TIME-AVERAGING OF PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS: EXPONENTIAL ESTIMATES

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Chapter 1

Introduction

In this thesis we discuss averaging up to exponentially small effects and backward error analysis for a class of parabolic partial differential equations. Our motivation is the success of these methods in ordinary differential equations. In this introduction we first present the situation for ordinary differential equations as a goal of our analysis for parabolic partial differential equations. Then we describe the general framework of the equations, that we consider. Afterwards we give the main results. Finally we outline some applications towards an improved geometric theory and backward error analysis for special numerical schemes.

1.1 Situation for ordinary differential equations

Consider first an ordinary differential equation

$$\begin{aligned} \dot{x} &= f(x), \\ x(0) &= x_0 \in \mathbf{R}^m. \end{aligned} \tag{1.1}$$

We want to compare the flow $\Phi(t,u)$ with the evolution defined by a nonautonomous time-periodic perturbation

$$\begin{aligned} \dot{x} &= f(x) + h^p g(x, \frac{t}{h}, h), \\ x(0) &= x_0 \in \mathbf{R}^m. \end{aligned} \tag{1.2}$$

The flow is also compared with numerical methods e.g. some Runge-Kutta method with step-size h, which defines a local diffeomorphism

$$x_{n+1} = \Phi(h, x_n). \tag{1.3}$$

Our main intention is not to give best estimates to compare these perturbations with the original flow of (1.1). Instead we want to compare them to some modified equation $\dot{x} = \tilde{f}(x)$, with $\tilde{f}$ inside the same class as f. In the case of periodic forcing this will lead to averaging methods and in the case of numerical schemes to backward error analysis.

For ordinary differential equations there is a strong relationship between backward error analysis of numerical schemes and the averaging of nonautonomous equations. One-step methods for equation (1.1) with step-size h can be interpreted as the exact time-h-map of a rapidly forced ordinary differential equation as in (1.2) with $g(.,\tau,.)$ periodic in $\tau = \frac{t}{h}$ with period 1, see [FS96].

For such rapidly forced equations like (1.2) Neishtadt proved:

Theorem (Neishtadt,1984)[Nei84] *Let f and g be real analytic and bounded on a complex extension of some domain D. Then there exists a time-periodic, real analytic near-identity coordinate change*

$$x = y + hU(y, \frac{t}{h}, h), \tag{1.4}$$

such that the transformed equation has only exponentially small nonautonomous remainder terms

$$\dot{y} = F(y) + \alpha(y, \frac{t}{h}, h) \tag{1.5}$$

with $\|\alpha\| < c_2 \exp(-c_1/h)$ and $\|F - f\| < c_3 h^p$ uniformly on a complex extension of D.

For the backward error analysis of Runge-Kutta methods the question is, if there is a modified vector field $\tilde{f}(h,\tilde{x})$ near f, such that the exact solution of $\dot{\tilde{x}} = \tilde{f}(h,\tilde{x})$ and x_n, defined by (1.3), coincide at times $n \cdot h$. Although this cannot hold exactly in general, there is a satisfying answer. One could go via the embedding into nonautonomous vector fields and then use the Neishtadt theorem. For real analytic f bounded on some polydisc around x_0, Hairer and Lubich [HL97, theorem 1] construct directly a $\tilde{f}(h,\tilde{x})$ such that

$$\|x_1 - \tilde{x}(h)\| \leq c_1 \exp(-\frac{c_2}{h}) \tag{1.6}$$

for $h < h_0$ with explicit estimates of c_1, c_2 and h_0. If $x(t)$ remains inside some compact set K, where f is also real analytic and bounded, then such an extremely good approximation also holds for finite times $T = O(h^{-1})$

$$x_n - \tilde{x}(nh) = O(\exp(-\frac{c}{h})) \text{ for } nh \leq T. \tag{1.7}$$

These exponential estimates can be used to explain, why several qualitative effects are well replicated by numerical schemes. Some of these properties are local behaviour near steady states, stable periodic orbits and energy conservation for symplectic integrators [BG94]. There is a vast literature on different aspects of backward error analysis for ordinary differential equations, see [HL97], [Rei99] and the review article [HL99] and references therein.

When considering special solutions we can also explain, why there has to be still some error $\|x_n - \tilde{x}(nh)\|$ for any $\tilde{f}$ for most numerical methods Φ. One example is the discretization of homoclinic orbits, where a solution $\Gamma(t)$ converges to a hyperbolic steady state $\bar{x}$ for $t \to \pm\infty$. There are generically effects which belong genuinely to time discrete dynamical systems [FS96]. In these systems homoclinic orbits are usually transversal and consist out of discrete points instead of a continuous arc. The stable and unstable manifold intersect transversally at some homoclinic point and give rise to shift type dynamics. Several quantities describing these effects were shown by Fiedler and Scheurle [FS96] to be exponentially small, but generically to be positive.

1.2 Semilinear parabolic equations

Averaging up to exponentially small remainder terms and backward error analysis are until now completely unknown in the literature of infinite dimensional dynamical system or partial differential equations. In this thesis we will give results for semilinear parabolic partial differential equations. The general setting for these equations is given for example by Henry [Hen83]

$$\frac{\partial}{\partial t} U + AU = F(U) \tag{1.8}$$

with initial conditions

$$U(0) = U_0 \in X^\alpha$$

where A is a sectorial operator on the Banach space X with domain $D(A)$, X^α some fractional power space, $F : X^\alpha \to X$ a Lipschitz continuous nonlinearity. For details one should consult [Hen83]. Equation (1.8) then defines a continuous local semiflow $S(t, U_0)$, such that $S(0, U_0) = U_0$, $S(t, U_0) \in D(A)$ and $\frac{d}{dt}U(t)$ exists for $t > 0$. Hence the right hand side of the equation (1.8) is defined for $t > 0$. Several partial differential equations like reaction-diffusion equations, nonlinear heat equations, Hodgekin-Huxley equation and the Navier-Stokes equation fit in this framework.

Some general results on averaging of such rapidly forced partial differential equations are given for example in Henry [Hen83, theorem 3.4.9] and Ilyin [Il98]. In these articles the main subject is continuous dependence on h, hence averaging in the 0^{th} order in h. Our aim is averaging beyond any finite order for some concrete examples, where is enough regularity in space and time for our analysis. We will consider a system of reaction-diffusion equations with periodic boundary conditions as a model.

We discus possible generalisations including the Navier-Stokes equation and other nonlinearities in chapter 6.

Letting $U = (u_1, \ldots, u_n)$, $D = \mathrm{diag}(d_1, \ldots, d_n)$ and $F = (f_1, \ldots, f_n)$, we consider the following system of reaction diffusion equations

$$\frac{\partial}{\partial t} U(x,t) - D\Delta U(x,t) = F(U(x,t)) \tag{1.9}$$

with periodic boundary conditions on $\Omega = [0,l]^d, d = 1,2,3$ and initial condition

$$U(0) = U_0 \in H^s_{per}(\Omega, \mathbf{R}^n)$$

with $\frac{d}{2} < s < 2$. The definition of $H^s_{per}(\Omega, \mathbf{R}^n)$ with non-integer s can be given in terms of the Fourier series expansion, see [Tem88, p.48]:

$$H^s_{per}(\Omega, \mathbf{R}) = \{u \in L^2(\Omega, \mathbf{R}) | \ |u|^2_{H^s_{per}} = \sum_{j \in \mathbf{Z}^d} (1 + |j|^2)^s (u^j)^2 < \infty\}, \tag{1.10}$$

where the u^j are the Fourier coefficients of the spatial Fourier series expansion of u into $\exp(i\frac{2\pi}{l} jx), j \in \mathbf{Z}^d$. The d_k are diffusion rates and we assume $d_k > 0$ for $k = 1, \ldots, n$. The nonlinearity $F(U)(x) = (f_1(U(x)), \ldots, f_n(U(x)))$ is of local superposition-type, with each $f_k : \mathbf{R}^n \to \mathbf{R}$. Let f_k be entire on $\mathbf{R}^n$, i.e. it is given by a power series, which is convergent on whole $\mathbf{R}^n$.

Note that $H^s_{per}(\Omega, \mathbf{R}^n) = X^{\frac{s}{2}}$ with $X = L^2(\Omega, \mathbf{R}^n)$ and $D(A) = H^2_{per}(\Omega, \mathbf{R}^n)$ for $A = -\mathrm{diag}(d_1, \ldots, d_n)\Delta$. Then again by [Hen83] there exists a local semiflow. The exponent $s > \frac{d}{2}$ is chosen such that $H^s_{per}(\Omega, \mathbf{R}^n)$ is an algebra and has an continuous embedding into bounded continuous functions $BC(\Omega, \mathbf{R}^n)$, see [Ad75, theorem 7.5.7], i.e.

$$|u|_{BC} \leq C_{emb} |u|_{H^s_{per}}. \tag{1.11}$$

Furthermore we gain much more regularity in this setting. There is high regularity in space. The solutions lie in Gevrey classes which we define by:

$$G^{\frac{s}{2}}_\sigma(\Omega, \mathbf{R}^n) = D\left(A^{\frac{s}{2}} \exp(\sigma(-\Delta)^{\frac{1}{2}})\right). \tag{1.12}$$

A norm is given by

$$|v|_{G^{\frac{s}{2}}_\sigma} = \left(\sum_{k=1}^n \sum_{j \in \mathbf{Z}^d} v^j_k v^j_k (1 + d_k \|j\|^2)^s \exp(2\sigma(1 + \|j\|^2)^{\frac{1}{2}}) \right)^{\frac{1}{2}}. \tag{1.13}$$

where v^j_k is the k^{th} component of the Fourier coefficient of the Fourier expansion into $\exp(i\frac{2\pi}{l} j \cdot x)$ of the periodic functions v. Note that $G^{\frac{s}{2}}_0(\Omega, \mathbf{R}^n) = H^s_{per}(\Omega, \mathbf{R}^n)$. For $\sigma > 0$ the Fourier coefficients V_j of $V \in G^{\frac{s}{2}}_\sigma(\Omega, \mathbf{R}^n)$ decay like $\|j\|^{-s} \exp(-\sigma\|j\|)$.

In the sequel we will usually drop for notational convenience the domain and the range of function spaces. Generalising a result of Ferrari and Titi [FT98, theorem 1] in a straight forward way, there is a time T^* such that as long $S(t, U_0)$ remains bounded in some ball inside H^s_{per}

$$S(t, U_0) \in G_t^{\frac{s}{2}} \text{ for } 0 \leq t \leq T^* \tag{1.14}$$

$$S(t, U_0) \in G_{T^*}^{\frac{s}{2}} \text{ for } t \geq T^*. \tag{1.15}$$

Hence the Fourier coefficients decay exponentially fast for positive times. This will be a by-product of showing similar results for more general and time-dependent equations in chapter 2.

1.3 Main results

Using the spatial regularity results we will prove a result like the Neishtadt theorem for a rapidly forced version of (1.9) with $G = (g_1, \ldots, g_n)$:

$$\frac{\partial}{\partial t} U(x,t) - D\Delta U(x,t) = F(U(x,t)) + hG(U(x,t), \frac{t}{h}, h \tag{1.16}$$

with periodic boundary conditions on $\Omega = [0,l]^d, d = 1,2,3$ and $U(0) = U_0 \in H^s_{per}(\Omega, \mathbf{R}^n)$. We denote the ball of radius R in the Banach space X by $B_X(R, 0)$.

Theorem 1 *Suppose that the $F : B_{\mathbf{R}^n}(R,0) \to \mathbf{R}^n, G : B_{\mathbf{R}^n}(R,0) \times \mathbf{R} \times [0, h_0) \to \mathbf{R}^n$ are entire on $B_{\mathbf{R}^n}(R,0)$, i.e. real analytic in U and given by a power series which is convergent on whole $B_{\mathbf{R}^n}(R,0)$. Let G be continuous with respect to t and periodic in $\tau = \frac{t}{h}$ with period 1. Let $F = (f_1, \ldots, f_n)$ and $G = (g_1, \ldots, g_n)$ be bounded on the complex δ-neighbourhood $B_{\mathbf{R}^n}(R,0) + \delta$ with bounds independent of t and h:*

$$\sup_{B_{\mathbf{R}^n}(R,0)+\delta} \|F\| \leq B_1, \qquad \sup_{B_{\mathbf{R}^n}(R,0)+\delta \times \mathbf{R}^+ \times [0,h_0)} \|G\| \leq B_2. \tag{1.17}$$

Furthermore assume that for initial values U_0 with $|U_0|_{H^s_{per}} \leq M_1$ the forward orbit remains bounded for $t \in [0,T]$, i.e. $|S(t,U_0)|_{H^s_{per}} \leq M_0$ with $M_0 \leq \frac{R}{C_{emb}}$.

Then there exists a real analytic and time-periodic change of coordinates for $0 < h < h_1$ of (1.16)

$$U = V + hW(V, \frac{t}{h}, h) \tag{1.18}$$

with W bounded on H^s. Its image $W(H^s_{per})$ is finite dimensional for fixed h and $W(.,0,.) = 0$. The transformed nonautonomous terms $\alpha = (\alpha_1, \ldots, \alpha_n)$ are exponentially small after a short transient, but the equation may contain non-local terms

$\bar{F} = (\bar{f}_1, \dots, \bar{f}_n)$:

$$\frac{\partial}{\partial t} V(x,t) - D\Delta V(x,t) = F(V(x,t)) + \bar{F}(V(t),h)(x) + \alpha(V(t), \frac{t}{h}, h)(x), \quad (1.19)$$

with

$$\sup_{V(0)\in B_{H^s_{per}}(M_1,0)} |\bar{F}(V(t))|_{H^s_{per}} \leq c_3 h + c_4 \exp(-\min(t,T^*)h^{-\frac{1}{3}}), \quad (1.20)$$

$$\sup_{V(0)\in B_{H^s_{per}}(M_1,0)\times\mathbf{R}^+\times[0,h_1)} |\alpha|_{H^s_{per}} \leq c_2 h \exp(-c_1 h^{-\frac{1}{3}}) + c_5 h \exp(-\min(t,T^*)h^{-\frac{1}{3}}) \quad (1.21)$$

where c_1, c_2, c_3, h_1 do not depend on U_0. The time T^* and c_4, c_5 depend on U_0 only via M_0.

Some remarks discussing this theorem are appropriate:

1. The transformed equation leaves the class of local superposition-type nonlinearities. The transformed nonlinearity $\bar{F}$ and α can only be evaluated nonlocally in space like $\bar{F}(V(t),h)(x)$, whereas F and G can be computed by point evaluation $F(V(x,t))$. A different version of theorem 1, where the assumptions F and G are formulated in terms of Gevrey norms, can be found in theorem 1' in chapter 3. There the transformed equation remains in the same class as the original one.

2. Our construction is discontinuous in h. But if G is smooth in h, then $\bar{F}$ and α can be chosen to depend smoothly on h for $h > 0$. Smoothness may break down for $h \to 0$ in general. An exception is analysed in remark 3.4.

3. The transformed nonlinearities consist out of two parts. First there is a finite dimensional part, where a finite-dimensional averaging result is applied. Secondly we have to consider some further correction terms caused by this approximation of the infinite-dimensional problem. The formulas for the correction terms are given in (3.21,3.22). The additional infinite-dimensional terms give rise to the time dependent error terms $\exp(-\min(t,T^*)h^{-\frac{1}{3}})$. These errors are damped due to the parabolic regularisation.

4. The estimates on $\bar{F}, \alpha$ can be improved when we estimate the L^2 norm. Then the correction term $\bar{F}$ converges to 0 for $h \to 0$:

$$\sup_{V(0)\in B_{H^s_{per}}(M_1,0)} |\bar{F}(V(t))|_{L^2} \leq c_3 h + \tilde{c}_4 h^{\frac{s}{3}} \exp(-\min(t,T^*)h^{-\frac{1}{3}}). \quad (1.22)$$

For α we get

$$\sup_{V(0)\in B_{H^s_{per}}(M_1,0)} |\alpha(V(t),\frac{t}{h},h)|_{L^2} \leq c_2 h \exp(-c_1 h^{-\frac{1}{3}}) + \tilde{c}_5 h^{1+\frac{s}{3}} \exp(-\min(t,T^*)h^{-\frac{1}{3}}). \tag{1.23}$$

A proof is given in remark 3.3.

5. Usually one wants to compare the autonomous part of (1.19)

$$\frac{\partial}{\partial t}V(x,t) - D\Delta V(x,t) = F(V(x,t)) + \bar{F}(V(t),h)(x) \tag{1.24}$$

with the full equation. If we start at $t = 0$ we get an error estimate of order $h^{\frac{s+4}{3}}$, when using the above L^2 estimates and the variation of constants formula. But if we start dropping the nonautonomous part after some transient time, we get an exponential estimate: $h\exp(-ch^{-\frac{1}{3}})$, see lemma 5.1.

1.4 Applications

As an application to an improved geometric theory we analyse the influence of rapid forcing to hyperbolic equilibria and to homoclinic orbits in one-parameter families of equations. We compare the period h map of the semi-evolution $S^h(h,0,.)$ defined by the forced equation (1.16) with the time-h map of the semiflow defined by the averaged equation (1.24). Then we can show several effects to be exponentially small.

When the original unforced equation (1.9) possesses a hyperbolic equilibrium, then (1.19) possesses a hyperbolic periodic orbit $\tilde{u}_h$ of period h and thus $\tilde{S}^h(h,0,.)$ a hyperbolic fixed point. We will show, that $\tilde{u}_h$ is exponentially close in h to a hyperbolic equilibrium of the averaged equation (1.24) and the difference of their phase portrait is also exponentially small, when being close enough to these points. Then the splitting of homoclinic orbits can only happen in exponentially small wedges in parameter space. In the special situation of a one-dimensional unstable manifold we can estimate the size of the splitting at least at some points of a possibly discrete homoclinic orbit.

As another application we will construct modified equations of (1.9) describing a very special discretization scheme up to exponentially small errors. We have to stress that the analysed scheme will be hardly used in practise due to very strict rules on the size of the time-steps. Anyway, this scheme will give a first example of a numerical scheme in parabolic partial differential equations for which a backward error analysis is possible. A scheme of full discretization of (1.9) in space and time is analysed. In space we use a Galerkin spectral method, which defines a system of analytic ordinary differential equations for

$$U_N(x,t) = \sum_{||j||\leq N} \begin{pmatrix} U_N^{1,j}(t) \\ \vdots \\ U_N^{n,j}(t) \end{pmatrix} \exp(i\frac{2\pi}{l}jx).$$

with multi-index $j \in \mathbf{Z}^d$. The equation is given by

$$\begin{aligned} \dot{U}_N + AU_N &= P_N F(U_N) \\ U_N(0) &= P_N U_0 \end{aligned} \tag{1.25}$$

where P_N denotes the L^2 projection on the real Galerkin approximation space H_N, which is spanned by eigenfunctions of A. We describe H_N in complex notation, but $U_N(x) \in \mathbf{R}^n$ for all $x \in \Omega$ and $U_N \in H_N$:

$$H_N = \{ \sum_{j\in\mathbf{Z}^d, ||j||\leq N} \begin{pmatrix} U^{1,j} \\ \vdots \\ U^{n,j} \end{pmatrix} \exp(i\frac{2\pi}{l}jx) | U^{l,j} \in \mathbf{C}, U^{l,j} = \overline{U^{l,-j}} \}$$

The ordinary differential equation (1.25) defines a local flow on H_N denoted by $S_N(t, U_N)$.

Let λ_N be the largest eigenvalue of $-Id\Delta$ restricted to H_N. Equation (1.25) is then discretised in time by some analytic one-step method (e.g. Runge-Kutta) with step size h and order p. The full discretization defines a local discrete semigroup on H_N which is analytic in U_0 and its j^{th} iterate is denoted by

$$\Phi^j_{h,N}(U_0), j = 0, 1, 2, \ldots$$

We will couple the step size h and the number of Galerkin modes via

$$h \cdot \lambda_N^{1+\beta} \leq C \tag{1.26}$$

for $\beta > 0$. Our main theorem for this kind of discretization is the infinite-dimensional counterpart of (1.7).

Theorem 2 *Consider equation (1.9) with the same assumptions on F as in theorem 1. Suppose, that the local semiflow $S(t, U_0)$, defined by (1.9), remains inside the ball $B_{H^s_{per}}(M_0, 0)$ for $t < \tilde{T} \leq \infty$ and some fixed $M_0 \leq \frac{R}{C_{emb}}$.*

Let the full discretization $\Phi_{h,N}$ be real analytic in U. Let the discretization be of order p for a complex extended domain with $N(h)$ defined by (1.26), i.e. there is $C : \mathbf{R}^+ \to \mathbf{R}^+$ with $C(r)$ bounded for $r \to 0$ independent of σ

$$|\Phi_{h,N}(U_0) - S_N(h, U_0)|_{G^{s/2}_\sigma} \leq C(|U_N(0)|_{G^{s/2}_\sigma})h^{p+1} \tag{1.27}$$

for U_N in a complex extension of $H_N \cap B_{G_\sigma^{s/2}}(R,0)$.

Then there exist h^*, t^*, T and a modified equation on $B_{H^s_{per}}(R,0)$

$$\dot{\bar{U}} + A\bar{U} = F(\bar{U}) + \bar{F}(\bar{U}, h, N(h)) \tag{1.28}$$

$$\bar{U}(0) = U_0 \tag{1.29}$$

which defines a local semiflow $\bar{S}_{h,N}(t, U_0)$, such that for $0 < h \leq h^*$ and $N(h)$ defined by (1.26):

$$|\Phi^j_{h,N}(U_0) - \bar{S}_{h,N}(j \cdot h, U_0)| \leq C\left(exp\left(-c_1 \min(t,t^*)\sqrt{\lambda_N}\right) + exp\left(-c_2 h^{-\frac{\beta}{1+\beta}}\right)\right) \tag{1.30}$$

for $0 < t = j \cdot h < T$ with

$$T = \begin{cases} \tilde{T} & , \text{ if } \tilde{T} < \infty \\ O(h^{-\frac{\beta}{1+\beta}}, \sqrt{\lambda_N}) & , \text{ if } \tilde{T} = \infty \end{cases}$$

The possible non-local perturbation

$$\bar{F} : H^s_{per} \times [0, h^*] \times \mathbf{N} \to H^s_{per}$$

is uniformly bounded in $h \to 0, N = N(h) \to \infty$ for $U \in B_{H^s}(M_0, 0)$:

$$|\bar{F}(U)|_{H^s} \leq C_1 h^p. \tag{1.31}$$

Thus we have a backward error analysis result here. There are some differences to theorem 1.

1. A modified autonomous differential equation is directly given, which corresponds more to the truncated equation (1.24) than to the transformed equation (1.19) in theorem 1.

2. Exponential estimates hold without a transient time, even if the estimates are weaker for small times t.

3. The assumptions on $\Phi_{h,N}$ are fulfilled, when discritising in time by some implicit discretization schemes (e.g. higher order Radau methods). But the order p, that these methods have as integrators of ordinary differential equations cannot be achieved, see section 4.1.

We will prove this theorem in chapter 4. The main ingredient will be the embedding of the numerical scheme into a nonautonomous equation. Then we can apply the averaging techniques of chapter 3.

