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Asymptotically cylindrical 7-manifolds of holonomy G_2 with applications to compact irreducible G_2-manifolds

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Abstract We construct examples of exponentially asymptotically cylindrical (EAC) Riemannian 7-manifolds with holonomy group equal to G_2. To our knowledge, these are the first such examples. We also obtain EAC coassociative calibrated submanifolds. Finally, we apply our results to show that one of the compact G_2-manifolds constructed by Joyce by desingularisation of a flat orbifold T^7/Γ can be deformed to give one of the compact G_2-manifolds obtainable as a generalized connected sum of two EAC $SU(3)$-manifolds via the method of Kovalev (J Reine Angew Math 565:125–160, 2003).

Keywords Special holonomy · G_2-manifolds · Asymptotically cylindrical manifolds · Moduli spaces · Coassociative submanifolds

1 Introduction

The Lie group G_2 occurs as the holonomy group of the Levi–Civita connection on some Riemannian 7-dimensional manifolds. The possibility of holonomy G_2 was suggested in Berger's classification of the Riemannian holonomy groups [3], but finding examples of metrics with holonomy exactly G_2 is an intricate task. The first local examples were constructed by Bryant [5] using the theory of exterior differential systems, and complete examples were constructed by Bryant and Salamon [6] and by Gibbons, Page and Pope [10]. The first compact examples were constructed by Joyce [15] by resolving singularities of finite quotients of flat tori, and the method was further developed in [16].

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Later the first author [18] gave a different method of producing new compact examples of 7-manifolds with holonomy G_2 by gluing pairs of *asymptotically cylindrical* manifolds. More precisely, a Riemannian manifold is exponentially asymptotically cylindrical (EAC) if outside a compact subset it is diffeomorphic to $X \times \mathbb{R}_{>0}$ for some compact X, and the metric is asymptotic to a product metric at an exponential rate. An important part of the method in [18] is the proof of a version of the Calabi conjecture for manifolds with cylindrical ends producing EAC Ricci-flat Kähler 3-folds W with holonomy $SU(3)$. The product EAC metric on a 7-manifold $W \times S^1$ then also has holonomy $SU(3)$, a maximal subgroup of G_2, and is induced by a torsion-free G_2-structure. In fact, $W \times S^1$ cannot have an EAC metric with holonomy equal to G_2 by [29, Theorem 3.8] because the fundamental group of $W \times S^1$ is not finite.

The purpose of this article is to construct examples of exponentially asymptotically cylindrical manifolds whose holonomy is exactly G_2. To our knowledge these are first such examples. Note that the metrics with holonomy G_2 in [6] are asymptotically conical and not EAC.

It is by now a standard fact that a metric with holonomy G_2 on a 7-manifold can be defined in terms of a 'stable' differential 3-form φ equivalent to a G_2-*structure*. More generally, any G_2-structure φ determines a metric since G_2 is a subgroup of $SO(7)$. This metric will have holonomy in G_2 if the G_2-structure is *torsion-free*. The latter condition is equivalent to the defining 3-form φ being closed and coclosed, a nonlinear first-order PDE. A 7-manifold endowed with a torsion-free G_2-structure is called a G_2-*manifold*. Thus a G_2-manifold is a Riemannian manifold with holonomy contained in G_2. For compact or EAC G_2-structures there is a simple topological criterion to determine if the holonomy is exactly of G_2. See §2 for the details.

Joyce finds examples of G_2-structures on compact manifolds that have small torsion by resolving singularities of quotients of a torus T^7 equipped with a flat G_2-structure by suitable finite groups Γ. The proof in [16, Chap. 11] of the existence result for torsion-free G_2-structures on compact 7-manifolds is carefully written to use the compactness assumption as little as possible. A large part of the proof can therefore be used in the EAC setting too. The main additional difficulty in this case is to show that the G_2-structure constructed has the desired exponential rate of decay to its cylindrical asymptotic model. This task is accomplished by our first main result Theorem 3.1. In §4 we apply this result and explain how, in one particular example, one can cut T^7/Γ into two pieces along a hypersurface, attach a semi-infinite cylinder to each half, and resolve the singularities to form EAC G_2-structures satisfying the hypotheses of Theorem 3.1. In a similar way, we obtain in §5 more examples of EAC G_2-manifolds which are simply-connected with a single end and, therefore, have holonomy exactly G_2 by [29, Theorem 3.8] (see §2.2). This includes examples both where the holonomy of the cross-section is $SU(3)$ and where it reduces to (a finite extension of) $SU(2)$ or is flat. We explain how to compute their Betti numbers, and find some examples of asymptotically cylindrical coassociative minimal submanifolds.

