_t(x)}\,\dot{W}_t^{(2)}(x), \end{cases}$$

with suitable nonnegative initial conditions $u_0(x) \geq 0$, $v_0(x) \geq 0$, $x \in \mathbb{R}$. Here, $\gamma > 0$ is the branching rate, and $(\dot{W}^{(1)}, \dot{W}^{(2)})$ is a pair of correlated standard Gaussian white noises on $\mathbb{R}_+ \times \mathbb{R}$ with correlation $\varrho \in [-1, 1]$, that is, for $t_1, t_2 \geq 0$,

$$\text{(2)}\qquad \mathbb{E}\big[W_{t_1}^{(i)}(A_1)W_{t_2}^{(j)}(A_2)\big] = \begin{cases} (t_1 \wedge t_2)\ell(A_1 \cap A_2), & i = j, \\ \varrho(t_1 \wedge t_2)\ell(A_1 \cap A_2), & i \neq j, \end{cases}$$

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where ℓ denotes the Lebesgue measure and A_1, A_2 are Borel sets. Solutions of this model have been considered rigorously in the framework of the corresponding martingale problem in Theorem 4 of [11], which states that, under natural conditions on the initial conditions $u_0(\cdot), v_0(\cdot)$, a solution exists for all $\varrho \in [-1, 1]$. Further, the martingale problem is well-posed for all $\varrho \in [-1, 1)$, which implies the strong Markov property except in the boundary case $\varrho = 1$. The model interpolates between several well-known examples of spatial population models. Indeed, for $\varrho = -1$ and $u_0 = 1 - v_0$, the system reduces to the continuous-space *stepping stone model*, discussed, for example, by Tribe in [27]. For $\varrho = 0$, the system is the so-called *mutually catalytic model* of Dawson and Perkins [8], and for $\varrho = 1$ and $u_0 = v_0$, an instance of the *parabolic Anderson model*; see, for example, [21].

Natural questions about such (systems of) SPDEs are related to their long-term behavior, for example, the limiting shape of the interface for suitable initial conditions. For us, of particular interest are "complementary Heaviside initial conditions," that is,

$$u_0(x) = \mathbf{1}_{\mathbb{R}^-}(x) \quad \text{and} \quad v_0(x) = \mathbf{1}_{\mathbb{R}^+}(x), \qquad x \in \mathbb{R}.$$

DEFINITION 1.1. The interface at time t of a solution $(u_t, v_t)_{t \geq 0}$ of the symbiotic branching model $\mathrm{cSBM}(\varrho, \gamma)_{u_0, v_0}$ with $\varrho \in [-1, 1]$, $\gamma > 0$ is defined as

$$\mathrm{Ifc}_t = \mathrm{cl}\{x \in \mathbb{R} : u_t(x) v_t(x) > 0\},$$

where $\mathrm{cl}(A)$ denotes the closure of the set A in $\mathbb{R}$.

The main question addressed by Etheridge and Fleischmann [11] is whether for the above initial conditions, the "compact interface property" holds, that is, whether the interface is compact at each time almost surely. This is answered affirmatively in their Theorem 6, together with the assertion that the interface propagates with at most linear speed, that is, there exists a constant $c = c(\gamma)$ such that for each $\varrho \in [-1, 1]$, there is a (almost-surely) finite random time T_0 such that, almost surely, for all $T \geq T_0$,

$$\bigcup_{t \leq T} \mathrm{Ifc}_t \subseteq [-cT, cT]. \tag{3}$$

However, due to the scaling property of the symbiotic branching model [see (4) below], one might expect that the fluctuations of the position of the interface should be of order $t^{1/2}$. Indeed, Blath, Döring and Etheridge [3], Theorem 2.11, strengthen the linear propagation bounds (3) for a (rather small) parameter range:

THEOREM 1.2 ([3]). *There exists $\varrho_0 > -1$ such that the following holds: Suppose $(u_t, v_t)_{t \geq 0}$ is a solution to $\mathrm{cSBM}(\varrho, \gamma)_{1_{\mathbb{R}^-}, 1_{\mathbb{R}^+}}$ with $-1 < \varrho < \varrho_0$. Then there is a constant $C(\gamma, \varrho) > 0$ and a finite random time T_0 such that almost surely*

$$\bigcup_{t \leq T} \mathrm{Ifc}_t \subseteq \left[-C\sqrt{T \log(T)}, C\sqrt{T \log(T)}\right],$$

for all $T > T_0$.

The restriction to $\varrho < \varrho_0$ seems artificial and comes from the technique of the proof. Although the value of $\varrho_0 \approx -0.9958$ is rather close to -1, the result is remarkable, since it shows that sub-linear speed of propagation is not restricted to situations in which solutions are uniformly bounded as, for instance, for $\varrho = -1$. The proof is based on the "dyadic grid technique" in [27] together with improved bounds on the moments of the symbiotic branching model (see the "critical curve" in Theorem 1.3 below), circumventing the lack of uniform boundedness of the population sizes.

The symbiotic branching model exhibits the following fundamental *scaling property* (see Lemma 8 of [11]): If $(u_t, v_t)_{t\geq 0}$ is a solution to $\mathrm{cSBM}(\varrho, \gamma)_{u_0,v_0}$, then

$$(4) \qquad \big(u_t^{(K)}(x), v_t^{(K)}(x)\big) := \big(u_{K^2 t}(Kx), v_{K^2 t}(Kx)\big), \qquad x \in \mathbb{R}, K > 0$$

is a solution to $\mathrm{cSBM}(\varrho, K\gamma)_{u_0^{(K)}, v_0^{(K)}}$ with initial states $(u_0^{(K)}, v_0^{(K)})$ transformed accordingly. Note that complementary Heaviside initial conditions $(u_0, v_0) = (\mathbb{1}_{\mathbb{R}^-}, \mathbb{1}_{\mathbb{R}^+})$ are invariant under this rescaling. Thus in this case letting $K \to \infty$ in (4) is equivalent to increasing the branching rate $\gamma \to \infty$.

In light of the scaling property (4), one might hope that (at least for a suitable range of parameters) a diffusive rescaling could lead to an interesting scaling limit. In fact, the program of letting the branching rate tend to infinity has been carried out for the *discrete space* version of (1). For the mutually catalytic model (the case $\varrho = 0$), Klenke and Mytnik construct in a series of papers [14–16] a nontrivial limiting process for $\gamma \to \infty$ (on the lattice) and study its long-term properties. This limit is called the "infinite rate mutually catalytic branching process." Moreover, Klenke and Oeler [17] give a Trotter-type approximation and *conjecture* that, under suitable assumptions, a nontrivial interface for the limiting process exists; see page 485, before Corollary 1.2. Recently, analogous results have been derived by Döring and Mytnik in the case $\varrho \in (-1, 1)$ in [9, 10].

Returning to the continuous-space set-up, for $\varrho = -1$ (the stepping stone model) Tribe [27] proves a "functional limit theorem": For a pair of (continuous) functions (u, v), define

$$(5) \qquad R(u, v) := \sup\{x : u(x) > 0\}, \qquad L(u, v) = \inf\{x : v(x) > 0\}.$$

Note that for a solution $(u_t, v_t)_{t\geq 0}$ of the symbiotic branching model, the interface at time t is contained in the interval $[L(u_t, v_t), R(u_t, v_t)]$. It is proved in [27] for $\varrho = -1$ and for continuous initial conditions $u_0 = 1 - v_0$ which satisfy $-\infty < L(u_0, v_0) \leq R(u_0, v_0) < \infty$ that under Brownian rescaling, the motion of the position of the right endpoint of the interface $t \mapsto \frac{1}{n} R(u_{n^2 t}, 1 - u_{n^2 t}), t \geq 0$, converges to a Brownian motion as $n \to \infty$.

The above results suggest the existence of an interesting diffusive scaling limit for the continuous-space symbiotic branching model (and its interface) for $\varrho > -1$. This is the starting point of our investigation. However, compared to the case $\varrho = -1$, the situation is more involved here: For example, the total mass of the solution is not necessarily bounded, and in particular, moments of the solution may diverge as $t \to \infty$, depending on ϱ. For instance, second moments diverge for $\varrho \geq 0$. In order to state this result, which was obtained in [3], we define the *critical curve* $p : (-1, 1) \to (1, \infty)$ of the symbiotic branching model by

$$p(\varrho) = \frac{\pi}{\arccos(-\varrho)}, \tag{6}$$

and denote its inverse by $\varrho(p) = -\cos(\frac{\pi}{p})$ (for $p > 1$). This curve separates the upper right quadrant into two areas: below the critical curve, where moments remain bounded, and above the critical curve, where moments increase to infinity as $t \to \infty$:

THEOREM 1.3 ([3], Theorem 2.5). *Suppose $(u_t, v_t)_{t\geq 0}$ is a solution to the symbiotic branching model with initial conditions $u_0 = v_0 \equiv 1$. Let $\varrho \in (-1, 1)$ and $\gamma > 0$. Then, for every $x \in \mathbb{R}$,*

$$\varrho < \varrho(p) \quad \textit{iff} \quad \mathbb{E}_{1,1}\big[u_t(x)^p\big] \qquad \textit{is bounded uniformly in all } t \geq 0.$$

In particular, if $\varrho < \varrho(p)$, there exists a constant $C(\varrho)$ so that, uniformly for all $x \in \mathbb{R}$ and $t \geq 0$,

$$\mathbb{E}_{1,1}\big[u_t(x)^p\big] \leq C(\varrho), \qquad t \geq 0.$$

REMARK 1.4. (i) Of course, due to symmetry, the same result holds for the v population. The existence of a finite bound which is independent of x follows from the fact that the system is translation invariant under the $(\mathbf{1}, \mathbf{1})$ starting condition.

(ii) In particular, for $\varrho < \varrho(4) = -\frac{1}{\sqrt{2}}$ and any $x_1, \dots, x_4$ we have by the generalized Hölder inequality that

$$\mathbb{E}_{1,1}\big[u_t(x_1)u_t(x_2)v_t(x_3)v_t(x_4)\big] \leq \max_{i=1,\dots,4} \mathbb{E}_{1,1}\big[u_t(x_i)^4\big] \leq C(\varrho),$$

and similarly if some of the v's are replaced by u (and vice versa).

The main tool of our approach is the use of several *dual processes* for the symbiotic branching model. For the case $\varrho = -1$ (heat equation with Wright–Fisher noise), Tribe [27] uses the duality with coalescing Brownian motions. In our case, we have to use instead a duality due to [11] with a system of colored Brownian particles with an exponential correction term, involving collision local times. Moreover, we will rely on an exponential *self-duality* for uniqueness. These dualities will be explained in detail below.

1.2. *Main results and open problems.* We define the measure-valued processes

$$\mu_t^{(n)}(dx) := u_{n^2 t}(nx)\,dx, \qquad \nu_t^{(n)}(dx) := v_{n^2 t}(nx)\,dx, \tag{7}$$

obtained by taking the diffusively rescaled solutions of cSBM(ϱ, γ) as densities. We consider the pair $(\mu_t^{(n)}, \nu_t^{(n)})_{t\geq 0}$ as random elements of $\mathcal{C}_{[0,\infty)}(\mathcal{M}_{\text{tem}}^2)$, the space of continuous processes taking values in the space of (pairs of) tempered measures endowed with the Skorokhod topology. Loosely speaking, $\mathcal{M}_{\text{tem}}$ contains all the measures whose integral against any nonnegative function that is decaying exponentially fast at $\pm\infty$ is finite. We recall the precise definition of $\mathcal{M}_{\text{tem}}$ and all other necessary spaces of functions and measures in Appendix A.1. Our first main result reads as follows:

THEOREM 1.5. *Assume* $\varrho < \varrho(4) = -\frac{1}{\sqrt{2}}$. *Let* $(u_t, v_t)_{t\geq 0}$ *be a solution to* cSBM$(\varrho, \gamma)_{u_0, v_0}$ *with complementary Heaviside initial conditions* $(u_0, v_0) = (\mathbb{1}_{\mathbb{R}^-}, \mathbb{1}_{\mathbb{R}^+})$. *Then the processes* $(\mu_t^{(n)}, \nu_t^{(n)})_{t\geq 0}$ *converge weakly in* $\mathcal{C}_{[0,\infty)}(\mathcal{M}_{\text{tem}}^2)$ *to a limit* $(\mu_t, \nu_t)_{t\geq 0}$ *which has the following properties*:

- *Absolute continuity*: *For each fixed* $t > 0$, μ_t *and* ν_t *are absolutely continuous w.r.t. the Lebesgue measure* ℓ, $\mathbb{P}$-*a.s.*,

$$\mu_t(dx) = \mu_t(x)\,dx, \qquad \nu_t(dx) = \nu_t(x)\,dx, \qquad \mathbb{P}\text{-}a.s.^{2}$$

- *Separation of types*: *For each fixed* $t > 0$ *the* (*absolutely continuous*) *measures* μ_t *and* ν_t *are mutually singular*: *We have*

$$\mu_t(\cdot)\nu_t(\cdot) = 0, \qquad \mathbb{P}\otimes\ell\text{-}a.s. \tag{8}$$

REMARK 1.6. (a) *Identification of the limit.* For $\varrho = -1$, Tribe [27] shows that the process $(\mu_t^{(n)}, \nu_t^{(n)})_{t\geq 0}$ converges weakly to

$$(\mathbb{1}_{\{x\leq B_t\}}\,dx, \mathbb{1}_{\{x\geq B_t\}}\,dx)_{t\geq 0},$$

for $(B_t)_{t\geq 0}$ a standard Brownian motion. In our case, however, that is, for $\varrho \in (-1, -\frac{1}{\sqrt{2}})$, one can show that the limit $(\mu_t, \nu_t)_{t\geq 0}$ cannot be of the form

$$(\mathbb{1}_{\{x\leq I_t\}}\,dx, \mathbb{1}_{\{x\geq I_t\}}\,dx)_{t\geq 0}$$

for a semimartingale $(I_t)_{t\geq 0}$; see Remark 1.14 below for more details. Moreover, we remark that the limiting process in Theorem 1.5 is also not trivial, that is, nondeterministic: If it were, then by the Green function representation of the limit (see Corollary A.4 below) it would have to be given by

$$(\mu_t, \nu_t) = (S_t u_0, S_t v_0),$$

[2] For an absolutely continuous measure, we will usually use the same symbol to denote the measure and its density.

which, however, violates the "separation of types" condition (8).

(b) *Restrictions on ϱ and initial conditions.* Note that the restrictions on the range of parameters only comes from our proof of tightness for the rescaled solutions. The decisive step is an estimate on the second moment of the integral $\int u_t(x)v_t(x)\,dx$ that is uniform in time. It is here that both assumptions $\varrho < -\frac{1}{\sqrt{2}}$ and complementary Heaviside initial conditions are essential. The restriction on ϱ comes from the fact that the second moment of the product $u_t v_t$ is really a fourth moment, and we recall from Theorem 1.3 that only for $\varrho < \varrho(4) = -\frac{1}{\sqrt{2}}$ fourth moments (at a single location) remain bounded in time. In fact, our technique would work in principle if we could control pth moments for $p > 2$, but our integer-moment particle system duality in combination with the Burkholder–Davis–Gundy inequality requires mixed fourth moments; see Lemma 3.4.

Similarly, the restriction to Heaviside initial conditions is due to the technique of proof. Only in this case can we control the expression obtained via the particle dual. Roughly speaking, we need the "simple" shape of the initial conditions to be able to reduce the (spatial) integrals to pointwise estimates that can be controlled via Theorem 1.3.

More generally, it seems conceivable (although probably technically much more involved) that one can deal with initial conditions of the type $u_{0,n} = \mathbb{1}_{(-\infty,an]} + \mathbb{1}_{[bn,cn]}$, $v_{0,n} = \mathbb{1}_{[an,bn]} + \mathbb{1}_{[cn,\infty)}$ (and its obvious generalizations to several blocks; cf. also [27] for $\varrho = -1$). It seems difficult to go beyond this class, and it is clear that the "overlap" of the support of the initial conditions needs to vanish sufficiently quickly (for $n \to \infty$) for the moment bound to hold.

Note that we can relax both assumptions to any $\varrho < 0$ and general initial conditions if we allow a weaker topology than the Skorokhod topology on $\mathcal{C}_{[0,\infty)}$; see Theorem 1.10 below.

Unfortunately, we do not yet have a fully explicit representation of the limiting process of Theorem 1.5 as in [27]. We can, however, characterize it as the unique solution to a certain martingale problem. For the (standard) notation we again refer the reader to Appendix A.1.

DEFINITION 1.7 [Martingale problem $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$]. Fix $\varrho \in [-1, 1]$ and (possibly random) initial conditions $(\mu_0, \nu_0) \in \mathcal{M}^2_{\mathrm{tem}}$ (resp., $\mathcal{M}^2_{\mathrm{rap}}$). A continuous $\mathcal{M}^2_{\mathrm{tem}}$-valued (resp., $\mathcal{M}^2_{\mathrm{rap}}$-valued) stochastic process $(\mu_t, \nu_t)_{t\geq 0}$ is called a solution to the martingale problem $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$ if there exists a continuous $\mathcal{M}_{\mathrm{tem}}$-valued (resp., $\mathcal{M}_{\mathrm{rap}}$-valued) process $(\Lambda_t)_{t\geq 0}$ such that for each test function $\phi \in \mathcal{C}^{(2)}_{\mathrm{rap}}$ (resp., $\phi \in \mathcal{C}^{(2)}_{\mathrm{tem}}$), the process $(M(\phi), N(\phi))$ defined by

$$
\begin{aligned}
M(\phi)_t &:= \langle \mu_t, \phi\rangle - \langle \mu_0, \phi\rangle - \int_0^t \left\langle \mu_s, \frac{1}{2}\Delta\phi\right\rangle ds, \\
N(\phi)_t &:= \langle \nu_t, \phi\rangle - \langle \nu_0, \phi\rangle - \int_0^t \left\langle \nu_s, \frac{1}{2}\Delta\phi\right\rangle ds
\end{aligned}
\tag{9}
$$

is a pair of continuous square-integrable martingales null at zero with covariance structure

$$\begin{aligned} &[M(\phi), M(\phi)]_t = [N(\phi), N(\phi)]_t = \langle \Lambda_t, \phi^2 \rangle, \\ &[M(\phi), N(\phi)]_t = \varrho \langle \Lambda_t, \phi^2 \rangle. \end{aligned} \tag{10}$$

Observe that if there exists a process Λ controlling the correlation as in Definition 1.7, then it is uniquely determined by (μ, ν) via the martingales in (9). Obviously, $\Lambda_0 = 0$ and Λ has to be an increasing process in the sense that $(\langle \Lambda_t, \phi \rangle)_{t \geq 0}$ is increasing for all $\phi \geq 0$. Also, condition (10) implies that the martingale measure M (and similarly N) is orthogonal in the sense of [28]; that is, for all test functions ϕ, ψ with $\phi\psi \equiv 0$ we have $[M(\phi), M(\psi)]_t = \langle \Lambda_t, \phi\psi \rangle = 0$.

It is important to note that in the definition of the martingale problem $(\mathbf{MP})^{\varrho}_{\mu_0, \nu_0}$, we do not specify the measure-valued process Λ more explicitly. As a consequence, it is not surprising that the martingale problem $(\mathbf{MP})^{\varrho}_{\mu_0, \nu_0}$ is not well posed without any further conditions.

Indeed, assume that the initial conditions are absolutely continuous with densities $(u_0, v_0) \in (\mathcal{B}^+_{\mathrm{tem}})^2$, and let $\gamma > 0$ be arbitrary. If we denote by $(u^{[\gamma]}_t, v^{[\gamma]}_t)$ the symbiotic branching process with finite branching rate γ, then $(u^{[\gamma]}_t, v^{[\gamma]}_t)$, considered as measure-valued processes, is a solution to $(\mathbf{MP})^{\varrho}_{\mu_0, \nu_0}$ with $\Lambda = \Lambda^{[\gamma]}$ given by

$$\Lambda^{[\gamma]}_t(dx) := \gamma \int_0^t ds\, u^{[\gamma]}_s(x) v^{[\gamma]}_s(x)\, dx, \tag{11}$$

for *every* $\gamma > 0$; see Theorem 4 in [11].

Certainly uniqueness in the martingale problem $(\mathbf{MP})^{\varrho}_{\mu_0, \nu_0}$ can thus be achieved by *prescribing* an explicit correlation structure as in (11). However, in order to characterize the limiting object in Theorem 1.5, we proceed differently, and we only require (a slightly stronger version of) the "separation of types" property (8) for uniqueness.

Our uniqueness argument relies on a variant of the self-duality à la Mytnik [23]. Also, instead of requiring that the dual process lives in the space of continuous measure-valued processes, we can relax this condition, and we will construct the dual for a large class of initial conditions by approximations in the (less restrictive) Meyer–Zheng topology.

Recall the self-duality function employed in [11]: let $\varrho \in (-1, 1)$ and if either $(\mu, \nu, \phi, \psi) \in \mathcal{M}^2_{\mathrm{tem}} \times \mathcal{B}^2_{\mathrm{rap}}$ or $(\mu, \nu, \phi, \psi) \in \mathcal{M}^2_{\mathrm{rap}} \times \mathcal{B}^2_{\mathrm{tem}}$, denote

$$\langle\langle \mu, \nu, \phi, \psi \rangle\rangle_{\varrho} := -\sqrt{1-\varrho}\langle \mu + \nu, \phi + \psi \rangle + i\sqrt{1+\varrho}\langle \mu - \nu, \phi - \psi \rangle. \tag{12}$$

Then we define the *self-duality function* F as

$$F(\mu, \nu, \phi, \psi) := \exp\langle\langle \mu, \nu, \phi, \psi \rangle\rangle_{\varrho}. \tag{13}$$

With this notation, we define another (somewhat weaker) martingale problem, which is tailored for an application of the self-duality.

DEFINITION 1.8 [Martingale problem $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$]. Fix $\varrho \in (-1,1)$ and (possibly random) initial conditions $(\mu_0,\nu_0) \in \mathcal{M}^2_{\mathrm{tem}}$ (resp., $\mathcal{M}^2_{\mathrm{rap}}$). A càdlàg $\mathcal{M}^2_{\mathrm{tem}}$-valued (resp., $\mathcal{M}^2_{\mathrm{rap}}$-valued) stochastic process $(\mu_t,\nu_t)_{t\geq 0}$ is called a solution to the martingale problem $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$ if the following holds: There exists an increasing càdlàg $\mathcal{M}_{\mathrm{tem}}$-valued (resp., $\mathcal{M}_{\mathrm{rap}}$-valued) process $(\Lambda_t)_{t\geq 0}$ with $\Lambda_0 = 0$ and

$$\mathbb{E}_{\mu_0,\nu_0}\big[\Lambda_t(dx)\big] \in \mathcal{M}_{\mathrm{tem}} \qquad \big(\text{resp., } \mathbb{E}_{\mu_0,\nu_0}\big[\Lambda_t(dx)\big] \in \mathcal{M}_{\mathrm{rap}}\big) \tag{14}$$

for all $t > 0$, such that for all test functions $\phi, \psi \in (\mathcal{C}^{(2)}_{\mathrm{rap}})^+$ [resp., $\phi,\psi \in (\mathcal{C}^{(2)}_{\mathrm{tem}})^+$], the process

$$\begin{aligned} &F(\mu_t,\nu_t,\phi,\psi) - F(\mu_0,\nu_0,\phi,\psi) \\ &\quad - \frac{1}{2}\int_0^t F(\mu_s,\nu_s,\phi,\psi)\langle\langle \mu_s,\nu_s,\Delta\phi,\Delta\psi\rangle\rangle_{\varrho}\, ds \\ &\quad - 4\big(1-\varrho^2\big)\int_{[0,t]\times\mathbb{R}} F(\mu_s,\nu_s,\phi,\psi)\phi(x)\psi(x)\Lambda(ds,dx) \end{aligned} \tag{15}$$

is a martingale.

In (15) we have interpreted the right-continuous and increasing process $t \mapsto \Lambda_t(dx)$ as a (locally finite) measure $\Lambda(ds,dx)$ on $\mathbb{R}^+ \times \mathbb{R}$, via

$$\Lambda\big([0,t]\times B\big) := \Lambda_t(B).$$

REMARK 1.9. Note that in contrast to Definition 1.7, we do not require a solution of $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$ to be continuous, but only càdlàg. Hence we can construct solutions to $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$ via approximations in the weaker Meyer–Zheng "pseudo-path" topology (see [19] and [18] and cf. Appendix A.1), which allows us to work with second instead of fourth moment bounds and more general initial conditions.

We do not include the boundary cases $\varrho = \pm 1$, since either the real or imaginary part in (12) vanishes, and we cannot use the resulting F for our approach, showing uniqueness via self-duality.

As in Definition 1.7, the martingale problem of Definition 1.8 is not well posed: In fact, by Corollary A.6 any solution to $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$ is also a solution to $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$; this is a simple application of Itô's formula. In particular, for any $\gamma > 0$ the solution to the finite rate symbiotic branching model $\mathrm{cSBM}(\varrho,\gamma)_{u_0,v_0}$ also solves the martingale problem $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$.

Somewhat surprisingly, even without prescribing Λ we can (at least for $\varrho < 0$) still prove self-duality and thus uniqueness, as long we require a certain "separation of types" property. We denote by $(S_t)_{t\geq 0}$ the usual heat semigroup.

THEOREM 1.10. *Fix absolutely continuous initial conditions with densities which are tempered or rapidly decreasing functions, that is,* $(\mu_0,\nu_0) \in (\mathcal{B}^+_{\mathrm{tem}})^2$, *respectively,* $(\mu_0,\nu_0) \in (\mathcal{B}^+_{\mathrm{rap}})^2$. *Assume that* $\varrho \in (-1,0)$.

(i) *There exists a unique solution* $(\mu_t, \nu_t)_{t\geq 0}$ *to the martingale problem* $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$ *that is characterized by the following "separation of types" property*: *For all* $t \in (0,\infty)$, $x \in \mathbb{R}$ *and* $\varepsilon > 0$ *we have*

$$S_{t+\varepsilon}\mu_0(x) S_{t+\varepsilon}\nu_0(x) \geq \mathbb{E}_{\mu_0,\nu_0}\big[S_\varepsilon \mu_t(x) S_\varepsilon \nu_t(x)\big] \stackrel{\varepsilon\downarrow 0}{\to} 0. \tag{16}$$

(ii) *Moreover, for each* $\gamma > 0$ *denote by* $(\mu_t^{[\gamma]}, \nu_t^{[\gamma]})_{t\geq 0}$ *the solution to* $\mathrm{cSBM}(\varrho,\gamma)_{\mu_0,\nu_0}$, *considered as measure-valued processes. Then, as* $\gamma \uparrow \infty$, *the sequence of processes* $(\mu_t^{[\gamma]}, \nu_t^{[\gamma]})_{t\geq 0}$ *converges in law in* $D_{[0,\infty)}(\mathcal{M}^2_{\mathrm{tem}})$, *respectively, in* $D_{[0,\infty)}(\mathcal{M}^2_{\mathrm{rap}})$, *equipped with the Meyer–Zheng "pseudo-path" topology to the unique solution of the martingale problem* $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$ *satisfying* (16).

We call the unique solution to the martingale problem $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$ satisfying (16) the *continuous-space infinite rate symbiotic branching process*.

Note that *if* the measures μ_t and ν_t are absolutely continuous for some $t > 0$, then by a simple application of Fatou's lemma, condition (16) implies mutual singularity of the measures, that is, the separation of types in the more intuitive sense (8); see also the proof of Corollary 4.5.