The rest of the thesis is organised as follows. In chapter 2 we collect properties of Gevrey classes and prove the Gevrey regularity of solutions of (1.16) and for equations with quite general nonlinearities. In chapter 3 we show theorem 1 by proving a theorem 1' in a wider class of nonlinearities. The proof of theorem 2 is in chapter 4. In chapter 5 the local averaging theorem 1 is applied to describe the qualitative behaviour near equilibria and homoclinic orbits. A discussion of the assumptions, possible generalisations and related results can be found in chapter 6.

Throughout the thesis C will denote a constant, which might change its value from one line to another.

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Chapter 2

Gevrey Regularity

In this chapter we discuss the regularity of solutions of equations like (1.16) for $U = (u_1, \ldots, u_n)$:

$$\begin{aligned}\frac{\partial}{\partial t}U(x,t) - D\Delta U(x,t) &= F(U(x,t)) + G(U(x,t),t,h) && (2.1)\\ U(t_0) &= U_0 && (2.2)\end{aligned}$$

We extend results of Ferrari and Titi [FT98] to show that the solutions lie in some Gevrey class under appropriate conditions on the nonlinearities $F = (f_1, \ldots, f_n)$ and $G = (g_1, \ldots, g_n)$. We will state the conditions on the nonlinearities F and G in terms of the Gevrey classes defined in (1.12). In the next section we collect some preliminaries about Gevrey classes. Then we proceed to give a uniqueness and existence result for regular solutions using Galerkin approximations. Solutions will be shown to be in Gevrey classes. This extremely high regularity of the solutions and the exponential decay of the Fourier coefficients will be a key ingredient for the proof of theorem 1.

2.1 Properties of Gevrey classes

We list some properties of $G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$ defined in (1.12). Most of the proofs are adapted from [FT98]. The Gevrey class $G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$ is a Hilbert space with scalar product

$$(v,w)_{G_\sigma^{\frac{s}{2}}} = \sum_{k=1}^{n} \sum_{j \in \mathbf{Z}^d} v_k^j w_k^j (1 + d_k|j|^2)^s \exp(2\sigma(1+|j|^2)^{\frac{1}{2}}) \qquad (2.3)$$

where v^j and w^j are the Fourier coefficients in $\mathbf{R}^n$ of the periodic functions v and w.

Lemma 2.1 *For $s > \frac{d}{2}$ and $\sigma > 0$ all functions $U : \Omega \to \mathbf{R}^n, U \in G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$ are real analytic in the spatial variable $x \in \Omega$.*

Proof: To show, that $U = (u_1, \dots, u_n) \in G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$ is real analytic in $x \in \Omega$, we estimate for each u_k the derivative $\sup_{x\in\Omega} |D^\beta u_k(x)|$. We drop the index. When we show that there are M and r, such that for all multi-indices $\beta \in \mathbf{N}_0^d$

$$\sup_{x\in\Omega} |D^\beta u(x)| \leq M|\beta|! r^{-|\beta|},$$

then by e.g. [RR94, theorem 2.16] u is real analytic. This can be done using

$$\sup_{x\in\Omega} |D^\beta u(x)| = |D^\beta U|_{C^0} \leq C_{emb} |D^\beta U|_{H^s}.$$

So we have to estimate - for simplicity with the size of the domain $l = 1$

$$|D^\beta u|_{H^s} \leq \left(\sum_{j\in\mathbf{Z}^d} |j|^{2|\beta|+2s} |u^j|^2 \right)^{\frac{1}{2}},$$

using $|U|_{G_\sigma^{s/2}} \leq C$ we get

$$|u^j|^2 \leq C|j|^{-2s} \exp(-2\sigma|j|).$$

Thus we obtain

$$|D^\beta u|_{H^s} \leq C \left(\sum_{j\in\mathbf{Z}^d} |j|^{2|\beta|} \exp(-2\sigma|j|) \right)^{\frac{1}{2}} \leq \left(\sum_{m\in\mathbf{N}} 2m^{2|\beta|+2} \exp(-2\sigma m) \right)^{\frac{1}{2}}$$

Noting $m^{2|\beta|+2} \exp(-\sigma m) \leq \left(\frac{2|\beta|+2}{\sigma}\right)^{2|\beta|+2} \exp(-2|\beta| - 2)$ for all $m \in N$ we get

$$|D^\beta u|_{H^s} \leq C \left(\frac{|\beta|+1}{\sigma} \right)^{|\beta|+1} \exp(-|\beta| - 1) \left(\sum_{m\in\mathbf{N}} \exp(-\sigma m) \right)^{\frac{1}{2}} \leq M|\beta|! r^{-|\beta|}.$$

For the last step we used the Stirling formula. This gives the needed estimate on $|D^\beta U|_{C^0}$, thus the u_k and therefore $U \in G_\sigma^{s/2}$ are real analytic for $\sigma > 0$. $\square$

For $s > \frac{d}{2}$ the Gevrey classes are Banach algebras under point-wise multiplication. The constants are even independent of the exponent σ:

Lemma 2.2 *If $u, v \in G_\sigma^{s/2}$ with $s > \frac{d}{2}$ then w defined by some point-wise multiplication is in $G_\sigma^{s/2}$: Let $w(x) = \begin{pmatrix} u^{l_1}(x) \cdot v^{m_1}(x) \\ \vdots \\ u^{l_n}(x) \cdot v^{m_n}(x) \end{pmatrix}$ for arbitrary components*

$l_k, m_k \in 1, \ldots, n$ and there exists a constant C_s not depending on u, v and σ, such that

$$|w|_{G_\sigma^{s/2}} \leq C_s |u|_{G_\sigma^{s/2}} |v|_{G_\sigma^{s/2}}. \tag{2.4}$$

Proof: This result is a direct extension of the scalar case ($n = 1$) of [FT98, lemma 1]. This lemma yields for each component $w^k = u^{l_k} v^{m_k} \in D\left((-\Delta)^{\frac{s}{2}} \exp(\sigma(-\Delta)^{\frac{1}{2}})\right)$ with $|w^k|_{G_\sigma^{s/2}} \leq C_s |u^{l_k}|_{G_\sigma^{s/2}} |v^{m_k}|_{G_\sigma^{s/2}}$. Summing these inequalities leads to (2.4). □

Using this lemma we can show, that a big class of superposition-type nonlinearities map $G_\sigma^{s/2}$ into itself. The function F is required to be real analytic on some ball $B_{\mathbf{R}^n}(R, 0)$ and that it is given by power series which is convergent on whole $B_{\mathbf{R}^n}(R, 0)$, i.e. $F(U) = \sum_{|j|=0}^{\infty} \beta_j U^j$ with multi-indices $j \in \mathbf{N}_0^n$ and $\beta_j \in \mathbf{R}^n$. Then we can explicitly choose a majorising function a given by a power series $\sum_{l=0}^{\infty} a_l s^l$ with $a_l = \sum_{|j|=l} \|\beta_j\|$, such that

$$\|F(U)\| \leq a(\|U\|) \text{ for all } U \in B_{\mathbf{R}^n}(R, 0). \tag{2.5}$$

Lemma 2.3 *Let $F : B_{\mathbf{R}^n}(R, 0) \to \mathbf{R}^n$ be real analytic and given by a power series, which is convergent on whole $B_{\mathbf{R}^n}(R, 0)$ with a majorising function as in (2.5). If $u \in G_\sigma^{s/2}$ with $|u|_{G_\sigma^{s/2}} \leq R_0$ and $s > \frac{d}{2}$, such that $|u|_{BC} \leq \frac{R}{C_s}$, then*

$$|F(U)|_{G_\sigma^{s/2}} \leq (1 + C_s^{-1}) a(C_s |U|_{G_\sigma^{s/2}}). \tag{2.6}$$

Furthermore $F : B_{G_\sigma^{s/2}}(R, 0) \to G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$ is differentiable and $F'(U)$ is a uniformly bounded linear mapping in $L^2(\Omega, \mathbf{R}^n)$ for U bounded in H_{per}^s.

Proof: The first part is an easy generalisation of the scalar case in [FT98]. We use the Taylor series expansion of F and estimate partial sums:

$$\begin{aligned} \left| \sum_{|k|=0}^{N} \beta_k U^k \right|_{G_\sigma^{s/2}} &\leq \sum_{|k|=0}^{N} \|\beta_k\| |U^k|_{G_\sigma^{s/2}} \leq \|\beta_0\| + \sum_{|k|=1}^{N} \|\beta_k\| C_s^{|k|-1} |U|_{G_\sigma^{s/2}}^k \\ &\leq (1 + C_s^{-1}) \sum_{|k|=0}^{N} \|\beta_k\| C_s^{|k|} |U|_{G_\sigma^{s/2}}^k \leq (1 + C_s^{-1}) a(C_s |U|_{G_\sigma^{s/2}}). \end{aligned}$$

As $\left|\sum_{|k|=M}^{N} \beta_k U^k\right|_{G_\sigma^{s/2}} \leq (1 + C_s^{-1}) \sum_{|k|=M}^{N} \|\beta_k\| C_s^{|k|} |U|_{G_\sigma^{s/2}}^k \to 0$ holds for $N, M \to \infty$, when passing to the limit for $N \to \infty$ we get $\sum_{|k|=0}^{N} \beta_k U^k \to \tilde{F}(.) \in G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$ by the completeness of $G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$. The norm of $G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$ is stronger than the uniform convergence norm, since even $G_0^{s/2}(\Omega, \mathbf{R}^n) = H_{per}^s(\Omega, \mathbf{R}^n)$ embeds continuously into $BC(\Omega, \mathbf{R}^n)$. Hence the analyticity of $F(u)$ in u and the uniform boundedness of U in Ω imply $\sum_{|k|=0}^{N} \beta_k U^k \to F(U(.))$. Therefore $\tilde{F}(x) = F(U(x))$.

To show, that F is differentiable as a map from $B_{G_\sigma^{s/2}}(R,0)$ to $G_\sigma^{s/2}$, we show

$$|F(U(.)+hV(.)) - F(U(.)) - F'(U(.))hV(.)|_{G_\sigma^{s/2}} = o(h)$$

for $U, V \in G_\sigma^{s/2}$. $F : B_{\mathbf{R}^n}(R,0) \to \mathbf{R}^n$ is differentiable, since F is analytic. Furthermore each component of F', given by the partial derivative $\frac{\partial f_k}{\partial u_l}$, is majorised by a power series $\tilde{a}$. Hence applying the first part of the lemma gives, that $\frac{\partial f_k}{\partial u_l}(U) \in G_\sigma^{s/2}$ and thus $F'(U(.))hV(.) \in G_\sigma^{s/2}$. Taking a standard representation of the remainder r of the Taylor expansion $r(U(.), hV(.)) = F(U(.)+hV(.)) - F(U(.)) - F'(U(.))hV(.)$ gives

$$r(U(.), hV(.)) = \int_0^1 F''(U(.)+thV(.))(hV(.), hV(.))dt.$$

As F'' is given by a convergent power series, each $F''(U(.)+thV(.))(V(.), V(.)) \in G_\sigma^{s/2}$ for $U, V \in G_\sigma^{s/2}$ and thus $|r(U, hV)|_{G_\sigma^{s/2}} \le h^2 C$. Hence $F : B_{G_\sigma^{s/2}}(R,0) \to G_\sigma^{s/2}$ is differentiable and its derivative is given by $F'(U)$.

Then we can also estimate $|F'(U)V|_{L^2} = |F'(U(.))V(.)|_{L^2}$. As $H^s_{per} = G_0^{s/2}$ embeds continuously into $BC(\Omega, \mathbf{R}^n)$ and $F'(U(x))$ is bounded for $U(x) \in B_{\mathbf{R}^n}(R,0)$, we obtain $|F'(U(.))V(.)|_{L^2} \le M|V(.)|_{L^2}$ and the lemma is proved. $\square$

2.2 Existence and uniqueness of regular solutions

The setting of Nemitski operators $F : G_\sigma^{s/2} \to G_\sigma^{s/2}$ defined by superposition-type nonlinearities as in lemma 2.3 leads to the general setting of differentiable operators $G_\sigma^{s/2} \to G_\sigma^{s/2}$. We prove for this kind of nonlinearities an existence and uniqueness result for regular solutions. So we will first define, what we mean by a solution of the initial value problem (2.1) and (2.2):

Definition 2.4 *Let $U_0 \in H^s_{per}(\Omega, \mathbf{R}^n)$ with $s > \frac{d}{2}$. A function*

$$U \in C([t_0,t_1], H^s_{per}(\Omega, \mathbf{R}^n)) \cap L^2([t_0,t_1], D(A))$$

is called a regular solution of (2.1) and (2.2) on $[t_0,t_1]$, if $\frac{dU}{dt} \in L^2([t_0,t_1], L^2(\Omega, \mathbf{R}^n))$ and

$$(\frac{dU}{dt}, \phi)_{L^2} + (A^{\frac{1}{2}}U, A^{\frac{1}{2}}\phi)_{L^2} - (F(U), \phi)_{L^2} - (G(U, \frac{t}{h}, h), \phi)_{L^2} = 0 \tag{2.7}$$

for every $\phi \in H^1_{per}(\Omega, \mathbf{R}^n)$ and almost all $t \in [t_0,t_1]$.

We rewrite (2.1) with general nonlinear operators

$$\begin{aligned} \frac{\partial}{\partial t}U + AU &= F(U) + G(U,t) \\ U(0) &= U_0 \end{aligned} \tag{2.8}$$

with $A = -\text{diag}(d_1, \ldots, d_n)\Delta$ on $\Omega = [0, l]^d$ with periodic boundary conditions. Let $d_{min} = \min(d_1, \ldots, d_n) > 0$. The initial time $t_0 = 0$ is used just for notational simplicity. For U_0, F, G we use the following assumptions

$$\begin{array}{l} U_0 \in H^s_{per} \text{ with } s > \frac{d}{2} \\ F(.), G(., t) : B_{G^{s/2}_\sigma}(R, 0) \to G^{s/2}_\sigma \text{ differentiable in } U \\ F'(U), G'(U) \in L(L^2(\Omega, \mathbf{R}^n), L^2(\Omega, \mathbf{R}^n)) \text{ for } U \text{ bounded in } H^s_{per} \\ G \text{ continuous in } t \\ F(.), G(.) \text{ bounded with constants not depending on } \sigma, t : \\ \ |F(U)|_{G^{s/2}_\sigma} \leq C_s a(C_s |U|_{G^{s/2}_\sigma}) \\ \ |G(U,t)|_{G^{s/2}_\sigma} \leq C_s b(C_s |U|_{G^{s/2}_\sigma}) \\ \ \text{with } a, b \text{ monotone increasing on } [0, \frac{R}{C_s}) \end{array} \tag{H.1}$$

This hypothesis is fulfilled by all nonlinearities fulfilling the assumptions of lemma 2.3. But we might have to decrease the radius R.

Proposition 2.5 *Let the hypothesis (H.1) be satisfied. Also assume* $|U|_{H^s_{per}} \leq M_0 < \frac{R}{C_{emb}}$ *for some* $M_0 > 0$. *Then there exists a constant* $T^*(M_0)$ *such that equation (2.8) has a unique regular solution on* $[0, T^*)$ *with initial value* U_0. *Furthermore* $U(., t) \in G^{s/2}_t$ *and* $|U(., t)|_{G^{s/2}_t} \leq 2C_0 |U(0)|_{H^s} + 1$ *for* $t \in [0, T^*)$.

Proof:

Using results in Henry [Hen83] we could prove local uniqueness and existence of solutions for a different definition of a solution. Both coincide when F and G are highly regular, such that we have classical solutions, but we will follow [FT98]. Their existence proof will lead directly to the higher space regularity. We start by proving uniqueness.

Assume, that U is a regular solution, then $AU \in L^2(\Omega, \mathbf{R}^n)$. From definition 2.4 we have

$$\frac{d}{dt}(U, \phi)_{L^2} = (-AU + F(U) + G(U, t), \phi)_{L^2}$$

for almost all $t \in [0, T)$ and all $\phi \in H^1_{per}(\Omega, \mathbf{R}^n)$. By regularity of U and (H.1) we have

$$-AU(t) + F(U(t)) + G(U(t), t) \in L^2([0, T), L^2(\Omega, \mathbf{R}^n)) \subset L^1([0, T), L^2(\Omega, \mathbf{R}^n)).$$

Hence we get e.g. by [Tem77, lemma 1.1, ch.3], that

$$\frac{d}{dt}U(t) + AU(t) = F(U(t)) + G(U(t), t) \tag{2.9}$$

holds in $L^2(\Omega, \mathbf{R}^n)$ for almost every $t \in [0,T]$.

Letting U, V be any two regular solutions on $[0,T]$ with initial value U_0. By equation (2.9) we have

$$\frac{d}{dt}(U-V)(t) + A(U(t)-V(t)) = F(U(t)) - F(V(t)) + G(U(t),t) - G(V(t),t)$$

Since $(U-V)(t) \in H^s_{per}(\Omega, \mathbf{R}^n) \subset L^2(\Omega, \mathbf{R}^n)$ we can take the scalar product with $(U-V)(t)$ to get

$$\begin{aligned} & (\frac{d}{dt}(U-V)(t), (U-V)(t))_{L^2} + (A(U(t)-V(t)), U-V)_{L^2} \\ = \; & (F(U(t)) - F(V(t)), (U-V)(t))_{L^2} + (G(U(t),t) - G(V(t),t), (U-V)(t))_{L^2} \end{aligned}$$

As $\frac{d}{dt}(U-V)(t) \in L^2([0,T), L^2(\Omega, \mathbf{R}^n))$ we have that the first term on the left hand side equals $\frac{1}{2}\frac{d}{dt}|(U-V)(t)|_{L^2}$. Using the mean value theorem and the boundedness of $F'(U)$ and $G'(U,t)$ as a linear mapping from $L^2(\Omega, \mathbf{R}^n)$ to $L^2(\Omega, \mathbf{R}^n)$ we get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|(U-V)(t)|^2_{L^2} + |A^{\frac{1}{2}}(U-V)(t)|^2_{L^2} \\ \leq \; & (|F(U(t)) - F(V(t))|_{L^2} + |G(U(t),t) - G(V(t),t)|_{L^2})\, |(U-V)(t)|_{L^2} \\ \leq \; & \Bigg(\sup_{|W|_{H^s_{per}} \leq K} |F'(W)|_{L(L^2,L^2)} |(U-V)(t)|_{L^2} \\ & + \sup_{|W|_{H^s_{per}} \leq K} |G'(W,t)|_{L(L^2,L^2)} |(U-V)(t)|_{L^2} \Bigg) \, |(U-V)(t)|_{L^2} \end{aligned}$$

with $K \leq 2\max(\sup |U(t)|_{H^s_{per}}, \sup |V(t)|_{H^s_{per}})$, hence we obtain

$$\frac{1}{2}\frac{d}{dt}|(U-V)(t)|^2_{L^2} + \frac{1}{2}|A^{\frac{1}{2}}(U-V)(t)|^2_{L^2} \leq C_K |(U-V)(t)|^2_{L^2}$$

and by the Gronwall inequality

$$|U(t) - V(t)|^2_{L^2} \leq |U(0) - V(0)|^2_{L^2} e^{C_K t} = 0.$$

Thus U and V coincide and the uniqueness is proven.

To show existence we use a Galerkin approximation as an analytic tool with a similar setting as for the numerical scheme discussed in chapter 4. So

$$U_N(x,t) = \sum_{\|j\| \leq N} \begin{pmatrix} \alpha_N^{1,j}(t) \\ \vdots \\ \alpha_N^{n,j}(t) \end{pmatrix} \exp(i\frac{2\pi}{l} jx).$$

with multi-index $j \in \mathbf{Z}^d$ defines a sequence of approximative solutions and fulfils the ordinary differential equation

$$\begin{aligned} \dot{U}_N + AU_N &= P_N F(U_N) + P_N G(U_N, t) \\ U_N(0) &= P_N U_0 \end{aligned} \tag{2.10}$$

This is a well-posed ordinary differential equation with a unique solution for every N for some interval $[0, T_N]$, because the vector field in (2.10) is locally Lipschitz in U_N and continuous in time t. To show convergence for the approximating solutions in $G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$ we derive a priori estimates for the U_N, not depending on N. Note that $U_N(t) \in G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$ for all $t \in [0, T_N]$ and $\sigma \geq 0$.