In §6 we study a kind of inverse of the above construction. Given a pair of EAC G_2-manifolds with asymptotic cylindrical models matching via an orientation-reversing isometry, one can truncate their cylindrical ends after some large length L and identify their boundaries to form a generalized connected sum, a compact manifold with an approximately cylindrical neck of length approximately $2L$. This compact 7-manifold inherits from the pair of EAC G_2-manifolds a well-defined G_2-structure and the gluing theorem in [18, §5] asserts that when L is sufficiently large this G_2-structure can be perturbed to a torsion-free one.

Our method of constructing EAC G_2-manifolds by resolving 'half' of T^7/Γ produces them in such matching pairs. The connected sum of the pair is topologically the same as the compact G_2-manifold (M, φ) obtained by resolving the initial orbifold T^7/Γ. We show in our second main result Theorem 6.3 that φ can be continuously deformed to the torsion-free G_2-structures obtained by gluing a pair of EAC G_2-manifolds as in [18]. In other words, the G_2-structures produced by the connected-sum method lie in the same connected component of the moduli space of torsion-free G_2-structures as the ones originally constructed by Joyce. Informally, the path connecting φ to the connected-sum G_2-structures is given by increasing the length of one of the S^1 factors in T^7 before resolving T^7/Γ. In this sense, the EAC G_2-manifolds are obtained by 'pulling apart' the compact G_2-manifold (M, φ).

In §7 we consider one pulling-apart example in detail and identify the two EAC manifolds as products of S^1 and a complex 3-fold. The latter complex 3-folds were studied in [19] obtained from K3 surfaces with non-symplectic involution, and the gluing produces a compact G_2-manifold according to the method of [18]. Thus the compact 7-manifold M admits a path $g(t), 0 < t < \infty$ of metrics with holonomy G_2 so that the limit as $t \to 0$ corresponds to an orbifold T^7/Γ and the limit as $t \to \infty$ corresponds to a disjoint union of EAC G_2-manifolds of the form $W_j \times S^1$, $j = 1, 2$, where each W_j is an EAC Calabi–Yau complex 3-fold with holonomy $SU(3)$. To the authors' knowledge, $g(t)$ is the first example of G_2-metrics on a compact manifold exhibiting two geometrically different types of deformations, related to different constructions [16,18] of compact irreducible G_2-manifolds. (Demonstrating that two constructions produce distinct examples of G_2-manifolds can often be accomplished by checking that these have different Betti numbers, a rather easier task.)

For the examples in this article, we mostly restrict attention to one compact 7-manifold underlying the G_2-manifolds constructed in [15, I §2]. However, our techniques can be extended with more or less additional work to construct more examples of EAC G_2-manifolds from other G_2-manifolds, including those obtained in [16] by resolving more complicated singularities. The authors hope to develop this in a future article.

2 Preliminaries

2.1 Torsion-free G_2-structures and the holonomy group G_2

The group G_2 can be defined as the automorphism group of the normed algebra of octonions. Equivalently, G_2 is the stabiliser in $GL(\mathbb{R}^7)$ of

$$\varphi_0 = \mathrm{d}x^{123} + \mathrm{d}x^{145} + \mathrm{d}x^{167} + \mathrm{d}x^{246} - \mathrm{d}x^{257} - \mathrm{d}x^{347} - \mathrm{d}x^{356} \in \Lambda^3(\mathbb{R}^7)^*, \tag{1}$$

where $\mathrm{d}x^{ijk} = \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$ [5, pp. 539–541]. A *G_2-structure* on a 7-manifold M may therefore be induced by a choice of a differential 3-form φ such that $\iota_p^*(\varphi(p)) = \varphi_0$, for each $p \in M$ for some linear isomorphism $\iota_p : \mathbb{R}^7 \to T_pM$ smoothly depending on p. Every such 3-form on M will be called *stable*, following [14], and we shall, slightly inaccurately, say that φ *is* a G_2-structure. As $G_2 \subset SO(7)$, a G_2-structure induces a Riemannian metric g_φ and an orientation on M, and thus also a Levi–Civita connection ∇_φ and a Hodge star $*_\varphi$.