REMARK 1.11. *Comparison to the discrete-space infinite rate model*. The martingale problem $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$ may be regarded as a continuous-space analogue of the martingale problem employed in [15], Theorem 1.1, to characterize the discrete-space infinite rate mutually catalytic branching model. In the discrete case, uniqueness is achieved by prescribing the condition that $u_t(k)v_t(k) = 0$ for all k in the state space, and it suffices to consider test functions ϕ, ψ with disjoint support (i.e., $\phi\psi \equiv 0$). Consequently, the last term in (15) vanishes, and Λ does not appear. Also, it is not possible to copy the self-duality proof from [15], Proposition 4.7, since unlike in the discrete-space context in continuous space, we cannot apply the Laplacian directly to the solutions. We have to "smooth out" the solutions (see the proof of Proposition 5.1 below), which, however, destroys the disjoint support property, giving the additional term in (15) involving the correlation structure Λ.

A similarity to the discrete model is that we formulate the convergence in the weaker Meyer–Zheng topology. This allows us to work with very general initial conditions and to relax the condition on the correlation to $\varrho < 0$. Unfortunately, we cannot show convergence for all $\varrho \in (-1,1)$ as for the discrete model, as we do need bounded second moments.

Finally, in the discrete model the limiting object can be described by a system of stochastic differential equations with jumps. We do not yet have such an explicit description of the limit and we will describe possible approaches to this problem in Remark 1.16 below.

We return to the symbiotic branching model with complementary Heaviside initial conditions for some fixed branching rate $\gamma > 0$, and to the corresponding diffusively rescaled solutions, considered as measure-valued processes $(\mu_t^{(n)}, \nu_t^{(n)})_{t\geq 0}$

as in (7). From the scaling property (4) it follows that $(\mu_t^{(n)}, \nu_t^{(n)})_{t\geq 0}$ are in law equal to the nonrescaled system $(\mu_t^{[n\gamma]}, \nu_t^{[n\gamma]})_{t\geq 0}$ with branching rate γn. In particular, Theorem 1.10(ii) shows that $(\mu_t^{(n)}, \nu_t^{(n)})_{t\geq 0}$ converges in law in the Meyer–Zheng "pseudo-path" topology for any $\varrho < 0$.

However, in Theorem 1.5 we have stated convergence in the stronger Skorokhod topology on $C_{[0,\infty)}(\mathcal{M}_{\text{tem}}^2)$ (albeit for a smaller range of the parameter ϱ). As we have explained in Remark 1.6, this indicates that some extra input is needed. We now state the full version of our main result, which generalizes Theorem 1.5 by characterizing the limit for $\varrho > -1$ as the continuous-space infinite rate symbiotic branching process; recall that for $\varrho = -1$, the limit is characterized by the results in [27].

THEOREM 1.12. *Assume $\varrho \in (-1, -\frac{1}{\sqrt{2}})$. Let $(u_t, v_t)_{t\geq 0}$ be a solution to* $\mathrm{cSBM}(\varrho,\gamma)_{\mu_0,\nu_0}$ *with complementary Heaviside initial conditions $(\mu_0, \nu_0) = (\mathbb{1}_{\mathbb{R}^-}, \mathbb{1}_{\mathbb{R}^+})$. Then the sequence of processes $(\mu_t^{(n)}, \nu_t^{(n)})_{t\geq 0}$ converges in $\mathcal{C}_{[0,\infty)}(\mathcal{M}_{\text{tem}}^2)$ w.r.t. the Skorokhod topology to the unique solution $(\mu_t, \nu_t)_{t\geq 0}$ of the martingale problem $(\mathbf{MP}')_{\mu_0,\nu_0}^{\varrho}$ satisfying (16) from Theorem 1.10. Moreover, the limit has the following properties*:

- *It is also the unique solution to the martingale problem $(\mathbf{MP})_{\mu_0,\nu_0}^{\varrho}$ of Definition 1.7 with the property (16).*
- *Absolute continuity: For each fixed $t > 0$, μ_t and ν_t are absolutely continuous w.r.t. the Lebesgue measure ℓ,*

$$\mu_t(dx) = \mu_t(x)\,dx, \qquad \nu_t(dx) = \nu_t(x)\,dx, \qquad \mathbb{P}\text{-a.s.}$$

- *The "separation of types" property holds also in the sense (8), that is, for each $t > 0$, the (absolutely continuous) measures μ_t and ν_t are mutually singular: We have*

$$\mu_t(\cdot)\nu_t(\cdot) = 0, \qquad \mathbb{P}\otimes\ell\text{-a.s.}$$

REMARK 1.13. Note that our result state convergence in the Skorokhod topology, which is stronger than the Meyer–Zheng topology employed in Theorem 1.10 and also in the discrete-space model. For the continuous model, we believe that the stronger result should also be true for a larger range of parameters. In contrast, in the discrete model, the limit is given by a system of stochastic differential equations with jumps that is not continuous and so cannot be the Skorokhod limit of continuous processes.

REMARK 1.14. With the help of the characterization in Theorem 1.12, one can now show that unlike in the stepping stone case considered in [27], for $\varrho > -1$ the limit cannot be of the form

$$(\mu_t, \nu_t)_{t\geq 0} = (\mathbb{1}_{\{x\leq I_t\}}\,dx, \mathbb{1}_{\{x\geq I_t\}}\,dx)_{t\geq 0}, \tag{17}$$

for a semimartingale $(I_t)_{t\geq 0}$. Indeed, suppose that (μ_t, ν_t) is of this form and satisfies the martingale problem $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$. First of all, since the limiting measure-valued processes are continuous, this forces $(I_t)_{t\geq 0}$ to be a continuous semimartingale. Moreover, the initial conditons tell us that $I_0 = 0$. Therefore, we can write $I_s = M_s + A_s$ for a continuous local martingale M_t (with $M_0 = 0$) and a continuous, adapted process A_t that is of locally finite variation (and $A_0 = 0$). Now, let $\phi \in \mathcal{C}^{(2)}_{\mathrm{rap}}$. Then, by Itô's formula, we have that

$$\begin{aligned}\langle \mu_t, \phi\rangle &= \int_{-\infty}^{I_t} \phi(x)\,dx = \langle \mathbb{1}_{\mathbb{R}^-}, \phi\rangle + \int_0^t \phi(I_s)\,dI_s + \frac{1}{2}\int_0^t \phi'(I_s)\,d[I]_s \\ &= \langle \mathbb{1}_{\mathbb{R}^-}, \phi\rangle + \int_0^t \phi(I_s)\,dM_s + \int_0^t \phi(I_s)\,dA_s + \frac{1}{2}\int_0^t \langle \mu_s, \Delta\phi\rangle\,d[I]_s.\end{aligned}$$

Thus, by the first condition (9) of $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$, we can deduce that

$$\int_0^t \phi(I_s)\,dA_s + \frac{1}{2}\int_0^t \langle \mu_s, \Delta\phi\rangle\,d[I]_s - \frac{1}{2}\int_0^t \langle \mu_s, \Delta\phi\rangle\,ds$$

is a local martingale. Since it is continuous and of locally finite variation, the expression has to be constant equal to 0. Moreover, since ϕ was arbitrary, this allows us to conclude that A_t is identically 0 and $[I]_t = t$. Hence, I_t is a Brownian motion by Lévy's characterization and thus

$$\big[\langle \mu_\cdot, \phi\rangle, \langle \mu_\cdot, \phi\rangle\big]_t = \int_0^t \phi(I_s)^2\,ds.$$

Finally, we note that

$$\langle \nu_t, \phi\rangle = \int_{I_t}^{\infty} \phi(x)\,dx = \int \phi(x)\,dx - \langle \mu_t, \phi\rangle.$$

In particular, we find that

$$\big[\langle \mu_\cdot, \phi\rangle, \langle \nu_\cdot, \phi\rangle\big]_t = -\big[\langle \mu_\cdot, \phi\rangle, \langle \mu_\cdot, \phi\rangle\big]_t.$$

This contradicts the second condition (10) of $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$ (since we assume $\varrho \neq -1$), so that the limit cannot be of the form given in (17).

In the case $\varrho = -1$, the authors in [22] exploit the corresponding fourth moment bound to get an estimate on the moments of the width of the interface $|R(u_t, v_t) - L(u_t, v_t)|$, *without* any rescaling [here we use the notation (5)]. However, this estimate heavily relies on the fact that there are "no holes" in the system where both u and v are zero. In our case, we can imitate the reasoning to get an estimate for the approximate interface defined in the following way. For any $\varepsilon > 0$, define an approximate left endpoint of the interface as

$$L_t(\varepsilon) = \inf\left\{x \in \mathbb{R} : \int_{-\infty}^{x} u_t(y)v_t(y)\,dy \geq \varepsilon\right\} \wedge R(u_t, v_t)$$

and similarly for the right endpoint

$$R_t(\varepsilon) = \sup\left\{x \in \mathbb{R} : \int_x^\infty u_t(y)v_t(y)\,dy \geq \varepsilon\right\} \vee L(u_t, v_t).$$

Since $|R(u_t, v_t)|$, $|L(u_t, v_t)|$ are almost surely finite, $R_t(\varepsilon)$, $L_t(\varepsilon)$ are well defined. Our final result states that this width of the approximate interface remains small uniformly in t in the following way.

THEOREM 1.15. *Suppose* $(u_0, v_0) = (\mathbb{1}_{\mathbb{R}^-}, \mathbb{1}_{\mathbb{R}^+})$, (u_t, v_t) *is a solution of* $\mathrm{cSBM}(\varrho, \gamma)_{u_0, v_0}$ *and* $\varepsilon > 0$. *Then, for any* $\varrho < \varrho(4) = -\frac{1}{\sqrt{2}}$, $p \in (0, 1)$ *and any* $\delta \in (0, 2(1-p))$, *there exists a constant* $C = C(\varrho, \delta, p)$ *such that for all* $t > 0$,

$$\mathbb{E}_{\mathbb{1}_{\mathbb{R}^-}, \mathbb{1}_{\mathbb{R}^+}}\big(\big(R_t(\varepsilon) - L_t(\varepsilon)\big)^+\big)^p \leq C\varepsilon^{-2+\delta}\gamma^{-(2+p-\delta)}.$$

REMARK 1.16. *Open problems.* Ideally, one would like to characterize the limiting process (μ, ν) in Theorems 1.10 and 1.12 in an explicit way. A first approach toward a better understanding of the limit would be the identification of the quadratic (co-)variation of the limit martingales. Indeed, using the same method as in [5], Lemma 41, it should be in principle possible to "compute" the limit of the processes $\Lambda^{[\gamma]}$ from (11) as $\gamma \uparrow \infty$, for general initial conditions and all $\varrho < 0$. The resulting expression will (as the "collision local time" in [5]) involve a spatial smoothing of the limit densities and can then be used to specify the process Λ in the martingale problems $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$ and $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$. Remarkably, it seems that proving the self-duality (Proposition 5.1 below), and hence uniqueness, using this specification of Λ turns out to be technically substantially more involved than using the "separation of types" approach. In fact, the strength of our approach is that we can show uniqueness while leaving the process Λ largely unspecified.

Nevertheless, it is promising to again specialize to the case of complementary Heaviside initial conditions. In this case, we first note that the constant on the right-hand side of Theorem 1.15 tends to 0 as $\gamma \uparrow \infty$. This strongly suggests that the interface of the diffusively rescaled processes shrinks to a single point in the limit. That is, we expect the limit densities to be of the form

$$\mu_t(x) = \mu_t(x)\mathbb{1}_{\{x < I_t\}}, \qquad \nu_t(x) = \nu_t(x)\mathbb{1}_{\{x > I_t\}}, \qquad x \in \mathbb{R},$$

with $I_t := \sup\{x \in \mathbb{R} : \mu_t(x) > 0\} = \inf\{x \in \mathbb{R} : \nu_t(x) > 0\}$ denoting the position of the interface. In fact, if we assume this and moreover that the densities are sufficiently regular at the point of the interface, then the expression for Λ given in terms of the above spatial smoothing (analogous to [5]) simplifies considerably. Indeed, preliminary calculations suggest that under these assumptions,

$$\Lambda_t(dx) = \frac{1}{|\varrho|}\int_0^t ds\, \mu_s(I_s-)\nu_s(I_s+)\delta_{I_s}(dx).$$

Note that this is in line with the stepping stone case $\varrho = -1$ considered in [27], and that the expression blows up as $\varrho \uparrow 0$. However, especially the assumption that the densities are sufficiently regular at the interface is rather strong. In particular, it seems likely that for a proof one would have to go beyond the measure-valued approach of the present paper. At the moment, we do not even know whether the densities are locally bounded or not; recall, for example, that the densities of the *two-dimensional* finite rate continuous mutually catalytic branching model considered in [5] are locally unbounded.

In order to prove that the interface shrinks to one point, a possible line of attack would be to establish stationarity of the interface without any rescaling as in [22]. Another approach, which might also shed some light on the question of an explicit equation for the limit, could be to diffusively rescale the discrete-space infinite rate model and to investigate whether it converges to our limit process. This is also supported by the conjecture of Klenke and Oeler [17], that for the discrete-space infinite rate model, the interface is essentially a single point. However, carrying out this rather ambitious program is clearly beyond the scope of the present paper, and will be taken up in future research.

1.3. *Strategy of proof and organization of the paper.* The proof of our main result, Theorem 1.12, splits into two parts: The first step is to show tightness, while the second step is to find a property that uniquely identifies the limit points. In our case, we can show that any limit point satisfies the martingale problem $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$ and also the additional "separation of types" property (16) which by Theorem 1.10 gives uniqueness. More concretely the *proof of Theorem* 1.12 is obtained by combining the following results:

- Tightness in $\mathcal{C}_{[0,\infty)}(\mathcal{M}^2_{\mathrm{tem}})$ is proved in Proposition 3.6.
- In Proposition 4.1, we show that any limit point satisfies $(\mathbf{MP})^{\varrho}_{\mathbb{1}_{\mathbb{R}^-},\mathbb{1}_{\mathbb{R}^+}}$ and therefore by Corollary A.6 also $(\mathbf{MP}')^{\varrho}_{\mathbb{1}_{\mathbb{R}^-},\mathbb{1}_{\mathbb{R}^+}}$. To guarantee uniqueness, we also check in Lemma 4.4 that the "separation of types" condition (16) is satisfied.
- Finally, we note that the absolute continuity of the limit is proved in Proposition 4.2, from which together with Lemma 4.4 we obtain also the separation of types in the form of (8); see Corollary 4.5.

The *proof of Theorem* 1.10 relies on a strong interplay between parts (i) and (ii). More precisely, we will proceed as follows:

- We show tightness (in the Meyer–Zheng sense) for $(\mu^{[\gamma]},\nu^{[\gamma]})$ as $\gamma\to\infty$ starting with general initial conditions in $\mathcal{B}^+_{\mathrm{tem}}$ or $\mathcal{B}^+_{\mathrm{rap}}$ for any $\varrho<0$; see Proposition 3.8.
- Next, we show in Proposition 4.3 and Lemma 4.4 that any limit point satisfies the martingale problem $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$ and also property (16).
- These first two steps cover the existence statement of part (i). Moreover, they are also essential for the uniqueness as stated in Proposition 5.2. Indeed, the uniqueness proof relies on a self-duality argument, where we need the existence

of the dual process, which in our case is the infinite rate symbiotic branching model with rapidly decreasing initial conditions.

- Part (ii) of Theorem 1.10 is now a corollary of what we have already shown. Indeed, we have covered the tightness and proved that any limit point satisfies $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$ including (16) so that uniqueness follows immediately.

The structure of the remaining paper is as follows: In Section 3, we show tightness for complementary Heaviside initial conditions and $\varrho < -\frac{1}{\sqrt{2}}$ on $\mathcal{C}_{[0,\infty)}$ in the Skorokhod sense, and for general initial conditions and all $\varrho < 0$ on $D_{[0,\infty)}$ in the Meyer–Zheng sense. Next, we consider in Section 4 the properties of limit points in both topologies. Furthermore, we prove uniqueness of the martingale problems in Section 5. In Section 2, we provide a missing ingredient for the proof of tightness in the strong sense, namely an estimate on integrated fourth mixed moments. Finally, in Section 6 we prove Theorem 1.15 as a corollary to the fourth moment bound.

Many of the basic techniques, such as using duality to show uniqueness and deducing tightness from moments estimates, are standard in the literature for measure-valued processes. Also, the Meyer–Zheng topology has been used for the *discrete* infinite rate symbiotic branching model, because in this topology, tightness relies only on relatively weak moment bounds. However, we would like to highlight two novelties in our approach: In our Theorem 1.5 we claim convergence in the Skorokhod topology, which is stronger than convergence in the Meyer–Zheng sense. For our result, we use the Meyer–Zheng topology only to construct the dual process that then yields uniqueness; cf. also Theorem 1.10. This approach allows us to construct the dual process for a large class of initial conditions, which is essential, since only then duality can be used to identify the law of the original process.

The second novelty is to show uniqueness without specifying the correlation $(\Lambda_t)_{t\geq 0}$ in the martingale problem. This should be compared to a similar situation in [7], where the authors show uniqueness for the two-dimensional equivalent of the mutually catalytic branching model, which satisfies a similar "separation of types" property. In their case, they identify the correlation as an intersection local time and only then deduce uniqueness.

One further important contribution is the integrated fourth moment bound of Proposition 2.2 below which is essential for tightness in the $\mathcal{C}_{[0,\infty)}$-sense. Its derivation relies on careful estimates of intersection local times together with (uniformly) bounded fourth moments, which explains the restriction on ϱ.

Notation: We have collected some of the standard facts and notation about measure-valued processes in Appendix A.1. In Appendix A.2 we recall the martingale problem formulation of the finite rate symbiotic branching model $\mathrm{cSBM}(\varrho,\gamma)_{u_0,v_0}$ and deduce some consequences of the martingale problem $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$ of Definition 1.7. Finally, Appendix A.3 is a collection of estimates for Brownian motion and its local time. Throughout this paper, we will denote by

c, C generic constants whose value may change from line to line. If the dependence on parameters is essential, we will indicate this correspondingly.

2. A bound on integrated fourth mixed moments. The first step is a bound on integrated fourth mixed moments that will allow us to prove tightness of the sequence (7) of rescaled processes along the lines of [27]; see the next section. For this estimate, we heavily use that the symbiotic branching model is dual to a system of colored particles via a moment duality due to [11] that we explain now.

We aim to describe the asymptotic behavior of mixed moments of the form

$$\mathbb{E}_{u_0,v_0}\big[u_t(x_1)\cdots u_t(x_n)v_t(x_{n+1})\cdots v_t(x_{n+m})\big].$$

For $\varrho \in [-1, 1]$, the dual works as follows: Consider $n + m$ particles in $\mathbb{R}$ which can take on two colors, say `red` and `blue`. Each particle moves like a Brownian motion independently of all other particles. At time 0, we place n `red` particles at positions $x_1, \dots, x_n$, respectively, and m `blue` particles at positions $x_{n+1}, \dots, x_{n+m}$. As soon as two particles meet, they start collecting collision local time. If both particles are of the same color, one of them changes color when their collision local time exceeds an (independent) exponential time with parameter γ. Denote by $L_t^{=}$ the total collision local time collected by all pairs of the same color up to time t, and let $L_t^{\neq}$ be the collected local time of all pairs of different color up to time t. Finally, let $l_t := (l_t^{\mathtt{red}}, l_t^{\mathtt{blue}})$, $t \ge 0$, be the corresponding particle process, that is, $l_t^{\mathtt{red}}(x)$ denotes the number of `red` particles at x at time t, and $l_t^{\mathtt{blue}}(x)$ is defined accordingly for `blue` particles. Our mixed moment duality function will then be given, up to an exponential correction involving both $L_t^{=}$ and $L_t^{\neq}$, by a moment duality function

$$(u, v)^{l_t} := \prod_{\substack{x\in\mathbb{R}:\\ l_t^{\mathtt{red}}(x) \text{ or } l_t^{\mathtt{blue}}(x)\neq 0}} u(x)^{l_t^{\mathtt{red}}(x)} v(x)^{l_t^{\mathtt{blue}}(x)}.$$

Note that since there are only $n + m$ particles, the potentially uncountably infinite product is actually a finite product and hence well defined. The following lemma is taken from [11], Proposition 12.

LEMMA 2.1. *Let $(u_t, v_t)_{t\ge 0}$ be a solution to* $\mathrm{cSBM}(\varrho, \gamma)_{u_0,v_0}$ *with $\varrho \in [-1, 1]$. Then, for any $x \in \mathbb{R}$ and $t \ge 0$,*

$$\mathbb{E}_{u_0,v_0}\big[u_t(x_1)\cdots u_t(x_n)v_t(x_{n+1})\cdots v_t(x_{n+m})\big] = \mathbb{E}\big[(u_0, v_0)^{l_t} e^{\gamma(L_t^{=}+\varrho L_t^{\neq})}\big], \tag{18}$$

where the dual process $(l_t)_{t\ge 0}$ behaves as explained above, starting in $l_0 = (l_0^{\mathtt{red}}, l_0^{\mathtt{blue}})$ with `red` *particles located in $(x_1, \dots, x_n)$ and* `blue` *particles in $(x_{n+1}, \dots, x_{n+m})$, respectively.*

Note that if $u_0 = v_0 \equiv 1$, the first factor in the expectation of the right-hand side equals 1. Also note that for second mixed moments, the duality simplifies considerably: In this case, the dual process is started from two particles of different color, which by the definition of the process will retain their respective color for all time (color changes can only occur if two particles of the same color meet). Introducing two independent Brownian motions $(B_t^i)_{t\geq 0}$, $i = 1, 2$, with intersection local time $(L_t^{1,2})_{t\geq 0}$, equation (18) can thus be written as

$$\mathbb{E}_{u_0,v_0}\big[u_t(x)v_t(y)\big] = \mathbb{E}_{x,y}\big[u_0\big(B_t^1\big)v_0\big(B_t^2\big)e^{\gamma\varrho L_t^{1,2}}\big], \tag{19}$$

where here and in the following we will label the Brownian motions according to their starting positions from left to right.

We now state the fourth (mixed) moment estimate announced above:

PROPOSITION 2.2 (Mixed moments). *Let $(u_t, v_t)_{t\geq 0}$ be a solution to* $\text{cSBM}(\varrho, \gamma)_{u_0,v_0}$ *with initial values $(u_0, v_0) = (\mathbb{1}_{\mathbb{R}^-}, \mathbb{1}_{\mathbb{R}^+})$. Then, for $\varrho < \varrho(4) = -\frac{1}{\sqrt{2}}$,*

$$\mathbb{E}_{u_0,v_0}\left[\int\int u_t(x)u_t(y)v_t(x)v_t(y)\,dx\,dy\right] \leq C(u_0, v_0; \gamma, \varrho)$$

uniformly for all $t \geq 0$.

Note that by Fubini's theorem and a simple substitution, it is sufficient to prove that for $z > 0$,

$$\mathbb{E}_{u_0,v_0}\left[\int u_t(x)u_t(x-z)v_t(x)v_t(x-z)\,dx\right]$$

is integrable in z. Our Ansatz is to use the moment duality from Lemma 2.1 and combine it with the moment bounds of Theorem 1.3. However, Theorem 1.3 requires constant initial conditions, which simplifies the moment duality considerably.

In our case, the duality in (18) reads

$$\begin{aligned}&\mathbb{E}_{\mathbb{1}_{\mathbb{R}^-},\mathbb{1}_{\mathbb{R}^+}}\big[u_t(x)u_t(x-z)v_t(x)v_t(x-z)\big]\\ &\quad = \mathbb{E}_{l_0^{\texttt{red}}=(x,x-z),l_0^{\texttt{blue}}=(x,x-z)}\big[(u_0, v_0)^{l_t}e^{\gamma(L_t^{=}+\varrho L_t^{\neq})}\big].\end{aligned}$$

To describe the dynamics of $(l_t)_{t\geq 0}$, we introduce a system of four independent Brownian motions $\{B_t^i, i = 1, \dots, 4\}$ with respective colors $c_i(t) \in \{\texttt{red}, \texttt{blue}\}$ at time t. We label the Brownian motions according to their starting positions $B_0^1 = 0$, $B_0^2 = 0$, $B_0^3 = z$, $B_0^4 = z$ in increasing order, and we set their initial colors to be $c_1(0) = c_3(0) = \texttt{red}$, while $c_2(0) = c_4(0) = \texttt{blue}$. Defining

$$f^{\texttt{red}} := u_0 = \mathbb{1}_{\mathbb{R}^-}, \qquad f^{\texttt{blue}} := v_0 = \mathbb{1}_{\mathbb{R}^+},$$

we can rewrite the duality as

$$\mathbb{E}_{u_0,v_0}\big[u_t(x)v_t(x)u_t(x-z)v_t(x-z)\big]$$
$$= \mathbb{E}_{l_0^{\texttt{red}}=(0,z),l_0^{\texttt{blue}}=(0,z)}\left[\prod_{i=1}^{4} f^{c_i(t)}\big(x-B_t^i\big)e^{\gamma(L_t^{=}+\varrho L_t^{\neq})}\right].$$

We now integrate over x and estimate the integral. Note that the exponential term does not depend on x. Hence, we may restrict our attention to

$$\int \prod_{i=1}^{4} f^{c_i(t)}\big(x-B_t^i\big)\,dx, \tag{20}$$

for different color configurations. First observe that

$$f^{\texttt{red}}(x-B_t) = \mathbb{1}_{\{x<B_t\}} \quad \text{and} \quad f^{\texttt{blue}}(x-B_t) = \mathbb{1}_{\{x>B_t\}}, \tag{21}$$

so that one should think of the integral in (20) as an integral over a product of Heaviside functions centered at B_t^i, where the color determines the shape.

Now denote by $r(t)$ the index of the left-most red Brownian motion at time t, that is, $c^{r(t)}(t) = \texttt{red}$ and

$$B_t^{r(t)} \le B_t^i \qquad \text{for all } i \text{ such that } c^i(t) = \texttt{red},$$

where we choose the smaller index to resolve ties. Similarly, we denote by $\ell(t)$ the index of the right-most blue Brownian motion, that is, $c^{\ell(t)}(t) = \texttt{blue}$ and

$$B_t^{\ell(t)} \ge B_t^i \qquad \text{for all } i \text{ such that } c^i(t) = \texttt{blue}$$

(with the smaller index to resolve ties).

Observe that, due to the definition of our dual particle system $(l_t)_{t\ge 0}$, if we start with four particles and two colors, there will always be at least one red particle and at least one blue particle around at any time, no matter what the actual color changes were (color changes can only occur if two particles of the same color meet). Moreover, with the above notation, the integral in (20) is 0 unless $B_t^{r(t)} > B_t^{\ell(t)}$ (see Figure 1), and since the product is either 0 or 1, we obtain

$$\int \prod_{i=1}^{4} f^{c_i(t)}\big(x-B_t^i\big)\,dx = \big(B_t^{r(t)} - B_t^{\ell(t)}\big)^+;$$

see also Figure 2.