We apply $A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}}$ to the differential equation (2.10) and take the L^2 scalar product with $A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t)$ to get

$$\begin{aligned} & \left(A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} \frac{d}{dt} U_N(t) + A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} AU_N(t), A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t) \right)_{L^2} \\ = & \left(A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} P_N F(U_N(t)) + A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} P_N G(U_N(t), t), A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t) \right)_{L^2}. \end{aligned}$$

Using the equality

$$\begin{aligned} & \left(A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} \frac{d}{dt} U_N(t), A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t) \right)_{L^2} \\ = & \frac{1}{2} \frac{d}{dt} \left| A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t) \right|_{L^2}^2 - \left(A^{\frac{s}{2}} (-\Delta)^{\frac{1}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t), A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t) \right)_{L^2} \end{aligned}$$

we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left| A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t) \right|_{L^2}^2 + \left(A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} AU_N(t), A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t) \right)_{L^2} \\ = & \left(A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} P_N F(U_N(t)), A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t) \right)_{L^2} \\ & + \left(A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} P_N G(U_N(t), t), A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t) \right)_{L^2} \\ & + \left(A^{\frac{s}{2}} (-\Delta)^{\frac{1}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t), A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t) \right)_{L^2} \end{aligned}$$

Then we use the symmetry of $A^{\frac{1}{2}}$ and for the last term on the right hand side the Cauchy-Schwarz inequality and the Young inequality $ab < \epsilon a^2 + C_\epsilon b^2$ with $C_\epsilon = \frac{1}{4\epsilon}$. Thus we estimate

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |U_N(t)|^2_{G_\sigma^{s/2}} + |U_N(t)|^2_{G_\sigma^{(s+1)/2}} \\ \leq & \epsilon |A^{\frac{s}{2}} (-\Delta)^{\frac{1}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t)|^2_{L^2} + C_\epsilon |U_N(t)|^2_{G_t^{s/2}} \end{aligned}$$

$$+|(A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}P_N F(U_N(t)), A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t))_{L^2}|$$
$$+|(A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}P_N G(U_N(t),t), A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t))_{L^2}|$$

As $|\Delta V|_{L^2} \leq \frac{1}{d_{min}}|AV|_{L^2}$ with $d_{min} = \min(d_1, \ldots, d_n)$ we have

$$|A^{\frac{s}{2}}(-\Delta)^{\frac{1}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t)|^2_{L^2} \leq \frac{1}{d_{min}}|U_N(t)|^2_{G_t^{(s+1)/2}}$$

and then

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|U_N(t)|^2_{G_t^{s/2}} + (1 - \frac{\epsilon}{d_{min}})|U_N(t)|^2_{G_t^{(s+1)/2}} \\ \leq \quad & C_\epsilon|U_N(t)|^2_{G_t^{s/2}} + |(A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}P_N F(U_N(t)), A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t))_{L^2}| \\ + \quad & |(A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}P_N G(U_N(t),t), A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t))_{L^2}| \end{aligned} \tag{2.11}$$

Using the Cauchy-Schwarz inequality and dividing by $|U_N(t)|_{G_t^{s/2}}$ in inequality (2.11), choosing $\epsilon = \frac{d_{min}}{2}$ and using

$$\begin{aligned} |P_N F(U_N)|_{G_\sigma^{s/2}} & \leq |F(U_N)|_{G_\sigma^{s/2}} \text{ and} \\ |P_N G(U_N,t)|_{G_\sigma^{s/2}} & \leq |G(U_N,t)|_{G_\sigma^{s/2}} \end{aligned}$$

we get

$$\frac{d}{dt}|U_N(t)|_{G_t^{s/2}} + \frac{1}{2}|U_N(t)|_{G_t^{s+1/2}} \leq \frac{1}{d_{min}}C|U_N(t)|_{G_t^{s/2}} + |F(U_N)|_{G_t^{s/2}} + |G(U_N,t)|_{G_t^{s/2}}.$$

The nonlinearities can be estimated using (H.1) and by integrating time

$$\begin{aligned} & |U_N(t)|_{G_t^{s/2}} + \frac{1}{2}\int_0^t |U_N(\theta)|_{G_\theta^{(s+1)/2}}d\theta \\ \leq \quad & |P_N U_0|_{G_0^{s/2}} + \int_0^t \frac{1}{d_{min}}C|U_N(\theta)|_{G_\theta^{s/2}}d\theta + \int_0^t C_s a(C_s|U_N(\theta)|_{G_\theta^{s/2}})d\theta \\ + \quad & \int_0^t C_s b(C_s|U_N(\theta)|_{G_\theta^{s/2}})d\theta \end{aligned} \tag{2.12}$$

Now we try to bound the first term on the left hand side. As $|P_N U_0|_{G_0^{s/2}} \leq |U_0|_{G_0^{s/2}} \leq C_0|U_0|_{H^s_{per}}$ and as $|U_N(t)|_{G_t^{s/2}}$ is continuous in t, the norm of the solution can be bounded for some time $T_N > 0$:

$$|U_N(t)|_{G_t^{s/2}} \leq 2|U_0|_{G_0^{s/2}} + 1 \leq 2C_0|U_0|_{H^s_{per}} + 1 = \tilde{M}_1$$

As the majorising functions are increasing in $\mathbf{R}_+$, we have for $t \in [0, T_N]$ by (2.12):

$$\begin{aligned} & |U_N(t)|_{G_t^{s/2}} \\ \leq \quad & C_0|U_0|_{H^s_{per}} + t\left[\frac{1}{d_{min}}C\tilde{M}_1 + C_s a(C_s\tilde{M}_1) + C_s b(C_s\tilde{M}_1)\right] \end{aligned} \tag{2.13}$$

So we have a bound on $|U_N(t)|_{G_t^{s/2}}$ not depending on N, but we still have to show that T_N is bounded away from 0 for all N. The inequality (2.13) gives also some estimate. When considering $|U_N(t)|_{G_t^{s/2}} \leq 2C_0|U_0|_{H^s_{per}} + 1 = \tilde{M}_1$ we get

$$T^*(|U_0|_{H^s_{per}}) = \frac{C_0|U_0|_{H^s_{per}} + 1}{\frac{1}{d_{min}}C\tilde{M}_1 + C_s a(C_s\tilde{M}_1) + C_s b(C_s\tilde{M}_1)} \tag{2.14}$$

and thus $0 < T^* \leq T_N$ for all N. Hence we have now for $t \in [0, T^*]$

$$\sup_{0\leq t\leq T^*} |U_N(t)|_{G_t^{s/2}} \leq C. \tag{2.15}$$

with C only depending on $|U_0|_{H^s_{per}}$. Next we have to show, that the U_N converge to a limit function U. Using (2.12) we also have

$$\int_0^{T^*} |U_N(\theta)|_{G_\theta^{(s+1)/2}} d\theta \leq C. \tag{2.16}$$

By the Sobolev embedding theorem (see e.g. [Ad75, ch. VII]) we get

$$\sup_{0\leq t\leq T^*} |U_N(t)|_{L^\infty} \leq C. \tag{2.17}$$

Using the last inequalities (2.15, 2.16, 2.17) and (2.10) we also obtain

$$\int_0^{T^*} |\frac{d}{d\theta}U_N(\theta)|_{G_\theta^{(s+1)/2}} d\theta \leq C$$

By standard compactness results see e.g. the book by Constantin and Foias [CF88, ch 8] there exists a subsequence of U_N denoted by U_N that converges to some limit function U strongly in $L^2((0,T^*), H^s_{per})$ and weakly in $L^2((0,T^*), H^{s+1}_{per})$. By subtracting another subsequence, we may assume $U_N(t) \to U(t)$ for almost all $t \in [0, T^*]$. Next we have to show that U satisfies (2.8).

Integrating equation (2.10) in $0 \leq t_0 < t \leq T^*$ and taking the scalar product with some $\phi \in H^1_{per}$ yields for almost all t_0, t:

$$\begin{aligned} &(U_N(t), \phi)_{L^2} - (U_N(t_0), \phi)_{L^2} + \int_{t_0}^t (A^{\frac{1}{2}}U_N(\theta), \phi)_{L^2} d\theta \\ = &\int_{t_0}^t (P_N F(U_N(\theta)) + G(U_N(\theta), \theta), \phi)_{L^2} d\theta \end{aligned}$$

By the convergence of $U_N(\theta) \to U(\theta)$ for almost all $\theta \in [0, T^*]$, we get that the left hand side converges to $(U_N(t), \phi)_{L^2} - (U_N(t_0), \phi)_{L^2} + \int_{t_0}^t (A^{\frac{1}{2}}U_N(\theta), \phi)_{L^2} d\theta$. For the right hand side we use the following estimate for F:

$$\begin{aligned} &|P_N F(U_N(\theta)) - F(U(\theta))|_{L^2} \\ \leq\ &|P_N F(U_N(\theta)) - P_N F(U(\theta))|_{L^2} + |P_N F(U(\theta)) - F(U(\theta))|_{L^2} \\ \leq\ &\sup_{|W|_{H^s_{per}} \leq C} |F'(W)|_{L(L^2,L^2)} |U_N - U|_{L^2} + |P_N F(U(\theta)) - F(U(\theta))|_{L^2}, \end{aligned}$$

which converges to zero for $N \to \infty$ and almost all θ, because by (H.1) $F(U) \in H^s_{per}$. These estimates hold in the same way for G. Hence U fulfils equation (2.7). Using further estimates and arguments as in [Tem88, ch III, sec 1.1] the solution is continuous in time, even $U \in C([0,T^*], H^s_{per}(\Omega, \mathbf{R}^n)) \cap L^2([0,T^*], D(A)$. Hence U satisfies definition 2.4 and is the unique solution.

It remains to show, that $U(t) \in G_t^{s/2}(\Omega, \mathbf{R}^n)$ is bounded for $t \in [0, T^*]$. We will prove this by contradiction using the estimate (2.15) and the uniqueness of the solution. So suppose $U(t) \in G_t^{s/2}(\Omega, \mathbf{R}^n)$ is not bounded, then there exists an interval $[t_0, t_1]$ and m such that $|P_m U(t)|_{G_t^{s/2}} > C + \epsilon$ but $|U_N(t)|_{G_t^{s/2}} < C$ for all N and $t \in [t_0, t_1]$. Thus we have

$$
\begin{aligned}
C(t_1 - t_0) &\geq \int_{t_0}^{t_1} |U_N(t)|_{G_t^{s/2}} dt \geq \int_{t_0}^{t_1} |P_m U_N(t) - P_m U(t) + P_m U(t)|_{G_t^{s/2}} dt \\
&\geq \int_{t_0}^{t_1} |P_m U(t)|_{G_t^{s/2}} dt - \int_{t_0}^{t_1} |P_m U_N(t) - P_m U(t)|_{G_t^{s/2}} dt \\
&> (C+\epsilon)(t_1 - t_0) - \int_{t_0}^{t_1} \exp(2t_1\sqrt{1+m^2}) |P_m U_N(t) - P_m U(t)|_{H^s_{per}} dt \\
&> (C + \frac{\epsilon}{2})(t_1 - t_0)
\end{aligned}
$$

for N large enough as $U_N \to U$ in $L^2([t_0,t_1], H^s_{per})$ and hence in $L^1([t_0,t_1], H^s_{per})$, thus there is a contradiction. Therefore the solution U is bounded in Gevrey classes by $|U(t)|_{G_t^{s/2}} < C = 2C_0 |U_0|_{H^s_{per}} + 1$. □

For large time intervals we get the following easy corollary as in [FT98].

Corollary 2.6 *If $|U(t)|_{H^s_{per}} \leq M_0$ for $t \in [0,T]$, then $U(t) \in G_t^{s/2}(\Omega, \mathbf{R}^n)$ for $t \in [0,T^*]$ and $U(t) \in G_{T^*}^{s/2}(\Omega, \mathbf{R}^n)$ for $t \in [T^*, T]$ with T^* given explicitly by (2.14).*

Proof : We show $U(T^* + t) \in G_{T^*}^{s/2}$ for $0 \leq t \leq \min(T^*, T - T^*)$, the corollary follows by induction. We define $V(\theta) = U(\theta + t)$ for $0 \leq t, \theta \leq T^*$. Then as $|V(0)|_{H^s_{per}} \leq M_0$ we can apply proposition 2.5 to V gives $V(\theta) \in G_\theta^{s/2}(\Omega, \mathbf{R}^n)$ and hence $U(T^* + t) = V(T^*) \in G_{T^*}^{s/2}(\Omega, \mathbf{R}^n)$. □

Thus we have high regularity in space, as U is real analytic in the spatial variable $x \in \Omega$ by lemma 2.1. We could also prove high regularity in time following Promislow [Pro91]. If the nonlinearities F, G are analytic in U and t, then U is analytic in t. Time can be extended to a complex strip, but we will not use this high regularity in this paper.

Chapter 3

Averaging

We prove theorem 1 using an adaption of Neishtadt's methods [Nei84] and applying the high regularity in space proved in proposition 2.5. Let us describe the idea of the proof first. We split the phase space into a finite dimensional part, where we use a Neishtadt-like coordinate change, and into an infinite dimensional part. The finite dimensional part will be the Galerkin approximation space H_N, where N is chosen depending on h with $N \to \infty$ for $h \to 0$. Using the averaging of the Neishtadt result, the forcing term can be removed up to exponentially small terms with constants depending on N. Due to Gevrey regularity the influence of the higher modes is also exponentially small with $N \to \infty$. We will prove the theorem by taking both exponential smallness results together for an appropriate coupling of N and h.

In the proof we do not directly use, that the nonlinearities are of superposition-type. In fact we prove a version of theorem 1 in a larger class of nonlinearities F and G. This is also useful for the proof of theorem 2. We consider nonlinearities F and G, which are operators on Gevrey classes fulfilling some analyticity assumptions.

We will first describe these general assumptions. Then we state theorem 1' which is the version of the main result for the larger class of nonlinearities. Before proving theorem 1' we show that theorem 1' implies theorem 1. We end this chapter by giving two remarks on L^2 estimates and smooth dependence on the parameter h.

We consider an equation like (2.1) but with small rapid forcing.

$$
\begin{aligned}
\frac{\partial}{\partial t}U - D\Delta U &= F(U) + h^p G(U, \frac{t}{h}, h) && (3.1)\\
U(t_0) &= U_0 && (3.2)
\end{aligned}
$$

Additional to the assumption (H.1) given on page 18 we will need further assumptions on the smoothness of F and G. They are defined on the ball $B_{G_\sigma^{s/2}}(R, 0)$. Let F and G be real analytic, when restricted to the finite dimensional approximation spaces $H_N \cap B_{G_\sigma^{s/2}}(R,0)$, i.e. the restricted and projected nonlinearities

$$P_N F, P_N G : D_N = H_N \cap B_{G_\sigma^{s/2}}(R,0) \to H^N$$

are real analytic taking $H_N \cong \mathbf{R}^k$ for $k = dim_{\mathbf{R}}(H_N)$. Moreover we assume that the analytic continuation to a complex δ-neighbourhood

$$D_N + \delta := \{U = \sum_{j \in Z^d, \|j\| \le N} U^j \exp(i\frac{2\pi}{l} jx) | U^j \in \mathbf{C}^n, \inf_{V \in D_N} |U - V|_{G_\sigma^{s/2}(\Omega, \mathbf{C}^n)} < \delta\}$$

is uniformly bounded in σ, N, t and h. To summarise

$$\begin{array}{l} P_N F, P_N G : D_N \to H_N \text{ is real analytic} \\ \sup_{U_N \in D_N + \delta} |P_N F(U_N)|_{G_\sigma^{s/2}} \le B_1 \\ \sup_{U_N \in D_N + \delta, t \in \mathbf{R}} |P_N G(U_N, t, h)|_{G_\sigma^{s/2}} \le B_2 \end{array} \tag{H.2}$$

Theorem 1' *Assume that the hypotheses (H.1) and (H.2) hold. Let G be periodic in $\tau = \frac{t}{h}$ with period 1 and $p > 0$. Suppose for initial values U_0 with $|U_0|_{H^s} \le M_1$, that the forward orbit remains bounded, i.e. $|S(U_0,t)|_{H^s} \le M_0 < R$ for $t \in [0,T)$. Then there exists a real analytic and time-periodic change of coordinates for $0 < h < h_1$*

$$U = V + hW(V, \frac{t}{h}, h) \tag{3.3}$$

with W bounded on H^s. Its image $W(H^s_{per}, \frac{t}{h}, h)$ is finite dimensional for fixed h and $W(.,0,.) = 0$. The transformed nonautonomous terms are exponentially small after a transient, but the equation may contain additional non-local terms $\bar{F} = (\bar{f}_1, \dots, \bar{f}_n)$:

$$\frac{\partial}{\partial t} U(x,t) - D\Delta U(x,t) = F(U(t))(x) + \bar{F}(U(t),h)(x) + \alpha(U(t), \frac{t}{h}, h)(x), \tag{3.4}$$

with

$$\sup_{V(0) \in B_{H^s_{per}}(M_1,0)} |\bar{F}(V(t))|_{H^s_{per}} \le c_3 h^p + c_4 \exp(-\min(t,T^*) h^{-\frac{1}{3}}), \tag{3.5}$$

$$\sup_{V(0) \in B_{H^s_{per}}(M_1,0) \times \mathbf{R}^+ \times [0,h_1)} |\alpha|_{H^s_{per}} \le c_2 h^p \exp(-c_1 h^{-\frac{1}{3}})$$

$$+ c_5 h^p \exp(-\min(t,T^*) h^{-\frac{1}{3}}) \tag{3.6}$$

where the constants c_1, c_2, c_3, h_1 only depend on B_1 and B_2 and the constants T^*, c_4, c_5 depend on the majorising functions a and b in $(H.1)$ and M_0.

Remark 3.1 *Theorem 1' implies theorem 1.*

Proof : We have to check that the nonlinearities F and G described in theorem 1 fulfil hypotheses (H.1) and (H.2).

F and G satisfy the assumptions of lemma 2.3. A majorising power series can be chosen as in equation (2.5) for F and for G uniformly in t. Hence F and G fulfil hypothesis (H.1) with a changed radius for the domain $B_{G_\sigma^{s/2}}(R,0)$.

When we restrict and project F and G to H_N, they are power series of the Fourier-coefficients $\alpha_j^N = (\alpha_{j,1}^N, \ldots, \alpha_{j,n}^N)^T \in \mathbf{R}^n, j \in \mathbf{Z}^d, \|j\| \leq N$ of the functions

$$U_N = \sum_{\|j\|\leq N} \alpha_j^N \exp(i\frac{2\pi}{l}jx).$$

This can be seen, when we insert U_N into F:

$$F(U_N) = \sum_{k\in\mathbf{N}_0^n} \beta_k U_N^k$$

Taking a single monomial $\beta_k U_N^k$ first, we derive

$$\beta_k U_N^k = \beta_k \sum_{j\in\mathbf{Z}^d} \Big(\sum_{\substack{j_1,\ldots,j_n \in \mathbf{Z}^d \\ \|j_1\|,\ldots,\|j_n\| \leq N \\ k_1 j_1 + \ldots + k_n j_n = j}} (\alpha_{j_1,1}^N)^{k_1} \cdot \ldots \cdot (\alpha_{j_n,n}^N)^{k_1} \exp(i\frac{2\pi}{l}jx)\Big)$$

with $|U_N^k|_{G_\sigma^{s/2}} \leq C_s^{|k|} |U_N|_{G_\sigma^{s/2}}^{|k|}$ by lemma 2.2. When considering the projection to a single Fourier coefficient in the range we get

$$\begin{aligned}
&< F(U_N), \exp(-i\frac{2\pi}{l}jx) > \\
= & \sum_{k\in\mathbf{N}_0^n} \beta^k \left(\sum_{j_1,\ldots,j_n\in\mathbf{Z}^d, \|j_1\|,\ldots,\|j_n\|\leq N, k_1 j_1+\ldots+k_n j_n = j} (\alpha_{j_1,1}^N)^{k_1} \cdot \ldots \cdot (\alpha_{j_n,n}^N)^{k_n} \right)
\end{aligned}$$

This power series converges for $|\alpha_{j,l}^N| \leq R$ by lemma 2.3. Hence $P_N F$ is real analytic when restricted to $H_N \cap B_{G_\sigma^{s/2}}(R,0)$. In the same manner this can be shown for G.

As $F, G : B_{\mathbf{R}^n}(R,0) \to \mathbf{R}^n$ are supposed to be bounded on a complex neighbourhood of $B_{\mathbf{R}^n}(R,0)$, we can extend the domain of the α_j^N to a complex neighbourhood of $D_N = B_{G_\sigma^{s/2}}(R,0) \cap H_N$. For the bounds of F and G in $G_\sigma^{s/2}$, we just have to take $G_\sigma^{s/2}$-norms in the domain using lemma 2.3. The inequality $|F(U)|_{G_\sigma^{s/2}} \leq C_s a(C_s |U|_{G_\sigma^{s/2}})$ also holds for complex extended U. Then $|F(U_N)|_{G_\sigma^{s/2}}$ is bounded on $D_N + \delta$ by $C_s a(C_s(|U|_{G_\sigma^{s/2}} + \delta))$. Similarly we get a bound for G. Thus $(H.2)$ also holds. The assertions of theorem 1' imply those of theorem 1. $\square$

3.1 Proof of Theorem 1'

The proof can be divided into four steps. In the first two steps we analyse a finite dimensional problem by reducing to the Galerkin approximation

$$\begin{aligned} \dot{U}_N + AU_N &= P_N F(U_N) + h^p P_N G(U_N, \frac{t}{h}, h) \\ U_N(0) &= P_N U_0 \in H_N, \end{aligned}$$

where we choose the parameter N depending on h. When λ_N is the largest eigenvalue of $-Id\Delta$ restricted to the Galerkin approximation space H_N, we couple N and h by $\lambda_N^{1+\beta} h \leq 1$, where $\beta > 0$ is to be chosen later. Step 1 and step 2 are adapted from the Neishtadt theorem for the finite dimensional case. In step 1 we make successive formal coordinate changes, such that the nonautonomous terms are formally of higher order in h in the transformed equation

$$\dot{V}_N + AV_N = P_N F(V_N) + \bar{F}(V_N, h) + \alpha(V_N, \frac{t}{h}, h). \tag{3.7}$$

Step 2 will give estimates uniform in h and $N(h)$. We will perform $r \sim \frac{1}{h^\gamma}$ successive coordinate changes, where γ is chosen in the proof. In the transformed equation the nonautonomous terms are exponentially small in h. In step 3 and step 4 we will consider again the full infinite dimensional problem. We do the formal coordinate change in step 3. In step 4 we will finally prove the estimates and hence justify the use of the finite dimensional approximation.

Step 1: Formal coordinate changes

We describe the formal coordinate changes needed to remove nonautonomous terms. For a moment we suppress the dependence of U on N. The situation after j coordinate changes is given by

$$\dot{U} + AU = P_N F(U) + \bar{F}_j(U, h) + \alpha_j(U, \frac{t}{h}, h) \tag{3.8}$$

with $< \alpha_j > (U, h) = \int_0^h \alpha_j(U, \frac{\theta}{h}, h) d\theta = 0$ and U in a complex extended domain D_N^j with $D_N + \frac{\delta}{2} \subset D_N^j \subset D_N + \delta$, see figure 3.1. Before performing the first coordinate change, we have $\bar{F}_0 = h^p < G >$ and $\alpha_0 = h^p G - h^p < G >$.

Starting with (3.8) the next coordinate change is given by

$$U = V + hW_j(V, \tau, h) \tag{3.9}$$

with W periodic in $\tau = \frac{t}{h}$ with period 1. Substitution into (3.8) yields to

$$\begin{aligned} &\dot{V} + h\frac{\partial}{\partial V} W_j(V, \tau, h)\dot{V} + \frac{\partial}{\partial \tau} W_j(V, \tau, h) + A(V + hW_j(V, \tau, h)) \\ = \;& P_N F(V + hW_j(V, \tau, h)) + \bar{F}_j(V + hW_j(V, \tau, h), h) + \alpha_j(V + hW_j(V, \tau, h), \tau, h) \end{aligned}$$

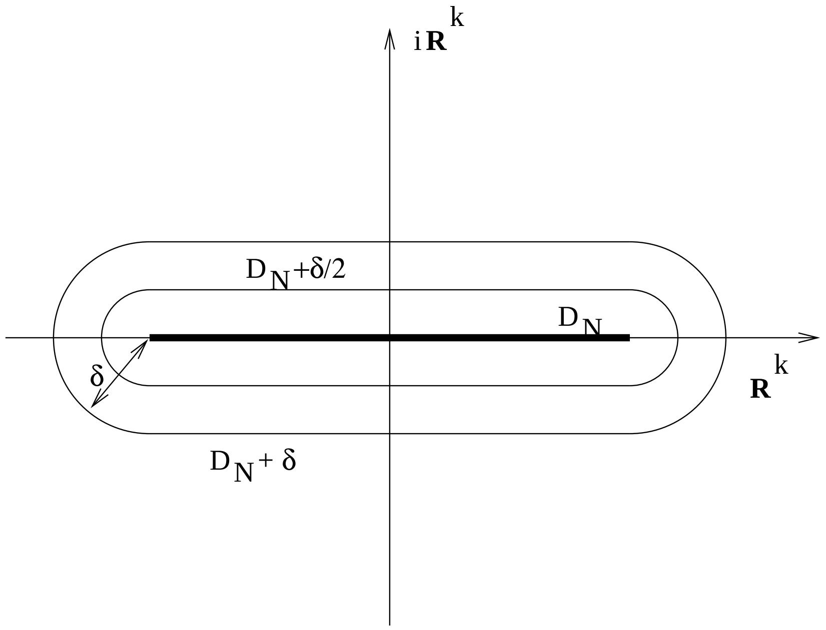

Figure 3.1: The domain D_N and its complex extensions.