The holonomy group of a connected Riemannian manifold M is defined up to isomorphism as the group of isometries of a tangent space at $p \in M$ generated by parallel transport, with respect to the Levi–Civita connection, around closed curves based at p. Parallel tensor fields on a manifold correspond to invariants of its holonomy group and the holonomy of g_φ on M will be contained in G_2 if and only if $\nabla_\varphi \varphi = 0$. A G_2-structure satisfying this latter condition is called *torsion-free* and by a result of Gray [31, Lemma 11.5] this is equivalent to

$$\mathrm{d}\varphi = 0 \quad \text{and} \quad \mathrm{d}{*_\varphi}\varphi = 0.$$

We call a 7-dimensional manifold equipped with a torsion-free G_2-structure a G_2*-manifold*. We call a G_2-manifold *irreducible* of the holonomy of the induced metric is all of G_2 (i.e. not a proper subgroup). A compact G_2-manifold is irreducible if and only if its fundamental group is finite [16, Proposition 10.2.2].

More generally, the only connected Lie subgroups of G_2 that can arise as holonomy of the Riemannian metric on a G_2-manifold are G_2, $SU(3)$, $SU(2)$ and $\{1\}$ [16, Theorem 10.2.1].

We call a G_2-structure φ_X on a product manifold $X^6 \times \mathbb{R}$ *cylindrical* if it is translation-invariant in the second factor and defines a product metric $g_M = dt^2 + g_X$, where t denotes the coordinate on $\mathbb{R}$. Then $\frac{\partial}{\partial t}$ is a parallel vector field on $X^6 \times \mathbb{R}$. The stabiliser in G_2 of a vector in $\mathbb{R}^7$ is $SU(3)$, so the Riemannian product of X^6 with $\mathbb{R}$ has holonomy contained in G_2 if and only if the holonomy of X is contained in $SU(3)$. The latter condition means that X is a complex 3-fold with a Ricci-flat Kähler metric and admits a nowhere-vanishing holomorphic (3,0)-form, i.e. X is a *Calabi–Yau 3-fold*. More explicitly, we can write

$$\varphi_X = \Omega + \mathrm{d}t \wedge \omega, \quad \text{where } \omega = \tfrac{\partial}{\partial t} \lrcorner \varphi_X \text{ and } \Omega = \varphi_X|_{X\times\{\mathrm{pt}\}}. \tag{2}$$

Then ω is the Kähler form on X and Ω is the real part of a holomorphic $(3,0)$-form on X, whereas $g(\varphi_X) = \mathrm{d}t^2 + g_X$. It can be shown that a pair (Ω, ω) of closed differential forms obtained from a torsion-free G_2-structure as in (2) determines a Calabi–Yau structure on X (cf. [13, Lemma 6.8] and [16, Proposition 11.1.2]).

If the cross-section is itself a Riemannian product $X = S^1 \times S^1 \times D$ then D is a Calabi–Yau complex surface with holonomy in $SU(2) \cong Sp(1)$, with Kähler form κ_I and holomorphic (2,0)-form $\kappa_J + i\kappa_K$. Alternatively, D may be described as a *hyper-Kähler 4-manifold*, so D has three integrable complex structures I, J, K satisfying quaternionic relations $IJ = -JI = K$ and a metric which is Kähler with respect to all three. The $\kappa_I, \kappa_J, \kappa_K$ are the respective Kähler forms and this triple of closed real 3-forms in fact determines the hyper-Kähler structure (see [12, p. 91]). Denote by x^1, x^2 the coordinates on the two S^1 factors of X. Then the cylindrical torsion-free G_2-structure on $\mathbb{R} \times S^1 \times S^1 \times D$ corresponding to a hyper-Kähler structure on D is

$$\varphi_D = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}t + \mathrm{d}x^1 \wedge \kappa_I + \mathrm{d}x^2 \wedge \kappa_J + \mathrm{d}t \wedge \kappa_K. \tag{3}$$

It induces a product metric $g(\varphi_D) = \mathrm{d}t^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + g_D$ (cf. [16, Proposition 11.1.1]).