Altogether, we arrive at

$$\begin{aligned}&\mathbb{E}_{u_0,v_0}\int u_t(x)u_t(x-z)v_t(x)v_t(x-z)\,dx\\ &\quad= \mathbb{E}_{(0,z),(0,z)}\big[\big(B^{r(t)} - B^{\ell(t)}\big)^+ e^{\gamma(L_t^{=}+\varrho L_t^{\neq})}\big]\end{aligned} \tag{22}$$

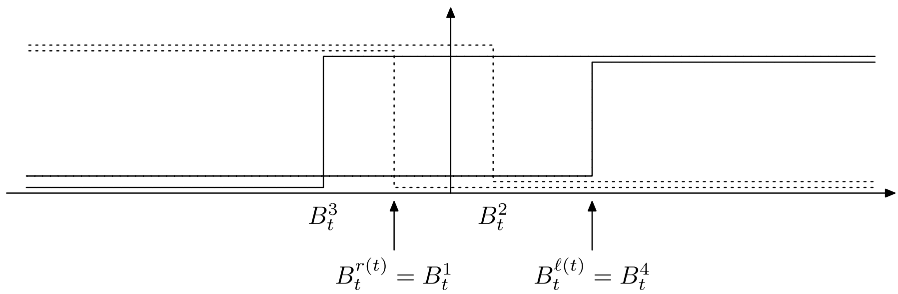

FIG. 1. *An illustration of the four factors in the product in* (20) (*drawn slightly shifted for illustration*). *The Heaviside functions are centred at the positions of the Brownian motions, and the color determines the shape* (`red` *is dotted, and* `blue` *is drawn in black*). *Here*, $c_1(t) = c_2(t) =$ `red` *and* $c_3(t) = c_4(t) =$ `blue`. *In this case, the product of all four factors is zero, since* $B_t^{r(t)} < B_t^{\ell(t)}$.

and need to show that, for $z > 0$, this expression is integrable in z. We prepare this with a lemma which covers the important case where the two particles that are initially in the middle start in the same location.

LEMMA 2.3. *Assume that* $\varrho < \varrho(4)$, *and let* $-\infty < x < y < z < \infty$ *and* $\delta \in (0, \frac{1}{2})$. *Then, for any initial configuration* $l_0 = \underline{x}$ *that contains four particles in positions* x, y, y, z *and two of each color, that is*,

$$\underline{x} \in \{(x, y), (y, z); (y, z), (x, y); (x, z), (y, y); (y, y), (x, z)\},$$

we have

$$\mathbb{E}_{\underline{x}}\big[\big(B_t^{r(t)} - B_t^{\ell(t)}\big)^+ e^{\gamma(L_t^= + \varrho L_t^{\neq})}\big] \\ \le C(\varrho, \gamma, \delta) \min\left\{\frac{(z-y+1)(y-x+1)}{t^{1/2-\delta}}, 1 \vee t^{\delta}\right\}.$$

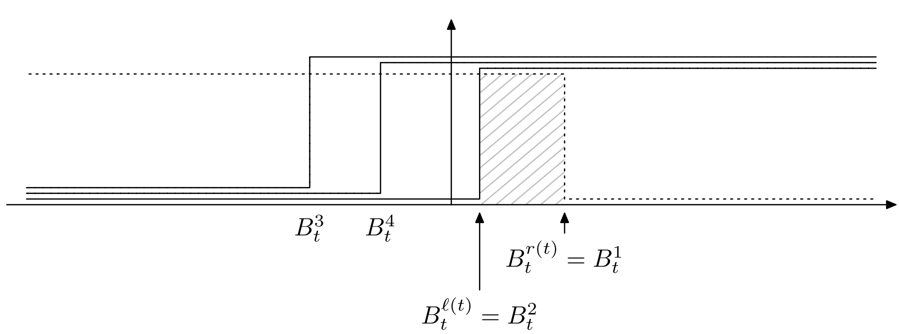

FIG. 2. *In this scenario*, $c_1(t) =$ `red`, $c_2(t) = c_3(t) = c_4(t) =$ `blue`. *Since* $B_t^{\ell(t)} < B_t^{r(t)}$, *the integral gives a nonzero contribution corresponding to the shaded area.*

PROOF. Pick ϱ' so that $\varrho < \varrho' < \varrho(4)$, and let $\delta \in (0, \frac{1}{2})$. Using the (generalized) Hölder inequality twice for $p_1, p_2, p_3 \geq 1$ with $p_3 = (1 - \frac{\delta}{2})^{-1}$ and $p_1 = p_2$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, we obtain

$$
\begin{aligned}
&\mathbb{E}_{\underline{x}}\big[\big(B^{r(t)} - B^{\ell(t)}\big)^+ e^{\gamma(L_t^{=} + \varrho L_t^{\neq})}\big] \\
&\quad \leq \mathbb{E}_{\underline{x}}\big[\big(\big(B^{r(t)} - B^{\ell(t)}\big)^+\big)^{p_1}\big]^{1/p_1} \mathbb{E}_{\underline{x}}\big[e^{p_2\gamma(L_t^{=} + \varrho' L_t^{\neq})}\big]^{1/p_2} \\
&\qquad \times \mathbb{E}_{\underline{x}}\big[e^{-p_3\gamma(\varrho' - \varrho)L_t^{\neq}}\big]^{1/p_3}.
\end{aligned}
\tag{23}
$$

By the moment duality (18), the second expectation in (23) corresponds to the fourth mixed moment of a system with branching rate $p_2\gamma$, correlation parameter ϱ' and constant initial conditions. Since $\varrho' < \varrho(4)$, this expression is bounded by a constant (depending only on ϱ') uniformly in $t \geq 0$; see Theorem 1.3 and also Remark 1.4.

For the first expectation on the right-hand side in (23), we claim that

$$
\mathbb{E}_{\underline{x}}\big[\big(\big(B_t^{r(t)} - B_t^{\ell(t)}\big)^+\big)^{p_1}\big]^{1/p_1} \leq C(p_1)t^{1/2}. \tag{24}
$$

The claim follows if we can show that the expectation on the left-hand side does not depend on the distances of the starting points $z - y, y - x$. We recall that the particles are labeled from left to right according to the initial positions. In particular $2, 3$ are the labels of the particles started in y. Also, we can always assume that $B_t^{\ell(t)} < B_t^{r(t)}$ since this is the only scenario when we observe a positive contribution to the expectation.

Denote by $\tau_{i,j}$ the first collision time of particles i, j. We claim that if $B_t^{\ell(t)} < B_t^{r(t)}$, then there exist $i, j \in \{1, \dots, 4\}, i \neq j$, such that $c_i(t) \neq c_j(t)$ and $\tau_{ij} \leq t$. Indeed, suppose first that no color change occurs up to time t. Then, if particles 2 and 3 (both started in y) have different colors, $\tau_{2,3} = 0$, and the claim holds. Conversely, if 2 and 3 have the same color, there has to be a collision between particles of different colors before time t, since the condition $B_t^{\ell(t)} < B_t^{r(t)}$ implies that both `blue` particles are to the left of the `red` particles at time t; see Figure 3 for an illustration.

Moreover, if there is a color change before time t, we can consider particle i that has changed its color last before time t (out of all particles), say at time σ^i. Then by construction of the particle process, the color change happened through the interaction with particle j, which just before the change had the same color, but now satisfies $c_i(\sigma_i) \neq c_j(\sigma_i)$ and also $\tau_{ij} \leq \sigma_i \leq t$. However, since i was the last particle to change color, it follows that $c_j(t) = c_j(\sigma_i) \neq c_i(\sigma_i) = c_i(t)$; see also Figure 4.

Consequently, in order to show (24) we can assume that i, j are such that $\tau_{ij} \leq t$ and $c_i(t) \neq c_i(t)$. Again note that if $B_t^{r(t)} - B_t^{\ell(t)} > 0$, all `blue` particles are to the

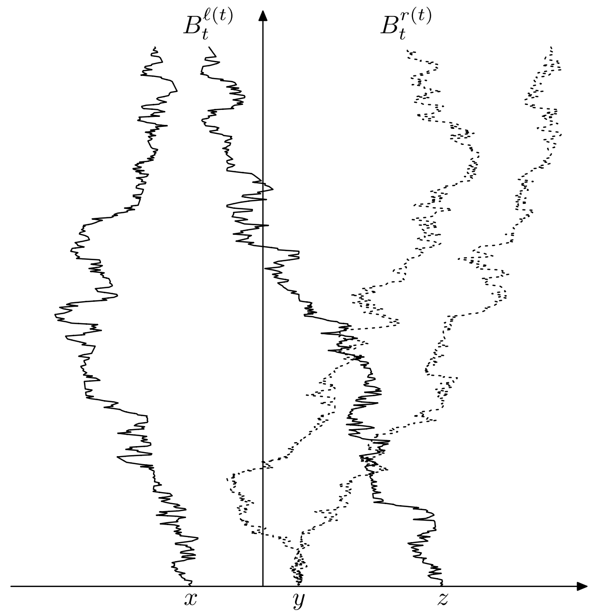

FIG. 3. *No color change occurs up to time t and two* `red` *particles* (*dotted line*) *start in* y, *while two* `blue` *particles* (*black line*) *start in* x *and* z, *respectively. Moreover at time* t, $B_t^{r(t)} > B_t^{\ell(t)}$ *so that particles of distinct colors must have crossed.*

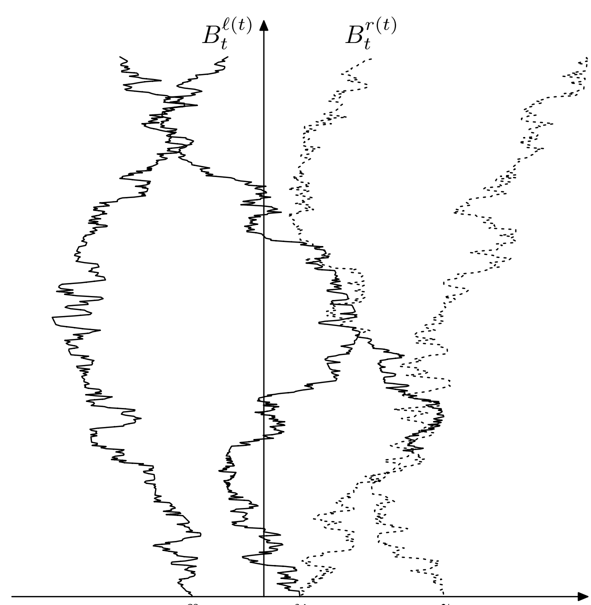

FIG. 4. *If a color change occurs, at least two particles of distinct colors at time t must have met before.*

left of `red` particles, so that since particles i and j have different colors, we find that $(B_t^{r(t)} - B_t^{\ell(t)})^+ \le |B_t^i - B_t^j|$. Therefore, by the strong Markov property,

$$\begin{aligned}
&\mathbb{E}_{\underline{x}}\bigl[\mathbb{1}_{\{\tau_{i,j}\le t, c_i(t)\neq c_j(t)\}}\bigl((B_t^{r(t)} - B_t^{\ell(t)})^+\bigr)^{p_1}\bigr]^{1/p_1} \\
&\quad\le \mathbb{E}_{\underline{x}}\bigl[\mathbb{1}_{\{\tau_{i,j}\le t\}}\bigl|B_t^i - B_t^j\bigr|^{p_1}\bigr]^{1/p_1} \\
&\quad\le \mathbb{E}_{\underline{x}}\Bigl[\mathbb{1}_{\{\tau_{i,j}\le t\}}\mathbb{E}\Bigl[\sup_{0\le s\le t-\tau_{i,j}}\bigl|B^i_{\tau_{i,j}+s} - B^j_{\tau_{i,j}+s}\bigr|^{p_1}\big|\mathcal{F}_{\tau_{i,j}}\Bigr]\Bigr]^{1/p_1} \\
&\quad\le \mathbb{E}_{0,0}\Bigl[\sup_{0\le s\le t}\bigl|B_s^1 - B_s^2\bigr|^{p_1}\Bigr]^{1/p_1} \le C(p_1)t^{1/2}.
\end{aligned}\tag{25}$$

By summing over all distinct pairs i, j, we thus obtain (24) (where we again make use of the convention that the value of unspecified constants may change from line to line).

Thus we can conclude from (23) that

$$\mathbb{E}_{\underline{x}}\bigl[\bigl(B^{r(t)} - B^{\ell(t)}\bigr)^+ e^{\gamma(L_t^= + \varrho L_t^{\neq})}\bigr] \le C(p_1, p_2, \gamma, \varrho)t^{1/2}\mathbb{E}_{\underline{x}}\bigl[e^{-\gamma p_3(\varrho'-\varrho)L_t^{\neq}}\bigr]^{1/p_3}.$$

Recalling that $\frac{1}{p_3} = 1 - \frac{\delta}{2}$, we see that in order to complete the proof it suffices to show that for any $s > 0$, there is a constant $C = C(s)$ such that for all $t \ge 0$,

$$\begin{aligned}
&\mathbb{E}_{\underline{x}}\bigl[e^{-sL_t^{\neq}}\bigr] \\
&\quad\le C\min\biggl\{\frac{(z - y + \log(t\vee e))(y - x + \log(t\vee e))}{t}, \bigl(\log(t\vee e)\bigr)t^{-1/2}\biggr\},
\end{aligned}\tag{26}$$

where we note that the term $\log(t \vee e)$ can be bounded by $t^{\delta'} \vee 1$ for any $\delta' > 0$. Also note that (26) holds trivially for $t \le 1$. Thus we will assume $t \ge 1$ throughout the rest of the proof.

First, recall that for the collision local time $L_t^{1,2}$ up to time t of two independent Brownian motions, started in positions $x \le y$, we have the classical bound that for all $t \ge 1$,

$$\mathbb{P}_{x,y}\bigl\{L_t^{1,2} \le \alpha\log t\bigr\} \le \frac{1}{\sqrt{\pi}}(2\alpha\log t + y - x)t^{-1/2}, \qquad \alpha > 0;\tag{27}$$

see, for example, Corollary A.9. Now fix $s > 0$, and let $c = \frac{2}{s}$. We distinguish the three cases:

(i) $L_t^{\neq} \ge c\log t$,
(ii) $L_t^{\neq} < c\log t$, but $L_t^{\mathrm{tot}} := L_t^= + L_t^{\neq} \ge 2c\log t$,
(iii) $L_t^{\neq} < c\log t$ and $L_t^{\mathrm{tot}} < 2c\log t$.

Regarding (i), we can estimate

$$\mathbb{E}_{\underline{x}}\bigl[e^{-sL_t^{\neq}}\mathbb{1}_{\{L_t^{\neq}\ge c\log t\}}\bigr] \le t^{-sc} = t^{-2},$$

by our choice of $c = \frac{2}{s}$.

For (ii), we have in particular that $L_t^= \geq c \log t$. Now, from the fourth moment bounds (Theorem 1.3 and Remark 1.4 for the system with branching rate $\frac{s}{|\varrho|}$) together with the moment duality (18) for constant initial conditions, we can deduce that

$$\begin{aligned}
&\mathbb{E}_{\underline{x}}\big[e^{-sL_t^{\neq}} \mathbb{1}_{\{L_t^{\neq} < c\log t, L^{\mathrm{tot}} \geq 2c\log t\}}\big] \\
&\quad \leq t^{-cs/|\varrho|}\mathbb{E}_{\underline{x}}\big[e^{s/|\varrho|(L_t^= + \varrho L_t^{\neq})} \mathbb{1}_{\{L_t^{\neq} < c\log t, L^{\mathrm{tot}} \geq 2c\log t\}}\big] \\
&\quad \leq t^{-cs/|\varrho|}\mathbb{E}_{\underline{x}}\big[e^{s/|\varrho|(L_t^= + \varrho L_t^{\neq})}\big] \\
&\quad \leq C(\varrho)t^{-cs/|\varrho|} \leq C(\varrho)t^{-cs} = C(\varrho)t^{-2}.
\end{aligned}$$

Finally, consider case (iii). Here, note that if the total collision local time is small, then in particular the collision local time between the two Brownian motions started at y is small. That is, using (27),

$$\mathbb{E}_{\underline{x}}\big[e^{-sL_t^{\neq}} \mathbb{1}_{\{L_t^{\neq} < c\log t, L_t^{\mathrm{tot}} < 2c\log t\}}\big] \leq \mathbb{P}_{y,y}\{L_t^{1,2} \leq 2c\log t\} \leq \frac{4c}{\sqrt{\pi}}(\log t)t^{-1/2}.$$

A different bound can be reached by considering the collision local times between each pair of Brownian motions started in y, z and y, x respectively, leading to [again using (27)]

$$\begin{aligned}
\mathbb{E}_{\underline{x}}\big[e^{-sL_t^{\neq}} \mathbb{1}_{\{L_t^{\neq} < c\log t, L_t^{\mathrm{tot}} < 2c\log t\}}\big] &\leq \mathbb{P}_{x,y}\{L_t^{1,2} \leq 2c\log t\}\mathbb{P}_{y,z}\{L_t^{1,2} \leq 2c\log t\} \\
&\leq \frac{1}{\pi}(4c\log t + y - x)(4c\log t + z - y)t^{-1}.
\end{aligned}$$

Hence, we can take the minimum of the two bounds for (iii). Then, we notice that since we are assuming that $t \geq 1$, cases (i) and (ii) are dominated by the contribution of (iii), so that we obtain (26). □

PROOF OF PROPOSITION 2.2. Fix $0 < \varepsilon < \frac{1}{2}$. By (22), it suffices to show that there exists a constant $C = C(\gamma, \varrho, \varepsilon)$ such that for all $z > 0$,

$$\mathbb{E}_{(0,z),(0,z)}\big[\big(B_t^{r(t)} - B_t^{\ell(t)}\big)^+ e^{\gamma(L_t^= + \varrho L_t^{\neq})}\big] \leq C\big(1 \wedge z^{-2(1-\varepsilon)}\big), \tag{28}$$

which is clearly integrable in z.

We condition on the time of the first collision of certain pairs of the four Brownian motions. Indeed, let $\tau_{i,j}$ denote the first hitting time of the Brownian motions with index i and j, and consider the stopping time

$$\tau := \tau_{1,3} \wedge \tau_{1,4} \wedge \tau_{2,3} \wedge \tau_{2,4},$$

which is the first time that a motion started in 0 meets with a motion started in z.

Note that we can always assume that $\tau \le t$, for otherwise the expectation in (28) is zero. Then, if $(\mathcal{F}_t)_{t\ge 0}$ denotes the filtration of the dual process, we can apply the strong Markov property and use that up to time τ there are no particles of the same color that accumulate local time. In particular, none of the particles have switched color up to time τ, so the positions of B^i_τ at time τ and the color configuration at time τ satisfy the assumptions of Lemma 2.3. Thus choosing $\delta := \frac{\varepsilon}{8}$ in Lemma 2.3, we obtain that there exists a constant $C(\varrho,\gamma,\varepsilon)$ such that

$$\begin{aligned}
&\mathbb{E}_{(0,z),(0,z)}\big[\big(B_t^{r(t)}-B_t^{\ell(t)}\big)^+ e^{\gamma(L_t^{=}+\varrho L_t^{\neq})}\big]\\
&\quad=\mathbb{E}_{(0,z),(0,z)}\big[\mathbb{E}\big[\big(B_t^{r(t)}-B_t^{\ell(t)}\big)^+ e^{\gamma(L_t^{=}-L_\tau^{=}+\varrho(L_t^{\neq}-L_\tau^{\neq}))}|\mathcal{F}_\tau\big]e^{\gamma(L_\tau^{=}+\varrho L_\tau^{\neq})}\big]\\
&\quad\le 4C(\varrho,\gamma,\varepsilon)\mathbb{E}_{(0,z),(0,z)}\\
&\qquad\times\bigg[\mathbb{1}_{\{\tau=\tau_{2,3}\le t\}}\min\bigg\{\frac{(B_\tau^4-B_\tau^3+1)(B_\tau^2-B_\tau^1+1)}{(t-\tau)^{1/2-\delta}},\\
&\qquad\qquad (t-\tau)^{\delta}\vee 1\bigg\}e^{\varrho\gamma(L_\tau^{1,2}+L_\tau^{3,4})}\bigg].
\end{aligned}\tag{29}$$

Here, we also used that the four possible cases $\tau=\tau_{1,3},\tau_{1,4},\tau_{2,3},\tau_{2,4}$ are all equally likely, and in all cases we obtain the same bound from Lemma 2.3. Moreover, in this scenario $L_\tau^{\neq}=L_\tau^{1,2}+L_\tau^{3,4}$.

In the following, we will use repeatedly the fact that for a standard Brownian motion $(B_t)_{t\ge 0}$ with maximum process $(M_t)_{t\ge0}$ and local time $(L_t^0)_{t\ge0}$ at zero, by Lévy's equivalence (see, e.g., Lemma A.7) we have $L_t^0\overset{d}{=}M_t\overset{d}{=}|B_t|$ for all $t>0$, implying that for any $s>0$ there exists a constant $C=C(s)$ such that for $t>0$,

$$\mathbb{E}_0\big[e^{-sL_t^0}\big]=\mathbb{E}_0\big[e^{-s|B_t|}\big]=\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}e^{-x^2/(2t)}e^{-s|x|}\,dx\le C\big(1\wedge t^{-1/2}\big).\tag{30}$$

In the analysis of the right-hand side of (29), we distinguish four cases (where we always assume $\tau\le t$):

(i) $\tau\le z^{2-\varepsilon}$;
(ii) $\tau>z^{2-\varepsilon}$ and ($z^{2-\varepsilon}>t^{1/4}$ or $t\le 2$);
(iii) $\tau>z^{2-\varepsilon}$, but $z^{2-\varepsilon}\le t^{1/4}$ and $\tau\le t^{1/2-\delta}$, $t\ge 2$;
(iv) $\tau>z^{2-\varepsilon}$, $z^{2-\varepsilon}\le t^{1/4}$, but $\tau>t^{1/2-\delta}$, $t\ge 2$.

Case (i). On the event that $\tau\le z^{2-\varepsilon}\wedge t$, we obtain

$$\begin{aligned}
&\mathbb{E}_{(0,z),(0,z)}\bigg[\mathbb{1}_{\{\tau=\tau_{2,3}\le z^{2-\varepsilon}\wedge t\}}\min\bigg\{\frac{(B_\tau^4-B_\tau^3+1)(B_\tau^2-B_\tau^1+1)}{(t-\tau)^{1/2-\delta}},\\
&\qquad\qquad (t-\tau)^{\delta}\vee 1\bigg\}e^{\varrho\gamma L_\tau^{\neq}}\bigg]\\
&\quad\le\mathbb{E}_{(0,z),(0,z)}\big[\mathbb{1}_{\{\tau=\tau_{2,3}\le z^{2-\varepsilon}\}}\big(B_\tau^4-B_\tau^3+1\big)\big(B_\tau^2-B_\tau^1+1\big)\big]
\end{aligned}$$

$$
\begin{aligned}
&\le \mathbb{E}_{0,0}\Big[\max_{s\le z^{2-\varepsilon}}\big(B_s^2-B_s^1+1\big)^2\Big]\mathbb{P}_{0,z}\big\{\tau_{1,2}\le z^{2-\varepsilon}\big\}^{1/2}\\
&\le C\big(1\vee z^{2-\varepsilon}\big)\mathbb{P}_{0,z}\big\{\tau_{1,2}\le z^{2-\varepsilon}\big\}^{1/2},
\end{aligned}
$$

where we used the Cauchy–Schwarz inequality in the penultimate step. In order to estimate the first collision time, denoting by $\tau(0)$ the first hitting time of 0 for a single Brownian motion B started at z, we observe that

$$
\begin{aligned}
\mathbb{P}_{0,z}\big\{\tau_{1,2}\le z^{2-\varepsilon}\big\} &= \mathbb{P}_z\big\{\tau(0)\le 2z^{2-\varepsilon}\big\}\\
&= \mathbb{P}_0\Big\{\max_{s\le 2z^{2-\varepsilon}} B_s\ge z\Big\}\\
&= 2\mathbb{P}_0\{B_{2z^{2-\varepsilon}}\ge z\}\\
&\le 1\wedge\bigg(\frac{2}{\sqrt{\pi}}z^{-(1/2)\varepsilon}e^{-z^{\varepsilon}/4}\bigg),
\end{aligned}
$$

where we used the reflection principle and a standard Gaussian estimate; see, for example, [20], Remark 2.22. Combining the previous two displays shows that in case (i) we obtain an upper bound

$$
C\big(1\vee z^{2-\varepsilon-(1/4)\varepsilon}\big)e^{-(1/8)z^{\varepsilon}}
$$

on the right-hand side of (29), which in turn can be estimated by the right-hand side of (28).

Case (ii). In this scenario, we can find an upper bound on the expectation on the right-hand side in (29) by

$$
\begin{aligned}
&\mathbb{E}_{(0,z),(0,z)}\bigg[\mathbb{1}_{\{z^{2-\varepsilon}<\tau=\tau_{2,3}\le t\}}\\
&\qquad\times\min\bigg\{\frac{(B_\tau^4-B_\tau^3+1)(B_\tau^2-B_\tau^1+1)}{(t-\tau)^{1/2-\delta}},(t-\tau)^{\delta}\vee 1\bigg\}e^{\varrho\gamma(L_\tau^{1,2}+L_\tau^{3,4})}\bigg]\\
&\quad\le \mathbb{E}_{(0,z),(0,z)}\big[\mathbb{1}_{\{z^{2-\varepsilon}<\tau_{2,3}=\tau\le t\}}\big(1\vee t^{\delta}\big)e^{\gamma\varrho(L_\tau^{1,2}+L_\tau^{3,4})}\big]\\
&\quad\le \big(1\vee t^{\delta}\big)\mathbb{E}_{0,0}\big[\exp\big(\gamma\varrho L^{1,2}_{z^{2-\varepsilon}}\big)\big]^2\le C\big(1\vee t^{\delta}\big)\big(1\wedge z^{-2+\varepsilon}\big),
\end{aligned}
$$

where we used the independence of the two pairs of Brownian motions and then (30). Since we assume $t\le 2$ or $z^{2-\varepsilon}>t^{1/4}$, this latter expression can be bounded by $C(1\wedge z^{-2+\varepsilon+4\delta(2-\varepsilon)})$, which by our choice of $\delta=\frac{\varepsilon}{8}$ is of the required form.