A formal Taylor expansion in V gives

$$\begin{aligned}
\dot{V} \;&=\; \left(I + h\frac{\partial}{\partial V}W_j(V,\tau,h)\right)^{-1}\Big\{P_N F(V) + \frac{\partial P_N F(V)}{\partial V}hW_j(V,\tau,h) + \bar{F}_j(V,h) \\
&\quad + \frac{\partial \bar{F}_j(V,h)}{\partial V}hW_j(V,\tau,h) - AV - hAW_j(V,\tau,h) \\
&\quad + \alpha_j(V,\tau,h) + \frac{\partial \alpha_j(V,\tau,h)}{\partial V}hW_j(V,\tau,h) + \text{h.o.t.} - \frac{\partial}{\partial \tau}W_j(V,\tau,h)\Big\} \\
&=: \; -AV + P_N F(V) + \bar{F}_j(V) + a
\end{aligned} \tag{3.10}$$

The term of lowest order in h, which are time-dependent, is $\alpha_j(V,\tau,h)$. To remove this term we choose

$$W_j(V,\tau,h) = \int_0^\tau \alpha_j(V,\theta,h)d\theta.$$

Then we choose

$$\begin{aligned}
\bar{F}_{j+1}(V) \;&=\; \bar{F}_j(V) + \langle a \rangle \;\text{ and} \\
\alpha_{j+1}(V,\tau,h) \;&=\; a - \langle a \rangle\,.
\end{aligned} \tag{3.11}$$

Step 2: Estimates for the finite dimensional system

We give rigorous estimates for the formal procedure of step 1. Suppose r substitutions are made altogether for a fixed h. The domain D_N^j after j substitutions is given by $D_N^j = D_N^0 - jK(h)h$ where $K(h)$ is chosen later and $D_N^0 = D_N + \delta$.

We use the notation $\|f\|_{G_\sigma^{s/2},D+\gamma} = \sup_{U\in D+\gamma}|f(U)|_{G_\sigma^{s/2}}$ for some complex extension $D+\gamma$ of $D\subset G_\sigma^{s/2}(\Omega,\mathbf{R})$. Then by construction we have

$$\begin{aligned}\|\alpha_0\|_{G_\sigma^{s/2},D_N+\delta} &\leq 2B_2h^p\\ \|\bar{F}_0\|_{G_\sigma^{s/2},D_N+\delta} &\leq B_2h^p\end{aligned}$$

uniformly in σ. We will show inductively for $1\leq j\leq r$:

$$\|\bar{F}_j\|_{G_\sigma^{s/2},D_N+\delta/2} \leq B_1 \tag{3.12}$$

$$\|\alpha_j\|_{G_\sigma^{s/2},D_N+\delta/2} \leq M_j \text{ with } M_j = 2^{-j}B_2h^p. \tag{3.13}$$

We will choose h_1 and K such that r substitutions are defined for $0<h<h_1$, $V\in D_N^{r+1}=D_N^r-Kh\neq\emptyset$ and the inductive assumptions (3.12) and (3.13) are fulfilled.

For the estimates we need the lemma about Cauchy estimates with arbitrary norms:

Lemma 3.2 *Let $f:\Omega\subset\mathbf{C}^k\to\mathbf{C}^k$ be analytic and $\|f\|_\Omega=\sup_{u\in\Omega}\|f(u)\|$. Then $\|\frac{\partial f}{\partial x}\|_{\Omega-\delta}\leq\frac{M}{\delta}$ for any norm $\|.\|$ on $\mathbf{C}^k$.*

The lemma follows directly from the usual one-dimensional Cauchy formula. For any $u\in\Omega-\delta$ we take a circle in the complex plane defined by x, letting without restriction $\|x\|=1$. Then we have

$$\begin{aligned}f(u) &= \frac{1}{2\pi i}\oint_{y-u=zx,z\in\mathbf{C},|z|=\delta}\frac{f(y)}{u-y}dy\\ \frac{\partial f}{\partial x}(u) &= \frac{1}{2\pi i}\oint_{y-u=zx,z\in\mathbf{C},|z|=\delta}f(y)\frac{\partial}{\partial x}\left(\frac{1}{u-y}\right)dy.\end{aligned}$$

Thus

$$\|\frac{\partial f}{\partial x}\|_{\Omega-\delta}\leq\frac{1}{2\pi}2\pi\delta\frac{\|f\|_\Omega}{\|y-u\|^2}=\frac{M}{\delta}$$

and the lemma is proved. □

To continue the proof of the theorem, assume for induction that (3.12) and (3.13) hold for j. To simplify notation we suppress the arguments of W and the dependence on time and parameters N,h in the functions. All norms are $G_\sigma^{s/2}$ norms.

Then for the j^{th} coordinate change we obtain

$$\|hW_j\|_{D_N^j}\leq M_jh, \tag{3.14}$$

which yields by lemma 3.2 to

$$\|h\frac{\partial W_j}{\partial u}\|_{D_N^j-Kh}\leq\frac{M_j}{K}. \tag{3.15}$$

We estimate the higher order term a in (3.9).

$$
\begin{aligned}
\|a\|_{D_N^j - Kh} \leq\ & \| \left[I + h \frac{\partial}{\partial V} W_j \right]^{-1} \{ P_N F(V + hW_j) + \bar{F}_j(V + hW_j) \\
& + \alpha_j(V + hW_j) - \alpha_j(V) - A(V + hW_j) \} \\
& - [-AV + P_N F(V) + \bar{F}_j(V)] \, \|_{D_N^j - Kh} \\
\leq\ & \| \left[I + h \frac{\partial}{\partial V} W_j \right]^{-1} \{ -hAW_j + P_N F(V + hW_j) - P_N F(V) \\
& + \bar{F}_j(V + hW_j) - \bar{F}_j(V) + \alpha_j(V + hW_j) - \alpha_j(V) \\
& + h \frac{\partial}{\partial V} W_j \, [-AV + P_N F(V) + \bar{F}_j(V)] \} \, \|_{D_N^j - Kh} \\
& \text{Neumann series and mean value theorem give} \\
\leq\ & \sum_{k=0}^{\infty} \| h \frac{\partial}{\partial V} W_j \|^k_{D_N^j - Kh} \Big\{ \| hW_j \|_{D_N^j} \Big[\lambda_N + \| \frac{\partial}{\partial V} P_N F \|_{D_N^j - Kh} \\
& + \| \frac{\partial}{\partial V} \bar{F}_j \|_{D_N^j - Kh} + \| \frac{\partial}{\partial V} \alpha_j \|_{D_N^j - Kh} \Big] \\
& + \| h \frac{\partial}{\partial V} W_j \|_{D_N^j - Kh} \Big[\lambda_N \| V \|_{D_N^j} + \| P_N F \|_{D_N^j} + \| \bar{F}_j \|_{D_N^j} + \| \alpha_j \|_{D_N^j} \Big] \Big\} \\
& \text{lemma 3.2 gives} \\
\leq\ & 2hM_j \left[\lambda_N + \frac{B_1}{Kh} + \frac{2B_1}{Kh} \right] + \frac{M_j}{K} [\lambda_N R + B_1 + 2B_1]
\end{aligned}
$$

Using $\lambda_N^{1+\beta} h \leq 1$, which is equivalent to $\lambda_N \leq h^{\frac{-1}{1+\beta}}$, and setting

$$K(h) = \tilde{K} h^{\frac{-1}{1+\beta}} \tag{3.16}$$

we get

$$\|a\|_{D_N^j - Kh} \leq M_j \left[2hh^{\frac{-1}{1+\beta}} + 6 \frac{B_1}{\tilde{K} h^{\frac{-1}{1+\beta}}} + \frac{R}{\tilde{K}} + 3 \frac{B_1}{\tilde{K} h^{\frac{-1}{1+\beta}}} \right] \leq \frac{1}{4} M_j$$

for $\tilde{K}$ large enough. Therefore

$$\|\alpha_{j+1}\|_{D_N^{j+1}} < \frac{M_j}{2} = M_{j+1}$$

and

$$\|\bar{F}_{j+1} - \bar{F}_j\|_{D_N^{j+1}} < \frac{M_j}{4}.$$

Hence

$$
\begin{aligned}
\|\bar{F}_{j+1}\|_{D_N^{j+1}} &\leq \sum_{k=0}^{j} \|\bar{F}_{k+1} - \bar{F}_k\|_{D_N^{j+1}} + \|\bar{F}_0\|_{D_N^{j+1}} \\
&\leq \frac{1}{4} \sum_{k=0}^{j} M_k + B_2 h^p \leq \frac{1}{2} B_2 h^p + B_2 h^p \leq B_1
\end{aligned} \tag{3.17}
$$

for h_1 small enough. Thus the inductive statements (3.12) and (3.13) are fulfilled for $j+1$ for such a h_1 and the above choice of K . So we can carry out the coordinate changes as long as $D_N^r \neq \emptyset$, i.e.

$$r = \frac{\delta}{2K(h)h} = \frac{\delta}{2\tilde{K}h^{\frac{-1}{1+\beta}}h} = \frac{\delta}{2\tilde{K}h^{\frac{\beta}{1+\beta}}}$$

Letting $\alpha_* = \alpha_{r(h)}(V, \frac{t}{h}, h, N(h))$ and $\bar{F}_* = \bar{F}_{r(h)}(V, h, N(h))$ we get estimates for $G_\sigma^{s/2}$-norms uniform in $h \to 0, N(h) \to \infty$:

$$\|\alpha_*\|_{D_N+\delta/2, G_\sigma^{s/2}} < 2^{-r}B_2h^p < c_2h^p \exp(-c_1h^{-\frac{\beta}{1+\beta}}), \tag{3.18}$$

$$\|\bar{F}_*\|_{D_N+\delta/2, G_\sigma^{s/2}} < Ch^p. \tag{3.19}$$

Step 3: Transformation of the complete system

Next we deal with the complete infinite dimensional system. We linearly extend the coordinate change to other Galerkin modes by

$$U = V + hW(P_NV, \frac{t}{h}, h, N)$$

where $W(P_NV, \frac{t}{h}, h, N) \in H_N$ and $N = N(h)$ is chosen maximally such that $\lambda_N^{1+\beta}h \leq 1$ holds. Thus in the new coordinates we get, when suppressing arguments $\frac{t}{h}$ and h of W

$$\begin{aligned} &\dot{V} + h\frac{\partial}{\partial P_NV}W(P_NV)\dot{V} + \frac{\partial}{\partial \tau}W(P_NV) + A(V + hW(P_NV)) \\ = \;& F(V + hW(P_NV)) + h^pG(V + hW(P_NV), \frac{t}{h}, h). \end{aligned}$$

Solving for $\dot{V}$ gives

$$\begin{aligned} \dot{V} \; = \; & \left[I + h\frac{\partial}{\partial P_NV}W(P_NV)\right]^{-1}\{-A(V + hW(P_NV)) + F(V + hW(P_NV)) \\ & + h^pG(V + hW(P_NV), \frac{t}{h}, h) - \frac{\partial}{\partial \tau}W(P_NV, \frac{t}{h}, h)\}. \end{aligned}$$

Splitting $G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$ into H_N and

$$H_N^\perp = \{U \in G_\sigma^{s/2} | U = \sum_{||j||>N} U^j \exp\left(i\frac{2\pi}{l}j.x\right)\}$$

and using

$$\left[I + h\frac{\partial}{\partial P_NV}W(P_NV)\right]^{-1}_{|H_N^\perp} = I_{|H_N^\perp}$$

we obtain

$$\begin{aligned}
&\dot V + A(I-P_N)V \\
= \;& (I-P_N)\left\{F(V+hW(P_NV)) + h^pG(V+hW(P_NV),\frac{t}{h},h)\right\} \\
&+P_N\left[I+h\frac{\partial}{\partial P_NV}P_NW(P_NV)\right]^{-1} P_N\{-A(V+hW(P_NV)) \\
&\quad +F(V+hW(P_NV)) + h^pG(V+hW(P_NV),\frac{t}{h},h) - \frac{\partial}{\partial\tau}P_NW(V,\frac{t}{h},h)\} \\
= \;& (I-P_N)\left\{F(V+hW(P_NV)) + h^pG(V+hW(P_NV),\frac{t}{h},h)\right\} \\
&+P_N\left\{-AV+F(V)+\bar F_*(V,h,N)+\alpha_*(V,\frac{t}{h},h,N)\right\} \\
&+P_N\left[I+h\frac{\partial}{\partial P_NV}W(P_NV)\right]^{-1} P_N\{F(V+hW(P_NV)) \\
&\quad +h^pG(V+hW(P_NV),\frac{t}{h},h) - F(P_NV+hW(P_NV)) \\
&\quad -h^pG(P_NV+hW(P_NV),\frac{t}{h},h)\}.
\end{aligned}$$

Thus we have an equation as in (1.19)

$$\dot V + AV = F(V) + \bar F(V,h) + \alpha(V,\frac{t}{h},h) \tag{3.20}$$

with

$$\begin{aligned}
\bar F(V,h) &= \bar F_*(V,h,N(h)) + < b_1+b_2 > & (3.21)\\
\alpha(V,\frac{t}{h},h) &= \alpha_*(V,\frac{t}{h},h,N(h)) + b_1 + b_2 - < b_1+b_2 >, & (3.22)
\end{aligned}$$

where we get additional correction terms. First there is an error in the higher Galerkin modes ($> N$), as they were neglected in the coordinate transformation

$$b_1 = (I-P_N)\left\{F(V+hW(P_NV)) - F(V) + h^pG(V+hP_NW(V),\frac{t}{h},h)\right\}. \tag{3.23}$$

Secondly there is an error in the lower modes ($\leq N$) due to the neglected influence of the higher Galerkin modes to the lower ones

$$\begin{aligned}
b_2 \;=\; & \left[I+h\frac{\partial}{\partial P_NV}W(P_NV)\right]^{-1} P_N\left\{F(V+hW(P_NV)) - F(P_NV+hW(P_NV))\right. \\
&+h^pG(V+hW(P_NV),\frac{t}{h},h) - h^pG(P_NV+hW(P_NV),\frac{t}{h},h)\}. \qquad (3.24)
\end{aligned}$$

Step 4: Estimating the complete system

To prove the estimate (3.5) and (3.6) we will show, that (3.20) has Gevrey regular solutions. Thus we have to check the assumptions (H.1) of proposition 2.5 for the nonlinearities $\bar{F}(V,h,N)+\alpha(V,\frac{t}{h},h,N)+b_1+b_2$.

The nonlinearities $\bar{F}_*(V,h,N)+\alpha_*(V,\frac{t}{h},h,N)$, which were defined by the finite dimensional part, fulfil these estimates on $B_{G_\sigma^{\frac{s}{2}}}(R,0)$ by (3.19) and (3.18).

It remains to show, that

$$F(.+hW(P_N.)),G(.+hP_NW(.),\frac{t}{h},h):B_{G_\sigma^{\frac{s}{2}}}(R,0)\to G_\sigma^{\frac{s}{2}}(\Omega,\mathbf{R}^n)$$

is continuously differentiable and bounded. $V+hW(V)$ is given by the successive coordinate changes $U=V+hW_j(V)$. Using (3.14) we get that $V+hW(V)$ is a near-identity and differentiable coordinate change uniformly for every $G_\sigma^{\frac{s}{2}}$ norm. The linear mapping

$$\left[I+h\frac{\partial}{\partial P_NV}P_NW(V)\right]^{-1}P_N$$

in b_2 is near to the identity by (3.15). Thus the estimates on F and G hold for b_1 and b_2 too.

Hence (3.20) has Gevrey regular solutions and we can estimate $|\alpha(V(t),\frac{t}{h},h)|_{H^s_{per}}$ and $|\bar{F}(V(t),h)|_{H^s_{per}}$. We start with $\bar{F}(V,h)=\bar{F}_*(V,h,N)+<b_1+b_2>$. For the finite dimensional part we have $|\bar{F}_*(V,h,N)|_{H^s_{per}}\le c_3h^p$ by (3.19). Estimating $<b_1>$ we obtain for $V\in H^s_{per}(\Omega,\mathbf{R}^n)$:

$$\begin{aligned}
&|<b_1>|_{H^s_{per}}\\
\le\ & |(I-P_N)\left\{F(V+hW(P_NV))-F(V)+h^pG(V+hP_NW(V),\frac{t}{h},h)\right\}|_{H^s_{per}}\\
\le\ & \sup_{U=V+rW(P_NV),r\in[0,h]}|F'(U)|_{L(H^s,H^s)}|hW(P_NV)|_{H^s_{per}}+h^pB_1\le Ch^p
\end{aligned}$$

because $|hW(P_NV)|\le Ch^{p+1}$. Next we deal with the term $<b_2>$ created by the Galerkin approximation in the finite dimensional part. Letting $V=V(t)$ and $K=\{rV+(1-r)P_NV(t)+hW(P_NV(t)),r\in[0,1]\}$, we get

$$\begin{aligned}
&|<b_2>|_{H^s_{per}}\\
\le\ & |\left[I+h\frac{\partial}{\partial P_NV}P_NW(V)\right]^{-1}P_N\left\{F(V+hW(V))+h^pG(V+hP_NW(V),\frac{t}{h},h)\right.\\
&\left.-F(P_NV+hW(P_NV))-h^pG(P_NV+hW(P_NV),\frac{t}{h},h)\right\}|_{H^s_{per}}\\
\le\ & 2(\sup_K|F'(U)|_{L(H^s,H^s)}|V(t)-P_NV(t)|_{H^s_{per}}\\
&+\sup_K|G'(U(t))|_{L(H^s,H^s)}|V(t)-P_NV(t)|_{H^s_{per}})
\end{aligned}$$

$$\leq \quad C|V(t) - P_N V(t)|_{H^s_{per}} \leq C \exp\left(-\min(t,T^*)\sqrt{\lambda_{N+1}}\right),$$

where we used the boundedness of $V(t)$ in the Gevrey norm $G^{s/2}_{\min(t,T^*)}$ and thus if we subtract the projection to H_N

$$|V(t) - P_N V(t)|_{H^s_{per}} \leq |V(t)|_{G^{s/2}_{\min(t,T^*)}} \exp\left(-\min(t,T^*)\sqrt{\lambda_{N+1}}\right). \tag{3.25}$$

Next we need a more careful analysis for the nonautonomous part:

$$|\alpha(V(t), \frac{t}{h}, h)|_{H^s_{per}} = |\alpha_*(V, \frac{t}{h}, h, N) + b_1 + b_2 - < b_1 + b_2 > |_{H^s_{per}}.$$

For the first part we have by (3.18):

$$|\alpha_*(V, \frac{t}{h}, h, N)|_{H^s_{per}} \leq c_2 h^p \exp(-c_1 h^{-\frac{\beta}{1+\beta}})$$

For $b_1 - < b_1 >$ we use the same analysis as for $< b_1 >$ above, except we note that $V(t)$ is bounded in the Gevrey norm $G^{s/2}_{\min(t,T^*)}$. Hence $b_1(V(t)) \in G^{s/2}_{\min(t,T^*)}(\Omega, \mathbf{R}^n)$ and thus with $K = \{V + rW(P_N V), r \in [0,h]\}$ and $\sigma(t) = \min(t, T^*)$

$$\begin{aligned}
|b_1 - < b_1 > |_{H^s_{per}} &\leq 2|(I - P_N)\{F(V(t) + hW(P_N V(t))) \\
&\quad -F(V(t)) + h^p G(V(t) + hW(P_N V(t)), \frac{t}{h}, h)\}|_{H^s_{per}} \\
&\leq \left(\sup_{U \in K} |F'(U)|_{L(G^{s/2}_{\sigma(t)}, G^{s/2}_{\sigma(t)})} |hW(P_N V)|_{G^{s/2}_{\sigma(t)}} + h^p B_1\right) \\
&\quad \cdot \exp\left(-\sigma(t)\sqrt{\lambda_{N+1}}\right) \\
&\leq Ch^p \exp\left(-\sigma(t)\sqrt{\lambda_{N+1}}\right) = Ch^p \exp\left(-\min(t,T^*)\sqrt{\lambda_{N+1}}\right).
\end{aligned}$$

For $b_2(V(t)) - < b_2(V(t)) >$ we get

$$\begin{aligned}
&|b_2 - < b_2 > |_{H^s_{per}} \\
\leq \; & | \left[I + h\frac{\partial}{\partial P_N V} W(P_N V, \frac{t}{h}, h)\right]^{-1} \\
&\cdot P_N \Big\{ F(V + hW(P_N V)) + h^p G(V + hW(P_N V), \frac{t}{h}, h) \\
&-F(P_N V + hW(P_N V)) - h^p G(P_N V + hW(P_N V), \frac{t}{h}, h)\Big\} - \int_0^1 b_2(V, \theta) d\theta|_{H^s_{per}}
\end{aligned}$$

adding and subtracting a partially averaged term gives

$$= \; | \left[I + h\frac{\partial}{\partial P_N V} W(P_N V, \frac{t}{h}, h)\right]^{-1}$$

$$
\begin{aligned}
& \cdot P_N \left\{ F(V + hW(P_N V)) + h^p G(V + hW(P_N V), \frac{t}{h}, h) \right. \\
& \quad \left. - F(P_N V + hW(P_N V)) - h^p G(P_N V + hW(P_N V), \frac{t}{h}, h) \right\} \\
& - \int_0^1 \left[I + h \frac{\partial}{\partial P_N V} W(P_N V, \theta, h) \right]^{-1} d\theta \\
& \cdot P_N \left\{ F(V + hW(P_N V)) + h^p G(V + hW(P_N V), \frac{t}{h}, h) \right. \\
& \quad \left. - F(P_N V + hW(P_N V)) - h^p G(P_N V + hW(P_N V), \frac{t}{h}, h) \right\} \\
& + \int_0^1 \left[I + h \frac{\partial}{\partial P_N V} W(P_N V, \theta, h) \right]^{-1} d\theta \\
& \cdot P_N \left\{ F(V + hW(P_N V)) + h^p G(V + hW(P_N V), \frac{t}{h}, h) \right. \\
& \quad \left. - F(P_N V + hW(P_N V)) - h^p G(P_N V + hW(P_N V), \frac{t}{h}, h) \right\} \\
& - \int_0^1 \left[I + h \frac{\partial}{\partial P_N V} W(P_N V, \theta, h) \right]^{-1} \\
& \cdot P_N \{ F(V + hW(P_N V)) + h^p G(V + hW(P_N V), \theta, h) \\
& \quad - F(P_N V + hW(P_N V)) - h^p G(P_N V + hW(P_N V), \theta, h) \} \, d\theta |_{H^s_{per}} \\
\leq \; & \left| \left[I + h \frac{\partial}{\partial P_N V} W(P_N V, \frac{t}{h}, h) \right]^{-1} - \int_0^1 \left[I + h \frac{\partial}{\partial P_N V} W(P_N V(t), \theta, h) \right]^{-1} d\theta \right|_{H^s_{per}} \\
& \cdot \left| P_N \left\{ F(V + hW(P_N V)) + h^p G(V + hW(P_N V), \frac{t}{h}, h) \right. \right. \\
& \quad \left. \left. - F(P_N V + hW(P_N V)) - h^p G(P_N V + hW(P_N V), \frac{t}{h}, h) \right\} \right|_{H^s_{per}} \\
& + \left| \int_0^1 \left[I + h \frac{\partial}{\partial P_N V} W(P_N V, \theta, h) \right]^{-1} \right. \\
& \cdot P_N \left\{ F(V + hW(P_N V, \frac{t}{h}, h)) + h^p G(V + hW(P_N V, \frac{t}{h}, h), \frac{t}{h}, h) \right. \\
& \quad - F(P_N V + hW(P_N V, \frac{t}{h}, h)) - h^p G(P_N V + hW(P_N V, \frac{t}{h}, h), \frac{t}{h}, h) \\
& \quad - F(V + hW(P_N V, \theta, h)) - h^p G(V + hW(P_N V, \theta, h), \theta, h) \\
& \quad \left. + F(P_N V + hW(P_N V, \theta, h)) + h^p G(P_N V + hW(P_N V, \theta, h), \theta, h) \right\} d\theta |_{H^s_{per}} \\
\leq \; & C h^p |b_2|_{H^s_{per}} \\
& + \left| \int_0^1 \left[I + h \frac{\partial}{\partial P_N V} W(P_N V, \theta, h) \right]^{-1} P_N \{ h^p G(V + hW(P_N V), \frac{t}{h}, h) \right. \\
& - h^p G(P_N V + hW(P_N V), \frac{t}{h}, h) - h^p G(V + hW(P_N V), \theta, h) \\
& + h^p G(P_N V + hW(P_N V), \theta, h)
\end{aligned}
$$

$$
\begin{aligned}
&+(F(V+hW(P_NV,\frac{t}{h},h))-F(V+hW(P_NV,\theta,h)))\\
&-(F(P_NV+hW(P_NV,\frac{t}{h},h))-F(P_NV+hW(P_NV,\theta,h)))\}d\theta|_{H^s_{per}}
\end{aligned}
$$

the last two lines are both of order h^p and the difference of these lines can still be estimated as in the analysis of $\bar{F}$

$$
\leq \; Ch^p \exp\left(-\min(t,T^*)\sqrt{\lambda_{N+1}}\right) = Ch^p \exp\left(-\min(t,T^*)h^{\frac{-1}{2(1+\beta)}}\right).
$$