2.2 Asymptotically cylindrical manifolds

A non-compact manifold M is said to have *cylindrical ends* if M is written as a union of a compact manifold M_0 with boundary ∂M_0 and a half-cylinder $M_\infty = \mathbb{R}_+ \times X$, the two pieces identified via the common boundary $\partial M_0 \cong \{0\} \times X \subset M_\infty$. The manifold X is assumed compact without boundary and is called the *cross-section* of M. Let t be a smooth real function on M which coincides with the $\mathbb{R}_+$-coordinate on M_∞, and is negative on the interior of M_0. A metric g on M is called *exponentially asymptotically cylindrical (EAC) with rate* $\delta > 0$ if the functions $e^{\delta t}\|\nabla_\infty^k(g - (\mathrm{d}t^2 + g_X))\|$ on the end M_∞ are bounded for all $k \geq 0$, where the point-wise norm $\|\cdot\|$ and the Levi–Civita connection ∇_∞ are induced by some product Riemannian metric $\mathrm{d}t^2 + g_X$ on $\mathbb{R}_+ \times X$. A Riemannian manifold (with cylindrical ends) with an EAC metric will be called an *EAC manifold*.

We can use ∇_∞ to define *translation-invariant* tensor fields on an EAC manifold M as tensor fields whose restrictions to M_∞ are independent of t. A tensor field s on M is said to be *exponentially asymptotic* with rate $\delta > 0$ to a translation-invariant tensor s_∞ on M_∞

if $e^{\delta t}\|\nabla_\infty^k(s-s_\infty)\|$ are bounded on M_∞ for all $k \geq 0$. A G_2-structure is said to be *EAC* if it is exponentially asymptotic to a cylindrical G_2-structure on $\mathbb{R}_+ \times X$. It is not difficult to check that each EAC G_2-structure φ induces an EAC metric $g(\varphi)$. The asymptotic limit of a torsion-free EAC G_2-structure then defines a Calabi–Yau structure on the cross-section X.

We shall need a topological criterion for an EAC G_2-manifold to be irreducible.

Theorem 2.1 **([29, Theorem 3.8])** *Let (M^7, φ) be an EAC G_2-manifold. Then the induced metric g_φ has full holonomy G_2 if and only if the fundamental group $\pi_1(M)$ is finite and neither M nor any double cover of M is homeomorphic to a cylinder $\mathbb{R} \times X^6$.*

Corollary 2.2 *Every simply-connected EAC G_2-manifold with a single end (i.e. a connected cross-section X) is irreducible.*

Remark 2.3 As every G_2-manifold is Ricci-flat, the Cheeger–Gromoll line splitting theorem [8] implies that a connected EAC G_2-manifold either has just one end or two ends. In the latter case, the EAC G_2-manifold is necessarily a cylinder $\mathbb{R} \times X$ with a product metric and cannot have full holonomy G_2.

On an asymptotically cylindrical manifold M it is useful to introduce *weighted Sobolev norms*. Let E be a vector bundle on M associated to the tangent bundle, $k \geq 0$ and $\delta \in \mathbb{R}$. We define the $L^2_{k,\delta}$-norm of a section s of E in terms of the usual Sobolev norm by

$$\|s\|_{L^2_{k,\delta}} = \|e^{\delta t}s\|_{L^2_k}. \tag{4}$$

Denote the space of sections of E with finite $L^2_{k,\delta}$-norm by $L^2_{k,\delta}(E)$. Up to Lipschitz equivalence the weighted norms are independent of the choice of asymptotically cylindrical metric, and of the choice of t on the compact piece M_0. In particular, the topological vector spaces $L^2_{k,\delta}(E)$ are independent of these choices. As any asymptotically cylindrical manifold M clearly has bounded curvature and injectivity radius bounded away from zero, the Sobolev embedding $L^2_k \subset C^r$ is still valid whenever $r < k - 7/2$ [2, § 2.7]. It follows that $L^2_{k,\delta}$ consists of sections decaying (when $\delta > 0$) with all derivatives of order up to r at the rate $O(e^{-\delta t})$ as $t \to \infty$.