Case (iii). In this case, we assume in particular that $t\ge 2$ and $z^{2-\varepsilon}<\tau\le t^{1/2-\delta}$, so that we can estimate

$$
(t-\tau)^{-(1/2-\delta)}\le\big(t-t^{1/2-\delta}\big)^{-(1/2-\delta)}\le Ct^{-1/2+\delta}.
$$

Hence, we can deduce from (29) that

$$\begin{aligned}
&\mathbb{E}_{(0,z),(0,z)}\bigg[\mathbb{1}_{\{z^{2-\varepsilon}<\tau=\tau_{2,3}\le t^{1/2-\delta}\}}\\
&\qquad\times\min\bigg\{\frac{(B^4_\tau-B^3_\tau+1)(B^2_\tau-B^1_\tau+1)}{(t-\tau)^{1/2-\delta}},(t-\tau)^\delta\vee 1\bigg\}e^{\varrho\gamma(L^{1,2}_\tau+L^{3,4}_\tau)}\bigg]\\
&\quad\le\mathbb{E}_{(0,z),(0,z)}\bigg[\mathbb{1}_{\{z^{2-\varepsilon}<\tau=\tau_{2,3}\le t^{1/2-\delta}\}}\\
&\qquad\qquad\times\frac{(B^4_\tau-B^3_\tau+1)(B^2_\tau-B^1_\tau+1)}{(t-\tau)^{1/2-\delta}}e^{\gamma\varrho(L^{1,2}_\tau+L^{3,4}_\tau)}\bigg]\\
&\quad\le Ct^{-1/2+\delta}\mathbb{E}_{(0,z),(0,z)}\Big[\max_{s\le t^{1/2-\delta}}\big(|B^4_s-B^3_s|+1\big)\big(|B^2_s-B^1_s|+1\big)\\
&\qquad\qquad\qquad\times\exp\big(\gamma\varrho\big(L^{1,2}_{z^{2-\varepsilon}}+L^{3,4}_{z^{2-\varepsilon}}\big)\big)\Big].
\end{aligned}$$

Now, applying Hölder's inequality with $p=\frac{1}{1-\varepsilon/2}$ and q its conjugate, and then using the independence of the two pairs of Brownian motions, we obtain an upper bound

$$\begin{aligned}
&Ct^{-1/2+\delta}\mathbb{E}_{(0,z),(0,z)}\Big[\max_{s\le t^{1/2-\delta}}\big(|B^4_s-B^3_s|+1\big)\big(|B^2_s-B^1_s|+1\big)\\
&\qquad\qquad\qquad\times\exp\big(\gamma\varrho\big(L^{1,2}_{z^{2-\varepsilon}}+L^{3,4}_{z^{2-\varepsilon}}\big)\big)\Big]\\
&\quad\le Ct^{-1/2+\delta}\mathbb{E}_{(0,0)}\Big[\max_{s\le t^{1/2-\delta}}\big(|B^2_s-B^1_s|+1\big)^q\Big]^{2/q}\mathbb{E}_{0,0}\big[\exp\big(\gamma\varrho pL^{1,2}_{z^{2-\varepsilon}}\big)\big]^{2/p}\\
&\quad\le Ct^{-1/2+\delta}\mathbb{E}_{(0,0)}\Big[\max_{s\le 1}\big(t^{(1/2)(1/2-\delta)}|B^2_s-B^1_s|+1\big)^q\Big]^{2/q}\\
&\qquad\times\mathbb{E}_{0,0}\big[\exp\big(\gamma\varrho pL^{1,2}_{z^{2-\varepsilon}}\big)\big]^{2/p}\\
&\quad\le C\big(1\wedge z^{-(2-\varepsilon)(1/p)}\big),
\end{aligned}$$

where we used Brownian scaling (and $t\ge 2$) to estimate the first term and (30) for the second term. In particular, we obtain that the latter expression is bounded by $C(1\wedge z^{-(2-\varepsilon)(1/p)})\le C(1\wedge z^{-2(1-\varepsilon)})$, by our choice of p.

Case (iv). For the remaining case (where we can assume $t\ge 2$), we use (29) and the independence of the Brownian motions to get an upper bound

$$\begin{aligned}
&\mathbb{E}_{(0,z),(0,z)}\bigg[\mathbb{1}_{\{t^{1/2-\delta}<\tau=\tau_{2,3}\le t\}}\min\bigg\{\frac{(B^4_\tau-B^3_\tau+1)(B^2_\tau-B^1_\tau+1)}{(t-\tau)^{1/2-\delta}},\\
&\qquad\qquad\qquad(t-\tau)^\delta\vee 1\bigg\}e^{\varrho\gamma(L^{1,2}_\tau+L^{3,4}_\tau)}\bigg]
\end{aligned}$$

$$\le (1 \vee t^{\delta})\mathbb{E}_{(0,z),(0,z)}\big[\exp\big(\gamma\varrho\big(L^{1,2}_{t^{1/2-\delta}} + L^{3,4}_{t^{1/2-\delta}}\big)\big)\big]$$
$$\le C\big(1 \wedge t^{-1/2+2\delta}\big) \le C\big(1 \wedge z^{4(2-\varepsilon)(-1/2+2\delta)}\big)$$

on (29), where we used again (30) and finally that $z^{2-\varepsilon} \le t^{1/4}$. Since $2\delta = \frac{1}{4}\varepsilon < \frac{1}{4}$, the resulting expression is of the form (28).

These cases exhaust all possibilities so that the Lemma is proved via (29). $\square$

3. Tightness. Recall that for initial conditions $(u_0, v_0) \in (\mathcal{B}^+_{\mathrm{rap}})^2$, respectively, $(\mathcal{B}^+_{\mathrm{tem}})^2$, we denote by $(u^{[\gamma]}_t, v^{[\gamma]}_t)_{t\ge 0} \in \mathcal{C}_{(0,\infty)}(\mathcal{C}^+_{\mathrm{rap}})^2$, respectively, $\mathcal{C}_{(0,\infty)}(\mathcal{C}^+_{\mathrm{tem}})^2$ the solution to $\mathrm{cSBM}(\varrho,\gamma)_{u_0,v_0}$ with these initial conditions and finite branching rate $\gamma > 0$. Also recall that by the scaling property (4), this includes the framework of diffusively rescaled solutions with complementary Heaviside initial conditions as considered in (7). We consider the measure-valued processes

$$\mu^{[\gamma]}_t(dx) := u^{[\gamma]}_t(x)\,dx, \qquad \nu^{[\gamma]}_t(dx) := v^{[\gamma]}_t(x)\,dx, \tag{31}$$

$$\Lambda^{[\gamma]}_t(dx) := \gamma\int_0^t ds u^{[\gamma]}_s(x)v^{[\gamma]}_s(x)\,dx. \tag{32}$$

In this section, we will prove tightness of the above processes on the space of paths taking values in the space of rapidly decreasing, respectively tempered, measures. For $\varrho < -\frac{1}{\sqrt{2}}$ and complementary Heaviside initial conditions, we obtain tightness with respect to the Skorokhod topology on the space of continuous paths. For $\varrho < 0$ and general initial conditions, we can still obtain tightness in the weaker Meyer–Zheng "pseudopath" topology on the space of càdlàg paths introduced by [19]; see also the end of Appendix A.1 for a brief description of this topology.

For tightness w.r.t. the Skorokhod topology, a nice exposition of the general strategy in the same setting of measure-valued processes can be found in [5], Section 4.1. We refer the reader to Appendix A.1 for a discussion of the spaces of functions and measures that are employed in the following.

3.1. *Some preliminary estimates.* In this subsection, we derive some estimates which are essential for establishing tightness in both the Skorokhod and the Meyer–Zheng sense. Let $(u_0, v_0) \in (\mathcal{B}^+_{\mathrm{rap}})^2$ [resp., $(\mathcal{B}^+_{\mathrm{tem}})^2$]. Recall that by the Green function representation for $\mathrm{cSBM}(\varrho,\gamma)_{u_0,v_0}$ (see [11], Corollary 19, or Corollary A.4 in the Appendix), we have for every $\gamma > 0$ and $\phi \in \bigcup_{\lambda>0}\mathcal{C}_{-\lambda}$ (resp., $\phi \in \bigcup_{\lambda>0}\mathcal{C}_{\lambda}$) that

$$M^{[\gamma]}_t(\phi) := \langle u^{[\gamma]}_t, \phi\rangle - \langle u_0, S_t\phi\rangle, \qquad N^{[\gamma]}_t(\phi) := \langle v^{[\gamma]}_t, \phi\rangle - \langle v_0, S_t\phi\rangle \tag{33}$$

are martingales with quadratic (co-)variation

$$
\begin{aligned}
&\big[M^{[\gamma]}(\phi), M^{[\gamma]}(\phi)\big]_t \\
&\quad = \big[N^{[\gamma]}(\phi), N^{[\gamma]}(\phi)\big]_t \\
&\quad = \gamma \int_0^t \int_{\mathbb{R}} S_{t-r}\phi(x)^2 u_r^{[\gamma]}(x) v_r^{[\gamma]}(x)\, dx\, dr, \\
&\big[M^{[\gamma]}(\phi), N^{[\gamma]}(\psi)\big]_t \\
&\quad = \varrho\gamma \int_0^t \int_{\mathbb{R}} S_{t-r}\phi(x) S_{t-r}\psi(x) u_r^{[\gamma]}(x) v_r^{[\gamma]}(x)\, dx\, dr.
\end{aligned} \tag{34}
$$

We start with the following lemma which shows in particular that the expectation of the previous display is bounded uniformly in $\gamma > 0$:

LEMMA 3.1. *Suppose $\varrho < 0$ and $(u_0, v_0) \in (\mathcal{B}^+_{\mathrm{rap}})^2$ [resp., $(\mathcal{B}^+_{\mathrm{tem}})^2$]. Then for all $t > 0$, $\gamma > 0$ and $\phi, \psi \in \bigcup_{\lambda>0} \mathcal{C}^+_{-\lambda}$ (resp., $\bigcup_{\lambda>0} \mathcal{C}^+_{\lambda}$), we have*

$$
\begin{aligned}
&\gamma \mathbb{E}_{u_0,v_0}\bigg[\int_0^t \int_{\mathbb{R}} S_{t-s}\phi(x) S_{t-s}\psi(x) u_s^{[\gamma]}(x) v_s^{[\gamma]}(x)\, dx\, ds\bigg] \\
&\quad = \frac{1}{|\varrho|} \iint \phi(x)\psi(y) \mathbb{E}_{x,y}\big[u_0\big(B_t^{(1)}\big) v_0\big(B_t^{(2)}\big)\big(1 - e^{\gamma\varrho L_t^{1,2}}\big)\big]\, dx\, dy \\
&\quad \uparrow \frac{1}{|\varrho|} \iint \phi(x)\psi(y) \mathbb{E}_{x,y}\big[u_0\big(B_t^{(1)}\big) v_0\big(B_t^{(2)}\big) \mathbb{1}_{\{L_t^{1,2}>0\}}\big]\, dx\, dy < \infty,
\end{aligned} \tag{35}
$$

as $\gamma \uparrow \infty$, where $B^{(1)}$, $B^{(2)}$ are independent Brownian motions with intersection local time $L^{1,2}$.

PROOF. First, note that the limit on the right-hand side of (35) holds by monotone convergence since $(1 - e^{\gamma\varrho L_t^{1,2}}) \uparrow \mathbb{1}_{L_t^{1,2}>0}$ as $\gamma \uparrow \infty$. Also observe that the right-hand side is finite under our assumptions since it is bounded by

$$
\begin{aligned}
&\frac{1}{|\varrho|} \iint \phi(x)\psi(y) \mathbb{E}_{x,y}\big[u_0\big(B_t^{(1)}\big) v_0\big(B_t^{(2)}\big)\big]\, dx\, dy \\
&\qquad = \frac{1}{|\varrho|} \langle \phi, S_t u_0\rangle \langle \psi, S_t v_0\rangle < \infty;
\end{aligned} \tag{36}
$$

see, for example, Lemma A.1(a).

In order to show the first equality in (35), we adapt and elaborate an argument from the proof of [27], Lemma 4.4: For a suitable process X, denote by $(L_t^{x,X})_{t\ge 0}$ the local time of X at $x \in \mathbb{R}$. Let $B^{(1)}$, $B^{(2)}$ be independent Brownian motions. Then by a change of variables $s \mapsto t - s$, Fubini's theorem and the colored particle moment duality, we have

$$\gamma \mathbb{E}_{u_0,v_0}\left[\int_0^t ds \int_{\mathbb{R}} dx\, S_{t-s}\phi(x) S_{t-s}\psi(x) u_s^{[\gamma]}(x) v_s^{[\gamma]}(x)\right]$$

$$(37) \qquad = \gamma \int_0^t ds \int_{\mathbb{R}} dx\, S_s\phi(x) S_s\psi(x) \times \mathbb{E}_{0,0}\big[u_0\big(B_{t-s}^{(1)}+x\big) v_0\big(B_{t-s}^{(2)}+x\big) \exp\big(\gamma \varrho L_{t-s}^{0,B^{(2)}-B^{(1)}}\big)\big].$$

Writing $B := (B^{(1)}, B^{(2)})$ and denoting by $(\mathcal{F}_s)_{s\geq 0}$ the natural filtration of B, we use that (by the independence and stationarity of the increments) for functionals $f(B.)$ of the two-dimensional Brownian path, we have

$$\mathbb{E}_{0,0}\big[f(B_{\cdot+s}-B_s)|\mathcal{F}_s\big] \equiv \mathbb{E}_{0,0}\big[f(B.)\big]$$

for each fixed time $s \geq 0$. Applying this with the functional

$$f(B.) := u_0\big(B_{t-s}^{(1)}+x\big) v_0\big(B_{t-s}^{(2)}+x\big) \exp\big(\gamma \varrho L_{t-s}^{0,B^{(2)}-B^{(1)}}\big)$$

for $s \in [0,t]$ and then shifting the dx-integral (change of variables $y := -B_s^{(2)} + x$), we see that (37) is equal to

$$\begin{aligned}
&\gamma \int_0^t ds \int_{\mathbb{R}} dx \mathbb{E}_{0,0}\big[\phi\big(-B_s^{(1)}+x\big)\psi\big(-B_s^{(2)}+x\big) \\
&\qquad \times \mathbb{E}_{(0,0)}\big[u_0\big(B_t^{(1)}-B_s^{(1)}+x\big) v_0\big(B_t^{(2)}-B_s^{(2)}+x\big) \\
&\qquad\qquad \times \exp\big(\gamma\varrho\big(L_t^{0,B^{(2)}-B_s^{(2)}-(B^{(1)}-B_s^{(1)})} \\
&\qquad\qquad\qquad - L_s^{0,B^{(2)}-B_s^{(2)}-(B^{(1)}-B_s^{(1)})}\big)|\mathcal{F}_s\big]\big] \\
&= \gamma \int_0^t ds \int_{\mathbb{R}} dy \mathbb{E}_{0,0}\big[\phi\big(B_s^{(2)}-B_s^{(1)}+y\big)\psi(y) \\
&\qquad \times u_0\big(B_t^{(1)}-B_s^{(1)}+B_s^{(2)}+y\big) v_0\big(B_t^{(2)}+y\big) \\
&\qquad \times \exp\big(\gamma\varrho\big(L_t^{0,B^{(2)}-B_s^{(2)}-(B^{(1)}-B_s^{(1)})} \\
&\qquad\qquad - L_s^{0,B^{(2)}-B_s^{(2)}-(B^{(1)}-B_s^{(1)})}\big)\big)\big] \\
&= \gamma \int_{\mathbb{R}} dy \psi(y) \mathbb{E}_{0,0}\Big[v_0\big(B_t^{(2)}+y\big) \int_0^t ds \phi\big(B_s^{(2)}-B_s^{(1)}+y\big) \\
&\qquad \times u_0\big(B_t^{(1)}+B_s^{(2)}-B_s^{(1)}+y\big) \\
&\qquad \times \exp\big(\gamma\varrho\big(L_t^{B_s^{(2)}-B_s^{(1)},B^{(2)}-B^{(1)}} \\
&\qquad\qquad - L_s^{B_s^{(2)}-B_s^{(1)},B^{(2)}-B^{(1)}}\big)\big)\Big].
\end{aligned}$$

Now for the inner integral $\int_0^t \cdots ds$, we apply Lemma A.10 in the Appendix and then another change of variables $x := y + z$ to see that the above equals

$$\int_{\mathbb{R}} dy \psi(y) \mathbb{E}_{0,0}\Big[v_0\big(B_t^{(2)} + y\big) \int_{\mathbb{R}} dz \phi(z+y) u_0\big(B_t^{(1)} + z + y\big) \\ \times \int_0^t dL_s^{z, B^{(2)} - B^{(1)}} \gamma \exp\big(\gamma \varrho \big(L_t^{z, B^{(2)} - B^{(1)}} - L_s^{z, B^{(2)} - B^{(1)}}\big)\big)\Big]$$

$$= \iint dx\, dy \phi(x) \psi(y) \\ \times \mathbb{E}_{0,0}\Big[u_0\big(B_t^{(1)} + x\big) v_0\big(B_t^{(2)} + y\big) \\ \times \int_0^t dL_s^{x-y, B^{(2)} - B^{(1)}} \gamma \exp\big(\gamma \varrho \big(L_t^{x-y, B^{(2)} - B^{(1)}} - L_s^{x-y, B^{(2)} - B^{(1)}}\big)\big)\Big]$$

$$= \iint dx\, dy \phi(x) \psi(y) \mathbb{E}_{0,0}\Big[u_0\big(B_t^{(1)} + x\big) v_0\big(B_t^{(2)} + y\big) \\ \times \frac{1}{|\varrho|}\big(1 - \exp\big(\gamma \varrho L_t^{x-y, B^{(2)} - B^{(1)}}\big)\big)\Big],$$

which gives the first equality in (35). $\square$

From the above estimate, we obtain a uniform bound on the first moment of $(u^{[\gamma]}, v^{[\gamma]}, \Lambda^{[\gamma]})$ integrated against suitable test functions:

LEMMA 3.2. *Suppose $\varrho < 0$ and $(u_0, v_0) \in (\mathcal{B}_{\mathrm{rap}}^+)^2$ [resp., $(\mathcal{B}_{\mathrm{tem}}^+)^2$]. Then for all $T > 0$ and $\phi \in \bigcup_{\lambda > 0} \mathcal{C}_{-\lambda}^+$ (resp., $\bigcup_{\lambda > 0} \mathcal{C}_{\lambda}^+$), we have*

$$\sup_{\gamma > 0} \mathbb{E}_{u_0, v_0}\Big[\sup_{0 \le t \le T} \langle u_t^{[\gamma]}, \phi \rangle\Big] < \infty, \qquad \sup_{\gamma > 0} \mathbb{E}_{u_0, v_0}\Big[\sup_{0 \le t \le T} \langle v_t^{[\gamma]}, \phi \rangle\Big] < \infty \tag{38}$$

and

$$\sup_{\gamma > 0} \mathbb{E}_{u_0, v_0}\Big[\sup_{0 \le t \le T} \langle \Lambda_t^{[\gamma]}, \phi \rangle\Big] < \infty. \tag{39}$$

PROOF. Suppose $(u_0, v_0) \in (\mathcal{B}_{\mathrm{rap}}^+)^2$. By (33) and (34), using the Burkholder–Davis–Gundy and Jensen inequalities as well as Lemma 3.1 [recall the upper bound (36)], we have

$$\mathbb{E}_{u_0, v_0}\Big[\sup_{0 \le t \le T} \langle u_t^{[\gamma]}, \phi \rangle\Big] \le \mathbb{E}_{u_0, v_0}\Big[\sup_{t \in [0,T]} \big|M_t^{[\gamma]}(\phi)\big|\Big] + \sup_{0 \le t \le T} \langle u_0, S_t \phi \rangle \\ \le C\big(\mathbb{E}_{u_0, v_0}\big[\big[M^{[\gamma]}(\phi), M^{[\gamma]}(\phi)\big]_T\big]\big)^{1/2} + \sup_{0 \le t \le T} \big|\langle u_0, S_t \phi \rangle\big| \\ \le C\bigg(\frac{1}{|\varrho|} \langle \phi, S_T u_0 \rangle \langle \phi, S_T v_0 \rangle\bigg)^{1/2} + \sup_{0 \le t \le T} \big|\langle u_0, S_t \phi \rangle\big| < \infty,$$

and analogously for $\tilde{v}^{[\gamma]}$. Since this bound is independent of γ, (38) follows. In order to show (39), assume without loss of generality that $\phi = \phi_\lambda$ with $\lambda < 0$. Since

$$\phi_\lambda(x) \le C(\lambda, T) \inf_{t\in[0,T]} S_t\phi_\lambda(x), \qquad x \in \mathbb{R}$$

[see Lemma A.1, estimate (69) in the Appendix], and again applying bound (36), we get

$$\begin{aligned}
\mathbb{E}_{u_0,v_0}\Big[\sup_{0\le t\le T}\langle\Lambda_t^{[\gamma]}, \phi_\lambda\rangle\Big] &= \mathbb{E}_{u_0,v_0}\big[\langle\Lambda_T^{[\gamma]}, \phi_{\lambda/2}^2\rangle\big] \\
&= \gamma\mathbb{E}_{u_0,v_0}\Big[\int_0^T\int_{\mathbb{R}} \phi_{\lambda/2}(x)^2 u_s^{[\gamma]}(x)v_s^{[\gamma]}(x)\,dx\,ds\Big] \\
&\le C\gamma\mathbb{E}_{u_0,v_0}\Big[\int_0^T\int_{\mathbb{R}} \big(S_{T-s}\phi_{\lambda/2}(x)\big)^2 u_s^{[\gamma]}(x)v_s^{[\gamma]}(x)\,dx\,ds\Big] \\
&\le \frac{C}{|\varrho|}\langle\phi_{\lambda/2}, S_T u_0\rangle\langle\phi_{\lambda/2}, S_T v_0\rangle < \infty
\end{aligned}$$

uniformly in $\gamma > 0$.

The proof for initial conditions in $(\mathcal{B}_{\text{tem}}^+)^2$ is completely analogous. $\square$

COROLLARY 3.3 (Compact containment). *Suppose $\varrho < 0$ and $(u_0, v_0) \in (\mathcal{B}_{\text{tem}}^+)^2$ [resp., $(\mathcal{B}_{\text{rap}}^+)^2$]. Then the* compact containment condition *holds for the family $(u_t^{[\gamma]}, v_t^{[\gamma]}, \Lambda_t^{[\gamma]})_{t\ge0}$; that is, for every $\varepsilon > 0$ and $T > 0$, there exists a compact subset $K = K_{\varepsilon,T} \subseteq \mathcal{M}_{\text{tem}}$ (resp., $\mathcal{M}_{\text{rap}}$) such that*

$$\inf_{\gamma>0}\mathbb{P}\big\{u_t^{[\gamma]} \in K_{\varepsilon,T} \text{ for all } t \in [0,T]\big\} \ge 1-\varepsilon,$$

and similarly for $v_t^{[\gamma]}$ and $\Lambda_t^{[\gamma]}$.

PROOF. Let $(u_0, v_0) \in (\mathcal{B}_{\text{tem}}^+)^2$. To check the compact containment condition, as in the proof of [5], Proposition 37, use compact subsets of $\mathcal{M}_{\text{tem}}$ of the form

$$K = K\big((c_m)_{m\in\mathbb{N}}\big) := \big\{\nu \in \mathcal{M}_{\text{tem}} : \langle\nu, \phi_{1/m}\rangle \le c_m \text{ for all } m \in \mathbb{N}\big\},$$

where $(c_m)_{m\in\mathbb{N}}$ is a sequence of positive numbers: Given $\varepsilon > 0$ and $T > 0$, for any $m \in \mathbb{N}$ we can find by Lemma 3.2 a number $c_m = c_m(\varepsilon, T) > 0$ such that for all $\gamma > 0$,

$$\mathbb{P}\Big\{\sup_{0\le t\le T}\langle u_t^{[\gamma]}, \phi_{1/m}\rangle \ge c_m\Big\} \le \frac{\varepsilon}{2^m}.$$

In particular, it follows that for all $\gamma > 0$,

$$\mathbb{P}\big\{u_t^{[\gamma]} \in K\big((c_m)_{m\in\mathbb{N}}\big) \text{ for all } t \in [0,T]\big\} \ge 1-\varepsilon. \tag{40}$$

The same reasoning shows that the compact condition also holds for $v^{[\gamma]}$ and $\Lambda^{[\gamma]}$.

The proof for rapidly decreasing initial conditions $(\tilde{u}_0, \tilde{v}_0) \in (\mathcal{B}^+_{\mathrm{rap}})^2$ is completely analogous, using compact subsets of $\mathcal{M}_{\mathrm{rap}}$ of the form

$$K = K\big((c_m)_{m\in\mathbb{N}}\big) := \big\{\nu \in \mathcal{M}_{\mathrm{rap}} : \langle \nu, \phi_{-m}\rangle \le c_m \text{ for all } m \in \mathbb{N}\big\}$$

together with Lemma 3.2. □

3.2. *Tightness in $\mathcal{C}$.* In this subsection we will prove tightness of the family of processes (31)–(32) with respect to the Skorokhod topology on the space of continuous paths. The proof relies on the fourth moment bound of Proposition 2.2 and thus requires complementary Heaviside initial conditions and the condition $\varrho < -\frac{1}{\sqrt{2}}$.

In the first step, we establish tightness of the above measures integrated against suitable test functions:

LEMMA 3.4. *Suppose $\varrho < -\frac{1}{\sqrt{2}}$ and $(u_0, v_0) = (\mathbb{1}_{\mathbb{R}^-}, \mathbb{1}_{\mathbb{R}^+})$. Then for all $\phi \in \bigcup_{\lambda>0} \mathcal{C}_\lambda$ the family of coordinate processes $(\langle \phi, u_t^{[\gamma]}\rangle, \langle \phi, v_t^{[\gamma]}\rangle, \langle \phi, \Lambda_t^{[\gamma]}\rangle)_{t\ge 0}$, considered as a family indexed over $\gamma > 0$, is tight in the space $\mathcal{C}_{[0,\infty)}(\mathbb{R}^3)$.*

Having established the fourth moment bound in Proposition 2.2, the proof of tightness follows closely the proof of [27], Lemma 4.1.