As $\sqrt{\lambda_{N+1}} \geq h^{\frac{-1}{2(1+\beta)}}$. When we balance $\exp(-c_1 h^{-\frac{\beta}{1+\beta}})$ and $\exp(-\min(t,T^*)h^{\frac{-1}{2(1+\beta)}})$ we are choosing $\beta=\frac{1}{2}$. Then the estimates are of order $\exp(-ch^{-\frac{1}{3}})$ and $N(h) = \sqrt{\lambda_{N(h)}} = \left[h^{-\frac{1}{3}}\right]$. This gives the estimates (3.5,3.6). □

Remark 3.3 *L^2 estimates of $\bar{F}$ and α.*

The above estimates on b_1 and b_2 can be improved, when we consider the L^2 norm instead of H^s norms. We use a variant of (3.25):

$$
\begin{aligned}
|V(t)-P_NV(t)|_{L^2} &\leq C\lambda_{N+1}^{-\frac{s}{2}}|V(t)-P_NV(t)|_{H^s_{per}}\\
&\leq C|V(t)|_{G^{s/2}_{\min(t,T^*)}}\lambda_{N+1}^{-\frac{s}{2}}\exp\left(-\min(t,T^*)\sqrt{\lambda_{N+1}}\right)\\
&= C|V(t)|_{G^{s/2}_{\min(t,T^*)}}h^{\frac{s}{3}}\exp\left(-\min(t,T^*)h^{-\frac{1}{3}}\right),
\end{aligned}
$$

where we also used the choice $\lambda_{N+1}^{-\frac{1}{2}} \leq h^{\frac{1}{3}}$. This proves (1.22,1.24).□

Remark 3.4 *If $V \in G^{s/2}_\sigma$ and G is smooth in h, then $\bar{F}(V,h)$ and $\alpha(V,\frac{t}{h},h)$ can be chosen to be smooth in h.*

The coordinate changes in the above proof are not continuous in h, even in the finite dimensional part $\bar{F}_*$ and α_*. The number of averaging steps $[r]$ with $r=\frac{\delta}{2\bar{K}}h^{-\frac{1}{1+\beta}}$ increases discontinuously for $h\to 0$, as $[.]$ denotes the discontinuous integer part of r. This problem can be solved, when the additional terms are introduced using smooth cut-off functions

$$
\chi_k(\tau)=\begin{cases} 0 & \text{for } \tau\leq k\\ 1 & \text{for } \tau \geq k+1.\end{cases}
$$

When we need $[r]$ averaging steps, then we compute an additional coordinate change $Id+\chi_{[r(h)]}(r(h))W_{[r(h)]+1}$, which only slightly affects the estimates. Then our coordinate change and hence the finite dimensional parts of $\bar{F},\alpha$ are smooth in h for $h>0$. Similarly we introduce the increasing number of Galerkin modes using smooth cut-off functions to get smoothness for $h > 0$ in the correction terms b_1, b_2 (3.23,3.24) too.

Next we consider the regularity at $h = 0$ for the finite dimensional part. We briefly sketch, how one can estimate $\left(\frac{d}{dh}\right)^m \bar{F}$. One uses similar estimates on $\left(\frac{d}{dh}\right)^m a$ as in step 2 for a to uniformly bound $\left(\frac{d}{dh}\right)^m \bar{F}_j$ and $\left(\frac{d}{dh}\right)^m \alpha_j$ in each step.

As an example we sketch the first derivative. First we change the estimates on $\|W_j\|_{D_N^j} \leq M_j = 2^{-j} B_2 h^p$. By performing one more standard averaging step, we get $\|W_j\|_{D_N^j} \leq M_j = 2^{-j} \tilde{B}_2 h^{p+1}$ maybe on a smaller domain. Then we will prove inductively

$$
\begin{aligned}
\|\frac{d}{dh}\bar{F}_j\|_{D_N+\frac{\delta}{2}} &\leq B_1 \\
\|\frac{d}{dh}\alpha_j\|_{D_N+\frac{\delta}{2}} &\leq \tilde{M}_j = 2^{-j}\tilde{B}_2 h^p
\end{aligned}
$$

just as in step 2 of the proof

$$
\begin{aligned}
\|\frac{d}{dh}a\|_{D_N^j-Kh} \leq\ & \| - \left[I + h\frac{\partial}{\partial V}W_j\right]^{-2}\left(\frac{\partial}{\partial V}W_j + h\frac{\partial}{\partial h}\frac{\partial}{\partial V}W_j\right) \\
& \cdot\{P_N F(V+hW_j) + \bar{F}_j(V+hW_j) + \alpha_j(V+hW_j) \\
& \quad -P_N F(V) - \bar{F}_j(V) - \alpha_j(V) - hAW_j\} \\
& + \left[I + h\frac{\partial}{\partial V}W_j\right]^{-1}\{D_V[P_N F(V+hW_j) + \bar{F}_j(V+hW_j) \\
& \quad +\alpha_j(V+hW_j) - A]\cdot\left(W_j + h\frac{\partial}{\partial h}W_j\right) \\
& + \frac{\partial}{\partial h}\left(\bar{F}_j(V+hW_j) + \alpha_j(V+hW_j)\right) - \frac{\partial}{\partial h}\left(\bar{F}_j(V) + \alpha_j(V)\right)\} \\
& + \left(\frac{\partial}{\partial V}W_j + h\frac{\partial}{\partial h}\frac{\partial}{\partial V}W_j\right)[-AV + P_N F(V) + \bar{F}_j(V)] \\
& + h\frac{\partial}{\partial V}W_j\frac{\partial}{\partial h}\bar{F}_j(V)\|_{D_N^j-Kh} \\
\leq\ & 2\left(\frac{\tilde{M}_j h}{Kh} + \frac{\tilde{M}_j h}{Kh}\right)h^2\tilde{M}_j\left[\lambda_N + \frac{B_1}{Kh} + \frac{2B_1}{Kh}\right] \\
& + 2\{\left(\frac{4B_1}{Kh}\right)\left(2h\tilde{M}_j\right) + h^2\tilde{M}_j\frac{4B_1}{Kh}\} \\
& + 2\frac{\tilde{M}_j}{K}[\lambda_N R + B_1 + 2B_1] + h\tilde{M}_j B_1 \\
\leq\ & \frac{\tilde{M}_j}{2}
\end{aligned}
$$

for K large enough. Then summing up as in (3.17) will bound the h-derivatives. This can be done for all derivatives $\left(\frac{d}{dh}\right)^m \bar{F}$. So all derivatives are bounded for

$h \to 0$ and thus we can smoothly extend down to $h = 0$. Hence we have smoothness in h for the finite dimensional part.

The smoothness of the complete terms $\bar{F}(V,h), \alpha(V, \frac{t}{h}, h)$ for $h = 0$ can easily by shown, if we assume $V \in G_\sigma^{s/2}$. This can be done, if we guarantee backward existence as on unstable manifolds. For fixed N, b_1 and b_2 can be chosen to be smooth in h, because W and G are smooth in h even for $h \to 0$ and $N(h) \to \infty$. It remains to analyse the influence of $P_{N(h)}$ for $h \to 0$. We have to bound $\frac{d}{dh} P_{N(h)} V$, where we introduce additional Galerkin modes by smooth cut-off functions. If $V \in G_\sigma^{s/2}$, then with $N(h) = h^{-\frac{1}{3}}$

$$|\frac{d}{dh} P_{N(h)} V|_{H^s} = h^{-\frac{4}{3}} \exp(-\sqrt{\lambda_{N(h)}}\sigma) = h^{-\frac{4}{3}} \exp(-h^{-\frac{1}{3}}\sigma) \to 0$$

for $h \to 0$ and similarly for the higher derivatives. Hence we can smoothly extend b_1 and b_2 down to $h = 0$, with $\left(\frac{d}{dh}\right)^n_{|h=0} b_1 = \left(\frac{d}{dh}\right)^n_{|h=0} b_2 = 0$. This proves the remark. $\square$

Chapter 4

Backward Error Analysis

We now prove theorem 2 and thus provide an example of backward error analysis. Our proof is an application of theorem 1'. We rewrite equation (1.9) as

$$\begin{aligned} \frac{\partial}{\partial t} U &= -AU + F(U) \\ U(0) &= U_0 \in H^s_{per}(\Omega, \mathbf{R}^n) \end{aligned} \tag{4.1}$$

with $A = -\mathrm{diag}(d_1\Delta, \ldots d_n\Delta)$ on $\Omega = [0,l]^d$ with $d = 1,2,3$, periodic boundary conditions and $s \geq \frac{d}{2}$. The nonlinearity F is assumed to be of local superposition-type $F(U)(x) = (f_1(U(x)), \ldots, f_n(U(x)))$ with each f_k given by a convergent power series on a large ball $B_{\mathbf{R}^n}(R, 0)$.

To compare the numerical scheme $\Phi_{h,N}$ with equations of type (4.1) we are going to embed it first into a rapidly forced ordinary differential equation. In section 4.2 this will yield to an equation

$$\begin{aligned} \frac{\partial}{\partial t} U_N &= -AU_N + P_N F(U_N) + h^p G(U_N, \frac{t}{h}, h, N) \\ U_N(0) &= P_N U_0 \end{aligned} \tag{4.2}$$

with evolution $\Psi(t, 0, U_N; h, N)$. The discretization scheme coincides exactly with the time-h map of (4.2):

$$\Phi^n_{h,N}(U_0) = \Psi(n \cdot h, 0, P_N U_0; h, N).$$

The nonautonomous ordinary differential equation (4.2) can easily be extended to a partial differential equation with nonlinearities, which are non-local

$$\begin{aligned} \frac{\partial}{\partial t} U &= -AU + P_N F(P_N U) + h^p G(P_N U, \frac{t}{h}, h, N) \\ U(0) &= P_N U_0. \end{aligned}$$

Applying theorem 1' to this equation and using its special structure, we will complete the proof of theorem 2 in section 4.3. Before giving the proof, we briefly review results on the convergence of Runge-Kutta methods for parabolic partial differential equations.

4.1 Error estimates

We give examples of discretization schemes, which fulfil the approximation assumption (1.27) of the theorem for some $p \geq 1$. It assumes the convergence of the method independent of N and independent of the Banach space $G_\sigma^{s/2}$. In the proof of theorem 2 we furthermore need estimates when extending $H_N \cap B_{G_\sigma^{s/2}}$ to a complex neighbourhood, i.e. we have to complexify $G_\sigma^{s/2}$ for $\sigma \geq 0$. Error estimates for strongly $A(\theta)$ stable Runge-Kutta methods were given by Lubich and Ostermann in [LO93] and [LO96] in a general setting of semilinear parabolic equations. The equation (1.9) fits in this setting even when posing it on Gevrey classes, since A is sectorial on $G_\sigma^{s/2}$ by the following lemma and the nonlinearity maps $G_\sigma^{s/2}$ to itself by lemma 2.3.

Lemma 4.1 *The operator*

$$A = -diag(d_1, \dots, d_n)\Delta$$

is sectorial on $G_\sigma^{s/2}(\Omega, \mathbf{C}^n)$ *with* $D(A) = G_\sigma^{s/2+1}(\Omega, \mathbf{C}^n)$.

Proof: We use definition 1.3.1 of Henry [Hen83]. A is densely defined on $G_\sigma^{s/2}$, as it is defined on finite Fourier series. To check closeness, let $u_k \overset{G_\sigma^{s/2}}{\to} u$ and $Au_k \overset{G_\sigma^{s/2}}{\to} v$. We have to show $u \in D(A)$ and $Au = v$. The convergence of Au_k in $G_\sigma^{s/2}$ gives, that u_k converges in $G_\sigma^{s/2+1}$, hence $u \in D(A)$. As A is closed on L^2, we have by $|v - Au|_{L^2} = 0$, that the Fourier series of v and Au coincide and thus $Au = v$ in $G_\sigma^{s/2}$.

As A is sectorial on L^2, the resolvent can be estimated. As $|(\lambda - A)^{-1}|_{L(L^2,L^2)} \leq M(\lambda)$, and as each Fourier monomial $\exp(i\frac{2\pi}{l}jx)$ is an eigenfunction of $(\lambda - A)^{-1}$, we have

$$|(\lambda - A)^{-1}\exp(i\frac{2\pi}{l}jx)|_{G_\sigma^{s/2}} \leq M(\lambda)|\exp(i\frac{2\pi}{l}jx)|_{G_\sigma^{s/2}}.$$

Thus for all $v \in G_\sigma^{s/2}(\Omega, \mathbf{C}^n)$ we get the resolvent estimate:

$$|(\lambda - A)^{-1}v|_{G_\sigma^{s/2}} \leq \frac{M}{|\lambda + 1|}|v|_{G_\sigma^{s/2}}$$

for λ in a sector $\{\lambda | \phi \leq |\arg(\lambda+1)| \leq \pi\}$ for some $\phi \in (0, \frac{\pi}{2})$.□

Althuogh the assumption (1.27) is standard for ordinary differential equations, it does not hold in general for parabolic partial differential equations. In this setting there are two different kind of error estimates. First there are nonsmooth error estimates for general initial values as in [LO96]. Uniform estimates can only be given after some transient: The constant in front of h^p will depend on the transient time and tend to infinity for small times and furthermore the order of convergence is too small. So they are not applicable to assure (1.27).

Secondly there are smooth error estimates. For higher order estimates we need smoothness in time starting at $t = 0$, which in general cannot be assumed for arbitrary initial conditions. Even then the full order of convergence known from the theory of ordinary differential equations is not realised in general. We will use these smooth error estimates, but instead of assuming smoothness we use the coupling of λ_N and h. We use results of Lubich and Ostermann [LO93, theorem4.1]. They consider a large class of semidiscretization in time (strongly $A(\theta)$ stable Runge-Kutta methods with internal stage order q and order p for $p \geq q+1$). Hence the estimates will also hold uniformly in N when using the full discretization.

$$|\Phi_{h,N}(U_0) - S_N(h, U_0)|_{G_\sigma^{s/2}} \leq C \cdot (|U_N^{(q+1)}(0)|_{G_\sigma^{s/2}} + \int_0^h |U_N^{(q+1)}(t)|_{G_\sigma^{s/2}} dt) h^{q+1},$$

where $U_N^{(q+1)}(t)$ denotes the $(q+1)^{\text{th}}$ time derivative, which can be bounded by $C\lambda_N^{q+1}$. Using $\lambda_N^{1+\beta} h \leq 1$ gives

$$|\Phi_{h,N}(U_0) - S_N(h, U_0)|_{G_\sigma^{s/2}} \leq C(\lambda_N h)^{q+1} = Ch^{\frac{\beta(q+1)}{1+\beta}}.$$

Hence assumption (1.27) holds for full discretisation with Galerkin discretisation in space and some higher order strongly $A(\theta)$ stable Runge-Kutta methods like higher order Radau methods in time coupled by $\lambda_N^{1+\beta} h \leq C$ with appropriate $\beta > 0$.

For linear parabolic equations with periodic boundary conditions with semigroup $S(t, U)$ and semidiscretization Φ of step size h, Lubich and Ostermann showed in [LO95] that the full order of convergence p can be achieved for finite times $t = nh \leq T$:

$$|S(n \cdot h, U_0) - \Phi^n(U_0)| \leq Ch^p$$

Convergence results for discretization in space by spectral Galerkin methods can be found e.g. in [Stu95] and [DT93]. In [DT93] the Gevrey regularity is used to show exponential convergence of the Galerkin approximation, if one already starts with Gevrey regular initial values.

4.2 Embedding

We consider the ordinary differential equation (1.25) with assoicated flow $S_N(t,U)$. As an adaption of [FS96, proposition 2.1] and its refinement in [ZB98] we construct a nonautonomous perturbation G as in (4.2) for our special discretization scheme with estimates uniform on its Gevrey norm in $h \to 0, N(h) \to \infty$.

Proposition 4.2 *Let the assumptions of theorem 2 on the nonlinearity F and the full discretisation scheme $\Phi_{h,N}$ hold: F is of superposition-type and real analytic in U and $\Phi_{h,N}$ is real analytic in U and h for fixed N. Furthermore N and h are coupled as in (1.26) by setting $\lambda_N^{1+\beta} h \leq C$ with $\beta > 0$.*

Then there exist $h^* > 0$, a nonautonomous vector field $G = G(U_N, \theta, h, N)$, a non-increasing continuous function $\rho : [0, h^*] \to (0, \infty]$ with $\rho(0) = \infty$ and a continuous non-decreasing function $r : \mathbf{R}^+ \to \mathbf{R}^+$ such that:

(i) $G(U_N, \theta, h, N) \in H_N$ *is defined for all real* θ *and* $0 \leq h \leq h^*$ *and* $U_N \in H_N$ *with* $|U_N|_{H^s-per} \leq \rho(h)$.

(ii) *For fixed* N *the perturbation* G *is* C^∞*-smooth in* θ *and analytic in* h, U_N.

(iii) G *is periodic in* θ *with period* 1.

(iv) $\Phi_{h,N}(h, U_0) = \Psi(h, 0, h, N, P_N U_0)$, *where* Ψ *is the evolution of (4.2).*

(v) $|G(U_N, \theta, h, N)|_{G_\sigma^{s/2}} \leq r(|U_N|_{G_\sigma^{s/2}})$ *independent of* σ, θ, h *and* N.

(vi) G *satisfies hypotheses (H.2): For* $D_N = H_N \cap B_{G_\sigma^{s/2}}(R, 0) \subset \mathbf{R}^k$ *the perturbation* G *can be extended to a complex* δ*-neighbourhood taking distances in the* $G_\sigma^{s/2}$ *norm:*

$$\sup_{U_N \in D_N + \delta, \theta \in \mathbf{R}} |G(U_N, \theta, h, N)|_{G_\sigma^{s/2}} \leq B_2.$$

Proof: The basic idea of the proof is to interpolate for fixed N between the identity and the numerical scheme Φ by a family $t \mapsto \Psi(t, 0, U; h, N)$ of diffeomorphisms with

$$\begin{aligned} \Psi(0,0,U;h,N) &= 0 \\ \Psi(h,0,U;h,N) &= \Phi_{h,N}(h,U). \end{aligned}$$

An evolution is then defined omitting the other arguments by

$$\Psi(t,s) = \Psi(t,0) \circ \Psi(s,0)^{-1}$$

extended to $t \notin [0, h]$ by iteration. We repeat the construction of [FS96]. This interpolation is done in step 1. In step 2 we will construct G, such that $-A + P_N F + G$

is the time derivative of Ψ. The properties (i)-(iv) will be checked in step 3. The estimates in (v) and (vi) will be derived in step 4.

Step 1: Interpolation

Choose $\chi : \mathbf{R} \to [0,1]$ a C^∞ cut-off function analytic in $\tau \neq 0,1$ with

$$\chi(\tau) = \begin{cases} 1 \text{ for } \tau \leq 0 \\ 0 \text{ for } \tau \geq 1. \end{cases}$$

Then we define Ψ by interpolation for $t \in [0,h]$

$$\Psi(t,0,U_N;h,N) := \chi(\frac{t}{h})S_N(t,U_N) + (1-\chi(\frac{t}{h}))S_N(t-h,\Phi_{h,N}(h,U_N)). \tag{4.3}$$

As S_N is only a local flow in time, we need the restriction $|U_N|_{H^s} \leq \rho(h)$, then

$$\begin{aligned} \Psi(0,0,U_N;h,N) &= S_N(0,U_N) = U_N && (4.4)\\ \Psi(h,0,U_N;h,N) &= S_N(0,\Phi_{h,N}(U_N)) = \Phi_{h,N}(U_N)). && (4.5) \end{aligned}$$

We can then extend the definition to all $t \geq 0$ by

$$\Psi(t,0,U_N;h,N) := \Psi(t - \left[\frac{t}{h}\right] h, 0, \Phi_{h,N}^{\left[\frac{t}{h}\right]}(U_N);h,N). \tag{4.6}$$

Thus Ψ is an interpolation of $\Phi(n\cdot h, U_N), n \in \mathbf{Z}$. Then

$$\Psi(t,0,\Psi(n\cdot h,0,U_N;h,N);h,N) = \Psi(t+n\cdot h,0,U_N;h,N)$$

and the curve $t \mapsto \Psi(t,0,U_N;h,N)$ is clearly analytic for $\frac{t}{h} \notin \mathbf{N}$. By the recursion formula above, it is enough to check C^∞-smoothness at $t = h$ to prove that $t \mapsto \Psi(t,0,U_N;h,N)$ is C^∞. We denote by $D_\pm^m$ the m^{th} t-derivative from above $(+)$ and from below $(-)$. Then we get for arbitrary m dropping dependence on h,N:

$$\begin{aligned} & D_+^m|_{t=h}\Psi(t,0,U_N) \\ = \ & D_+^m|_{t=0}\Psi(t,0,\Phi_{h,N}(U_N)) \\ = \ & D_+^m|_{t=0}\left(\chi(\frac{t}{h})S_N(t,\Phi_{h,N}(U_N)) + (1-\chi(\frac{t}{h}))S_N(t-h,\Phi_{h,N}^2(U_N))\right) \\ = \ & D_+^m|_{t=0}S_N(t,\Phi(h,U_N)) = D_-^m|_{t=0}S_N(t,\Phi(h,U_N)) \\ = \ & D_-^m|_{t=h}\left(\chi(\frac{t}{h})S_N(t,U_N) + (1-\chi(\frac{t}{h}))S_N(t-h,\Phi_{h,N}(U_N))\right) \\ = \ & D_-^m|_{t=h}\Psi(t,0,U_N). \end{aligned}$$

This proves C^∞-regularity of Ψ in t.