An important property of the weighted norms is that elliptic linear operators with asymptotically translation-invariant coefficients over M extend to Fredholm operators between δ-weighted spaces of sections, for 'almost all' choices of weight parameter δ [22,23,25]. In particular, this can be applied to the Hodge Laplacian of an EAC metric to deduce results analogous to Hodge theory for compact manifolds. In this article, we shall require only a result about *Hodge decomposition*. Let M^n be an EAC manifold with rate δ_0 and cross-section X. Abbreviate $\Lambda^m T^*M$ to Λ^m, and let

$$L^2_{k,\delta}\left[\mathrm{d}\Lambda^{m-1}\right], L^2_{k,\delta}\left[\mathrm{d}^*\Lambda^{m+1}\right] \subset L^2_{k,\delta}\left(\Lambda^m\right)$$

denote the subspaces of exact and coexact $L^2_{k,\delta}$ m-forms, respectively. Let $\mathcal{H}^m_+$ denote the space of L^2 harmonic forms on M, and $\mathcal{H}^m_\infty$ the space of translation-invariant harmonic forms on the product cylinder $X \times \mathbb{R}$. If $\rho : M \to [0,1]$ is a smooth cut-off function supported on the cylindrical ends M_∞ of M and such that $\rho \equiv 1$ in the region $\{t > 1\} \subset M$ then $\rho\mathcal{H}^m_\infty$ can be identified with a space of smooth m-forms on M. Suppose that $0 < \delta < \delta_0$ and that δ^2 is smaller than any positive eigenvalue of the Hodge Laplacian on $\oplus_m \Lambda^m T^*X$ for the asymptotic limit metric g_X on X. Then the elements of $\mathcal{H}^m_+$ are smooth and decay exponentially with rate δ [25].

Theorem 2.4 (cf. [29, p. 328]) *In the notation above, there is an L^2-orthogonal direct sum decomposition*

$$L^2_{k,\delta}(\Lambda^m) = \mathcal{H}^m_+ \oplus L^2_{k,\delta}\left[\mathrm{d}\Lambda^{m-1}\right] \oplus L^2_{k,\delta}\left[\mathrm{d}^*\Lambda^{m+1}\right]. \tag{5}$$

Furthermore, any element of $L^2_{k,\delta}\left[\mathrm{d}\Lambda^{m-1}\right]$ can be written as $\mathrm{d}\phi$, for some coexact form $\phi \in L^2_{k+1,\delta}\left(\Lambda^{m-1}\right) \oplus \rho\mathcal{H}^{m-1}_\infty$.

3 Existence of EAC torsion-free G_2-structures

We shall construct EAC manifolds with holonomy exactly G_2 by modifying Joyce's construction of compact G_2-manifolds. To this end, we shall obtain a one-parameter family of G_2-structures with 'small' torsion on a manifold with cylindrical end. More precisely, this family will satisfy the hypotheses of the following theorem, the main result of this section, which is an EAC version of [16, Theorem 11.6.1].

Theorem 3.1 *Let μ, ν, λ positive constants. Then there exist positive constants κ, K such that whenever $0 < s < \kappa$ the following is true.*

Let M be a 7-manifold with cylindrical end M_∞ and cross-section X^6, and suppose that a closed stable 3-form $\tilde{\varphi}$ defines on M a G_2-structure which is cylindrical and torsion-free on M_∞. Suppose that ψ is a smooth compactly supported 3-form on M satisfying $d^*\psi = d^*\tilde{\varphi}$, and let $r(\tilde{\varphi})$ and $R(\tilde{\varphi})$ be the injectivity radius and Riemannian curvature of the EAC metric $g_{\tilde{\varphi}}$ on M. If

(a)

$$\|\psi\|_{L^2} < \lambda s^4, \quad \|\psi\|_{C^0} < \lambda s^{1/2}, \quad \|\mathrm{d}^*\psi\|_{L^{14}} < \lambda, \tag{6}$$

(b) $r(\tilde{\varphi}) > \mu s$,
(c) $\|R(\tilde{\varphi})\|_{C^0} < \nu s^{-2}$,

then there is a smooth exact 3-form $\mathrm{d}\eta$ on M, exponentially decaying with all derivatives as $t \to \infty$, such that

$$\|\mathrm{d}\eta\|_{L^2} < Ks^4, \quad \|\mathrm{d}\eta\|_{C^0} < Ks^{1/2}, \quad \|\nabla\mathrm{d}\eta\|_{L^{14}} < K, \tag{7}$$

and $\varphi = \tilde{\varphi} + \mathrm{d}\eta$ is a torsion-free G_2-structure.