PROOF OF LEMMA 3.4. The Green function representation for cSBM$(\varrho, \gamma)_{u_0,v_0}$ (see, e.g., Corollary A.4) yields for $\phi \in \bigcup_{\lambda>0} \mathcal{C}_\lambda$ that

$$\langle \phi, u_t^{[\gamma]}\rangle = \langle \phi, S_t u_0\rangle + \int_{[0,t]\times\mathbb{R}} S_{t-r}\phi(x) M^{[\gamma]}(dr, dx), \tag{41}$$

where $(S_t)_{t\ge 0}$ denotes the heat semigroup, and $M^{[\gamma]}(dr, dx)$ is a zero-mean martingale measure with quadratic variation given by

$$\gamma \int_0^t \int_{\mathbb{R}} u_r^{[\gamma]}(x) v_r^{[\gamma]}(x) \big(S_{t-r}\phi(x)\big)^2 \, dx \, dr.$$

We check Kolmogorov's tightness criterion for the stochastic integral in (41). For $0 < s < t$, we have

$$\begin{aligned}
&\int_{[0,t]\times\mathbb{R}} S_{t-r}\phi(x) M^{[\gamma]}(dr, dx) - \int_{[0,s]\times\mathbb{R}} S_{s-r}\phi(x) M^{[\gamma]}(dr, dx) \\
&\quad = \int_{[s,t]\times\mathbb{R}} S_{t-r}\phi(x) M^{[\gamma]}(dr, dx) \\
&\qquad + \int_{[0,s]\times\mathbb{R}} \big(S_{t-r}\phi(x) - S_{s-r}\phi(x)\big) M^{[\gamma]}(dr, dx).
\end{aligned} \tag{42}$$

Consider the fourth moment of the first term on the right-hand side in (42): Using first the Burkholder–Davis–Gundy inequality, then Jensen's inequality, the scaling property (4) and finally the fourth moment bound of Proposition 2.2 for $\varrho < -\frac{1}{\sqrt{2}}$, we obtain

$$
\begin{aligned}
&\mathbb{E}_{u_0,v_0}\Big[\Big(\int_{[s,t]\times\mathbb{R}} S_{t-r}\phi(x)M^{[\gamma]}(dr,dx)\Big)^4\Big]\\
&\quad\le C\mathbb{E}_{u_0,v_0}\Big[\Big(\gamma\int_s^t\int_{\mathbb{R}} u_r^{[\gamma]}(x)v_r^{[\gamma]}(x)\big(S_{t-r}\phi(x)\big)^2\,dx\,dr\Big)^2\Big]\\
&\quad\le C\|\phi\|_\infty^4(t-s)^2\mathbb{E}_{u_0,v_0}\Big[\Big(\frac{1}{t-s}\int_s^t\int_{\mathbb{R}}\gamma u_r^{[\gamma]}(x)v_r^{[\gamma]}(x)\,dx\,dr\Big)^2\Big] \qquad (43)\\
&\quad\le C(\phi)(t-s)\mathbb{E}_{u_0,v_0}\Big[\int_s^t\Big(\int_{\mathbb{R}} u^{[1]}_{\gamma^2 r}(x)v^{[1]}_{\gamma^2 r}(x)\,dx\Big)^2\,dr\Big]\\
&\quad\le C(u_0,v_0,\phi,\varrho)(t-s)^2.
\end{aligned}
$$

Now consider the expectation of the fourth power of the second term on the right-hand side in (42): Again using the Burkholder–Davis–Gundy inequality and the elementary bound

$$
\|S_t\phi - S_s\phi\|_\infty \le 2\|\phi\|_\infty\big((t-s)s^{-1}\wedge 1\big),
$$

which follows from the estimate $\|\partial_r S_r\phi\|_\infty \le \|\phi\|_\infty\frac{1}{r}$ together with $\|S_r\phi\|_\infty \le \|\phi\|_\infty$ for any $r>0$, we have

$$
\begin{aligned}
&\mathbb{E}_{u_0,v_0}\Big[\Big(\int_{[0,s]\times\mathbb{R}}\big(S_{t-r}\phi(x)-S_{s-r}\phi(x)\big)M^{[\gamma]}(dr,dx)\Big)^4\Big]\\
&\quad\le C\|\phi\|_\infty^4\mathbb{E}_{u_0,v_0}\Big[\Big(\int_0^s\int_{\mathbb{R}}\gamma u_r^{[\gamma]}(x)v_r^{[\gamma]}(x) \qquad (44)\\
&\qquad\qquad\times\big((t-s)^2(s-r)^{-2}\wedge 1\big)\,dx\,dr\Big)^2\Big].
\end{aligned}
$$

Now defining

$$
f(r) := 1\wedge (t-s)^2(s-r)^{-2}, \qquad r\in[0,s],
$$

we can rewrite the right-hand side of (44) and then apply Jensen's inequality, the scaling property and finally the fourth moment bound to obtain

$$
\begin{aligned}
&C\|\phi\|_\infty^4\Big(\int_0^s f(r)\,dr\Big)^2\mathbb{E}_{u_0,v_0}\Big[\Big(\frac{1}{\int_0^s f(r)\,dr}\int_0^s\int_{\mathbb{R}}\gamma u_r^{[\gamma]}(x)v_r^{[\gamma]}(x)\,dx f(r)\,dr\Big)^2\Big]\\
&\quad\le C\|\phi\|_\infty^4\int_0^s f(r)\,dr\int_0^s\mathbb{E}_{u_0,v_0}\Big[\Big(\int_{\mathbb{R}} u^{[1]}_{\gamma^2 r}(x)v^{[1]}_{\gamma^2 r}(x)\,dx\Big)^2\Big]f(r)\,dr \qquad (45)\\
&\quad\le C(u_0,v_0,\phi,\varrho)\Big(\int_0^s f(r)\,dr\Big)^2.
\end{aligned}
$$

Now note that if $s \in [\frac{t}{2}, t)$, we have by an explicit calculation

$$\int_0^s f(r)\,dr = \int_0^{2s-t} \left(\frac{t-s}{s-r}\right)^2 dr + \int_{2s-t}^s 1\,dr = 2(t-s) - \frac{(t-s)^2}{s} \le 2(t-s).$$

On the other hand, if $s \in [0, \frac{t}{2}]$, we find that $f(r) = 1$ for all $r \in [0, s]$ and thus

$$\int_0^s f(r)\,dr = s \le t - s.$$

Thus in both cases we obtain from (45) that (44) is bounded by $4C(t-s)^2$.

Combining the fourth moment estimates of the two terms in (42), one can deduce that

$$\mathbb{E}_{u_0,v_0}\left[\left(\int_{[0,t]\times\mathbb{R}} S_{t-r}\phi(x) M^{[\gamma]}(dr, dx) - \int_{[0,s]\times\mathbb{R}} S_{s-r}\phi(x) M^{[\gamma]}(dr, dx)\right)^4\right] \\ \le C(t-s)^2,$$

confirming that the stochastic integral satisfies Kolmogorov's tightness criterion. The proof for tightness of $\langle \phi, v_t^{[\gamma]}\rangle$ is analogous. Finally, noting that

$$\mathbb{E}_{u_0,v_0}\big[\langle \Lambda_t^{[\gamma]} - \Lambda_s^{[\gamma]}, \phi\rangle^2\big] \le \|\phi\|_\infty^2 \mathbb{E}_{u_0,v_0}\left[\left(\gamma \int_s^t \int_{\mathbb{R}} u_r^{[\gamma]}(x) v_r^{[\gamma]}(x)\,dx\,dr\right)^2\right],$$

tightness of $\langle \phi, \Lambda_t^{[\gamma]}\rangle$ follows by the same argument as in (43). $\square$

REMARK 3.5. Note that for the application of Kolmogorov's tightness criterion, it would suffice to control pth moments for any $p > 2$ instead of $p = 4$ in the above proof. However, the duality technique only allows us to estimate integer (mixed) moments. This is the reason for the restriction $\varrho < \varrho(4) = -\frac{1}{\sqrt{2}}$ in the above approach. We believe this restriction to be due to the technique of the proof (duality), however, and expect the above results to hold for *all* $\varrho < \varrho(2) = 0$. Since our approach allows us to control second moments, we can at least show tightness in the weaker Meyer–Zheng topology for all $\varrho < 0$; see Proposition 3.8 below.

PROPOSITION 3.6. *Let* $\varrho < -\frac{1}{\sqrt{2}}$ *and* $(u_0, v_0) = (\mathbb{1}_{\mathbb{R}^-}, \mathbb{1}_{\mathbb{R}^+})$. *Then the family* $(\mu_t^{[\gamma]}, \nu_t^{[\gamma]}, \Lambda_t^{[\gamma]})_{t\ge 0}$ *of measure-valued processes is tight with respect to the Skorokhod topology on the space* $\mathcal{C}_{[0,\infty)}(\mathcal{M}_{\mathrm{tem}}^3)$.

PROOF. By a standard argument known as Jakubowski's criterion (see [13], Theorem 3.1 or [4], Theorem 3.6.4; see also [12], Theorem 3.9.1), tightness of the measure-valued processes follows from tightness of the coordinate processes together with the compact containment condition. We have already checked the latter in Corollary 3.3. Moreover, for each test function $\phi \in \bigcup_{\lambda>0} \mathcal{C}_\lambda^+$, the coordinate processes $\langle \phi, u_t^{[\gamma]} \rangle$, $\langle \phi, v_t^{[\gamma]} \rangle$ and $\langle \phi, \Lambda_t^{[\gamma]} \rangle$ are tight in $\mathcal{C}_{[0,\infty)}(\mathbb{R})$ by Lemma 3.4. Since the family of functions $\{\langle \phi, \cdot \rangle : \phi \in \bigcup_{\lambda>0} \mathcal{C}_\lambda^+\}$ is separating for $\mathcal{M}_{\text{tem}}$ [recall the definition of the topology of $\mathcal{M}_{\text{tem}}$ in (65)], an application of [13], Theorem 3.1, completes the proof. $\square$

REMARK 3.7. Note that the restriction to $\varrho < -\frac{1}{\sqrt{2}}$ and complementary Heaviside initial conditions in the previous proposition comes only from Lemma 3.4 (tightness of coordinate processes). The compact containment condition, on the other hand, holds for *all* $\varrho < 0$ and general initial conditions by Corollary 3.3. As a consequence, any generalization of Lemma 3.4 to other values of $\varrho < 0$ or to more general initial conditions would immediately result in a corresponding strengthening of the conclusion in Proposition 3.6.

3.3. *Meyer–Zheng tightness*. The approach of the previous subsection relies heavily on the assumption of complementary Heaviside initial conditions and that $\varrho < -\frac{1}{\sqrt{2}}$. In particular, only under those conditions we are able to establish the fourth moment bound of Proposition 2.2, which in turn is essential for proving tightness in the space of continuous paths w.r.t. the Skorokhod topology. We will see now that both assumptions can be weakened if we consider tightness w.r.t. the weaker Meyer–Zheng "pseudopath" topology on the space of càdlàg paths. This extension will be of crucial importance in the uniqueness proof in Section 5 below. Indeed, in order to show uniqueness of the limit point, we use self-duality for solutions of the martingale problem $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$. Therefore, we have to construct a dual process, that is, solutions to the martingale problem, for a sufficiently rich class of rapidly decreasing initial conditions. In particular, we will need solutions for initial conditions with nondisjoint support.

We now show that tightness of the family $(\mu_t^{[\gamma]}, \nu_t^{[\gamma]}, \Lambda_t^{[\gamma]})_{t\geq 0}$ of measure-valued processes in the Meyer–Zheng topology is a simple consequence of the estimates already derived in Section 3.1:

PROPOSITION 3.8. *Suppose $\varrho < 0$ and $(u_0, v_0) \in (\mathcal{B}_{\text{rap}}^+)^2$ [resp., $(\mathcal{B}_{\text{tem}}^+)^2$]. Then the family of processes $(\mu_t^{[\gamma]}, \nu_t^{[\gamma]}, \Lambda_t^{[\gamma]})_{t\geq 0}$ from (31)–(32) is tight with respect to the Meyer–Zheng topology on $D_{[0,\infty)}(\mathcal{M}_{\text{rap}}^3)$ [resp., $D_{[0,\infty)}(\mathcal{M}_{\text{tem}}^3)$].*

PROOF. Suppose $(u_0, v_0) \in (\mathcal{B}_{\text{rap}}^+)^2$. We aim at applying [18], Corollary 1.4, which requires us to check the Meyer–Zheng tightness condition [see, e.g., (71) in the Appendix] for the coordinate processes plus a compact containment condition. Let $\phi \in \mathcal{C}_{\text{tem}}^+$, and fix $T > 0$.

For $(\langle \phi, u_t^{[\gamma]} \rangle)_{t\geq 0}$, in view of (33) and since $t \mapsto \langle u_0, S_t\phi \rangle$ has finite variation on $[0, T]$, checking the Meyer–Zheng condition (71) amounts to showing that

$$\sup_{\gamma>0} \sup_{t\in[0,T]} \mathbb{E}_{u_0,v_0}\big[\langle \phi, u_t^{[\gamma]} \rangle\big] < \infty,$$

which is, however, implied immediately by Lemma 3.2. The same argument works for $(\langle\phi, \nu_t^{[\gamma]}\rangle)_{t\geq 0}$. For the increasing process $t\mapsto\langle\phi,\Lambda_t^{[\gamma]}\rangle$, condition (71) reduces to

$$\sup_{\gamma>0}\mathbb{E}_{u_0,v_0}\big[\langle\phi,\Lambda_T^{[\gamma]}\rangle\big]<\infty,$$

which is also ensured by Lemma 3.2.

This shows that the Meyer–Zheng tightness criterion is satisfied for the coordinate processes.

The compact containment condition has already been checked in Corollary 3.3. Applying [18], Corollary 1.4, we are done. The proof for initial conditions in $(\mathcal{B}_{\mathrm{tem}}^{+})^2$ is completely analogous. □

4. Properties of limit points. Having established tightness of our family (31)–(32) of measure-valued processes on path space, we turn to the investigation of the properties of limit points in the respective topologies. Our starting point is the observation that each limit point w.r.t. the Skorokhod topology on $\mathcal{C}$ satisfies the martingale problem $(\mathbf{MP})_{\mu_0,\nu_0}^{\varrho}$ from Definition 1.7. This implies in particular the absolute continuity of the limit measures which is part of our main result Theorem 1.12. We will also see that limit points w.r.t. the (weaker) Meyer–Zheng topology still satisfy the (weaker) martingale problem $(\mathbf{MP}')_{\mu_0,\nu_0}^{\varrho}$ from Definition 1.8, which will be used in the proof of self-duality and uniqueness later on.

The second fundamental observation is the fact that each Meyer–Zheng limit point has the "separation of types" property (16) (see Lemma 4.4 below), which will allow us to prove self-duality and uniqueness without having to specify the quadratic variation of the limit martingales in the martingale problem $(\mathbf{MP})_{\mu_0,\nu_0}^{\varrho}$. For Skorokhod limit points, this will also imply the separation of types in the more intuitive sense (8).

Suppose $(\mu_t,\nu_t,\Lambda_t)_{t\geq 0}$ is a limit point of the family (31)–(32) of measure-valued processes. [Recall that by Proposition 3.6, such a limit point exists under complementary Heaviside initial conditions $(u_0,v_0)=(\mathbb{1}_{\mathbb{R}^-},\mathbb{1}_{\mathbb{R}^+})$ whenever $\varrho<-\frac{1}{\sqrt{2}}$.] By the definition of the finite rate symbiotic branching model $\mathrm{cSBM}(\varrho,\gamma)_{u_0,v_0}$, we know that for every $\gamma>0$, $(\mu_t^{[\gamma]},\nu_t^{[\gamma]})_{t\geq 0}$ is a solution to the martingale problem $(\mathbf{MP})_{\mu_0,\nu_0}^{\varrho}$, with the covariation structure in (10) given by the measure $\Lambda^{[\gamma]}$ from (32). Thus it comes as no surprise that the limit point (μ,ν) of $(\mu^{[\gamma]},\nu^{[\gamma]})$ satisfies the same martingale problem, with the covariation now controlled by the limit point Λ of $\Lambda^{[\gamma]}$:

PROPOSITION 4.1. *Let $\varrho<0$ and $(u_0,v_0)\in(\mathcal{B}_{\mathrm{tem}}^{+})^2$ [resp., $(\mathcal{B}_{\mathrm{rap}}^{+})^2$]. If $(\mu_t,\nu_t,\Lambda_t)_{t\geq 0}\in\mathcal{C}_{[0,\infty)}(\mathcal{M}_{\mathrm{tem}}^3)$ [resp., $\mathcal{C}_{[0,\infty)}(\mathcal{M}_{\mathrm{rap}}^3)$] is a limit point with respect to the Skorokhod topology of the family $(\mu_t^{[\gamma]},\nu_t^{[\gamma]},\Lambda_t^{[\gamma]})_{t\geq 0}$, $\gamma>0$, then $(\mu_t,\nu_t)_{t\geq 0}$ satisfies the martingale problem $(\mathbf{MP})_{u_0,v_0}^{\varrho}$ with the covariation structure in (10) being given by the process $(\Lambda_t)_{t\geq 0}$.*

PROOF. We give the proof for $(u_0, v_0) \in (\mathcal{B}^+_{\mathrm{tem}})^2$, the proof for initial conditions in $\mathcal{B}^+_{\mathrm{rap}}$ being completely analogous.

Consider a sequence $\gamma_k \uparrow \infty$ such that

$$\big(\mu^{[\gamma_k]}_t, \nu^{[\gamma_k]}_t, \Lambda^{[\gamma_k]}_t\big)_{t\ge 0} \underset{k\to\infty}{\overset{\mathcal{L}}{\to}} (\mu_t, \nu_t, \Lambda_t)_{t\ge 0}$$

in $\mathcal{C}_{[0,\infty)}(\mathcal{M}^3_{\mathrm{tem}})$. Let $\phi \in \mathcal{C}^{(2)}_{\mathrm{rap}}$. Then we have also

$$\big(M^{[\gamma_k]}_t(\phi), N^{[\gamma_k]}_t(\phi), \Lambda^{[\gamma_k]}_t\big(\phi^2\big)\big)_{t\ge 0} \underset{k\to\infty}{\overset{\mathcal{L}}{\to}} \big(M_t(\phi), N_t(\phi), \Lambda_t\big(\phi^2\big)\big)_{t\ge 0}$$

in $\mathcal{C}_{[0,\infty)}(\mathbb{R}^3)$, where $(M^{[\gamma_k]}(\phi), N^{[\gamma_k]}(\phi))$ and $(M(\phi), N(\phi))$ denote the pairs of processes from (9) corresponding to $(\mu^{[\gamma_k]}, \nu^{[\gamma_k]})$ and (μ, ν), respectively. We already know that $M^{[\gamma_k]}(\phi)$ and $N^{[\gamma_k]}(\phi)$ are martingales. In order for the weak limit $(M(\phi), N(\phi))$ to be again a martingale, it suffices to show that $(M^{[\gamma_k]}_t(\phi))_{k\in\mathbb{N}}$ and $(N^{[\gamma_k]}_t(\phi))_{k\in\mathbb{N}}$ are uniformly integrable for every fixed t; see, e.g., [19], Theorem 11). Using the Burkholder–Davis–Gundy and Jensen inequalities as well as Lemma 3.2, we obtain for every $1 < p \le 2$ that

$$\begin{aligned}
\sup_{k\in\mathbb{N}} \mathbb{E}\big[\big|M^{[\gamma_k]}_t(\phi)\big|^p\big] &\le C_p \sup_{k\in\mathbb{N}} \mathbb{E}\big[\big(\big[M^{[\gamma_k]}(\phi), M^{[\gamma_k]}(\phi)\big]_t\big)^{p/2}\big] \\
&\le C_p \sup_{k\in\mathbb{N}} \big(\mathbb{E}\big[\big[M^{[\gamma_k]}(\phi), M^{[\gamma_k]}(\phi)\big]_t\big]\big)^{p/2} \\
&= C_p \sup_{k\in\mathbb{N}} \big(\mathbb{E}\big[\big\langle \Lambda^{[\gamma_k]}_t, \phi^2\big\rangle\big]\big)^{p/2} < \infty.
\end{aligned}$$

An analogous assertion holds for $N^{[\gamma_k]}(\phi)$. Hence the weak limit $(M(\phi), N(\phi))$ is again a martingale.

The quadratic (co)variation converges along with the sequence of martingales to the quadratic (co)variation of the limit martingales (see, e.g., [19], Theorem 12). Thus identity (10) on the covariation structure of the limit martingales follows directly from the corresponding identity for the finite rate model, which completes our proof. □

The fact that limit points w.r.t. the Skorokhod topology satisfy the martingale problem $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$ has important consequences: Namely, they also satisfy the (weaker) martingale problem $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$, which will be of crucial importance in the uniqueness proof in Section 5 below. Also, they admit a similar Green function representation as for the finite rate symbiotic branching model. Since these properties are true of *any* solution to the martingale problem $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$, not just limit points of our family of processes, and since the methods to prove them are standard, we have decided put the corresponding proofs into Appendix A.2; see Lemma A.3 and Corollary A.6. At this point, we only prove the absolute continuity of the limit measures which is part of our main result Theorem 1.12. This is in fact also true for any solution to the martingale problem $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$ and is a simple consequence of a general criterion for absolute continuity due to [6]:

PROPOSITION 4.2 (Absolute continuity). *Let* $\varrho \in (-1, 0]$, *and suppose* $(\mu_t, \nu_t)_{t\geq 0}$ *is any solution to the martingale problem* $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$. *Then for each fixed* $t > 0$, *the measures* μ_t *and* ν_t *are absolutely continuous w.r.t. Lebesgue measure*, $\mathbb{P}_{\mu_0,\nu_0}$-*a.s.*

PROOF. Fix $T > 0$. Using the same transformation as in [9], page 24, we define

$$\tilde{\mu}_t := \mu_t, \qquad \tilde{\nu}_t := \frac{1}{\sqrt{1-\varrho^2}}(\nu_t - \varrho\mu_t). \tag{46}$$

Then $(\tilde{\mu}_t, \tilde{\nu}_t)_{t\in[0,T]}$ is a continuous $\mathcal{M}^2$-valued process, where $\mathcal{M}$ denotes the space of Radon measures on $\mathbb{R}$ (note that $\varrho \leq 0$). Using the Green function representation of $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$ from Corollary A.4, it is easily checked that for all non-negative test functions $0 \leq \phi \in \mathcal{C}_c^\infty$, the processes

$$\tilde{M}(\phi)_t := \langle \tilde{\mu}_t, S_{T-t}\phi\rangle, \qquad \tilde{N}(\phi)_t := \langle \tilde{\nu}_t, S_{T-t}\phi\rangle, \qquad t \in [0, T]$$

are martingales with covariance structure

$$\big[\tilde{M}(\phi), \tilde{M}(\phi)\big]_t = \big[\tilde{N}(\phi), \tilde{N}(\phi)\big]_t, \qquad \big[\tilde{M}(\phi), \tilde{N}(\phi)\big]_t = 0, \qquad t \in [0, T].$$

Applying Theorem 57 in [6], we get a.s. absolute continuity of $\tilde{\mu}_T$ and $\tilde{\nu}_T$. Thus the same holds for $\mu_T = \tilde{\mu}_T$ and $\nu_T = \varrho\tilde{\mu}_T + \sqrt{1-\varrho^2}\tilde{\nu}_T$. $\square$

We remind the reader of our convention to use the same symbol for an absolutely continuous measure and its density. Thus if $(\mu_t, \nu_t)_{t\geq 0}$ is any limit point of the family (31), we will write

$$\mu_t(dx) = \mu_t(x)\,dx, \qquad \nu_t(dx) = \nu_t(x)\,dx.$$

Note, however, that although μ_t and ν_t are (as *measures*) elements of the space $\mathcal{M}_{\text{tem}}$ respectively $\mathcal{M}_{\text{rap}}$, their *densities* have no reason to be elements of the function space $\mathcal{B}_{\text{tem}}$, respectively $\mathcal{B}_{\text{rap}}$, let alone $\mathcal{C}_{\text{tem}}$, respectively $\mathcal{C}_{\text{rap}}$, as is the case for solutions to the finite rate symbiotic branching model $\text{cSBM}(\varrho, \gamma)_{u_0,v_0}$.

We now turn to limit points with respect to the Meyer–Zheng topology. It would be nice to prove that every Meyer–Zheng limit point satisfies also the martingale problem $(\mathbf{MP})^{\varrho}_{u_0,v_0}$, but unfortunately we have been unable to show an analogue of Proposition 4.1 for the Meyer–Zheng topology.[3] The reason is the following: While in that case we can still apply [19], Theorem 11, in order to show that the weak limit of the approximating martingales is again a martingale, it is no longer clear that the Meyer–Zheng limit Λ of the quadratic variation processes

[3] As a consequence, we also cannot show a Green function representation or absolute continuity of the limit measures μ_t and ν_t for fixed t.

$\Lambda^{[\gamma]}$ coincides with the quadratic variation of the limit martingales. (In order to apply [19], Theorem 12, we would have to know that Λ is continuous.)

However, we can still prove that any Meyer–Zheng limit point of the family (31)–(32) satisfies the weaker martingale problem $(\mathbf{MP}')^{\varrho}_{\tilde{u}_0,\tilde{v}_0}$ of Definition 1.8:

PROPOSITION 4.3. *Let $\varrho < 0$ and $(u_0, v_0) \in (\mathcal{B}^+_{\text{rap}})^2$ [resp., $(\mathcal{B}^+_{\text{tem}})^2$]. If $(\mu_t, \nu_t, \Lambda_t)_{t\geq 0} \in D_{[0,\infty)}(\mathcal{M}^3_{\text{rap}})$ [resp., $D_{[0,\infty)}(\mathcal{M}^3_{\text{tem}})$] is any limit point with respect to the Meyer–Zheng topology of the family $(\mu^{[\gamma]}_t, \nu^{[\gamma]}_t, \Lambda^{[\gamma]}_t)_{t\geq 0}$, $\gamma > 0$, then $(\mu_t, \nu_t)_{t\geq 0}$ solves the martingale problem $(\mathbf{MP}')^{\varrho}_{u_0,v_0}$, with the process $(\Lambda_t)_{t\geq 0}$ satisfying the requirements of Definition 1.8.*

PROOF. We give the proof for $(u_0, v_0) \in (\mathcal{B}^+_{\text{rap}})^2$.

First, we show that the limit point $(\Lambda_t)_{t\geq 0}$ of the family $(\Lambda^{[\gamma]}_t)_{t\geq 0}$ has the properties required in Definition 1.8. It is clear that $(\Lambda_t)_{t\geq 0}$ is increasing with $\Lambda_0 = 0$. We check condition (14): By [19], Theorem 5 (see also [18], Theorem 1.1(b)), we can find a sequence $\gamma_k \uparrow \infty$ and a set $I \subseteq (0, \infty)$ of full Lebesgue measure such that the finite dimensional distributions of $(\Lambda^{[\gamma_k]}_t)_{t\in I}$ converge weakly to those of $(\Lambda_t)_{t\in I}$ as $k \to \infty$. Fix $t \in I$. Then for all test functions $\phi \in \bigcup_{\lambda>0} \mathcal{C}^+_{-\lambda}$, by estimate (39) in Lemma 3.2 and Fatou's lemma, we have

$$\mathbb{E}_{u_0,v_0}\big[\langle \Lambda_t, \phi\rangle\big] \leq \liminf_{k\to\infty} \mathbb{E}_{u_0,v_0}\big[\big\langle \Lambda^{[\gamma_k]}_t, \phi\big\rangle\big] < \infty.$$

Now use right-continuity and monotonicity of $(\Lambda_t)_{t\geq 0}$ and another application of Fatou's lemma to extend this to all $t > 0$. This shows that $\mathbb{E}_{u_0,v_0}[\Lambda_t(dx)] \in \mathcal{M}_{\text{rap}}$ for all $t > 0$, that is, (14).

It remains to check that for all test functions $\phi, \psi \in (\mathcal{C}^{(2)}_{\text{tem}})^+$, the process

$$\begin{aligned} \tilde{M}_t &:= F(\mu_t, \nu_t, \phi, \psi) - F(\mu_0, \nu_0, \phi, \psi) \\ &\quad - \frac{1}{2}\int_0^t F(\mu_s, \nu_s, \phi, \psi)\langle\langle \mu_s, \nu_s, \Delta\phi, \Delta\psi\rangle\rangle_\varrho \, ds \\ &\quad - 4\big(1-\varrho^2\big)\int_{[0,t]\times\mathbb{R}} F(\mu_s, \nu_s, \phi, \psi)\phi(x)\psi(x)\Lambda(ds, dx), \qquad t \geq 0 \end{aligned} \tag{47}$$

is a martingale. Denote by $\tilde{M}^{[\gamma]}_t$ the same expression but with (μ, ν, Λ) replaced by $(\mu^{[\gamma]}, \nu^{[\gamma]}, \Lambda^{[\gamma]})$. Choosing a sequence $\gamma_k \uparrow \infty$ such that $(\mu^{[\gamma_k]}_t, \nu^{[\gamma_k]}_t, \Lambda^{[\gamma_k]}_t)_{t\geq 0}$ converges to $(\mu_t, \nu_t, \Lambda_t)_{t\geq 0}$ w.r.t. the Meyer–Zheng topology on $D_{[0,\infty)}(\mathcal{M}^3_{\text{rap}})$, we get that $(\tilde{M}^{[\gamma_k]}_t)_{t\geq 0}$ converges to $(\tilde{M}_t)_{t\geq 0}$ w.r.t. the Meyer–Zheng topology on $D_{[0,\infty)}(\mathbb{R})$ as $k \to \infty$. Moreover, by Corollary A.6 we know that $\tilde{M}^{[\gamma]}$ are martingales for each $\gamma > 0$ with quadratic variation

$$8\big(1-\varrho^2\big)\int_{[0,t]\times\mathbb{R}} F\big(\mu^{[\gamma]}_s, \nu^{[\gamma]}_s, \phi, \psi\big)^2\phi(x)\psi(x)\Lambda^{[\gamma]}(ds, dx). \tag{48}$$

Consequently, using the Burkholder–Davis–Gundy inequality and the fact that $|F(\cdot)| \le 1$, we have

$$\begin{aligned}\mathbb{E}_{u_0,v_0}\big[|\tilde{M}_t^{[\gamma]}|^2\big] &\le \mathbb{E}_{u_0,v_0}\big[[\tilde{M}^{[\gamma]}, \tilde{M}^{[\gamma]}]_t\big] \\ &\le 8(1-\varrho^2)\mathbb{E}_{u_0,v_0}\bigg[\int_{[0,t]\times\mathbb{R}} \phi(x)\psi(x)\Lambda^{[\gamma]}(ds,dx)\bigg] \\ &= 8(1-\varrho^2)\mathbb{E}_{u_0,v_0}\big[\langle \Lambda_t^{[\gamma]}, \phi\psi\rangle\big].\end{aligned}$$

By estimate (39) in Lemma 3.2, for each $T > 0$ the previous display is bounded uniformly in $\gamma > 0$ and $t \in [0, T]$. Hence we get

$$\sup_{\gamma>0}\sup_{t\in[0,T]} \mathbb{E}_{u_0,v_0}\big[|\tilde{M}_t^{[\gamma]}|^2\big] < \infty$$

for all $T > 0$. Applying [19], Theorem 11, we infer that the Meyer–Zheng limit $\tilde{M}$ is again a martingale, which completes our argument. $\square$

We now turn to proving the "separation of types" property, that is, the fact that for all limit points the measures μ_t and ν_t are mutually singular for each $t > 0$. In fact, we will prove a slightly stronger assertion, namely (49) below. Its proof relies on the colored particle moment duality of Lemma 2.1 applied to mixed second moments of $(\mu_t^{[\gamma]}, \nu_t^{[\gamma]})$.