Step 2: Construction of G

To construct the perturbation we scale time $\theta = \frac{t}{h}$, then we denote

$$\tilde{\Psi}(\theta,0,U_N;h,N) = \Psi(\theta h,0,U_N;h,N). \tag{4.7}$$

Thus we have for $\theta \in [0,1]$:

$$\tilde{\Psi}(\theta, 0, U_N; h, N) = \chi(\theta) S_N(h\theta, U_N) + (1 - \chi(\theta)) S_N(h\theta - h, \Phi_{h,N}(h, U_N)).$$

We define

$$G(U_N, \theta; h, N) := h^{-p} \left(h^{-1} D_\theta \tilde{\Psi}(\theta, 0, V_N; h, N) - (-AU_N + P_N F(U_N)) \right) \tag{4.8}$$

where $V_N = V_N(U_N, \theta; h, N)$ is defined implicitly by

$$U_N = \tilde{\Psi}(\theta, 0, V_N; h, N). \tag{4.9}$$

This can be solved for V_N by the implicit function theorem. $\tilde{\Psi}$ is analytic in h, V_N and smooth in Θ. For $h = 0$ the trivial solution is $V_N = V_N(U_N, \theta; 0, N) = U_N$ by (4.4) and the linearisation $D_{V_N} \tilde{\Psi}(\theta, 0, V_N; 0, N) = Id$ is invertible. Hence we can solve (4.9) for $V_N = V_N(U_N, \theta; h, N)$ uniformly for $0 \leq h \leq h^*$ as long as $|U_N| \leq \rho(h)$ and $\tilde{\Psi}$ is defined. Thus V_N will be defined on a reduced domain $0 \leq \theta \leq 2, 0 \leq h \leq h^*, |U_N| \leq \rho(h)$.

Step 3: Basic properties of G

First we extend the domain of G to all real θ by 1-periodic extension to show (iii). Thus we have to check periodicity for $0 \leq \theta \leq 2$. Let $0 \leq \theta \leq 1, h > 0$ then we get from (4.6)

$$\tilde{\Psi}(\theta, 0, \Phi_{h,N}(V_N(U_N, \theta{+}1; h, N)); h, N) = \tilde{\Psi}(\theta{+}1, 0, V_N(U_N, \theta{+}1; h, N); h, N) = U_N$$

which implies by applying $\tilde{\Psi}(\theta, 0)^{-1}$ on both sides

$$\Phi_{h,N}(V_N(U_N, \theta + 1; h, N)) = V_N(U_N, \theta; h, N).$$

Hence we get when dropping arguments h and N:

$$\begin{aligned}
& h^{p+1}(G(U_N, \theta + 1) - G(U_N, \theta)) \\
= \; & D_\theta \tilde{\Psi}(\theta + 1, 0, V_N(\theta + 1)) - D_\theta \tilde{\Psi}(\theta, 0, V_N(\theta)) \\
= \; & D_\theta \tilde{\Psi}(\theta, 0, \Phi_{h,N} V_N(\theta + 1)) - D_\theta \tilde{\Psi}(\theta, 0, V_N(\theta)) \\
= \; & D_\theta \tilde{\Psi}(\theta, 0, V_N(\theta)) - D_\theta \tilde{\Psi}(\theta, 0, V_N(\theta)) = 0,
\end{aligned}$$

proving periodicity for $h > 0$. Next we will extend G down to $h = 0$. Let d_j denote various remainder terms real analytic in $0 \leq h \leq h^0, |U_N| \leq \rho(h)$ and C^∞ in θ. By the p^{th} order approximation property we have

$$\Phi_{h,N}(U_N) = S_N(h, U_N) + h^{p+1} d_0$$

This implies

$$
\begin{aligned}
\tilde{\Psi}(\theta, U_N) &= \chi(\theta) S_N(h\theta, U_N) + (1-\chi(\theta)) S_N(h\theta - h, \Phi_{h,N}(h, U_N)) \\
&= \chi(\theta) S_N(h\theta, U_N) + (1-\chi(\theta)) S_N(h\theta - h, S_N(h, U_N)) \\
&\quad +(1-\chi(\theta)) \left[S_N(h\theta - h, \Phi_{h,N}(U_N)) - S_N(h\theta - h, S_N(h, U_N))\right] \\
&= S_N(h\theta, U_N) + h^{p+1} d_1. \qquad (4.10)
\end{aligned}
$$

Replacing U_N by V_N and using $\tilde{\Psi}(\theta, 0, V_N(\theta, U_N)) = U_N$ yields to

$$
V_N(\theta, U_N) = S_N(-h\theta, U_N) + h^{p+1} d_2. \qquad (4.11)
$$

From (4.10,4.11) we obtain

$$
\begin{aligned}
h^{-1} D_\theta \tilde{\Psi}(\theta, V_N(\theta, U_N)) &= -A S_N(h\theta, V_N) + P_N F(S_N(h\theta, V_N)) + h^{-1} D_\theta h^{p+1} d_1 \\
&= -A U_N + P_N F(U_N) + h^{p+1} d_3 + h^p D_\theta d_1
\end{aligned}
$$

where

$$
h^{p+1} d_3 = (-A + P_N F)(S_N(h\theta, S_N(-h\theta, U_N) + h^{p+1} d_2)) - (-A U_N + P_N F(U_N)). \qquad (4.12)
$$

Setting $G = h d_3 + D_\theta d_1$ guarantees the differentiability requirements in (ii) and by continuous extension to $h = 0$ also (iii) can be proved. Then G is defined on $[0, h^*]$ for $\theta \in \mathbf{R}$ and $|U_N| \leq \rho(h)$, thus proving (i). The interpolation property (iv) is fulfilled by construction of Ψ.

Step 4: Estimates uniform in $h, N(h)$

Properties (v) and (vi) are the essential new ingredients to [FS96, proposition2.1]. We show first the estimates for $G = h d_3 + D_\theta d_1$ in (v). Starting with $D_\theta d_1$, we obtain by (4.10):

$$
\begin{aligned}
h^{p+1} D_\theta d_1 &= D_\theta \left[(1-\chi(\theta)) \left(S_N(h\theta - h, \Phi_{h,N}(U_N)) - S_N(h\theta - h, S_N(h, U_N))\right)\right] \\
&= \chi'(\theta) \left(S_N(h\theta - h, \Phi_{h,N}(U_N)) - S_N(h\theta - h, S_N(h, U_N))\right) \\
&\quad +(1-\chi(\theta)) h \left[-A S_N(h\theta - h, \Phi_{h,N}(U_N))\right. \\
&\quad +P_N F(S_N(h\theta - h, \Phi_{h,N}(U_N))) + A S_N(h\theta - h, S_N(h, U_N)) \\
&\quad \left. -P_N F(S_N(h\theta - h, S_N(h, U_N)))\right].
\end{aligned}
$$

Using the mean value theorem we get, where

$$
K = \{V \in \eta\, \Phi_{h,N}(U_N) + (1-\eta) S_N(h, U_N), \eta \in [0,1]\},
$$

$$
\begin{aligned}
& h^{p+1} |D_\theta d_1|_{G_\sigma^{s/2}} \\
\leq\ & |\Phi_{h,N}(U_N) - S_N(h, U_N)|_{G_\sigma^{s/2}} \Big[|\chi'(\theta)| \sup_K |D S_N(h\theta - h, V)|_{L(G_\sigma^{s/2}, G_\sigma^{s/2})} \\
& + h\, |1-\chi(\theta)| \sup_K |D(A S_N(h\theta - h, V) + P_N F(S_N(h\theta - h, V))|_{L(G_\sigma^{s/2}, G_\sigma^{s/2})} \Big].
\end{aligned}
$$

So we need to estimate $|D_V S_N(h\theta - h, V)|_{L(G_\sigma^{s/2}, G_\sigma^{s/2})}$. To do this we use the variational equation, which is fulfilled by $W(t) = D_V S_N(t, V(t))$:

$$W_t + AP_N W = D_V P_N F(P_N V(t))W.$$

Thus we get

$$\begin{aligned} |D_V S_N(h\theta - h, V)|_{G_\sigma^{s/2}} &\leq |\exp(-A(h\theta - h))P_N|_{G_\sigma^{s/2}} \\ &+ \left| \int_0^{h\theta - h} \exp(-A(h\theta - h - \vartheta)) D_V P_N F(P_N V(\vartheta)) d\vartheta \right|_{G_\sigma^{s/2}}. \end{aligned} \tag{4.13}$$

To bound this expression, we first have to analyse

$$V(t) = \exp(-At)P_N V_0 + \int_0^t \exp(-A(t-\vartheta))P_N F(V(\vartheta))d\vartheta.$$

The linear part can be bounded using $h\lambda_n^{1+\beta} \leq C$. $P_N F$ is locally Lipschitz on H_N with constants $L(V_N)$ uniform bounded for $N \to \infty$ in the Gevrey norm of $G_\sigma^{s/2}$, thus we can bound $V(t)$ by a constant C for $|t| \leq h$ and $N < N(h)$ by the Gronwall inequality :

$$\begin{aligned} |V(t)|_{G_\sigma^{s/2}} &\leq C + \int_0^t C|P_N F(V(\vartheta))|d\vartheta \\ &\leq C + \int_0^t CL|V(\vartheta)|_{G_\sigma^{s/2}} d\vartheta \leq C. \end{aligned}$$

Then (4.13) is uniformly bounded when using $h\lambda_N^{1+\beta} \leq C$. Using the bound (1.27) for $|\Phi_{h,N}(U_N) - S_N(h, U_N)|_{G_\sigma^{s/2}}$ we have the estimate

$$h^{p+1}|D_\theta d_1|_{G_\sigma^{s/2}} \leq (c + hC\lambda_N)\, C(|U_N|_{H^s_{per}})h^{p+1} \leq CC(|U_N|_{H^s_{per}})h^{p+1} \tag{4.14}$$

for $\lambda^{1+\beta}h$ bounded.

Next we deal with

$$h^{p+1}d_3 = (-A + P_N F)(S_N(h\theta, S_N(-h\theta, U_N) + h^{p+1}d_2)) - (-AU_N + P_N F(U_N)),$$

hence we get an estimate depending on d_2, where

$$K = \{V \in S_N(-h\theta, U_N) + \eta h^{p+1} d_2, \eta \in [0,1]\}$$

$$\begin{aligned} h^{p+1}|d_3|_{G_\sigma^{s/2}} &\leq \sup_K |D(AS_N(h\theta, V) + P_N F(S_N(h\theta, V))|_{L(G_\sigma^{s/2}, G_\sigma^{s/2})} h^{p+1}|d_2|_{G_\sigma^{s/2}} \\ &\leq |\lambda_N + L|Ch^{p+1}|d_2|_{G_\sigma^{s/2}} \leq Ch^p|d_2|_{G_\sigma^{s/2}}. \end{aligned} \tag{4.15}$$

It remains to estimate d_2 given by (4.11)

$$h^{p+1}d_2(U_N,\theta) = V_N(\theta,U_N) - S_N(-h\theta,U_N).$$

From (4.10) we get

$$S_N(-h\theta,\tilde{\Psi}(\theta,U_N) - h^{p+1}d_1(U_N)) = U_N$$

replacing U_N by $V_N(U_N)$ yields to

$$S_N(-h\theta,U_N - h^{p+1}d_1(V_N)) = V_N.$$

Hence

$$h^{p+1}d_2(U_N,\theta) = S_N(-h\theta,U_N - d_1(V_N)) - S_N(-h\theta,U_N)$$

with

$$h^{p+1}|d_2(U_N,\theta)|_{G_\sigma^{s/2}} \leq Ch^{p+1}|d_1(V_N,\theta)|_{G_\sigma^{s/2}}.$$

$V_N(U_N,\theta)$ is defined by $\tilde{\Psi}(\theta,V_N(U_N,\theta),h) = U_N$, where $\tilde{\Psi}$ is a near-identity diffeomorphism. So V_N remains uniformly bounded per construction. Hence we can estimate d_1 in the way as $D_\theta d_1$ above and we get $h^{p+1}|d_1(V_N,\theta)|_{G_\sigma^{s/2}} \leq Ch^{p+1}$ and thus $h^{p+1}|d_3(U_N,\theta)|_{G_\sigma^{s/2}} \leq Ch^p$ for $U_N \in G_\sigma^{s/2}$ bounded. Hence for $G = hd_3 + D_\theta d_1$ we obtain

$$|G(U_N,\theta,h,N)|_{G_\sigma^{s/2}} \leq r(|U_N|_{G_\sigma^{s/2}})$$

and (v) is shown.

It remains to check (vi). We need uniform bounds when extending G to a complex domain $D_N+\delta$. First we check that $S_N(h\theta,U_N)$ does not leave $D_N+\delta_0$ for $\theta\in[0,1]$ and $U_N \in D_N+\delta$: For the imaginary part of U_N we get the following equation:

$$\frac{d}{dt}Im(U_N) = -AIm(U_N) + Im(F(U_N)).$$

As $Im(F(U_N)) = 0$ for $Im(U_N) = 0$, there is a linear factor of $Im(U_N)$ in $Im(F(U_N))$. Thus we can estimate

$$\begin{aligned}
|Im(S_N(h\theta,U_N))|_{G_\sigma^{s/2}} &\leq |e^{-h\theta A}Im(U_N(0)) + \int_0^{h\theta} e^{-(h\theta-\tau)A}Im(F(U_N(\tau)))d\tau|_{G_\sigma^{s/2}} \\
&\leq \delta + hC\delta \leq \delta + h_0C\delta = \delta_0,
\end{aligned}$$

and $S_N(h\theta,U_N)$ does not leave a slightly enlarged complex extended domain. Now all estimates for (v) still hold in the extended domain, essentially because the error estimate $|\Phi_{h,N}(U_0) - S_N(h,U_0)|_{G_\sigma^{s/2}} \leq C(|U_N(0)|_{G_\sigma^{s/2}})h^{p+1}$ given in (1.27) does extend to the complex extended domain by assumption. This completes the proof of the proposition. □

4.3 Proof of Theorem 2

We are now in the position to prove theorem 2. By proposition 4.2 the numerical scheme $\Phi_{h,N}$ coincides exactly with the time-h map of the ordinary differential equation

$$\dot{U}_N + AU_N = P_N F(U_N) + h^p G(U_N, \frac{t}{h}, h, N).$$

We extend this equation to a partial differential equation

$$\begin{aligned} \dot{\tilde{U}} + A\tilde{U} &= P_N F(\tilde{U}) + h^p G(\tilde{U}, \frac{t}{h}, h, N(h)) && (4.16) \\ \tilde{U}(0) &= U_0 && (4.17) \end{aligned}$$

with a local evolution $\tilde{S}(t, 0, U_0)$, such that the numerical scheme and $\tilde{S}(t, 0, P_N U_0)$ coincide. We want to apply theorem 1' to (4.16). By assumption F and therefore $P_N F$ fulfil (H.1) and (H.2). The perturbation term G satisfies (H.1) and (H.2) by proposition 4.2.

So we get by theorem 1' an averaged equation

$$\begin{aligned} \dot{V} + AV &= P_N F(V) + \bar{F}(V, h) + \alpha(V, \frac{t}{h}, h, N(h)) \\ V(0) &= U_0 \end{aligned}$$

with local evolution $\tilde{S}_V(t, 0, U_0)$ and even stronger estimates on the nonautonomous part:

$$\sup_{V(0) \in B_{H^s_{per}}(M_1, 0) \times \mathbf{R}^+ \times [0, h_1)} |\alpha|_{H^s_{per}} \leq c_2 h \exp(-c_1 h^{-\frac{\beta}{1+\beta}}).$$

As the nonlinearities in 4.16 are range only on a finite number of Galerkin modes the estimates in theorem 1' can be improved. We obtain $b_1 \equiv b_2 \equiv 0$ when using the same N here as in the proof of theorem 1', see (3.23,3.24). We also have

$$\tilde{S}_V(n \cdot h, 0, P_N U_0) = \Phi^n_{h,N}(U_0), \tag{4.18}$$

since the the coordinate change W is zero on multiples of the period h.

The modified equation (1.28) given in theorem 2 is then

$$\begin{aligned} \dot{\bar{U}} + A\bar{U} &= F(\bar{U}) + \bar{F}(\bar{U}, h, N(h)) && (4.19) \\ \bar{U}(0) &= U_0 && (4.20) \end{aligned}$$

which defines a local semiflow $\bar{S}_{h,N}(t, U_0)$ depending on N and h. The estimates on $\bar{F}$ in (1.31) then also follow from the proof of theorem 1', as $b_1 \equiv b_2 \equiv 0$ holds.

Estimating the difference between $\bar{U}(t) = \bar{S}_{h,N}(t, U_0)$ and $V(t) = \tilde{S}_V(t, 0, P_N U_0)$ will prove the desired exponential estimate (1.30) because of (4.18). We use $P_N V(\tau) =$

$V(\tau)$ and that $\exp(-A(t-\tau))U(\tau) \in G_t^{s/2}$, if $U(\tau) \in G_\tau^{s/2}$. The time T^* denotes the maximal Gevrey exponent given by proposition 2.5.

$$\begin{aligned}
& |\bar{U}(t) - V(t)|_{H^s} \\
= \ & |\bar{S}_{h,N}(t, U_0) - \tilde{S}_V(t, 0, P_N U_0)|_{H^s} \\
= \ & |\exp(-At)(Id - P_N)U_0|_{H^s} + |\int_0^t \exp(-A(t-\tau))\{F(\bar{U}(\tau)) + \bar{F}(\bar{U}(\tau)) \\
& -P_N F(P_N V(\tau))) - \bar{F}(V(\tau)) - \alpha(V(\tau), \frac{\tau}{h}, h, N(h))\} d\tau|_{H^s} \\
& \text{ adding inside the integral } 0 = -P_N F(\bar{U}(\tau)) + P_N F(P_N \bar{U}(\tau)) \\
\leq \ & C\exp(-\lambda_{N+1} t) + |\int_0^t \exp(-A(t-\tau))\{F(\bar{U}(\tau)) - P_N F(\bar{U}(\tau)) \\
& +P_N F(P_N \bar{U}(\tau)) - P_N F(P_N V(\tau))\} d\tau|_{H^s} \\
& +L\int_0^t |V(\tau) - \bar{U}(\tau)|_{H^s} d\tau + tC\exp(-c_1 h^{-\frac{\beta}{1+\beta}}) \\
\leq \ & C\exp\left(-\lambda_{N+1} t\right) + tC\exp\left(-\sqrt{\lambda_{N+1}}\min(t, T^*)\right) + tC\exp\left(-c_1 h^{-\frac{\beta}{1+\beta}}\right) \\
& +L\int_0^t |V(\tau) - \bar{U}(\tau)|_{H^s} d\tau.
\end{aligned}$$

For $t \leq \min(\frac{1}{2L}, T^*)$, where L is the Lipschitz constant of $\bar{F}(.) + P_N F(P_N .)$, the last inequality implies

$$y(t) = \max_{0 \leq \tau \leq t} |\bar{U}(\tau) - V(\tau)|_{H^s} \leq C\left(\exp\left(-\sqrt{\lambda_{N+1}} t\right) + \exp\left(-c_1 h^{-\frac{\beta}{1+\beta}}\right)\right).$$

For $t \geq t^* = \min(\frac{1}{2L}, T^*)$ we use the Gronwall inequality and get exponential estimates:

$$\begin{aligned}
|\bar{U}(t) - V(t)|_{H^s} \leq \ & C\left(\exp\left(-\sqrt{\lambda_{N+1}} t^*\right) + \exp\left(-c_1 h^{-\frac{\beta}{1+\beta}}\right)\right) \\
& \cdot \left(\exp(L(t-t^*)) + \frac{1}{L}(\exp(L(t-t^*)) - 1)\right) \\
\leq \ & \tilde{C}\left(\exp\left(-\sqrt{\lambda_{N+1}} t^*\right) + \exp\left(-c_1 h^{-\frac{\beta}{1+\beta}}\right)\right)\exp(L(t-t^*)).
\end{aligned}$$

This implies the desired exponetial estimate (1.30) on a time-scale $O\left(h^{-\frac{\beta}{1+\beta}}, \sqrt{\lambda_{N(h)}}\right)$ and the theorem is proved. □

Chapter 5

Global Errors

The aim of this chapter is to get some improved geometric theory by applying the averaging results. We describe the dynamics near forced equilibria and homoclinic orbits. The dynamics of the period-map $\tilde{S}^h(h,0,.)$ given by (1.19) and the time-h-map $\bar{S}^h(h,.)$ given by the semiflow of (1.24) are compared. By standard theory we get persistence of phase-portraits. For example the distance between the original equilibrium and the one of the forced system is of order $O(h)$. We want to improve results to exponential estimates, by comparing the dynamical systems given by the rapidly forced equation and the averaged equation. Hence we want to show that the dynamical effects of the rapid forcing are exponentially small in h.

When applying theorem 1 we have to circumvent the transient. The basic idea is given in the following lemma. We get exponential estimates when we can assure backward existence for the forced equation.

Lemma 5.1 *Let $\tilde{U}(t) \in H^s_{per}, t \in [-T^*, 0]$ be a solution of (1.19). Then the difference between $\bar{U}(t) = \bar{S}^h(t, U_0)$ and $\tilde{U}(t) = \tilde{S}^h(t, 0, U_0)$ with $U_0 = \tilde{U}(0)$ is exponentially small for finite times $[0, T]$: When the solutions remain in some ball $B_{H^s}(M_0, 0)$ then*

$$|\tilde{S}^h(t, 0, U_0) - \bar{S}^h(t, U_0)|_{H^s} \leq C \exp(-ch^{-\frac{1}{3}}). \tag{5.1}$$

Proof : We apply the variation of constant formula and estimate the nonautonomous part α of (1.19) using theorem 1. When the solution $\tilde{U}(\theta)$ starts at time $-T^*$, we get

$$|\alpha(\tilde{U}(\theta), \frac{\theta}{h}, h)| \leq C \exp(-ch^{-\frac{1}{3}})$$

for $\theta \geq 0$. Thus

$$|\tilde{U}(t) - \bar{U}(t)|_{H^s} \quad = \quad |\tilde{S}^h(t, 0, U_0) - \bar{S}^h(t, U_0)|_{H^s}$$

$$
\begin{aligned}
&= |\int_0^t \exp(-A(t-\theta))\{F(\tilde{U}(\theta)) + \bar{F}(\tilde{U}(\theta)) + \alpha(\tilde{U}(\theta)) \\
&\quad -(F(\bar{U}(\theta)) + \bar{F}(\bar{U}(\theta)))\}d\theta|_{H^s} \\
&\leq L\int_0^t |\tilde{U}(\theta) - \bar{U}(\theta))|_{H^s} d\theta + tCh\exp(-ch^{-\frac{1}{3}}),
\end{aligned}
$$

if $\tilde{U}(t), \bar{U}(t) \in B_{H^s}(M_0, 0)$ for $t \in [0, T]$. By the Gronwall inequality we get then the estimate (5.1) for such finite times $[0, T]$. $\Box$

In section 5.1 we describe the phase portrait near hyperbolic equilibria following [Stu95] and [AD91]. The rate of convergence is shown to be exponential. In section 5.2 we analyse the persistence of homoclinic orbits in one-parameter families of semiflows and give exponential estimates of the splitting due to the forcing.

5.1 Equilibria

Hyperbolic equilibria and the phase portrait near them persist under small perturbations of the semiflow, as shown in [Stu95] and [AD91] using quite general semigroup methods. We cannot apply these methods directly to get exponential estimates. Especially for the rapidly forced equation there is the problem of the transient. This problem is easier for the discretization analysed in theorem 2. Therefore we only consider the rapidly forced case and compare it to the averaged system. The backward error analysis is then similar for the special discretization scheme.