Remark 3.2 The difference between Theorem 3.1 and [16, Theorem 11.6.1] is that M is now non-compact with a cylindrical end and we made appropriate assumptions on $\tilde{\varphi}$, ψ away from a compact piece of M and are claiming an EAC property of the resulting φ. On the other hand, formally, taking the cross-section X^6 to be empty (hence M being compact) recovers the statement of [16, Theorem 11.6.1].

Remark 3.3 The fact that $\mathrm{d}\eta$ is *exponentially* decaying is more important than its precise rate of decay. We shall need to choose the rate $\delta > 0$ so that δ^2 is smaller than any non-zero eigenvalue of the Hodge Laplacian on X. It should be easy to modify the proof of the theorem to allow $\tilde{\varphi}$ to be EAC and ψ to be exponentially decaying. In that case one would also need δ to be smaller than the decay rates of $\tilde{\varphi}$ and ψ.

We wish to find an exact exponentially asymptotically decaying 3-form $\mathrm{d}\eta$ such that $\tilde{\varphi}+\mathrm{d}\eta$ is torsion-free. First, we show that for $\tilde{\varphi} + \mathrm{d}\eta$ to be torsion-free it suffices to show that η is a solution of a certain non-linear elliptic equation, which was also used by Joyce [16] in the compact case, and find a solution for this equation by a contraction-mapping argument. The details of this are complicated, but largely carry over from argument for the compact case worked out in [16, Chap. 11]. We initially obtain, adapting the method of [16, Chap. 11], a closed 3-form χ, so that $\phi + \chi$ is a torsion-free G_2-structure, and use elliptic regularity to show that the solution χ is smooth and uniformly decaying along the end M_∞ as $t \to \infty$. Then, and this is an additional argument required for an EAC manifold, we prove that the solution decays *exponentially*. This also ensures that χ is exact, which will complete the proof of Theorem 3.1.

3.1 Contraction-mapping argument

The proposition below is an asymptotically cylindrical version of [16, Theorem 10.3.7].

Proposition 3.4 *There is an absolute constant $\varepsilon_1 > 0$ such that the following holds. Let M^7 be an EAC manifold, $\tilde{\varphi}$ a closed EAC G_2-structure on M and ψ an exponentially decaying 3-form such that $\|\psi\|_{C^0} < \varepsilon_1$ and $d^*\psi = d^*\tilde{\varphi}$. Suppose that η is 2-form asymptotic to a translation-invariant harmonic form, and that $\|\mathrm{d}\eta\|_{C^0} < \varepsilon_1$. Suppose further that*

$$\triangle\eta = \mathrm{d}^*\psi + \mathrm{d}^*(f\psi) + *\mathrm{d}F(\mathrm{d}\eta), \tag{8}$$

where the function f is the point-wise inner product $\frac{1}{3}\langle \mathrm{d}\eta, \tilde{\varphi}\rangle$ and F denotes the quadratic and higher order parts, at $\tilde{\varphi}$, of the non-linear fibre-wise map $\Theta : \varphi \mapsto *_\varphi\varphi$ from G_2-structures to 4-forms. Then $\tilde{\varphi} + \mathrm{d}\eta$ is a torsion-free EAC G_2-structure on M.

Proof The proof for the compact case in [16] relies on integrating by parts. It is easy to check that, in the asymptotically cylindrical setting, the necessary integrals still converge provided that η is bounded and $\mathrm{d}\eta$ decays, so we can still use (8) as a sufficient condition for the torsion to vanish. □

A key part in the proof of the existence of solutions for (8) on a compact 7-manifold is the contraction-mapping argument [16, Proposition 11.8.1]. We observe that it can easily be adapted to the EAC case.