LEMMA 4.4 (Separation of types). *Let $\varrho < 0$ and $(u_0, v_0) \in (\mathcal{B}_{\mathrm{rap}}^+)^2$ [resp., $(u_0, v_0) \in (\mathcal{B}_{\mathrm{tem}}^+)^2$]. Suppose that $(\mu_t, \nu_t)_{t\ge 0} \in D_{[0,\infty)}(\mathcal{M}_{\mathrm{rap}}^2)$ [resp., $D_{[0,\infty)}(\mathcal{M}_{\mathrm{tem}}^2)$] is a limit point with respect to the Meyer–Zheng topology of the family of measure-valued processes $(\mu_t^{[\gamma]}, \nu_t^{[\gamma]})_{t\ge 0}$ from (31). Then for each $t > 0$, $x \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$S_{t+\varepsilon}u_0(x)S_{t+\varepsilon}v_0(x) \ge \mathbb{E}_{u_0,v_0}\big[S_\varepsilon\mu_t(x)S_\varepsilon\nu_t(x)\big] \overset{\varepsilon\downarrow 0}{\to} 0. \tag{49}$$

PROOF. We give the proof for $(u_0, v_0) \in (\mathcal{B}_{\mathrm{rap}}^+)^2$, the proof for initial conditions in $\mathcal{B}_{\mathrm{tem}}^+$ being completely analogous. Note that in either case, the left-hand side of (49) is finite by Lemma A.1(a).

Again using [19], Theorem 5, choose a sequence $\gamma_k \uparrow \infty$ and a set $I \subseteq (0, \infty)$ of full Lebesgue measure such that the finite dimensional distributions of $(\mu_t^{[\gamma_k]}, \nu_t^{[\gamma_k]})_{t\in I}$ converge weakly to those of $(\mu_t, \nu_t)_{t\in I}$ as $k \to \infty$. Fix $t \in I$. Then for all test functions ϕ, ψ we have weak convergence

$$\langle \mu_t^{[\gamma_k]}, \phi\rangle\langle \nu_t^{[\gamma_k]}, \psi\rangle \overset{k\uparrow\infty}{\to} \langle \mu_t, \phi\rangle\langle \nu_t, \psi\rangle \tag{50}$$

in $\mathbb{R}$. Thus for each $x \in \mathbb{R}$, letting $\phi(\cdot) := \psi(\cdot) := p_\varepsilon(x - \cdot)$ we obtain weak convergence

$$S_\varepsilon\mu_t^{[\gamma_k]}(x)S_\varepsilon\nu_t^{[\gamma_k]}(x) \overset{k\uparrow\infty}{\to} S_\varepsilon\mu_t(x)S_\varepsilon\nu_t(x).$$

Using Fatou's lemma and the colored particle moment duality in form (19) for mixed second moments, we get since $\varrho < 0$,

$$
\begin{aligned}
&\mathbb{E}_{u_0,v_0}\big[S_\varepsilon\mu_t(x)S_\varepsilon\nu_t(x)\big] \\
&\qquad \le \liminf_{k\to\infty}\mathbb{E}_{u_0,v_0}\big[S_\varepsilon\mu_t^{[\gamma_k]}(x)S_\varepsilon\nu_t^{[\gamma_k]}(x)\big] \\
&\qquad = \liminf_{k\to\infty}\iint dy\,dz p_\varepsilon(x-y)p_\varepsilon(x-z)\mathbb{E}_{u_0,v_0}\big[u_t^{[\gamma_k]}(y)v_t^{[\gamma_k]}(z)\big] \\
&\qquad = \liminf_{k\to\infty}\iint dy\,dz p_\varepsilon(x-y)p_\varepsilon(x-z)\mathbb{E}_{y,z}\big[u_0\big(B_t^{(1)}\big)v_0\big(B_t^{(2)}\big)e^{\gamma_k\varrho L_t^{1,2}}\big] \\
&\qquad = \iint dy\,dz p_\varepsilon(x-y)p_\varepsilon(x-z)\mathbb{E}_{y,z}\big[u_0\big(B_t^{(1)}\big)v_0\big(B_t^{(2)}\big)\mathbb{1}_{\{L_t^{1,2}=0\}}\big],
\end{aligned} \tag{51}
$$

for all $x\in\mathbb{R}$ and $t\in I$, where $(B_t^{(i)})_{t\ge 0}$, $i=1,2$ are independent Brownian motions started at y and z, respectively, and $(L_t^{1,2})_{t\ge 0}$ denotes their intersection local time. It is easy to see that the right-hand side of (51) is continuous in t. Using the fact that I has full Lebesgue measure together with right-continuity of the paths of $(\mu_t,\nu_t)_{t\ge 0}$ and Fatou's lemma, we get estimate (51) for *all* $t>0$. This implies in particular that

$$
\mathbb{E}_{u_0,v_0}\big[S_\varepsilon\mu_t(x)S_\varepsilon\nu_t(x)\big] \le S_{t+\varepsilon}u_0(x)S_{t+\varepsilon}v_0(x) < \infty, \qquad x\in\mathbb{R}, t>0.
$$

Moreover, using Hölder's inequality we have

$$
\begin{aligned}
&\mathbb{E}_{y,z}\big[u_0\big(B_t^{(1)}\big)v_0\big(B_t^{(2)}\big)\mathbb{1}_{\{L_t^{1,2}=0\}}\big] \\
&\qquad \le \big(\mathbb{E}_{y,z}\big[\big(u_0\big(B_t^{(1)}\big)v_0\big(B_t^{(2)}\big)\big)^2\big]\big)^{1/2}\big(\mathbb{P}_{y,z}\big\{L_t^{1,2}=0\big\}\big)^{1/2} \\
&\qquad = \big(S_t u_0^2(y)S_t v_0^2(z)\big)^{1/2}\big(\mathbb{P}_{y,z}\big\{L_t^{1,2}=0\big\}\big)^{1/2}.
\end{aligned}
$$

Observe that the right-hand side of the previous display tends to 0 as $(y,z)\to(x,x)$: Assume without loss of generality that $y<z$, and let B be a Brownian motion starting at $y-z<0$ with local time L^0 at 0. Using the fact that $L_t^{1,2}\stackrel{d}{=}\frac{1}{2}L_{2t}^0$ together with Lemma A.7 and the reflection principle (see, e.g., [20], Theorem 2.21), we obtain for $(y,z)\to(x,x)$ that

$$
\begin{aligned}
\mathbb{P}_{y,z}\big\{L_t^{1,2}=0\big\} &= \mathbb{P}_{y-z}\big\{L_{2t}^0=0\big\} = \mathbb{P}_{y-z}\big\{M_{2t}^+=0\big\} = \mathbb{P}_{y-z}\{M_{2t}\le 0\} \\
&= \mathbb{P}_0\{M_{2t}\le z-y\} = 1-2\mathbb{P}_0\{B_{2t}>z-y\} \\
&\to 1-2\mathbb{P}_0\{B_{2t}>0\} = 0.
\end{aligned}
$$

Since on the other hand clearly $\mathbb{E}_{x,x}[u_0(B_t^{(1)})v_0(B_t^{(2)})\mathbb{1}_{\{L_t^{1,2}=0\}}]=0$ for $t>0$, this shows that the mapping

$$
(y,z)\mapsto \mathbb{E}_{y,z}\big[u_0\big(B_t^{(1)}\big)v_0\big(B_t^{(2)}\big)\mathbb{1}_{\{L_t^{1,2}=0\}}\big]
$$

is continuous at all points (x, x) of the diagonal in $\mathbb{R}^2$, where it takes the value 0. As a consequence, the right-hand side of (51) converges to 0 as $\varepsilon \downarrow 0$, giving (49) for all $t > 0$ and $\in \mathbb{R}$. $\square$

Of course, Lemma 4.4 holds in particular also for limit points in the stronger Skorokhod topology. In this case, together with absolute continuity of the limiting measures from Proposition 4.2, it implies the separation of types in the intuitive sense (8), that is, the mutual singularity of the limiting measures (μ_t, ν_t) for fixed $t > 0$:

COROLLARY 4.5 (Separation of types). *Let $\varrho < 0$ and $(u_0, v_0) \in (\mathcal{B}^+_{\mathrm{rap}})^2$ [resp., $(u_0, v_0) \in (\mathcal{B}^+_{\mathrm{tem}})^2$]. If $(\mu_t, \nu_t)_{t\geq 0} \in \mathcal{C}_{[0,\infty)}(\mathcal{M}^2_{\mathrm{rap}})$ [resp., $\mathcal{C}_{[0,\infty)}(\mathcal{M}^2_{\mathrm{tem}})$] is a limit point with respect to the Skorokhod topology of the family (31), then for each $t > 0$ the measures μ_t and ν_t (which are known to be absolutely continuous by Proposition 4.2) are mutually singular: We have*

$$\mathbb{E}_{u_0,v_0}\left[\int_{\mathbb{R}} \mu_t(x)\nu_t(x)\,dx\right] = 0 \tag{52}$$

and thus also

$$\mu_t(\cdot)\nu_t(\cdot) = 0, \qquad \mathbb{P}_{u_0,v_0} \otimes \ell\text{-a.s.}$$

PROOF. Let $0 \leq \varphi \in \mathcal{C}_c^\infty$. By differentiation theory for measures (see, e.g., [25], Theorem 8.6), we have $\mathbb{P}_{u_0,v_0}$-a.s.

$$\big(S_\varepsilon\mu_t(x), S_\varepsilon\nu_t(x)\big) \overset{\varepsilon\downarrow 0}{\to} \big(\mu_t(x), \nu_t(x)\big) \qquad \text{for } \ell\text{-a.e. } x \in \mathbb{R}.$$

Using again Fatou's lemma and Fubini's theorem, we get

$$\begin{aligned} &\mathbb{E}_{u_0,v_0}\left[\int_{\mathbb{R}} \mu_t(x)\nu_t(x)\varphi(x)\,dx\right] \\ &\quad = \mathbb{E}_{u_0,v_0}\left[\int_{\mathbb{R}} \lim_{\varepsilon\downarrow 0} S_\varepsilon\mu_t(x)S_\varepsilon\nu_t(x)\varphi(x)\,dx\right] \\ &\quad \leq \liminf_{\varepsilon\downarrow 0} \int_{\mathbb{R}} \mathbb{E}_{u_0,v_0}[S_\varepsilon\mu_t(x)S_\varepsilon\nu_t(x)]\varphi(x)\,dx. \end{aligned} \tag{53}$$

By Lemma 4.4 the integrand in the integral $\int_{\mathbb{R}} \cdots dx$ on the right-hand side of the previous display converges to 0 as $\varepsilon \downarrow 0$ pointwise in $x \in \mathbb{R}$ and for $\varepsilon \in [0, 1]$ is dominated by the integrable function

$$\varphi(x) \sup_{s\in[0,t+1]} \{S_s u_0(x) S_s v_0(x)\}$$

[note Lemma A.1(a)]. By dominated convergence, $\mathbb{E}[\int_{\mathbb{R}} \mu_t(x)\nu_t(x)\varphi(x)\,dx] = 0$, and since $0 \leq \varphi \in \mathcal{C}_c^\infty$ was arbitrary, our proof is complete. $\square$

5. Self-duality and uniqueness. In this section, we establish uniqueness for the martingale problem $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$ [and thus also for the stronger martingale problem $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$] subject to the restriction that the solutions have the "separation of types" property (49). Recall from the Introduction that these martingale problems are not well posed without putting some restrictions on the solutions, and that for the finite rate symbiotic branching model $\mathrm{cSBM}(\varrho,\gamma)_{u_0,v_0}$ uniqueness is established by prescribing the structure of the quadratic variation process $(\Lambda)_{t\ge 0}$. In [11], Proposition 5, this is proved via an exponential self-duality. Our first goal in this section is to extend this self-duality to solutions of the martingale problem $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$ satisfying the said condition, circumventing an explicit specification of the quadratic variation. We have the following result:

PROPOSITION 5.1. *Let $\varrho\in(-1,1)$. Fix (possibly random) initial conditions $(\mu_0,\nu_0)\in\mathcal{M}^2_{\mathrm{tem}}$ and (deterministic) initial conditions $(\tilde\mu_0,\tilde\nu_0)\in(\mathcal{B}^+_{\mathrm{rap}})^2$. Suppose that $(\mu_t,\nu_t,)_{t\ge0}\in D_{[0,\infty)}(\mathcal{M}^2_{\mathrm{tem}})$ respectively $(\tilde\mu_t,\tilde\nu_t)_{t\ge0}\in D_{[0,\infty)}(\mathcal{M}^2_{\mathrm{rap}})$ are solutions to the martingale problem $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$ respectively $(\mathbf{MP}')^{\varrho}_{\tilde\mu_0,\tilde\nu_0}$. Further, assume that the solutions satisfy the "separation of types" property in the sense that for Lebesgue-a.e. $t\in(0,\infty)$ and all $x\in\mathbb{R}$, we have*

$$\mathbb{E}_{\mu_0,\nu_0}\big[S_\varepsilon\mu_t(x)S_\varepsilon\nu_t(x)\big]\overset{\varepsilon\downarrow0}{\to}0\quad\text{and}\quad\mathbb{E}_{\tilde\mu_0,\tilde\nu_0}\big[S_\varepsilon\tilde\mu_t(x)S_\varepsilon\tilde\nu_t(x)\big]\overset{\varepsilon\downarrow0}{\to}0.\tag{54}$$

Moreover, assume that for each $T>0$ we have

$$\begin{aligned}&\sup_{t\in[0,T],\varepsilon\in[0,1]}\mathbb{E}_{\mu_0,\nu_0}\big[S_\varepsilon\mu_t(\cdot)S_\varepsilon\nu_t(\cdot)\big]\in\mathcal{B}^+_{\mathrm{tem}},\\&\sup_{t\in[0,T],\varepsilon\in[0,1]}\mathbb{E}_{\tilde\mu_0,\tilde\nu_0}\big[S_\varepsilon\tilde\mu_t(\cdot)S_\varepsilon\tilde\nu_t(\cdot)\big]\in\mathcal{B}^+_{\mathrm{rap}}.\end{aligned}\tag{55}$$

Then the following approximate self-duality holds for the processes $(\mu_t,\nu_t)_{t\ge0}$ and $(\tilde\mu_t,\tilde\nu_t)_{t\ge0}$, involving the function F as in (13): for $T>0$,

$$\int_0^T\mathbb{E}\big[F(\mu_t,\nu_t,\tilde\mu_0,\tilde\nu_0)\big]\,dt=\lim_{\varepsilon\downarrow0}\int_0^T\mathbb{E}\big[F(S_\varepsilon\mu_0,S_\varepsilon\nu_0,\tilde\mu_t,\tilde\nu_t)\big]\,dt.\tag{56}$$

Moreover, for $(\mu_0,\nu_0)\in(\mathcal{B}^+_{\mathrm{tem}})^2$, we have the self-duality

$$\mathbb{E}_{\mu_0,\nu_0}\big[F(\mu_t,\nu_t,\tilde\mu_0,\tilde\nu_0)\big]=\mathbb{E}_{\tilde\mu_0,\tilde\nu_0}\big[F(\mu_0,\nu_0,\tilde\mu_t,\tilde\nu_t)\big],\qquad t\ge0.\tag{57}$$

The general strategy of the proof is similar to that of the results in [12], Section 4.4; however none of those results is directly applicable in our case. Also, we employ the same spatial smoothing procedure using the heat kernel as in the proof of [1], Proposition 1.

PROOF OF PROPOSITION 5.1. By Definition 1.8, there exist increasing processes $(\Lambda_t)_{t\ge0}\in D_{[0,\infty)}(\mathcal{M}_{\mathrm{tem}})$ and $(\tilde\Lambda_t)_{t\ge0}\in D_{[0,\infty)}(\mathcal{M}_{\mathrm{rap}})$, with $\Lambda_0=\tilde\Lambda_0=0$

and satisfying (14), such that for all test functions, expression (15) is a martingale. For the purposes of the proof, we may assume that (μ, ν, Λ) and $(\tilde{\mu}, \tilde{\nu}, \tilde{\Lambda})$ are defined on a common sample space Ω and are independent of each other. The corresponding probability, respectively expectation on Ω, will be denoted by $\mathbb{P}$, respectively $\mathbb{E}$.

Observe that by the definition of $\langle\langle\cdot\rangle\rangle_\varrho$ [recall (12)] and the symmetry of the heat kernel, we have for each $\varepsilon > 0$ and $\phi, \psi \in (\mathcal{C}^{(2)}_{\mathrm{rap}})^+$,

$$\langle\langle S_\varepsilon\mu_t, S_\varepsilon\nu_t, \phi, \psi\rangle\rangle_\varrho = \langle\langle \mu_t, \nu_t, S_\varepsilon\phi, S_\varepsilon\psi\rangle\rangle_\varrho,$$

$$\langle\langle S_\varepsilon\mu_t, S_\varepsilon\nu_t, \Delta\phi, \Delta\psi\rangle\rangle_\varrho = \langle\langle \mu_t, \nu_t, \Delta S_\varepsilon\phi, \Delta S_\varepsilon\psi\rangle\rangle_\varrho.$$

Thus by taking expectations in (15) with $(S_\varepsilon\phi, S_\varepsilon\psi)$ in place of (ϕ, ψ), we get

$$\begin{aligned}
&\mathbb{E}\big[F(S_\varepsilon\mu_t, S_\varepsilon\nu_t, \phi, \psi) - F(S_\varepsilon\mu_0, S_\varepsilon\nu_0, \phi, \psi)\big] \\
&\quad = \frac{1}{2}\mathbb{E}\bigg[\int_0^t F(S_\varepsilon\mu_s, S_\varepsilon\nu_s, \phi, \psi)\langle\langle S_\varepsilon\mu_s, S_\varepsilon\nu_s, \Delta\phi, \Delta\psi\rangle\rangle_\varrho\, ds\bigg] \\
&\qquad + 4(1-\varrho^2)\mathbb{E}\bigg[\int_{[0,t]\times\mathbb{R}} F(S_\varepsilon\mu_s, S_\varepsilon\nu_s, \phi, \psi) S_\varepsilon\phi(x) S_\varepsilon\psi(x)\Lambda(ds, dx)\bigg]
\end{aligned} \tag{58}$$

for all $\varepsilon > 0$ and $\phi, \psi \in (\mathcal{C}^{(2)}_{\mathrm{rap}})^+$. An analogous assertion holds for $(\tilde{\mu}, \tilde{\nu}, \tilde{\Lambda})$ if $\phi, \psi \in (\mathcal{C}^{(2)}_{\mathrm{tem}})^+$.

Now fix $T > 0$, and for $t, s \in [0, T]$, $\varepsilon > 0$, let

$$f_\varepsilon(t, s) := \mathbb{E}\big[F(S_\varepsilon\mu_t, S_\varepsilon\nu_t, S_\varepsilon\tilde{\mu}_s, S_\varepsilon\tilde{\nu}_s)\big].$$

Observe that this function is well defined since $S_\varepsilon\mu_t$ and $S_\varepsilon\nu_t$, respectively $S_\varepsilon\tilde{\mu}_t$ and $S_\varepsilon\tilde{\nu}_t$, are in $(\mathcal{C}^{(2)}_{\mathrm{tem}})^+$, respectively $(\mathcal{C}^{(2)}_{\mathrm{rap}})^+$; see, for example, Corollary A.2(b). Then

$$\begin{aligned}
&\int_0^T \big(f_\varepsilon(r, 0) - f_\varepsilon(0, r)\big)\, dr \\
&\quad = \int_0^T \big(f_\varepsilon(T-r, r) - f_\varepsilon(0, r)\big)\, dr - \int_0^T \big(f_\varepsilon(r, T-r) - f_\varepsilon(r, 0)\big)\, dr \\
&\quad = \int_0^T \big(\mathbb{E}\big[F(S_\varepsilon\mu_{T-r}, S_\varepsilon\nu_{T-r}, S_\varepsilon\tilde{\mu}_r, S_\varepsilon\tilde{\nu}_r) \\
&\qquad\qquad - F(S_\varepsilon\mu_0, S_\varepsilon\nu_0, S_\varepsilon\tilde{\mu}_r, S_\varepsilon\tilde{\nu}_r)\big]\big)\, dr \\
&\qquad - \int_0^T \big(\mathbb{E}\big[F(S_\varepsilon\mu_r, S_\varepsilon\nu_r, S_\varepsilon\tilde{\mu}_{T-r}, S_\varepsilon\tilde{\nu}_{T-r}) \\
&\qquad\qquad - F(S_\varepsilon\mu_r, S_\varepsilon\nu_r, S_\varepsilon\tilde{\mu}_0, S_\varepsilon\tilde{\nu}_0)\big]\big)\, dr.
\end{aligned}$$

Now we use (58) [resp., the analogous identity for $(\tilde{\mu}, \tilde{\nu}, \tilde{\Lambda})$] with t replaced by $T - r$ for each $r \in [0, T]$ and $(\phi, \psi) := (S_\varepsilon\tilde{\mu}_r, S_\varepsilon\tilde{\nu}_r)$ [resp., $(\phi, \psi) :=$

$(S_\varepsilon\mu_r, S_\varepsilon\nu_r)]$ to see that the previous display is equal to

$$\begin{aligned}
&\frac{1}{2}\int_0^T \mathbb{E}\Big[\int_0^{T-r} F(S_\varepsilon\mu_s, S_\varepsilon\nu_s, S_\varepsilon\tilde{\mu}_r, S_\varepsilon\tilde{\nu}_r)\langle\!\langle S_\varepsilon\mu_s, S_\varepsilon\nu_s, \Delta S_\varepsilon\tilde{\mu}_r, \Delta S_\varepsilon\tilde{\nu}_r\rangle\!\rangle_\varrho\, ds\Big]\, dr \\
&\qquad + 4(1-\varrho^2)\int_0^T \mathbb{E}\Big[\int_{[0,T-r]\times\mathbb{R}} F(S_\varepsilon\mu_s, S_\varepsilon\nu_s, S_\varepsilon\tilde{\mu}_r, S_\varepsilon\tilde{\nu}_r) \\
&\qquad\qquad\qquad\qquad \times S_{2\varepsilon}\tilde{\mu}_r(x)S_{2\varepsilon}\tilde{\nu}_r(x)\Lambda(ds,dx)\Big]\, dr \\
&\qquad - \frac{1}{2}\int_0^T \mathbb{E}\Big[\int_0^{T-r} F(S_\varepsilon\mu_r, S_\varepsilon\nu_r, S_\varepsilon\tilde{\mu}_s, S_\varepsilon\tilde{\nu}_s) \\
&\qquad\qquad\qquad \times \langle\!\langle \Delta S_\varepsilon\mu_r, \Delta S_\varepsilon\nu_r, S_\varepsilon\tilde{\mu}_s, S_\varepsilon\tilde{\nu}_s\rangle\!\rangle_\varrho\, ds\Big]\, dr \\
&\qquad - 4(1-\varrho^2)\int_0^T \mathbb{E}\Big[\int_{[0,T-r]\times\mathbb{R}} F(S_\varepsilon\mu_r, S_\varepsilon\nu_r, S_\varepsilon\tilde{\mu}_s, S_\varepsilon\tilde{\nu}_s) \\
&\qquad\qquad\qquad\qquad \times S_{2\varepsilon}\mu_r(x)S_{2\varepsilon}\nu_r(x)\tilde{\Lambda}(ds,dx)\Big]\, dr.
\end{aligned}$$

Observe that due to symmetry of the Laplacian and Fubini's theorem, the first and third term of the last display cancel. Thus we have shown that

$$\begin{aligned}
&\int_0^T \big(f_\varepsilon(r,0) - f_\varepsilon(0,r)\big)\, dr \\
&\quad = 4(1-\varrho^2)\Big(\int_0^T \mathbb{E}\Big[\int_{[0,T-r]\times\mathbb{R}} F(S_\varepsilon\mu_s, S_\varepsilon\nu_s, S_\varepsilon\tilde{\mu}_r, S_\varepsilon\tilde{\nu}_r) \\
&\qquad\qquad\qquad\qquad \times S_{2\varepsilon}\tilde{\mu}_r(x)S_{2\varepsilon}\tilde{\nu}_r(x)\Lambda(ds,dx)\Big]\, dr \\
&\qquad\quad - \int_0^T \mathbb{E}\Big[\int_{[0,T-r]\times\mathbb{R}} F(S_\varepsilon\mu_r, S_\varepsilon\nu_r, S_\varepsilon\tilde{\mu}_s, S_\varepsilon\tilde{\nu}_s) \\
&\qquad\qquad\qquad\qquad \times S_{2\varepsilon}\mu_r(x)S_{2\varepsilon}\nu_r(x)\tilde{\Lambda}(ds,dx)\Big]\, dr\Big).
\end{aligned}$$

We will show that each term in the difference on the right-hand side of the previous display converges to 0 as $\varepsilon \downarrow 0$. Consider the first term: Since $|F(\cdot)| \le 1$, it is bounded in absolute value up to a constant by

$$\begin{aligned}
&\int_0^T \mathbb{E}\Big[\int_{[0,T-r]\times\mathbb{R}} S_{2\varepsilon}\tilde{\mu}_r(x)S_{2\varepsilon}\tilde{\nu}_r(x)\Lambda(ds,dx)\Big]\, dr \\
&\quad = \int_0^T \mathbb{E}_{\mu_0,\nu_0}\Big[\int_{\mathbb{R}} \mathbb{E}_{\tilde{\mu}_0,\tilde{\nu}_0}\big[S_{2\varepsilon}\tilde{\mu}_r(x)S_{2\varepsilon}\tilde{\nu}_r(x)\big]\Lambda_{T-r}(dx)\Big]\, dr \\
&\quad \le \int_0^T \mathbb{E}_{\mu_0,\nu_0}\Big[\int_{\mathbb{R}} \mathbb{E}_{\tilde{\mu}_0,\tilde{\nu}_0}\big[S_{2\varepsilon}\tilde{\mu}_r(x)S_{2\varepsilon}\tilde{\nu}_r(x)\big]\Lambda_T(dx)\Big]\, dr.
\end{aligned}$$

By assumption (54), the integrand in the above display converges to 0 for all $x \in \mathbb{R}$ and almost all $r \in [0, T]$ as $\varepsilon \downarrow 0$. Hence using conditions (14) and (55) together with dominated convergence, we are done. The argument for the second term in the difference is completely analogous. Thus in view of the definition of f_ε, we have shown that

$$\lim_{\varepsilon \downarrow 0} \int_0^T \big(\mathbb{E}\big[F(\mu_t, \nu_t, S_\varepsilon\tilde{\mu}_0, S_\varepsilon\tilde{\nu}_0)\big] - \mathbb{E}\big[F(S_\varepsilon\mu_0, S_\varepsilon\nu_0, \tilde{\mu}_t, \tilde{\nu}_t)\big]\big)\, dt = 0. \tag{59}$$

Since $\tilde{\mu}_0$ and $\tilde{\nu}_0$ are assumed to be in $\mathcal{B}^+_{\mathrm{rap}}$, using estimate (68) in Lemma A.1(a) and dominated convergence, it is easy to see that

$$\int_0^T \mathbb{E}\big[F(\mu_t, \nu_t, S_\varepsilon\tilde{\mu}_0, S_\varepsilon\tilde{\nu}_0)\big]\, dt \to \int_0^T \mathbb{E}\big[F(\mu_t, \nu_t, \tilde{\mu}_0, \tilde{\nu}_0)\big]\, dt,$$

as $\varepsilon \downarrow 0$. [Note that the same argument cannot in general be employed for the second term in the difference in (59): Since μ_0 and ν_0 are only assumed to be in $\mathcal{M}_{\mathrm{tem}}$ and not in $\mathcal{B}^+_{\mathrm{tem}}$, we do not have (68) but only the weaker estimate (70) in Lemma A.1(b), which, however, is not sufficient for dominated convergence here.] Thus (56) is proved.