In ordinary differential equations the existence of phase portraits near equilibria is closely related to the Grobman-Hartman theorem. For specific examples, where a Grobman-Hartman theorem can be proved, see [BL94],[Lu91].

A point $u_0 \in H^s_{per}$ is called an equilibrium, when $S(t, u_0) = u_0, \forall t \geq 0$. It is a hyperbolic equilibrium, if for any fixed t the spectrum σ of the linearisation $DS(t, u_0)$ at u_0 splits into two parts by the unit circle: there exists $\alpha \in (0, 1)$ such that $\sigma = \sigma_s \cup \sigma_u$

$$\sigma_s = \{\lambda \in \sigma | |\lambda| < \alpha\} \text{ and } \sigma_u = \{\lambda \in \sigma | |\lambda| > \alpha^{-1}\}.$$

The stable and unstable eigenspaces of the linearisation at a hyperbolic equilibrium u_0 are denoted by E^s and E^u and orthogonal projections on them by P^s and P^u. We will set without restriction $u_0 = 0$. The local stable and unstable manifolds are then given by the graphs of

$$
\begin{aligned}
p_s : E^s \cap B_{H^s}(\rho, 0) &\to E^u \text{ for the stable and} \\
p_u : E^u \cap B_{H^s}(\rho, 0) &\to E^s \text{ for the unstable manifold.}
\end{aligned}
$$

Proposition 5.2 *Suppose the original equation (1.9) possesses a hyperbolic equilibrium at u_0. Then the forced equation (1.16) has a hyperbolic periodic orbit of period h: $\tilde{S}^h(h,0,\tilde{u}_h) = \tilde{u}_h$, which is exponentially close to a hyperbolic equilibrium $\bar{u}_h$ of the averaged equation (1.24) with:*

$$\begin{aligned} |\tilde{u}_h - \bar{u}_h|_{H^s} &\leq C\exp(-ch^{-\frac{1}{3}}) && (5.2)\\ |\bar{u}_h - u_0|_{H^s} &\leq Ch. && (5.3)\end{aligned}$$

All further results compare the discrete semigroup $\tilde{S}^h(j\cdot h,0,.), j \geq 0$ with $\bar{S}(t,.)$. There exists $\rho > 0$ such that the local stable and unstable manifolds of $\tilde{u}_h, \bar{u}_h$ are given by graphs of functions

$$\begin{aligned} \tilde{p}^h_s(.), \bar{p}^h_s(.) : E^s \cap B_{H^s}(\rho,0) &\to E^u\\ \tilde{p}^h_u(.), \bar{p}^h_u(.) : E^u \cap B_{H^s}(\rho,0) &\to E^s\end{aligned}$$

in the following way with $u_s \in E^s \cap B_{H^s}(\rho,0)$:

$$W^{s,loc}(\tilde{u}_h) = \tilde{u}_h + u_s + \tilde{p}^h_s(u_s)$$

and in the same way for the other manifolds too. Moreover the local unstable manifolds are exponentially close on $B_{H^s}(\rho,0)$:

$$\sup_{U\in P^u(B_{H^s}(\rho,0))} |(\tilde{p}^h_u(U)+\tilde{u}_h) - (\bar{p}^h_u(U)+\bar{u}_h)|_{H^s} \leq C\exp(-ch^{-\frac{1}{3}}) \tag{5.4}$$

and the local stable manifolds are exponentially close on $B_{H^s}(\rho,0)$ intersected with the image of $W^{s,loc}(\tilde{u}_h)$ under the regularising map $\tilde{S}^h(T^*,0,.)$:

$$\sup_{U\in P^s\left(B_{H^s}(\rho,0)\cap\tilde{S}^h(T^*,0,W^{s,loc}(\tilde{u}_h))\right)} |\tilde{p}^h_s(U)+\tilde{u}_h - (\bar{p}^h_s(U)+\bar{u}_h)|_{H^s} \leq C\exp(-ch^{-\frac{1}{3}}). \tag{5.5}$$

On $B_{H^s}(\rho,0)\cap\tilde{S}^h(T^*,0,B_{H^s}(\rho,0))$ the orbits of $\tilde{S}^h(t,0,.)$ are exponentially followed by orbits of $\bar{S}^h(t,.)$ in the following sense, taking all times including T^* as multiples of h: For all $U \in B_{H^s}(\rho,0)\cap\tilde{S}^h(T^*,0,B_{H^s}(\rho,0))$ there exists $V \in B_{H^s}(\rho,0)\cap \bar{S}^h(T^*,B_{H^s}(\rho,0))$, such that as long $\bar{S}^h(t,U) \in B_{H^s}(\rho,0)$ we have that

$$|\tilde{S}^h(t,0,U) - \bar{S}(t,V)|_{H^s_{per}} \leq C\exp(-ch^{-\frac{1}{3}}). \tag{5.6}$$

Remark 5.3 *It is also possible to compare $\tilde{S}^h(j\cdot h+\theta,\theta,.)$ with $\bar{S}^h(t,.)$, in the same way. The fixed point of $\tilde{S}^h(j\cdot h+\theta,\theta,.)$ is then $\tilde{u}_h(\theta) = \tilde{S}^h(\theta,0,\tilde{u}_h)$. All estimates above hold uniformly in θ. When considering the continuous time $\tilde{S}^h(t,0,.)$, then the stable and unstable manifolds will depend on time. The functions $\tilde{p}^h_s(\theta,.)$ and $\tilde{p}^h_u(\theta,.)$ are periodic in θ, but the estimates in (5.4) and (5.5) still hold.*

Remark 5.4 *The orbits of $\tilde{S}^h(t,0,.)$ also follow the orbits of $\bar{S}^h(t,.)$, but the author could not easily prove exponential closeness.*

Proof :

Step 1: Equilibrium

The averaged equation has a unique hyperbolic equilibrium $\bar{u}_h$, which is dependent on h near the original equilibrium u_0. This follows from [Stu95, theorem 4.3] or [AD91, theorem 2.2], since the semiflows $\bar{S}^h(t,u)$ given by (1.24) and $S(t,u)$ defined by (1.9) are close in $C^1(B_{H^s_{per}}(\rho,0),H^s_{per})$. The C^0 distance is given by

$$\begin{aligned} |\bar{S}^h(t,u)-S(t,u)|_{H^s_{per}} &\leq |\bar{S}^h(t,u)-\tilde{S}^h(t,u)|_{H^s_{per}}+|\tilde{S}^h(t,u)-S(t,u)|_{H^s_{per}} \\ &\leq Ch, \end{aligned}$$

where we used that the coordinate change in theorem 1 is $Id+O(h)$. Then we can estimate the distance $\bar{u}_h-u_0$ following [AD91]. This is based on the following idea: $-Id+S(t,.)$ is invertible for fixed t near 0 and its inverse is Lipschitz, thus

$$\begin{aligned} |\bar{u}_h-u_0|_{H^s_{per}} &\leq |\mathrm{Lip}(-Id+S(t,.))^{-1}(-\bar{u}_h+S(t,\bar{u}_h)-(-u_0+S(t,u_0)))|_{H^s_{per}} \\ &= \mathrm{Lip}(-Id+S(t,.))^{-1}|-\bar{u}_h+S(t,\bar{u}_h)|_{H^s_{per}} \\ &= \mathrm{Lip}(-Id+S(t,.))^{-1}|-\bar{S}^h(t,\bar{u}_h)+S(t,\bar{u}_h)|_{H^s_{per}} \\ &\leq Ch. \end{aligned}$$

Starting from hyperbolic equilibria of the averaged equation we are now going to analyse 'equilibria' of the forced equation, i.e. we consider periodic orbits of period h. For this we use similar methods as above. The existence of a periodic orbit with period h is given by [Stu95, theorem 4.4] with $\tilde{S}^h(h,0,\tilde{u}^h)=\tilde{u}^h$. Hyperbolicity follows by C^1 closeness of $\tilde{S}^h(h,0,.)$ and $\bar{S}^h(h,.)$. To estimate the difference we use again the same trick as above. $-Id+\bar{S}^h(t,.)$ is invertible for $t>0$ near $\bar{u}^h$ and its inverse is Lipschitz, then taking t as multiple of h

$$\begin{aligned} |\tilde{u}^h-\bar{u}_h|_{H^s_{per}} &\leq |\mathrm{Lip}(-Id+\bar{S}^h(t,.))^{-1}(-\tilde{u}^h+\bar{S}^h(t,\tilde{u}^h)-(-\bar{u}_h+\bar{S}^h(t,\bar{u}_h)))|_{H^s_{per}} \\ &= \mathrm{Lip}(-Id+\bar{S}^h(t,.))^{-1}|\tilde{S}^h(t,\tilde{u}_h)-\bar{S}^h(t,\tilde{u}_h)|_{H^s_{per}} \end{aligned}$$

As we can guarantee backward existence for $\tilde{u}_h$, we can apply lemma 5.1 and get exponential closeness:

$$|\tilde{u}^h-\bar{u}_h|_{H^s_{per}}\leq C\exp(-ch^{-\frac{1}{3}}).$$

Step 2: Stable and unstable manifold

Next we deal with the local stable and unstable manifolds. Inside some neighbourhood $B_{H^s_{per}}(\rho,0)$ of $u_0 = 0$ the existence and convergence of the local stable and unstable manifolds are essentially given by [Stu95, theorem 4.19, corollary 5.6] and [AD91, theorem 2.2]. We transform the forced and the averaged equations such that the equilibrium is at 0, i.e. $\bar{S}^h_0(t,0,.) = \bar{S}^h(t,0,.+\bar{u}^h) - \bar{u}^h$ and $\tilde{S}^h_0(t,0,.) = \tilde{S}^h(t,0,.+\tilde{u}_h) - \tilde{u}_h$, where $t = j\cdot h, j \in \mathbf{N}_0$. Then the local stable and unstable manifolds are given as graphs of the following functions:

$$\begin{aligned}
\tilde{p}^h_s(.), \bar{p}^h_s(.) : E^s \cap B_{H^s}(\rho,0) &\rightarrow E^u \\
\tilde{p}^h_u(.), \bar{p}^h_u(.) : E^u \cap B_{H^s}(\rho,0) &\rightarrow E^s.
\end{aligned}$$

The estimates can again be improved using lemma 5.1. So we can estimate $\tilde{p}^h_s(.) - \bar{p}^h_s(.)$ and $\tilde{p}^h_u(.) - \bar{p}^h_u(.)$ following e.g. [AD91]. We define

$$\begin{aligned}
C_0(H^s_{per}) &= \{(\gamma(j))_{j\in\mathbf{N}_0} | \gamma(j) \in H^s_{per}, \gamma(j) \rightarrow 0\} \\
C_{0,\rho}(H^s_{per}) &= \{(\gamma(j))_{j\in\mathbf{N}_0} \in C_0(H^s_{per}) | \|\gamma\|_\infty = \sup_{j\geq 0} |\gamma(j)|_{H^s_{per}} < \rho\}.
\end{aligned}$$

We start with the unstable manifold. In [AD91]

$$\begin{aligned}
T^u_h : C_{0,\rho}(H^s_{per}) &\rightarrow E^u \times C_0(H^s_{per}) \\
T^u_h((\gamma(j))_{j\in\mathbf{N}_0}) &= (\gamma(0)_u; \gamma(j) - \bar{S}^h(t,\gamma(j+1)), j \geq 0)
\end{aligned}$$

is shown to be be invertible with Lipschitz continuous inverse for fixed t. Hence taking $u_u \in E^u \cap B_{H^s}(\rho,0)$ and

$$\begin{aligned}
\gamma_1(j) &= \tilde{S}^h_0(-jt,0,u_u + \tilde{p}^h_u(u_u)) \\
\gamma_2(j) &= \bar{S}^h_0(-jt,0,u_u + \bar{p}^h_u(u_u))
\end{aligned}$$

gives

$$\begin{aligned}
T^u_h((\gamma_2(j))) &= (u_u, 0) \text{ and} \\
T^u_h((\gamma_1(j))) &= \Big(u_u, \tilde{S}^h_0\left(-jt,0,u_u + \tilde{p}^h_u(u_u)\right) \\
&\quad -\bar{S}^h_0\left(t, \tilde{S}^h_0(-(j+1)t,0,u_u + \tilde{p}^h_u(u_u))\right), j \geq 0\Big).
\end{aligned}$$

Then we get

$$\begin{aligned}
&|\tilde{p}^h_u(u_u) - \bar{p}^h_u(u_u)|_{H^s_{per}} \\
\leq\ & \|\gamma_1 - \gamma_2\|_\infty \leq 2\sigma \|T^u_h((\gamma_1(j))) - T^u_h((\gamma_2(j)))\|_\infty \\
\leq\ & 2\sigma \|T^u_h((\gamma_1(j)))\|_\infty \\
=\ & 2\sigma \|\tilde{S}^h_0(t,0,\tilde{S}^h_0(-(j+1)t,0,u_u + \tilde{p}^h_u(u_u))) \\
& -\bar{S}^h_0(t,\tilde{S}^h_0(-(j+1)t,0,u_u + \tilde{p}^h_u(u_u))), j \geq 0\|_\infty.
\end{aligned}$$

Each point $\tilde{S}_0^h(-(j+1)t, 0, u_u + \tilde{p}_u^h(u_u)) = \tilde{S}^h(-(j+1)t, 0, \tilde{u}_h + u_u + \tilde{p}_u^h(u_u))$ has a backward orbit in H^s_{per} under (1.19). Thus $\tilde{S}^h(-(j+1)t, 0, \tilde{u}_h + u_u + \tilde{p}_u^h(u_u)) = \tilde{S}^h(T^*, 0, u_j)$ for some $u_j \in B_{H^s}(\rho, 0)$. Hence for T^* a multiple of h

$$
\begin{aligned}
& \sup_{U \in P^u(B_{H^s}(\rho,0))} |\tilde{p}_u^h(U) + \tilde{u}_h - (\bar{p}_u^h(U) + \bar{u}_h))|_{H^s} \\
\leq \;& C \exp(-ch^{-\frac{1}{3}}) + 2\sigma \sup_{j \geq 0} |\tilde{S}_0^h(t, 0, \tilde{S}_0^h(T^*, 0, u_j)) - \bar{S}_0^h(t, \tilde{S}_0^h(T^*, 0, u_j))|_{H^s} \\
\leq \;& C \exp(-ch^{-\frac{1}{3}}),
\end{aligned}
$$

which shows (5.4). Next we deal with the stable manifolds, where the reasoning is similar. We use with t again a multiple of h

$$
\begin{aligned}
T_h^s : C_{0,\rho}(H^s_{per}) &\rightarrow E^s \times C_0(H^s_{per}) \\
T_h^s((\gamma(j))_{j \in \mathbf{N}}) &= (\gamma(0)_s; \gamma(j+1) - \bar{S}_0^h(t, \gamma(j)), j \geq 0)
\end{aligned}
$$

and with $u_s \in E^s$ we set

$$
\begin{aligned}
\gamma_1(j) &= \tilde{S}_0^h(jt, 0, u_s + \tilde{p}_s^h(u_s)) \\
\gamma_2(j) &= \bar{S}_0^h(jt, u_s + \bar{p}_s^h(u_s)).
\end{aligned}
$$

Again we have

$$
\begin{aligned}
& \sup_{U \in B_{H^s}(\rho, 0 \cap E^s} |\tilde{p}_s^h(U) + \tilde{u}_h - (\bar{p}_s^h(U) + \bar{u}_h)|_{H^s} \\
\leq \;& C \exp(-ch^{-\frac{1}{3}}) + 2\sigma \|T_h^s(\gamma_1) - T_h^s(\gamma_2)\|_\infty \\
\leq \;& C \exp(-ch^{-\frac{1}{3}}) + 2\sigma \sup_{j \geq 0} |\tilde{S}_0^h(t, \tilde{S}_0^h(jt, 0, u_s + \tilde{p}_s^h(u_s))) \\
& - \bar{S}_0^h(t, \tilde{S}_0^h(jt, 0, u_s + \tilde{p}_s^h(u_s)))|_{H^s} \\
\leq \;& 2C \exp(-ch^{-\frac{1}{3}}) + 2\sigma \sup_{j \geq 0} |\tilde{S}^h(t, \tilde{S}^h(jt, 0, \tilde{u}_h + u_s + \tilde{p}_s^h(u_s))) \\
& - \bar{S}^h(t, \tilde{S}^h(jt, 0, \tilde{u}_h + u_s + \tilde{p}_s^h(u_s)))|_{H^s}.
\end{aligned}
$$

The last term is exponentially small, if $u_s + \tilde{p}_s^h(u_s)$ has a backward orbit. This can be guaranteed, if $u_s \in P^s(B_{H^s}(\rho, 0) \cap \tilde{S}^h(T^*, 0, W^{s,loc}(\tilde{u}_h))$. This proves the estimate (5.5).

Step 3: Phase Portrait

The convergence of phase portraits can be obtained from [Stu95, theorem 4.18] and [AD91, theorem 2.2]. It is proved by solving a discrete boundary value problem. For initial value U_0 we prescribe the projection to the stable eigenspace $P^s U_0$ and for the last value U_J the projection to the unstable eigenspace $P^u U_J$. In [AD91] the mapping

$$\begin{aligned} H : B_{H^s}(\rho,0)^{J+1} &\to E^s \times (H^s_{per})^J \times E^u \\ \gamma &\mapsto (P^s(\gamma(0)); \gamma(j+1) - \bar{S}^h_0(t,\gamma(j)), 0 \le j \le J-1; P^u(\gamma(J))) \end{aligned}$$

is shown to be invertible with Lipschitz inverse. For all $u_s \in E^s \cap B_{H^s}(\rho_0, 0)$ and $u_u \in E^u \cap B_{H^s}(\rho_0, 0)$ for some smaller $0 < \rho_0 < \rho$ there exists $\gamma \in B_{H^s}(\rho, 0)^{J+1}$, such that $H(\gamma) = (u_s, 0, u_u)$. Here we now again assume all times to be multiples of the period h.

For the improved rates on $B_{H^s}(\rho,0) \cap \tilde{S}^h(T^*, 0, B_{H^s}(\rho, 0))$ we use again a similar argument as above: Let $\gamma_1 = \tilde{S}^h(j \cdot h, 0, U)$ given with $U \in B_{H^s}(\rho, 0) \cap \tilde{S}^h(T^*, 0, B_{H^s}(\rho, 0))$. Take J maximal such that all $\gamma_1(j) \in B_{H^s}(\rho, 0)$ for all $j \le J$. Then by [AD91] there exists $\gamma_2 \in B_{H^s}(\rho,0)^{J+1}$, which solves the boundary value problem for $\bar{S}^h$, i.e. with $H(\gamma_2) = (P^s U, 0, P^u \gamma_1(J))$. Choosing $V = P^s U + P^u \gamma_2(0))$ we get

$$\begin{aligned} & |\tilde{S}^h(t,0,U) - \bar{S}^h(t,V)|_{H^s_{per}} \le \sup_{j\ge 0} |\gamma_1(j) - \gamma_2(j)|_{H^s} \\ \le\ & C \sup_{j \ge 0} |H(\gamma_1)(j) - H(\gamma_2)(j)|_{H^s} = C \sup_{j\ge 0} |\gamma_1(j+1) - \bar{S}^h(h, \gamma_1(j))|_{H^s} \\ =\ & C \sup_{j\ge 0} |\tilde{S}^h((j+1)\cdot h, 0, U) - \bar{S}^h(h, \tilde{S}^h(j \cdot h, 0, U))|_{H^s} \le C \exp(-ch^{-\frac{1}{3}}), \end{aligned}$$

using lemma 5.1, as U has a backward orbit. This proves (5.6). $\square$

5.2 Homoclinic orbits

In this section we analyse the influence of rapid forcing on homoclinic orbits in semiflows defined by a parameter-dependent version of (1.9):

$$\frac{\partial}{\partial t} U(x,t) - D\Delta U(x,t) = F(U(x,t), \lambda) + hG(U(x,t), \frac{t}{h}, h, \lambda). \tag{5.7}$$

The nonlinearities $F = (f_1, \dots, f_n)$ and $G = (g_1, \dots, g_n)$ are supposed to depend smoothly on λ. All other assumptions of theorem 1' are also assumed. Homoclinic orbits $\Gamma(t)$ are solutions, which exist for positive and negative times and converge to the same equilibrium u_0 for $t \to \infty$ and $t \to -\infty$:

$$\lim_{t \to \pm\infty} \Gamma(t) = u_0.$$

Hence $\Gamma(t) \subset W^s(u_0) \cap W^u(u_0)$. Homoclinic orbits of the autonomous part (5.7 with $h = 0$) can be generically encountered in one parameter families of equations only, since the stable and the unstable manifold intersect nontransversally because of the direction of the homoclinic orbit.

The existence of homoclinic orbits can be deduced, for example, from the existence of Takens-Bogdanov bifurcation points in equations depending on two parameters (for a concrete example of reaction-diffusion equations see [Fie86]). In nonautonomous equations the stable and unstable manifold may intersect transversally at points of the homoclinic orbit. The stable and unstable manifolds split and the homoclinic orbit then consists out of discrete points instead of a continuous arc. We will show in the next proposition, that this effect only occurs in a exponentially small wedge in parameter space (λ, h). The splitting distance for one-dimensional unstable manifolds is also shown to be exponentially small in h.

Proposition 5.5 *Suppose the unforced equation (5.7 with $h = 0$) possesses a homoclinic orbit $\Gamma(t)$ for $\lambda_0 \in \Lambda \subset \mathbf{R}$, which is biasymptotic to the hyperbolic equilibrium u_0 with $dim(W^u(u_0)) = k < \infty$. Also assume that the forward orbit of the unstable manifold exists for all times. Assume that the homoclinic orbit is non-degenerate, i.e.*

$$(\lambda, T_U W^u(u_0, \lambda)) \pitchfork_{\lambda_0, \Gamma(t)} (\lambda, T_U W^s(u_0, \lambda)). \tag{5.8}$$

Then the averaged equation (1.24) possesses a homoclinic orbit $\Gamma^h(t)$ for some λ_h with $|\lambda_h - \lambda_0| \leq C h^{\frac{s+1}{3}}$. The nonautonomous equations (5.7) may possess transverse homoclinic orbits near $\Gamma^h(t)$ only for an exponentially small wedge $|\lambda - \lambda_h| \leq C \exp(-c h^{-\frac{1}{3}})$.

If the unstable manifold is one-dimensional ($k = 1$), then there exist homoclinic orbits. While there exists a transverse homoclinic orbit, the distance of the stable and the unstable manifolds is exponentially small when measuring the distance in points in $\bar{S}^h(T^*, W^{s,loc}(\tilde{u}_h))$.

Remark 5.6 *Such results can be shown for a larger class of parabolic equations, when using the functional analytic methods of [FS96] like Ljapunov-Schmidt reduction, extension to complex time and exponential dichotomies. For appropriate results on exponential dichotomies for semilinear parabolic equations, see [Bla86] and in particular [Pet93]. The size of the wedge in parameter space can then even estimated by $C \exp(-c h^{-1})$. This was first noted in [FS96, p.63].*

Proof :

We will first describe the geometric idea of the proof. The existence of the homoclinic orbit for the averaged system, which is a small perturbation of the original equation, is due to the non-degeneracy and transversality in the extended phase space $H^s_{per} \times \Lambda$. The result on exponential small splittings follows with proposition

5.2. The local stable and unstable manifolds are exponentially close to those of the averaged system, where is no splitting at all. When the local unstable manifold evolves forward in time, it only needs some finite time to reach the local stable manifold near the homoclinic orbit. In this finite time the unstable manifolds of $\bar{u}_h$ and $\tilde{u}_h$ remain exponentially close. So the stable and unstable manifolds can only intersect, when the distance between the stable and the unstable manifold of the averaged system is exponentially small. This holds only in an exponentially small wedge in parameter space due to transverse unfolding. The rigorous proof will be given in three steps.