If $(\mu_0, \nu_0) \in (\mathcal{B}^+_{\mathrm{tem}})^2$, we can again use estimate (68) and dominated convergence to conclude that also

$$\int_0^T \mathbb{E}\big[F(S_\varepsilon\mu_0, S_\varepsilon\nu_0, \tilde{\mu}_t, \tilde{\nu}_t)\big]\, dt \to \int_0^T \mathbb{E}\big[F(\mu_0, \nu_0, \tilde{\mu}_t, \tilde{\nu}_t)\big]\, dt$$

as $\varepsilon \downarrow 0$. Thus in this case we get from (59) that

$$\begin{aligned}
&\int_0^T \big(\mathbb{E}\big[F(\mu_t, \nu_t, \tilde{\mu}_0, \tilde{\nu}_0)\big] - \mathbb{E}\big[F(\mu_0, \nu_0, \tilde{\mu}_t, \tilde{\nu}_t)\big]\big)\, dt \\
&\quad = \lim_{\varepsilon \downarrow 0} \int_0^T \big(\mathbb{E}\big[F(\mu_t, \nu_t, S_\varepsilon\tilde{\mu}_0, S_\varepsilon\tilde{\nu}_0)\big] - \mathbb{E}\big[F(S_\varepsilon\mu_0, S_\varepsilon\nu_0, \tilde{\mu}_t, \tilde{\nu}_t)\big]\big)\, dt = 0
\end{aligned}$$

for each $T > 0$. Since the processes $(\mu_t, \nu_t)_{t \geq 0}$ and $(\tilde{\mu}_t, \tilde{\nu}_t)_{t \geq 0}$ are assumed càdlàg, it is readily checked that the same is true of the integrand in the last display. Differentiating, we obtain the self-duality (57) for all $t \geq 0$. $\square$

PROPOSITION 5.2 (Uniqueness). *Fix $\varrho \in (-1, 0)$ and (possibly random) initial conditions $(\mu_0, \nu_0) \in \mathcal{M}^2_{\mathrm{tem}}$ or $\mathcal{M}^2_{\mathrm{rap}}$. Then there is at most one solution (μ, ν, Λ) to the martingale problem $(\mathbf{MP}')^{\varrho}_{\mu_0, \nu_0}$ satisfying the "separation of types" property* (16).

PROOF. Let (μ, ν, Λ) and (μ', ν', Λ') be any two solutions to $(\mathbf{MP}')^{\varrho}_{\mu_0, \nu_0}$, with (possibly random) initial conditions $(\mu_0, \nu_0) \in \mathcal{M}^2_{\mathrm{tem}}$, which satisfy condition (16). By Propositions 3.8 and 4.3, we know that for any $(\tilde{u}_0, \tilde{v}_0) \in$

$(\mathcal{B}_{\mathrm{rap}}^{+})^2$, there exists a solution $(\tilde{\mu}_t, \tilde{v}_t)_t \in D_{[0,\infty)}(\mathcal{M}_{\mathrm{rap}}^2)$ of the martingale problem $(\mathbf{MP}')_{\tilde{u}_0,\tilde{v}_0}^{\varrho}$, which by Lemma 4.4 satisfies also the "separation of types" condition (16). Note that (16) ensures that both assumptions (54) and (55) of Proposition 5.1 hold [for (55), use Lemma A.1(a) in the Appendix]. Consequently, we can apply the self-duality of Proposition 5.1 to conclude that for all $(\tilde{u}_0, \tilde{v}_0) \in (\mathcal{B}_{\mathrm{rap}}^{+})^2$, we have

$$\begin{aligned}\int_0^T \mathbb{E}[F(\mu_t, \nu_t, \tilde{u}_0, \tilde{v}_0)]\,dt &= \lim_{\varepsilon\downarrow 0}\int_0^T \mathbb{E}[F(S_\varepsilon\mu_0, S_\varepsilon\nu_0, \tilde{\mu}_t, \tilde{v}_t)]\,dt\\ &= \int_0^T \mathbb{E}[F(\mu_t', \nu_t', \tilde{u}_0, \tilde{v}_0)]\,dt, \qquad T \ge 0.\end{aligned}$$

Differentiating, we get

$$\mathbb{E}[F(\mu_t, \nu_t, \tilde{u}_0, \tilde{v}_0)] = \mathbb{E}[F(\mu_t', \nu_t', \tilde{u}_0, \tilde{v}_0)] \tag{60}$$

first for Lebesgue-a.e. $t > 0$ and then, by right-continuity, for all $t > 0$. Since for $\varrho \in (-1, 1)$, the family of functions $\{F(\cdot, \cdot; \tilde{u}_0, \tilde{v}_0) : (\tilde{u}_0, \tilde{v}_0) \in (\mathcal{B}_{\mathrm{rap}}^{+})^2\}$ is measure-determining for $\mathcal{M}_{\mathrm{tem}}^2$ (see, e.g., [7], proof of Lemma 3.1), it follows that the one-dimensional distributions of (μ, ν) and (μ', ν') coincide. Arguing as in [2], proof of Theorem VI.3.2, this can be easily extended to the finite-dimensional distributions; thus (μ, ν) and (μ', ν') have the same law on $D_{[0,\infty)}(\mathcal{M}_{\mathrm{tem}}^2)$.

The proof for initial conditions in $\mathcal{M}_{\mathrm{rap}}$ is completely analogous. □

6. Bounds on the width of the interface. In this section, we will prove the pth moment estimate on the approximate width of the interface $(R_t(\varepsilon) - L_t(\varepsilon))$ of Theorem 1.15 using the fourth moment estimates established in Proposition 2.2. Since we are interested in the dependence of the constants on γ, we write as above $(u_t^{[\gamma]}, v_t^{[\gamma]})$ for a solution of cSBM(ϱ, γ) and moreover define

$$L_t^{[\gamma]}(\varepsilon) = \inf\left\{x : \int_{-\infty}^x u_t^{[\gamma]}(y)v_t^{[\gamma]}(y)\,dy \ge \varepsilon\right\} \wedge R(u_t^{[\gamma]}, v_t^{[\gamma]})$$

and

$$R_t^{[\gamma]}(\varepsilon) = \sup\left\{x : \int_x^{\infty} u_t^{[\gamma]}(y)v_t^{[\gamma]}(y)\,dy \ge \varepsilon\right\} \vee L(u_t^{[\gamma]}, v_t^{[\gamma]}).$$

PROOF OF THEOREM 1.15. First, we prove the statement for the case $\gamma = 1$, and at the end we will deduce the statement for general γ using a scaling argument. Therefore, we write $(u_t, v_t) := (u_t^{[1]}, v_t^{[1]})$ and $(R_t, L_t) := (R_t^{[1]}, L_t^{[1]})$. We recall from (22) and (28) in the proof of Proposition 2.2 (for the system with branching rate 1) that since $\varrho < -\frac{1}{\sqrt{2}}$, for any $\tilde{\varepsilon} \in (0, \frac{1}{2})$ there exists a constant $C(\varrho, \tilde{\varepsilon}) > 0$

such that for all $z > 0$ and $t \geq 0$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{1}_{\mathbb{R}^-},\mathbb{1}_{\mathbb{R}^+}}&\left[\int_{\mathbb{R}} u_t(x)v_t(x)u_t(x+z)v_t(x+z)\,dx\right]\\
&= \mathbb{E}_{\mathbb{1}_{\mathbb{R}^-},\mathbb{1}_{\mathbb{R}^+}}\left[\int_{\mathbb{R}} u_t(x)v_t(x)u_t(x-z)v_t(x-z)\,dx\right]\\
&\leq C(\varrho,\tilde{\varepsilon})\bigl(1\wedge z^{-2(1-\tilde{\varepsilon})}\bigr).
\end{aligned}$$

Defining for $q \in (0,1)$

$$I_q(t) := \int_{\mathbb{R}}\int_{\mathbb{R}} |x-y|^q u_t(x)v_t(x)u_t(y)v_t(y)\,dx\,dy$$

and choosing $\tilde{\varepsilon} = \frac{1}{4}(1-q)$, the estimate in (61) shows that for all $t \geq 0$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{1}_{\mathbb{R}^-},\mathbb{1}_{\mathbb{R}^+}}\bigl[I_q(t)\bigr] &= 2\int_0^\infty |z|^q \mathbb{E}_{\mathbb{1}_{\mathbb{R}^-},\mathbb{1}_{\mathbb{R}^+}}\left[\int_{\mathbb{R}} u_t(x)v_t(x)u_t(x+z)v_t(x+z)\,dx\right]dz\\
&\leq C\left(\varrho,\frac{1}{4}(1-q)\right)\int_0^\infty z^q\bigl(1\wedge z^{-2(1-\tilde{\varepsilon})}\bigr)\,dz\\
&\leq C\left(\varrho,\frac{1}{4}(1-q)\right)\left(1+\int_1^\infty z^{-2+2\tilde{\varepsilon}+q}\,dz\right)\\
&= C\left(\varrho,\frac{1}{4}(1-q)\right)\left(1+\frac{2}{1-q}\right) < \infty
\end{aligned}$$

since by our choice of $\tilde{\varepsilon}$ we have $2\tilde{\varepsilon}+q = \frac{1}{2}+\frac{1}{2}q < 1$. Fix $z > 0$. Then on the event that $R_t(\varepsilon) - L_t(\varepsilon) > z$, we can estimate using the definition of $L_t(\varepsilon)$, $R_t(\varepsilon)$ that

$$I_q(t) \geq z^q \int_{-\infty}^{L_t(\varepsilon)} u_t(x)v_t(x)\,dx \int_{R_t(\varepsilon)}^\infty u_t(y)v_t(y)\,dy \geq \varepsilon^2 z^q.$$

Hence we can conclude that

$$\begin{aligned}
\mathbb{P}_{\mathbb{1}_{\mathbb{R}^-},\mathbb{1}_{\mathbb{R}^+}}\bigl\{R_t(\varepsilon)-L_t(\varepsilon) > z\bigr\} &\leq \varepsilon^{-2}z^{-q}\mathbb{E}_{\mathbb{1}_{\mathbb{R}^-},\mathbb{1}_{\mathbb{R}^+}}\bigl[I_q(t)\mathbb{1}_{\{R_t(\varepsilon)-L_t(\varepsilon)>z\}}\bigr]\\
&\leq \varepsilon^{-2}z^{-q}\mathbb{E}_{\mathbb{1}_{\mathbb{R}^-},\mathbb{1}_{\mathbb{R}^+}}\bigl[I_q(t)\bigr] \leq \tilde{C}(\varrho,q)\varepsilon^{-2}z^{-q},
\end{aligned}$$

where we define $\tilde{C}(\varrho,q) := C(\varrho,\frac{1}{4}(1-q))(1+\frac{2}{1-q})$. Thus, we have by Fubini that for any $0 < p < q < 1$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{1}_{\mathbb{R}^-},\mathbb{1}_{\mathbb{R}^+}}\bigl[\bigl(\bigl(R_t(\varepsilon)-L_t(\varepsilon)\bigr)^+\bigr)^p\bigr] &= p\int_0^\infty z^{p-1}\mathbb{P}_{\mathbb{1}_{\mathbb{R}^-},\mathbb{1}_{\mathbb{R}^+}}\bigl\{R_t(\varepsilon)-L_t(\varepsilon) > z\bigr\}\,dz\\
&\leq p\int_0^\infty z^{p-1}\bigl(1\wedge \tilde{C}(\varrho,q)\varepsilon^{-2}z^{-q}\bigr)\,dz\\
&= p\bigl(\tilde{C}(\varrho,q)\varepsilon^{-2}\bigr)^{p/q}\int_0^\infty z^{p-1}\bigl(1\wedge z^{-q}\bigr)\,dz.
\end{aligned}$$

Therefore, for any $\delta \in (0, 2(1-p))$, by choosing $q = \frac{2p}{2-\delta} \in (p, 1)$ we can find a constant $C(\varrho, p, \delta)$ such that for all $t \geq 0$,

$$\mathbb{E}_{\mathbb{1}_{\mathbb{R}^-}, \mathbb{1}_{\mathbb{R}^+}}\big[\big(\big(R_t(\varepsilon) - L_t(\varepsilon)\big)^+\big)^p\big] \leq C(\varrho, p, \delta)\varepsilon^{-2+\delta}. \tag{61}$$

Finally, we return to the case of a general branching rate γ. Then, by the scaling property (4), we have that

$$\begin{aligned} L_t^{[\gamma]}(\varepsilon) &= \inf\left\{x : \int_{-\infty}^x u_t^{[\gamma]}(y) v_t^{[\gamma]}(y)\, dy \geq \varepsilon\right\} \wedge R\big(u_t^{[\gamma]}, v_t^{[\gamma]}\big) \\ &\stackrel{d}{=} \inf\left\{x : \int_{-\infty}^x u_{\gamma^2 t}^{[1]}(\gamma y) v_{\gamma^2 t}^{[1]}(\gamma y)\, dy \geq \varepsilon\right\} \wedge \frac{1}{\gamma} R\big(u_{\gamma^2 t}^{[1]}, v_{\gamma^2 t}^{[1]}\big) \\ &= \frac{1}{\gamma} L_{\gamma^2 t}^{[1]}(\gamma\varepsilon). \end{aligned}$$

Similarly, $R_t^{[\gamma]}(\varepsilon) \stackrel{d}{=} \frac{1}{\gamma} R_{\gamma^2 t}^{[1]}(\gamma\varepsilon)$. Hence, by (61) (which holds for branching rate 1), we can deduce that

$$\begin{aligned} &\mathbb{E}_{\mathbb{1}_{\mathbb{R}^-}, \mathbb{1}_{\mathbb{R}^+}}\big[\big(\big(R_t^{[\gamma]}(\varepsilon) - L_t^{[\gamma]}(\varepsilon)\big)^+\big)^p\big] \\ &\quad = \gamma^{-p}\mathbb{E}_{\mathbb{1}_{\mathbb{R}^-}, \mathbb{1}_{\mathbb{R}^+}}\big[\big(\big(R_{\gamma^2 t}^{[1]}(\gamma\varepsilon) - L_{\gamma^2 t}^{[1]}(\gamma\varepsilon)\big)^+\big)^p\big] \\ &\quad \leq C(\varrho, p, \delta)\varepsilon^{-2+\delta}\gamma^{-(2+p-\delta)}. \end{aligned}$$

□

APPENDIX

A.1. Notation and spaces of functions and measures. In this appendix, for the convenience of the reader, we have collected our notation, and we recall some well-known facts concerning the spaces of functions and measures employed throughout the paper. Most of the material in this subsection can be found, for example, in [5, 7] or [11].

For $\lambda \in \mathbb{R}$, let

$$\phi_\lambda(x) := e^{-\lambda|x|}, \qquad x \in \mathbb{R},$$

and for $f : \mathbb{R} \to \mathbb{R}$, define

$$|f|_\lambda := \|f/\phi_\lambda\|_\infty,$$

where $\|\cdot\|_\infty$ is the supremum norm. Let $\mathcal{B}_\lambda$ denote the space of all measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $|f|_\lambda < \infty$ and with the property that $f(x)/\phi_\lambda(x)$ has a finite limit as $|x| \to \infty$. Next, introduce the spaces

$$\mathcal{B}_{\mathrm{rap}} := \bigcap_{\lambda > 0} \mathcal{B}_\lambda \quad \text{and} \quad \mathcal{B}_{\mathrm{tem}} := \bigcap_{\lambda > 0} \mathcal{B}_{-\lambda} \tag{62}$$

of *rapidly decreasing* and *tempered* measurable functions, respectively.

We write $\mathcal{C}_\lambda, \mathcal{C}_{\text{rap}}, \mathcal{C}_{\text{tem}}$ for the subspaces of continuous functions in $\mathcal{B}_\lambda$, $\mathcal{B}_{\text{rap}}$, $\mathcal{B}_{\text{tem}}$, respectively. If we additionally require that all partial derivatives up to order $k \in \mathbb{N}$ exist and belong to $\mathcal{C}_\lambda, \mathcal{C}_{\text{rap}}, \mathcal{C}_{\text{tem}}$, we write $\mathcal{C}_\lambda^{(k)}, \mathcal{C}_{\text{rap}}^{(k)}, \mathcal{C}_{\text{tem}}^{(k)}$. We will also use the space $\mathcal{C}_c^\infty$ of infinitely differentiable functions with compact support. If $\mathcal{F}$ is any of the above spaces of functions, the notation $\mathcal{F}^+$ will refer to the subset of nonnegative elements of $\mathcal{F}$.

For each $\lambda \in \mathbb{R}$, the linear space $\mathcal{C}_\lambda$ endowed with the norm $|\cdot|_\lambda$ is a separable Banach space, and the space $\mathcal{C}_{\text{rap}}$ is topologized by the metric

$$d_{\text{rap}}^{\mathcal{C}}(f, g) := \sum_{n=1}^{\infty} 2^{-n}\big(|f-g|_n \wedge 1\big), \qquad f, g \in \mathcal{C}_{\text{rap}}, \tag{63}$$

which turns it into a Polish space. Analogously, $\mathcal{C}_{\text{tem}}$ is Polish if we topologize it with the metric

$$d_{\text{tem}}^{\mathcal{C}}(f, g) := \sum_{n=1}^{\infty} 2^{-n}\big(|f-g|_{-1/n} \wedge 1\big), \qquad f, g \in \mathcal{C}_{\text{tem}}. \tag{64}$$

Let $\mathcal{M}$ denote the space of (nonnegative) Radon measures on $\mathbb{R}$. For $\mu \in \mathcal{M}$ and a measurable function f, we will use any of the following notation:

$$\langle \mu, f\rangle, \qquad \int_{\mathbb{R}} \mu(dx) f(x), \qquad \int_{\mathbb{R}} f(x)\mu(dx)$$

to denote the integral of f with respect to the measure μ (if it exists). For integrals with respect to the Lebesgue measure ℓ on $\mathbb{R}$, we will simply write dx in place of $\ell(dx)$. If $\mu \in \mathcal{M}$ is absolutely continuous w.r.t. ℓ, we will identify μ with its density, writing

$$\mu(dx) = \mu(x)\, dx.$$

For $\lambda \in \mathbb{R}$, define

$$\mathcal{M}_\lambda := \{\mu \in \mathcal{M} : \langle \mu, \phi_\lambda\rangle < \infty\},$$

and introduce the spaces

$$\mathcal{M}_{\text{tem}} := \bigcap_{\lambda>0} \mathcal{M}_\lambda, \qquad \mathcal{M}_{\text{rap}} := \bigcap_{\lambda>0} \mathcal{M}_{-\lambda}$$

of *tempered* and *rapidly decreasing measures*, respectively. These spaces of measures are topologized as follows: Let d_0 be a complete metric on $\mathcal{M}$ inducing the vague topology, and define

$$d_{\text{tem}}^{\mathcal{M}}(\mu, \nu) := d_0(\mu, \nu) + \sum_{n=1}^{\infty} 2^{-n}\big(|\mu-\nu|_{1/n} \wedge 1\big), \qquad \mu, \nu \in \mathcal{M}_{\text{tem}}, \tag{65}$$

where we write

$$|\mu-\nu|_\lambda := \big|\langle \mu, \phi_\lambda\rangle - \langle \nu, \phi_\lambda\rangle\big|.$$

Note that with the above metric, $(\mathcal{M}_{\text{tem}}, d^{\mathcal{M}}_{\text{tem}})$ is also Polish, and it is easily seen that $\mu_n \to \mu$ in $\mathcal{M}_{\text{tem}}$ if and only if $\langle \mu_n, \varphi\rangle \to \langle \mu, \varphi\rangle$ for all $\varphi \in \bigcup_{\lambda>0} \mathcal{C}_\lambda$. Denote by $\mathcal{M}_f$ the space of finite measures on $\mathbb{R}$ endowed with the topology of weak convergence. Note that we have $\mathcal{M}_{\text{rap}} \subseteq \mathcal{M}_f$. The space $\mathcal{M}_{\text{rap}}$ is then topologized by saying that $\mu_n \to \mu$ in $\mathcal{M}_{\text{rap}}$ if and only if $\mu_n \to \mu$ in $\mathcal{M}_f$ (w.r.t. the weak topology) and $\sup_{n\in\mathbb{N}}\langle \mu_n, \phi_\lambda\rangle < \infty$ for all $\lambda < 0$; see [7], page 140. It is easy to see that this topology is also induced by the metric

$$d^{\mathcal{M}}_{\text{rap}}(\mu,\nu) := \widetilde{d}_0(\mu,\nu) + \sum_{n=1}^{\infty} 2^{-n}\big(|\mu-\nu|_{-n} \wedge 1\big), \qquad \mu,\nu \in \mathcal{M}_{\text{rap}}, \tag{66}$$

where $\widetilde{d}_0$ is a complete metric on $\mathcal{M}_f$ inducing the weak topology. Again, when endowed with this metric $(\mathcal{M}_{\text{rap}}, d^{\mathcal{M}}_{\text{rap}})$ becomes a Polish space.

It is clear that $\mathcal{C}^+_{\text{tem}}$ may be viewed as a subspace of $\mathcal{M}_{\text{tem}}$ by taking a function $u \in \mathcal{C}^+_{\text{tem}}$ as a density w.r.t. Lebesgue measure, that is, by identifying it with the measure $u(x)\,dx$. It is also clear that the topology of $\mathcal{M}_{\text{tem}}$ restricted to $\mathcal{C}^+_{\text{tem}}$ is weaker than the topology on $\mathcal{C}_{\text{tem}}$ introduced above. The same holds for the relation between $\mathcal{C}^+_{\text{rap}}$ and $\mathcal{M}_{\text{rap}}$. Thus we have *continuous* embeddings $\mathcal{C}^+_{\text{tem}} \hookrightarrow \mathcal{M}_{\text{tem}}$ and $\mathcal{C}^+_{\text{rap}} \hookrightarrow \mathcal{M}_{\text{rap}}$.

Let $(p_t)_{t\ge 0}$ denote the heat kernel in $\mathbb{R}$ corresponding to $\frac{1}{2}\Delta$,

$$p_t(x) = \frac{1}{(2\pi t)^{1/2}} \exp\left\{-\frac{|x|^2}{2t}\right\}, \qquad t > 0, x \in \mathbb{R}, \tag{67}$$

and write $(S_t)_{t\ge 0}$ for the associated heat semigroup (i.e., the transition semigroup of Brownian motion). For $\mu \in \mathcal{M}$ and $x \in \mathbb{R}$, let $S_t\mu(x) := \int_{\mathbb{R}} p_t(x-y)\mu(dy)$. The following estimates are well known and can be proved as in Appendix A of [7] (see also [26], Lemma 6.2(ii)):

LEMMA A.1. *Fix $\lambda \in \mathbb{R}$ and $T > 0$.*

(a) *For all $\varphi \in \mathcal{B}^+_\lambda$, we have*

$$\sup_{t\in[0,T]} S_t\varphi(x) \le C(\lambda, T)|\varphi|_\lambda \phi_\lambda(x), \qquad x \in \mathbb{R}. \tag{68}$$

Moreover, there is a positive constant $C'(\lambda, T) > 0$ such that we have a lower bound

$$\inf_{t\in[0,T]} S_t\phi_\lambda(x) \ge C'(\lambda, T)\phi_\lambda(x), \qquad x \in \mathbb{R}. \tag{69}$$

(b) *Let $0 < \varepsilon < T$. Then for all $\mu \in \mathcal{M}_\lambda$ we have*

$$\sup_{t\in[\varepsilon,T]} S_t\mu(x) \le C(\lambda, T, \varepsilon)\langle \mu, \phi_\lambda\rangle \phi_{-\lambda}(x), \qquad x \in \mathbb{R}. \tag{70}$$

In particular, the heat semigroup preserves the space $\mathcal{B}_\lambda$ and maps $\mathcal{M}_\lambda$ into $\mathcal{B}_\lambda$.

For $T > 0$ and $\lambda \in \mathbb{R}$, let $\mathcal{C}^{(1,2)}_{T,\lambda}$ denote the space of real-valued functions ψ defined on $[0, T] \times \mathbb{R}$ such that $t \mapsto \psi_t(\cdot)$, $t \mapsto \partial_t \psi_t(\cdot)$ and $t \mapsto \Delta\psi_t(\cdot)$ are continuous $\mathcal{C}_\lambda$-valued functions, and define

$$\mathcal{C}^{(1,2)}_{T,\mathrm{rap}} := \bigcap_{\lambda>0} \mathcal{C}^{(1,2)}_{T,\lambda}, \qquad \mathcal{C}^{(1,2)}_{T,\mathrm{tem}} := \bigcap_{\lambda>0} \mathcal{C}^{(1,2)}_{T,-\lambda}.$$

The following is a simple corollary of Lemma A.1:

COROLLARY A.2. *Fix $\lambda \in \mathbb{R}$ and $T > 0$.*

(a) *For all $\varphi \in \mathcal{C}^{(2)}_\lambda$, the function*

$$\psi_t(x) := S_{T-t}\varphi(x), \qquad t \in [0, T], x \in \mathbb{R}$$

is in $\mathcal{C}^{(1,2)}_{T,\lambda}$.

(b) *For all $\mu \in \mathcal{M}_\lambda$ and $\varepsilon > 0$, the function*

$$\psi_t(x) := S_{T-t}\mu(x), \qquad t \in [0, T-\varepsilon], x \in \mathbb{R}$$

is in $\mathcal{C}^{(1,2)}_{T-\varepsilon,\lambda}$.

For a Polish space E and $I \subseteq \mathbb{R}$, we denote by $D_I(E)$, respectively $\mathcal{C}_I(E)$, the space of càdlàg, respectively continuous, E-valued paths $t \mapsto f_t$, $t \in I$. (In our case, we will always have $I = [0, \infty)$ or $I = (0, \infty)$ and $E \in \{(\mathcal{C}^+_{\mathrm{tem}})^m, (\mathcal{C}^+_{\mathrm{rap}})^m, \mathcal{M}^m_{\mathrm{tem}}, \mathcal{M}^m_{\mathrm{rap}}\}$ for some power $m \in \mathbb{N}$.) Endowed with the usual Skorokhod (J_1)-topology, $D_I(E)$ is then also Polish. In this paper, we will use the Skorokhod topology only in restriction to $\mathcal{C}_I(E)$ where it coincides with the usual topology of locally uniform convergence.

For processes which are càdlàg but not continuous, we will instead use the weaker *Meyer–Zheng "pseudo-path" topology* on $D_{[0,\infty)}(E)$. To describe the Meyer–Zheng topology, introduced in [19], let $\lambda(dt) := \exp(-t)\,dt$, and let $w(t), t \in [0, \infty)$ be an E-valued Borel function. Then, a "pseudo-path" corresponding to w is the probability law ψ_w on $[0, \infty) \times E$ given as the image measure of λ under the mapping $t \mapsto (t, w(t))$. Note that two functions which are equal Lebesgue-a.e. give rise to the same pseudo-path. Further $w \mapsto \psi_w$ is one-to-one on the space of càdlàg paths $D_{[0,\infty)}(E)$, and thus yields an embedding of $D_{[0,\infty)}(E)$ into the space of probability measures on $[0, \infty) \times E$. The induced topology on $D_{[0,\infty)}$ is then called the pseudo-path topology. Very conveniently, convergence in this topology is equivalent to convergence in Lebesgue measure; see [19], Lemma 1.

For $E = \mathbb{R}$, [19], Theorem 4, provides a rather convenient sufficient condition for relative compactness of a sequence of stochastic processes on $D_{[0,\infty)}(E)$ equipped with this topology. The condition can be stated as follows: If $(X^{(n)}_t)_{t\geq 0}$,

$n \in \mathbb{N}$ is a sequence of càdlàg real-valued stochastic processes, with $(X_t^{(n)})_{t\geq 0}$ adapted to a filtration $(\mathcal{F}^{(n)})_{t\geq 0}$, then Meyer and Zheng require that

$$\sup_{n\in\mathbb{N}}\Big(V_T\big(X^{(n)}\big) + \sup_{t\leq T}\mathbb{E}\big[\big|X_t^{(n)}\big|\big]\Big) < \infty \tag{71}$$

for all $T > 0$. Here $V_T(X^{(n)}) := \sup\mathbb{E}[\sum_i |\mathbb{E}[X_{t_{i+1}}^{(n)} - X_{t_i}^{(n)}|\mathcal{F}_{t_i}^{(n)}]|]$, where the sup is taken over all partitions of the interval $[0, T]$, denotes the conditional variation of $X^{(n)}$ up to time T. In [18], this tightness criterion was extended to processes taking values in general separable metric spaces E, which is the version we need for our measure-valued processes. In fact, by [18], Corollary 1.4, we only have to check condition (71) for the coordinate processes and in addition a compact containment condition in order to obtain tightness of our measure-valued processes in the pseudopath topology (which again is equivalent to the topology of convergence in Lebesgue measure).[4]

A.2. Martingale problems and Green function representations. We define all stochastic processes over a sufficiently rich stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual hypotheses. If $Y = (Y_t)_{t\geq 0}$ is a stochastic process taking values in E and starting at $Y_0 = y \in E$, the law of Y is denoted $\mathbb{P}_y$, and we use $\mathbb{E}_y$ to denote the corresponding expectation.

Recall that solutions to the finite rate symbiotic branching model cSBM(ϱ, γ) are characterized by the martingale problem given in [11], Definition 3. Consequently, when the solutions are interpreted as densities w.r.t. Lebesgue measure, the corresponding measure-valued processes solve the martingale problem $(\mathbf{MP})_{\mu_0,\nu_0}^{\varrho}$ of Definition 1.7. In this appendix, we have collected some properties of solutions to this martingale problem which for the finite rate model cSBM(ϱ, γ) can already be found in [11]; however, they are in fact true for *any* solution to $(\mathbf{MP})_{\mu_0,\nu_0}^{\varrho}$. These are: an extended martingale problem for space–time functions which in turn implies a Green function representation, and a (weaker) martingale problem involving the self-duality function F from (13); see Proposition A.5 below. We include a proof only for the latter, in order to illustrate the point that the particular form of the quadratic variation process $(\Lambda_t)_{t\geq 0}$ from Definition 1.7 is irrelevant in this respect.

Recall that we consider the increasing process $t \mapsto \Lambda_t(dx)$ also as a (locally finite) measure $\Lambda(ds, dx)$ on $\mathbb{R}^+ \times \mathbb{R}$, via

$$\Lambda\big([0, t] \times B\big) := \Lambda_t(B).$$

The following "space–time version" of the martingale problem $(\mathbf{MP})_{\mu_0,\nu_0}^{\varrho}$ can be proved by standard arguments; see, for example, [5], Lemma 42:

[4]Note, however, that the main result in [18] is much stronger than just an extension of the Meyer–Zheng tightness criterion to a general state space E. Also note that in [18], equation (1.7), there seems to be missing a term $\sup_{s\leq t}\mathbb{E}[|f_i \circ X_s^{(n)}|]$; cf. equation (1.2).

LEMMA A.3. *Fix $\varrho \in [-1,1]$ and initial conditions $(\mu_0,\nu_0) \in \mathcal{M}_{\mathrm{tem}}^2$ (resp., $\mathcal{M}_{\mathrm{rap}}^2$). Let $T>0$. If $(\mu_t,\nu_t)_{t\ge 0} \in \mathcal{C}_{[0,\infty)}(\mathcal{M}_{\mathrm{tem}}^2)$ [resp., $\mathcal{C}_{[0,\infty)}(\mathcal{M}_{\mathrm{rap}}^2)$] is any solution to the martingale problem $(\mathbf{MP})_{\mu_0,\nu_0}^{\varrho}$, then for all test functions $\phi,\psi \in \mathcal{C}_{T,\mathrm{rap}}^{(1,2)}$ (resp., $\phi,\psi \in \mathcal{C}_{T,\mathrm{tem}}^{(1,2)}$) we have that*

$$\begin{aligned}
\langle \mu_t,\phi_t\rangle &= \langle \mu_0,\phi_0\rangle + \int_0^t \Big\langle \mu_s, \frac{1}{2}\Delta\phi_s + \frac{\partial}{\partial s}\phi_s\Big\rangle\,ds \\
&\quad + \int_{[0,t]\times\mathbb{R}} \phi_s(x) M\big(d(s,x)\big), \\
\langle \nu_t,\psi_t\rangle &= \langle \nu_0,\psi_0\rangle + \int_0^t \Big\langle \nu_s, \frac{1}{2}\Delta\psi_s + \frac{\partial}{\partial s}\psi_s\Big\rangle\,ds \\
&\quad + \int_{[0,t]\times\mathbb{R}} \psi_s(x) N\big(d(s,x)\big)
\end{aligned} \tag{72}$$

for $t\in[0,T]$, where $M(d(s,x))$ and $N(d(s,x))$ are zero-mean martingale measures with covariance structure

$$\begin{aligned}
&\Big[\int_{[0,\cdot]\times\mathbb{R}} f_s(x) M\big(d(s,x)\big)\Big]_t \\
&\qquad = \Big[\int_{[0,\cdot]\times\mathbb{R}} f_s(x) N\big(d(s,x)\big)\Big]_t = \int_{[0,t]\times\mathbb{R}} f_s^2(x)\Lambda(ds,dx), \\
&\Big[\int_{[0,\cdot]\times\mathbb{R}} f_s(x) M\big(d(s,x)\big), \int_{[0,\cdot]\times\mathbb{R}} g_s(x) N\big(d(s,x)\big)\Big]_t \\
&\qquad = \varrho \int_{[0,t]\times\mathbb{R}} f_s(x) g_s(x)\Lambda(ds,dx),
\end{aligned} \tag{73}$$

with Λ from (10). Here, f and g are predictable functions defined on $\Omega\times\mathbb{R}_+\times\mathbb{R}$ such that

$$\mathbb{E}_{\mu_0,\nu_0}\Big[\int_{[0,t]\times\mathbb{R}} f_s^2(x)\Lambda(ds,dx)\Big] < \infty, \qquad t\in[0,T]. \tag{74}$$

The previous lemma immediately implies a Green function representation for solutions to the martingale problem $(\mathbf{MP})_{\mu_0,\nu_0}^{\varrho}$ [recall that $(S_t)_{t\ge 0}$ denotes the heat semigroup]:

COROLLARY A.4 (Green function representation). *Under the assumptions of Lemma A.3, we have for all $T>0$ and test functions $\varphi \in \bigcup_{\lambda>0}\mathcal{C}_\lambda$ (resp., $\bigcup_{\lambda>0}\mathcal{C}_{-\lambda}$) that*

$$\begin{aligned}
\langle \mu_t, S_{T-t}\varphi\rangle &= \langle \mu_0, S_T\varphi\rangle + \int_{[0,t]\times\mathbb{R}} S_{T-s}\varphi(x) M(ds,dx), \\
\langle \nu_t, S_{T-t}\varphi\rangle &= \langle \nu_0, S_T\varphi\rangle + \int_{[0,t]\times\mathbb{R}} S_{T-s}\varphi(x) N(ds,dx)
\end{aligned} \tag{75}$$

for $t \in [0, T]$, *where* $M(d(s, x))$, $N(d(s, x))$ *are the martingale measures from Lemma* A.3. *In particular*, $(\langle \mu_t, S_{T-t}\varphi\rangle)_{t\in[0,T]}$ *and* $(\langle \nu_t, S_{T-t}\varphi\rangle)_{t\in[0,T]}$ *are martingales with covariance structure given by* (73) *with* $f_s(x) = g_s(x) = S_{T-s}\varphi(x)$.

PROOF. For $\varphi \in \mathcal{C}^{(2)}_{\mathrm{rap}}$ (resp., $\mathcal{C}^{(2)}_{\mathrm{tem}}$), this follows at once from the extended martingale problem of Lemma A.3 by putting $\phi_t := \psi_t := S_{T-t}\varphi$ for $t \in [0, T]$, observing that the latter function is in $\mathcal{C}^{(1,2)}_{T,\mathrm{rap}}$ for $\varphi \in \mathcal{C}^{(2)}_{\mathrm{rap}}$ (resp., in $\mathcal{C}^{(1,2)}_{T,\mathrm{tem}}$ for $\varphi \in \mathcal{C}^{(2)}_{\mathrm{tem}}$) by Corollary A.2, and that $(\frac{1}{2}\Delta + \frac{\partial}{\partial_s})S_{T-s}\varphi \equiv 0$. In order to extend (75) to more general φ, one uses simple approximation arguments involving monotone, respectively dominated, convergence. □

PROPOSITION A.5. *Fix* $\varrho \in (-1, 1)$ *and* $(\mu_0, \nu_0) \in \mathcal{M}^2_{\mathrm{tem}}$ (*resp*., $\mathcal{M}^2_{\mathrm{rap}}$). *Let* $(\mu_t, \nu_t)_{t\ge0} \in \mathcal{C}_{[0,\infty)}(\mathcal{M}^2_{\mathrm{tem}})$ [*resp*., $\mathcal{C}_{[0,\infty)}(\mathcal{M}^2_{\mathrm{rap}})$] *be any solution to the martingale problem* $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$. *Then the process* $(\Lambda_t)_{t\ge0} \in \mathcal{C}_{[0,\infty)}(\mathcal{M}_{\mathrm{tem}})$ [*resp*., $\mathcal{C}_{[0,\infty)}(\mathcal{M}_{\mathrm{rap}})$] *from Definition* 1.7, *governing the correlations of the martingales as in* (10), *is increasing with* $\Lambda_0 = 0$ *and satisfies condition* (14). *Moreover*, *for all* $T > 0$ *and* (*nonnegative*) *test functions* $0 \le \phi, \psi \in \mathcal{C}^{(1,2)}_{T,\mathrm{rap}}$ (*resp*., $\in \mathcal{C}^{(1,2)}_{T,\mathrm{tem}}$), *the process*

$$
\begin{aligned}
&F(\mu_t, \nu_t, \phi_t, \psi_t) - F(\mu_0, \nu_0, \phi_0, \psi_0) \\
&\quad - \int_0^t F(\mu_s, \nu_s, \phi_s, \psi_s)\left\langle\!\left\langle \mu_s, \nu_s, \left(\frac{1}{2}\Delta + \frac{\partial}{\partial s}\right)\phi_s, \left(\frac{1}{2}\Delta + \frac{\partial}{\partial s}\right)\psi_s \right\rangle\!\right\rangle_{\varrho} ds \qquad (76)\\
&\quad - 4(1-\varrho^2)\int_{[0,t]\times\mathbb{R}} F(\mu_s, \nu_s, \phi_s, \psi_s)\phi_s(x)\psi_s(x)\Lambda(ds, dx),
\end{aligned}
$$

$t \in [0, T]$, *is a martingale with quadratic variation given by*

$$
8(1-\varrho^2)\int_{[0,t]\times\mathbb{R}} F(\mu_s, \nu_s, \phi_s, \psi_s)^2\phi_s(x)\psi_s(x)\Lambda(ds, dx). \qquad (77)
$$

PROOF. In view of (10), it is clear that Λ is increasing and $\Lambda_0 = 0$. Moreover, since the martingales in Definition 1.7 are assumed square integrable, we have $\mathbb{E}_{\mu_0,\nu_0}[\langle \Lambda_t, \phi^2\rangle] = \mathbb{E}_{\mu_0,\nu_0}[M_t(\phi)^2] < \infty$ for all test functions $\phi \in \mathcal{C}^{(2)}_{\mathrm{rap}}$ (resp., $\mathcal{C}^{(2)}_{\mathrm{tem}}$). Thus (14) is satisfied.

The proof of (76) is basically a straightforward application of Itô's formula; cf. the proof of Proposition 5 in [11]. We sketch it here for the convenience of the reader and to make clear that the arguments in [11] do not rely on properties of the finite rate model, but actually work for any solution to the martingale problem $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$. Define

$$
\begin{aligned}
Y_t := \langle \mu_t + \nu_t, \phi_t + \psi_t\rangle &= \langle \mu_0 + \nu_0, \phi_0 + \psi_0\rangle \\
&\quad + \int_0^t \left\langle \mu_s + \nu_s, \left(\frac{1}{2}\Delta + \frac{\partial}{\partial s}\right)(\phi_s + \psi_s)\right\rangle ds
\end{aligned}
$$

$$+\int_{[0,t]\times\mathbb{R}}\big(\phi_s(x)+\psi_s(x)\big)(M+N)(ds,dx),$$

$$\begin{aligned} Z_t := \langle \mu_t-\nu_t,\phi_t-\psi_t\rangle &= \langle \mu_0-\nu_0,\phi_0-\psi_0\rangle \\ &\quad+\int_0^t\Big\langle \mu_s-\nu_s,\Big(\frac{1}{2}\Delta+\frac{\partial}{\partial s}\Big)(\phi_s-\psi_s)\Big\rangle ds \\ &\quad+\int_{[0,t]\times\mathbb{R}}\big(\phi_s(x)-\psi_s(x)\big)(M-N)(ds,dx), \end{aligned}$$

where M and N are the martingale measures from Lemma A.3. We observe that Y and Z are continuous real-valued semimartingales with covariance structure easily calculated as

$$\begin{aligned} [Y,Y]_t &= 2(1+\varrho)\int_{[0,t]\times\mathbb{R}}\big(\phi_s(x)+\psi_s(x)\big)^2\Lambda(ds,dx), \\ [Z,Z]_t &= 2(1-\varrho)\int_{[0,t]\times\mathbb{R}}\big(\phi_s(x)-\psi_s(x)\big)^2\Lambda(ds,dx), \\ [Y,Z]_t &= 0. \end{aligned}$$

Now define $H(y,z) := \exp(-\sqrt{1-\varrho}\,y + i\sqrt{1+\varrho}\,z)$, apply Itô's formula to the process $(H(Y_t,Z_t))_{t\ge 0}$ and use the trivial identity $(\phi+\psi)^2-(\phi-\psi)^2 = 4\phi\psi$ to obtain by a straightforward calculation that

$$\begin{aligned} &F(\mu_t,\nu_t,\phi_t,\psi_t) \\ &\quad= F(\mu_0,\nu_0,\phi_0,\psi_0) \\ &\qquad+\int_0^t F(\mu_s,\nu_s,\phi_s,\psi_s)\cdot\Big\langle\Big\langle \mu_s,\nu_s,\Big(\frac{\Delta}{2}+\frac{\partial}{\partial s}\Big)\phi_s,\Big(\frac{\Delta}{2}+\frac{\partial}{\partial s}\Big)\psi_s\Big\rangle\Big\rangle_\varrho ds \\ &\qquad+4(1-\varrho^2)\int_0^t\int_{\mathbb{R}} F(u_s,v_s,\phi_s,\psi_s)\phi_s(x)\psi_s(x)\Lambda(ds,dx) \\ &\qquad+\int_{[0,t]\times\mathbb{R}} F(u_s,v_s,\phi_s,\psi_s) \\ &\qquad\qquad\times\big(-\sqrt{1-\varrho}\big(\phi_s(x)+\psi_s(x)\big) \\ &\qquad\qquad\qquad+i\sqrt{1+\varrho}\big(\phi_s(x)-\psi_s(x)\big)\big)M(ds,dx) \\ &\qquad+\int_{[0,t]\times\mathbb{R}} F(u_s,v_s,\phi_s,\psi_s) \\ &\qquad\qquad\times\big(-\sqrt{1-\varrho}\big(\phi_s(x)+\psi_s(x)\big) \\ &\qquad\qquad\qquad-i\sqrt{1+\varrho}\big(\phi_s(x)-\psi_s(x)\big)\big)N(ds,dx). \end{aligned}$$

This gives (76), and computing the quadratic variation of the martingale term in the above display, we obtain (77). □

COROLLARY A.6. *Fix* $\varrho \in (-1, 1)$ *and* $(\mu_0, \nu_0) \in \mathcal{M}^2_{\mathrm{tem}}$ (*resp*., $\mathcal{M}^2_{\mathrm{rap}}$). *Then any solution* $(\mu_t, \nu_t)_{t\geq 0} \in \mathcal{C}_{[0,\infty)}(\mathcal{M}^2_{\mathrm{tem}})$ [*resp*., $\mathcal{C}_{[0,\infty)}(\mathcal{M}^2_{\mathrm{rap}})$] *to the martingale problem* $(\mathbf{MP})^{\varrho}_{\mu_0,\nu_0}$ *is also a solution to the martingale problem* $(\mathbf{MP}')^{\varrho}_{\mu_0,\nu_0}$.

A.3. Some facts on Brownian motion and its local time. In this subsection, we recall some of the standard facts (and their variations) on Brownian motion in a formulation adapted to our needs. In the following we will denote for any suitable process $(X_t)_{t\geq 0}$ its local time in x by $L^x_t := L^{x,X}_t$.

LEMMA A.7. *If* $(B_t)_{t\geq 0}$ *is a Brownian motion started at* $x \in \mathbb{R}$ *with local time* $(L^0_t)_{t\geq 0}$ *in* 0, *then*

$$(L^0_t)_{t\geq 0} \stackrel{d}{=} (M^+_t)_{t\geq 0},$$

where $(M_t)_{t\geq 0}$ *is the maximum process of a Brownian motion started at* $-|x|$.

PROOF. We adapt the proof of Theorem 7.38 in [20]. By Tanaka's formula [20], Theorem 7.33, we find that

$$|B_t| - |x| = \int_0^t \mathrm{sign}(B_s)\, dB_s + L^0_t.$$

By [20], Lemma 7.40, the stochastic integral is equal in distribution to a standard Brownian motion, so if we set

$$W_t = -\left(|x| + \int_0^t \mathrm{sign}(B_s)\, dB_s\right),$$

then W is a linear Brownian motion started at $-|x|$, and we have that

$$|B_t| = -W_t + L^0_t. \tag{78}$$

Let $(M_t)_{t\geq 0}$ denote the maximum process of $(W_t)_{t\geq 0}$. We want to show that for all $t \geq 0$, we have $M^+_t = L^0_t$. It follows immediately from (78) that $W_s \leq L^0_s \leq L^0_t$ for all $s \leq t$, so that by taking the maximum we obtain $M^+_t = 0 \vee M_t \leq L^0_t$.

Now suppose there exists a time t such that $M^+_t < L^0_t$. Let $u := \inf\{r \leq t : L^0_r = L^0_t\}$. Since L^0 only increases on the set $\{s : B_s = 0\}$, by continuity and since $L^0_t > 0$, we must have $B_u = 0$. In particular, from (78) we get $W_u = L^0_u$ with $u \leq t$. Thus, we can deduce that

$$M_u \geq W_u = L^0_u = L^0_t > M_t,$$

which yields a contradiction since $u \leq t$ and M is obviously increasing. Hence, $M^+_t = L^0_t$ as claimed. □

LEMMA A.8. *Let $(B_t)_{t\geq 0}$ be a Brownian motion started at $z\in\mathbb{R}$ with local time $(L_t^0)_{t\geq 0}$ in 0. Then for all $\alpha>0$ and $t\geq 1$,*

$$\mathbb{P}_z\{L_t^0\leq \alpha\log t\}\leq \sqrt{\frac{2}{\pi}}\frac{\alpha\log t+|z|}{t^{1/2}}.$$

PROOF. Using Lemma A.7, we find that if $(M_t)_{t\geq 0}$ denotes the maximum process of a Brownian motion started at $-|z|$, we can estimate

$$\begin{aligned}\mathbb{P}_z\{L_t^0\leq \alpha\log t\}&=\mathbb{P}_{-|z|}\{M_t^+\leq\alpha\log t\}=\mathbb{P}_0\{M_t\leq \alpha\log t+|z|\}\\&=\mathbb{P}_0\{|B_t|\leq \alpha\log t+|z|\}\leq\sqrt{\frac{2}{\pi}}\frac{\alpha\log t+|z|}{t^{1/2}},\end{aligned}$$

where we used the reflection principle; see, for example, [20], Theorem 2.21, in the second-to-last step. □

COROLLARY A.9. *Suppose that $(B_t^{(1)})_{t\geq 0}$ and $(B_t^{(2)})_{t\geq 0}$ are independent Brownian motions started at $x<y$, respectively, and denote their collision local time as $(L_t^{1,2})_{t\geq 0}$. Then for all $\alpha>0$ and $t\geq 1$,*

$$\mathbb{P}_{x,y}\{L_t^{1,2}\leq\alpha\log t\}\leq\frac{1}{\sqrt{\pi}}\frac{2\alpha\log t+y-x}{t^{1/2}}.$$

PROOF. This follows immediately from Lemma A.8. Note that $W_t:=B_t^{(2)}-B_t^{(1)}$, $t\geq 0$ is by definition a Brownian motion (with quadratic variation $2t$ and started at $y-x$), and thus $B_t:=W_{t/2}-(y-x)$, $t\geq 0$ is a standard Brownian motion. Moreover $L_t^{1,2}=L_t^{0,B^{(2)}-B^{(1)}}=L_t^{0,W}$. Now observe that

$$\begin{aligned}L_t^{0,W}&=\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t\mathbb{1}_{\{|W_s|\leq\varepsilon\}}\,ds=\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t\mathbb{1}_{\{|B_{2s}+y-x|\leq\varepsilon\}}\,ds\\&\stackrel{d}{=}\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t\mathbb{1}_{\{|\sqrt{2}B_s+y-x|\leq\varepsilon\}}\,ds=\frac{1}{\sqrt{2}}L_t^{(x-y)/\sqrt{2},B}.\end{aligned}$$

Hence by Lemma A.8,

$$\begin{aligned}\mathbb{P}_{x,y}\{L_t^{1,2}\leq\alpha\log t\}&=\mathbb{P}_0\{L_t^{(x-y)/\sqrt{2},B}\leq\sqrt{2}\alpha\log t\}\\&=\mathbb{P}_{(y-x)/\sqrt{2}}\{L_t^{0,B}\leq\sqrt{2}\alpha\log t\}\\&\leq\sqrt{\frac{2}{\pi}}\frac{\sqrt{2}\alpha\log t+(1/\sqrt{2})(y-x)}{t^{1/2}},\end{aligned}$$

which proves the corollary. □

The following is a slightly generalized version of Lemma 2 in [1]. It follows easily from the occupation times formula for Brownian local time.

LEMMA A.10. *Let $B^{(1)}$, $B^{(2)}$ be independent Brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then for every $h : \mathbb{R} \times \mathbb{R}^+ \times \Omega \to \mathbb{R}$ measurable and bounded or non-negative, we have*

$$\int_0^t h(B_s^{(2)} - B_s^{(1)}, s, \cdot)\, ds = \int_{\mathbb{R}} \int_0^t h(z, s, \cdot)\, dL_s^{z, B^{(2)} - B^{(1)}}\, dz, \qquad \mathbb{P}\text{-}a.s. \tag{79}$$

In [1], Lemma 2, this is stated for functions of the form $h(z, s, \omega) = f(z) Y_s(\omega)$, where it is assumed that f is continuous and (Y_s) is predictable. Neither of the two assumptions is really needed. Also note that the factor 2 in the statement of Lemma 2 in [1] seems to be incorrect.

PROOF OF LEMMA A.10. Let $X_s := B_s^{(2)} - B_s^{(1)}$. For $h(z, s, \omega) = f(z) \mathbb{1}_{(a,b]}(s) g(\omega)$, with f and g measurable bounded and $0 \le a < b < \infty$, (79) holds by the occupation times formula [24], Corollary VI.1.6, since for $\mathbb{P}$-almost all $\omega \in \Omega$,

$$\begin{aligned}
\int_0^t h(X_s(\omega), s, \omega)\, ds &= \left(\int_0^{b \wedge t} f(X_s(\omega))\, ds - \int_0^{a \wedge t} f(X_s(\omega))\, ds \right) g(\omega) \\
&= \left(\int_{\mathbb{R}} f(z) L_{b \wedge t}^{z, X}(\omega)\, dz - \int_{\mathbb{R}} f(z) L_{a \wedge t}^{z, X}(\omega)\, dz \right) g(\omega) \\
&= \int_{\mathbb{R}} f(z) \int_{a \wedge t}^{b \wedge t} dL_s^{z, X}(\omega)\, dz g(\omega) \\
&= \int_{\mathbb{R}} \int_0^t h(z, s, \omega)\, dL_s^{z, X}(\omega)\, dz.
\end{aligned}$$

Let $\mathcal{C}$ denote the class of all functions h of the above form. Clearly, $\mathcal{C}$ is closed under multiplication and generates the product σ-algebra $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F}$ on $\mathbb{R} \times \mathbb{R}^+ \times \Omega$. Moreover, let $\mathcal{H}$ denote the space of all bounded measurable functions $h : \mathbb{R} \times \mathbb{R}^+ \times \Omega \to \mathbb{R}$ for which (79) holds. Since (79) is stable under linear combinations and under monotone convergence, $\mathcal{H}$ is a monotone vector space of bounded measurable functions which contains $\mathcal{C}$. Hence by the monotone class theorem [24], Theorem 0.2.2, $\mathcal{H}$ contains all bounded measurable functions $h : \mathbb{R} \times \mathbb{R}^+ \times \Omega \to \mathbb{R}$, which proves the assertion. □

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