Step 1: Local coordinate transformation

We transform the semigroups in such a way, that the hyperbolic equilibrium of the averaged equation is at the origin 0 and that the local stable manifold coincides with E^s in $B_{H^s_{per}}(\frac{\rho}{2}, 0)$. Hence we first transform for all h and λ

$$\bar{S}^h_0(t, u) = \bar{S}^h(t, u + \bar{u}_h) - \bar{u}_h$$

In $B_{H^s_{per}}(\rho, 0)$ we use a smooth cut-off function χ on $\mathbf{R}^+_0$ defined by:

$$\chi(\tau) = \begin{cases} 1 \text{ for } 0 \le \tau \le \frac{1}{2} \\ 0 \text{ for } \tau \ge 1. \end{cases}$$

Then we let for $u = u_s + u_u$ with $u_s \in E^s$ and $u_u \in E^u$

$$\bar{S}^h_*(t, u_s + u_u) = \bar{S}^h_0(t, u_s + u_u + \chi(\frac{|u|_{H^s_{per}}}{\rho})\bar{p}^h_s(u_s)) \tag{5.9}$$

The representation of the local unstable manifold is also transformed by this local coordinate transform, but we will still denote the function representing the unstable manifold as a graph by $\bar{p}^h_u$. We use for the forced system the same transformation. We denote the transformed semi-evolution by $\tilde{S}^h_*(t, 0, .)$ and the transformed periodic orbit of period h of the forced system by U_h.

Step 2: Persistence of homoclinic orbits

We will apply the transversal isotopy theorem [AR67, theorem20.2] and the openness of transversal intersection to prove the persistence. The theorems in Abraham and Robbin state:

Theorem *Let $\mathcal{A}, X$ and Y be C^{r+1} manifolds ($r \ge 1$), $\phi : \mathcal{A} \times X \to Y$ a C^{r+1} representation of mappings, $W \subset Y$ a submanifold and $a_0 \in \mathcal{A}$ a point. For $a \in \mathcal{A}$ let $W_a = \phi_a^{-1}(W)$. Assume that*

1. *W is closed in Y,*
2. *X compact and C^{r+3},*

3. $\phi_{a_0} \pitchfork W$.

Then there is an open neighbourhood N of a_0 in $\mathcal{A}$, such that for $a \in N$ we have that $\phi_a \pitchfork W$ and W_a is C^r-isotopic to W_{a_0}. This implies, that there is a C^r-diffeomorphism $F_a : X \to X$ with $F_a(W_a) = W_{a_0}$ for $a \in N$.

We will choose $\mathcal{A} = [-h_0, h_0]$, $a_0 = 0$ and

$$\begin{aligned} X &= \Lambda \times \overline{(E^u \cap B_{H^s_{per}}(\rho, 0))} \\ Y &= \Lambda \times H^s_{per} \\ W &= \overline{E^s \cap B_{H^s_{per}}(\rho, 0)} \end{aligned}$$

which fulfil all assumptions of above. The smoothness assumptions are fulfilled trivially as all manifolds are linear spaces intersected with balls. X is compact as it is finite dimensional by assumption, bounded and closed.

ϕ describes the forward orbit of the transformed local unstable manifold:

$$\begin{aligned} \phi : [-h_0, h_0] \times \left(\Lambda \times \overline{(E^u \cap B_{H^s_{per}}(\rho, 0))}\right) &\to H^s_{per} \\ \phi_h(\lambda, u_u) &= \bar{S}^h_*(T, u_u + \bar{p}^h_u(u_u), \lambda), \quad (5.10) \end{aligned}$$

where T is chosen large enough, such that we can guarantee $\phi_0(X) \cap W \neq \emptyset$ for $h = 0, \lambda = \lambda_0$. This representation of mappings $\phi_h : X \to Y$ is C^2 in (λ, u_u). As the manifolds are smooth, the transformation in step 1 and $u_u + \bar{p}^h_u(u_u)$ are smooth. $F, \bar{F}$ depend smoothly on λ and u. We obtain by smooth dependence on parameters and initial values (see e.g. [Hen83, Corollary3.4.6]), that $\bar{S}^h_*(T, ., .)$ is smooth in (λ, u_u). The semiflow depends on W^u smoothly on h down to $h = 0$, when using remark 3.4. The last assumption of the theorem $\phi_0 \pitchfork W$ holds by (5.8).

Hence by the theorem there exists an interval $(-h_*, h_*)$ such that for all $h \in (-h_*, h_*)$: $\phi_h^{-1}(W)$ is diffeomorphic to $\phi_0^{-1}(W)$. Thus there is $\lambda_h \in \Lambda$ such that $W^s(0, \lambda_h, h) \cap W^u(0, \lambda_h, h)$ contains a one-dimensional arc, i.e. for this λ_h there exists a homoclinic orbit.

Step 3: Exponential smallness

Now we analyse the behaviour of the forced system. By proposition 5.2 the unstable manifolds of the forced and the averaged system are exponentially close in h. In the original coordinates they are given by

$$\begin{aligned} W^{u,loc}_h(\bar{u}_h) &= \{\bar{u}_h + u_u + \bar{p}^h_u(u_u), u_u \in E^u \cap B_{H^s_{per}}(\rho, 0)\} \\ W^{u,loc}_h(\tilde{u}_h(\theta)) &= \{\tilde{u}_h(\theta) + u_u + \tilde{p}^h_u(\theta, u_u), u_u \in E^u \cap B_{H^s_{per}}(\rho, 0)\} \end{aligned}$$

The exponential closeness does not change, when transforming the semigroups as in step 1, where $\bar{u}_h$ is transformed to the origin with and $\tilde{u}_h$ to U_h, see figure 5.1.

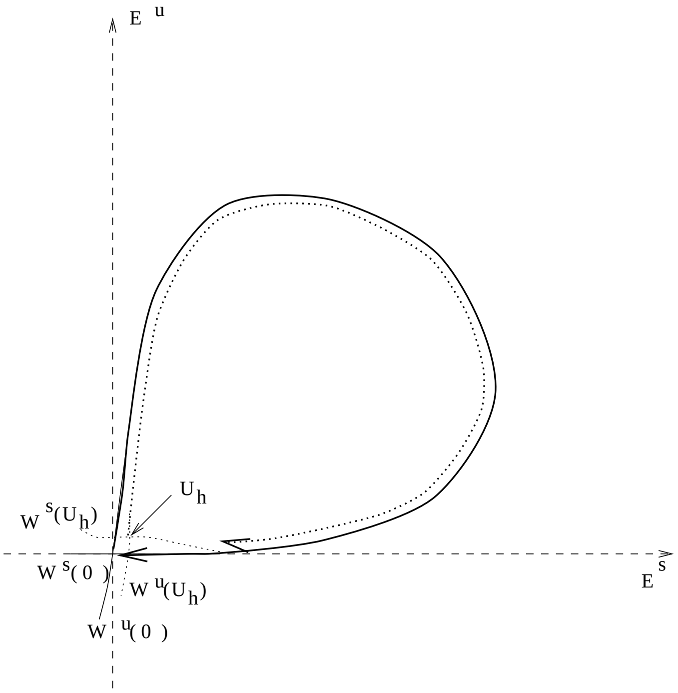

Figure 5.1: The homoclinic orbit to 0 in the averaged system is closely followed by the unstable manifold (dotted) of U_h, which is a point on the periodic orbit of period h of the forced system.

Next we consider the semiflow applied to the local unstable manifold. There is a time T as in (5.10), such that the forward orbit of the local unstable manifold intersects with the local stable manifold:

$$\bar{S}_*^h(T, W_h^{u,loc}(0)) \cap \bar{W}_h^{s,loc}(0) \neq \emptyset$$

After an additional time T^* we have

$$\bar{S}_*^h(T+T^*, W_h^{u,loc}(0)) \cap W_h^{s,loc}(0) \cap \bar{S}_*^h(T^*, W_h^{s,loc}(0)) \neq \emptyset,$$

where the local stable manifolds of 0 for the averaged system and of U_h for the periodically forced system are exponentially close.

Furthermore if the semiflows of the averaged system and the forced system evolve forward for the finite time $T+T^*$, they preserve the closeness of the unstable manifolds. Let $u_1 \in W_h^{u,loc}(U_h(\theta))$ and $u_2 \in W_h^{u,loc}(0)$ with $|u_1 - u_2| \leq C\exp(-ch^{-\frac{1}{3}})$. Then applying lemma 5.1 and using that the linear mapping $D\bar{S}_*^h(T+T^*, u)$ is bounded for $u \in K = \{ru_1 + (1-r)u_2 | r \in [0,1]\}$ we get

$$|\tilde{S}_*^h(\theta+T+T^*, \theta, u_1) - \bar{S}_*^h(T+T^*, u_2)|_{H^s_{per}}$$

$$\begin{aligned}
&\leq |\tilde{S}^h_*(T+T^*+\theta,\theta,u_1) - \bar{S}^h_*(T+T^*,u_1)|_{H^s_{per}} \\
&\quad +|\bar{S}^h_*(T+T^*,u_1) - \bar{S}^h_*(T+T^*,u_2)|_{H^s_{per}} \\
&\leq C\exp(-ch^{-\frac{1}{3}}) + \sup_{u\in K}|D\bar{S}^h_*(T+T^*,u)||u_1-u_2|_{H^s_{per}} \\
&\leq C\exp(-ch^{-\frac{1}{3}})
\end{aligned}$$

The transversality condition (5.8) also holds for $h \neq 0$ by step 2. Thus we can choose a special local coordinate system near points in $\bar{S}^h_*(T+T^*, W^{u,loc}_h(0)) \cap W^{s,loc}_h(0) \cap \bar{S}^h_*(T^*, W^{s,loc}_h(0))$. Then (5.8)

$$(\lambda, T_U W^u(u_0,\lambda)) \pitchfork_{\lambda_h,\Gamma_h(t)} (\lambda, T_U W^s(u_0,\lambda))$$

implies that

$$T_{\lambda,v}\bar{S}^h_*(T+T^*, W^u(0),\lambda) + E^s = H^s_{per},$$

where the tangential of the finite dimensional manifold $\bar{S}^h_*(T+T^*, W^u(U_h),\lambda)$ is taken at $\lambda_h, v \in \bar{S}^h_*(T+T^*, W^{u,loc}_h(0)) \cap W^{s,loc}_h(0)$. Then a coordinate system is given by

$$T_\lambda\bar{S}^h_*(T+T^*, W^u(0),\lambda) \oplus (T_v\bar{S}^h_*(T+T^*, W^u(0),\lambda) + E^s)$$

The stable eigenspace E^s is a subspace of codimension k. The tangent space $T_v\bar{S}^h_*(T+T^*, W^u(0),\lambda)$ is k dimensional, but has a non-zero intersection with E^s, hence $E^\perp = T_\lambda\bar{S}^h_*(T+T^*, W^u(0),\lambda)$ is not zero. Thus by the Taylor theorem $\bar{S}^h_*(T+T^*, W^u(0),\lambda_h+\epsilon)$ has a $E^\perp$ component of order ϵ. Then $\bar{S}^h_*(T+T^*, W^u(U_h),\lambda_h+\epsilon)$ has a $E^\perp$ component of order $\epsilon+\exp(-ch^{-\frac{1}{3}})$. Furthermore at points in $B_{H^s}(\rho,0) \cap \tilde{S}^h_*(T^*,0,B_{H^s}(\rho,0))$ the local stable manifold of the forced system $W^{s,loc}_h(U_h)$ is exponentially close of order $\exp(-ch^{-\frac{1}{3}})$ to $W^{s,loc}_h(0)$ by proposition 5.2. Thus the $E^\perp$ component is at most $\exp(-ch^{-\frac{1}{3}})$. Hence for $|\epsilon| > C\exp(-ch^{-\frac{1}{3}})$ there cannot be an intersection of $\tilde{S}^h_*(T+T^*, W^u(U_h),\lambda_h+\epsilon)$ with $W^{s,loc}_h(U_h)$ and therefore there is no transversal homoclinic orbit near $\Gamma_h(t)$.

Step 4: $\dim W^u(u_0) = 1$

To complete the proof we consider the special case of a one-dimensional unstable manifold of u_0, where we can give an easy argument for the persistence of the homoclinic orbit. The local coordinate system is given by $E^s \oplus E^\perp$ or alternatively by $E^s \oplus E^u$. E^s and also $W^{s,loc}(U_h)$ separate the phase space and for $|\epsilon|$ large enough $\tilde{S}^h_*(T+T^*+\theta,\theta,W^{u,loc}(U_h(\theta)),\lambda_h\pm\epsilon)$ will be on different sides of $W^{s,loc}(U_h)$. Due to the intermediate value theorem there has to be a $\tilde{\lambda}_h$, such that the semigroup $\tilde{S}^h_*(j\cdot h,0,.,\tilde{\lambda}_h), j\geq 0$ has a homoclinic orbit.

The exponential closeness of stable and unstable manifold of U_h in $B_{H^s}(\rho,0) \cap \bar{S}^h_*(T^*,B_{H^s}(\rho,0))$ can be shown in the following way. Let U be a homoclinic point of the forced equation, which is on the forward orbit of the local stable manifold

$$U \in W^{s,loc}(U_h) \cap \tilde{S}^h_*(T^*, 0, B_{H^s}(\rho, 0)) \cap \tilde{S}^h_*(T + T^*, W^{u,loc}(U_h)).$$

$W^{s,loc}(U_h)$ is exponentially close to E^s on $\tilde{S}^h_*(T^*, 0, B_{H^s}(\rho, 0))$ and $\{\tilde{S}^h_*(t, U), t \in (0, h)\}$ is the part of the unstable manifold that is between two consecutive homoclinic points. Hence it remains to show that $\tilde{S}^h_*(t, U)$ stays exponentially close to E^s, i.e. $P^u \tilde{S}^h_*(t, U)$ is exponentially small. This can be shown using lemma 5.1, the boundedness of $D\bar{S}^h(t, U)$ and that $E^s \cap B_{H^s}(\rho, 0)$ is invariant under $\bar{S}^h_*(t, .)$

$$\begin{aligned}
& |P^u \tilde{S}^h_*(t, U)|_{H^s} \\
\leq\ & |\tilde{S}^h_*(t, 0, U) - \bar{S}^h_*(t, P^s U)|_{H^s} \\
\leq\ & |\tilde{S}^h_*(t, 0, U) - \bar{S}^h_*(t, U)|_{H^s} + |\bar{S}^h_*(t, U) - \bar{S}^h_*(t, P^s U)|_{H^s} \\
\leq\ & C \exp(-ch^{-\frac{1}{3}}) + C|U - P^s U|_{H^s} \\
\leq\ & C \exp(-ch^{-\frac{1}{3}}).
\end{aligned}$$

This proves the exponential smallness of the splitting at points of the discrete transversal homoclinic orbit, which are on the forward orbit of the local stable manifold. □

Chapter 6

Discussion

This last chapter contains remarks outlining generalisations, variants, implications and limitations of the results above.

Different domains

Our particular choice of reaction-diffusion systems with periodic boundary conditions was made, because we need Gevrey regularity in our proofs. Other examples of equations with this extreme regularisation can be analysed in the same way. For scalar reaction-diffusion equations on the Sphere S^2 Cao, Rammaha and Titi [CRT99] show regularisation to Gevrey classes for initial values $u_0 \in H^s_{per}$ with $s \geq \frac{d}{2} + 1$. Ferrari and Titi [FT98] and Promislow [Pro91] also claim that their results hold on $\mathbf{R}^d$ too. Then it should be possible to extend our averaging to reaction-diffusions on these domains.

Gradient dependent nonlinearities

Gevrey regularity results are also possible for gradient dependent nonlinearities $F(u, \nabla u)$, see [FT98, theorem 3]. In our setting they will lead to Nemitski operators $F : G^{s/2}_\sigma \to G^{(s-1)/2}_\sigma$, using a lemma similar to lemma 2.3. To extend theorem 1 to this setting one needs some changes. In the estimates of α we will need a different coupling $\lambda_N^{1+\beta} h$, as we still have to estimate terms like $|\alpha|_{G^{s/2}_\sigma}$ despite we can only bound $|\alpha|_{G^{(s-1)/2}_\sigma}$. But when using $|P_N\alpha|_{G^{s/2}_\sigma} \leq \sqrt{\lambda_N}|P_N\alpha|_{G^{(s-1)/2}_\sigma}$ and compensating the additional unbounded factor $\sqrt{\lambda_N}$ by $K(h) = \tilde{K} h^{-\frac{3}{2+2\beta}}$ instead of $\tilde{K} h^{-\frac{1}{1+1\beta}}$ as in (3.16), then estimates with exponential terms like $\exp(-ch^{-\frac{1}{4}})$ can be shown.

Navier-Stokes equation

The same method of proof is also applicable to Navier-Stokes equations with periodic boundary conditions. In [FT89] Foias and Titi showed Gevrey regularity for this setting. Actually the Navier-Stokes equation was the first example of regularisation to Gevrey classes for semilinear parabolic equations.

Lower estimates

Another question is, if the derived exponential estimates are optimal. They are not optimal, when applied to the geometric theory of homoclinic orbits. As mentioned in remark 5.6 we expect estimates like $\tilde{C}\exp(\tilde{c}h^{-1})$ instead of exponentials of algebraic expressions of h^{-1} like $C\exp(\tilde{c}h^{-\frac{1}{3}})$.

For the general initial value problem consider the following linear scalar one-dimensional example, which fits in the assumptions of theorem 1'. It gives some insight, why probably we cannot improve much the estimates given in theorem 1 and theorem 1'.

$$\begin{aligned} u_t &= u_{xx} + h\sin\left(\frac{t}{h}\right) Lu \\ u(0) &= u_0 \in H^1_{per}(S^1, \mathbf{R}) \end{aligned}$$

with $S^1 = \mathbf{R}/\mathbf{Z}$ and with L a non-local linear operator defined on the Fourier modes $e_j = \exp(i2\pi jx), j \in \mathbf{Z}$:

$$Le_j = e_0, j \in \mathbf{Z}$$

with $|L|_{L(H^1,H^1)} = 1$. Considering a general near identity coordinate change

$$u = v + hW(v, \frac{t}{h}, h)$$

will lead to

$$v_t = (Id + hW(.))^{-1}[\Delta(Id + hW(.))v + h\sin\left(\frac{t}{h}\right) L(Id + hW(.)) - \dot{W}.$$

To try a first order averaging we set $W(t) = \int_0^t h\sin\left(\frac{s}{h}\right) dsL$, then

$$\begin{aligned} v_t &= (Id + hW(.))^{-1}[\Delta(Id + hW(.))v + h\sin\left(\frac{t}{h}\right) LhW(v, \frac{t}{h})] \\ &= v_{xx} + h\Delta W(v) + h\sin\left(\frac{t}{h}\right) LhW(v, \frac{t}{h}) \\ &\quad + \sum_{k=1}^{\infty}(-hW(.))^k[\Delta v + h\Delta W(v) + h\sin\left(\frac{t}{h}\right) LhW(v, \frac{t}{h})] \end{aligned}$$

As we want to keep the main part v_{xx}, we have to assure that $\sum_{k=1}^{\infty}(-hW(.))^k\Delta v + h\Delta W(v)$ is bounded as a linear map $H^1 \to H^1$. As L does not commute with Δ, W does not commute either, then we get the following problem. Consider for a moment only $M = h(W\Delta - \Delta W)$. To get this term at least of order h, it seems, that we have to restrict the number of Fourier modes:

$$
\begin{aligned}
Me_j &= h(W\Delta - \Delta W)e_j = hW\Delta e_j = h^2(1-\cos\left(\frac{t}{h}\right))L4\pi^2 j^2 e_j \\
&= h^2(1-\cos\left(\frac{t}{h}\right))4\pi^2 j^2 e_0
\end{aligned}
$$

gives that $|Me_j|_{L(H^s,H^s)} = O(h^2j^2)$. So only way to bound this, is to assume $hj^2 \leq C$. Then we can in general use Gevrey regularity to estimate the error in the truncated modes to get an error $C\exp(ch^{-\frac{1}{2}})$. There seems to be little hope in this example to get rid off this error. The error will even increase, if we perform more averaging steps.

So to summarise, the aim to keep the form $\Delta + L$, where L is bounded, in the transformed equation seems to force us to reduce the order of approximation. As the geometric behaviour does probably not depend on this form, the qualitative effects due to rapid forcing might have better exponential estimates.

More geometric theory

Equilibria and homoclinic orbits are only two examples of special solutions, where the averaging improves the understanding of nonautonomous effects. Future work should analyse e.g. as in [Il98] the influence on attractors and other invariant objects.

Approximation of homoclinic orbits

The ideas in proposition 5.5 can be used to prove the persistence of homoclinic orbits under perturbations of parameter dependent semigroups in a much more general framework. The persistence of phase portrait, local stable and unstable manifolds of hyperbolic equilibria is shown for many examples [Stu95, ch.3]. Then the forward orbit of the local unstable manifolds of the original and the perturbed semigroup will stay close. Under transversality assumptions like (5.8) one should be able to show the existence of a homoclinic orbit even for higher dimensional unstable manifolds.

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Appendix A

Deutsche Zusammenfassung

In dieser Dissertation wird die zeitliche Mittelung periodisch angeregter semilinear parabolischer partieller Differentialgleichungen betrachtet. Beispielhaft untersuchen wir ein System von schnell angeregten Reaktions-Diffusions-Gleichungen

$$\frac{\partial}{\partial t}u_k(x,t) - d_k \Delta u_k(x,t) = f_k(U(x,t)) + hg_k(U(x,t), \frac{t}{h}, h)$$

mit $k = 1, \ldots, n$, $U = (u_1, \ldots, u_n)$, g_k periodisch in $\tau = \frac{t}{h}$ mit Periode 1 und periodischen Randwerten auf $\Omega = [0, l]^d, d = 1, 2, 3$ und Anfangsbedingungen

$$U(0) = U_0 \in H^s_{per}(\Omega, \mathbf{R}^n)$$

mit $\frac{d}{2} < s < 2$. Durch zeitliche Mittelung auf endlichdimensionalen Galerkin-Approximationsräumen und unter Ausnutzung extremer Glättungseigenschaften dieser Gleichungen (Gevrey Regularität) wird ein dem Neishtadt-Satz für gewöhnliche Differentialgleichungen vergleichbares Resultat erreicht. Wir geben Koordinatenwechsel an, mit denen sich die nichtautonomen Anteile in der Gleichung bis auf exponentiell kleine Restterme wegtransformieren lassen.

Anwendungen werden auf ein erstes Beispiel einer Rückwärtsfehleranalyse für eine sehr spezielle Klasse von Diskretisierungsverfahren gegeben. Außerdem wird das Verhalten des angeregten Systems in der Nähe von Ruhelagen und homoklinen Orbits des ursprünglichen Systems untersucht und als exponentiell nah an einem gemittelten (autonomen) System erkannt.

Appendix B

Lebenslauf

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