Proposition 3.5 *Let (Ω, ω) be a Calabi–Yau structure on a compact manifold X^6 and μ, ν, λ be positive constants. Then there exist positive constants κ, K, C_1 such that whenever $0 < s < \kappa$ the following is true.*

Let M^7 be a manifold with cylindrical end and cross-section X, and $\tilde{\varphi}$ a closed EAC G_2-structure on M with asymptotic limit $\Omega + \mathrm{d}t \wedge \omega$. Suppose that ψ is a smooth exponentially decaying 3-form on M satisfying $\mathrm{d}^*\psi = \mathrm{d}^*\tilde{\varphi}$, and that

(a) $\|\psi\|_{L^2} < \lambda s^4$, $\|\psi\|_{C^0} < \lambda s^{1/2}$, $\|\mathrm{d}^*\psi\|_{L^{14}} < \lambda$,
(b) *the injectivity radius is $> \mu s$,*
(c) *the Riemannian curvature R satisfies $\|R\|_{C^0} < \nu s^{-2}$.*

Then there is a sequence $\mathrm{d}\eta_j$ of smooth exponentially decaying exact 3-forms with $\mathrm{d}\eta_0 = 0$ satisfying the equation

$$\triangle\eta_j = \mathrm{d}^*\psi + \mathrm{d}^*(f_{j-1}\psi) + *\mathrm{d}F(\mathrm{d}\eta_{j-1}), \tag{9}$$

where $f_j = \frac{1}{3}\langle \mathrm{d}\eta_j, \tilde{\varphi}\rangle$ for each $j > 0$. The solutions satisfy the inequalities

(i) $\|\mathrm{d}\eta_j\|_{L^2} < 2\lambda s^4$,
(ii) $\|\nabla\mathrm{d}\eta_j\|_{L^{14}} < 4C_1\lambda$,

(iii) $\|\mathrm{d}\eta_j\|_{C^0} < Ks^{1/2}$,
(iv) $\|\mathrm{d}\eta_{j+1} - \mathrm{d}\eta_j\|_{L^2} < 2^{-j}\lambda s^4$,
(v) $\|\nabla(\mathrm{d}\eta_{j+1} - \mathrm{d}\eta_j)\|_{L^{14}} < 4 \cdot 2^{-j}C_1\lambda$,
(vi) $\|\mathrm{d}\eta_{j+1} - \mathrm{d}\eta_j\|_{C^0} < 2^{-j}Ks^{1/2}$.

Proof The existence of the sequence $\mathrm{d}\eta_j$ and the inequalities (i)–(vi) are proved inductively. Take $\delta > 0$ smaller than the decay rates of $\tilde{\varphi}$ and ψ such that δ^2 is smaller than any positive eigenvalue of the Hodge Laplacian on X, and let ρ be a cut-off function for the cylinder on M. If $\mathrm{d}\eta_{j-1}$ exists and satisfies the uniform estimate (iii) then $F(\mathrm{d}\eta_{j-1})$ is well-defined, and the RHS of (9) is d^* of a 3-form that decays with exponential rate δ. The EAC Hodge decomposition Theorem 2.4 implies that there is a unique coexact solution $\eta_j \in L^2_{k,\delta}(\Lambda^2) \oplus \rho\mathcal{H}^2_\infty$ for all $k \geq 2$.

The induction step for the inequalities is proved using exactly the same argument as in [16, Proposition 11.8.1]. (i) and (iv) are proved using an integration by parts argument, and since each $\mathrm{d}\eta_j$ decays exponentially this is still justified when M has cylindrical ends.

(ii), (iii), (v) and (vi) are proved using interior estimates, which do not require compactness. □

It follows that if s is small, then $\mathrm{d}\eta_j$ is a Cauchy sequence, in each of the norms L^2, L^{14}_1 and C^0, and has a limit χ with

$$\|\chi\|_{L^2} < Ks^4, \quad \|\chi\|_{C^0} < Ks^{1/2}, \quad \|\nabla\chi\|_{L^{14}} < K, \tag{10}$$

for some $K > 0$. The form χ is closed, L^2-orthogonal to the space of decaying harmonic forms $\mathcal{H}^3_+$ and satisfies the equation

$$\mathrm{d}^*\chi = \mathrm{d}^*\psi + \mathrm{d}^*(f\psi) + *\mathrm{d}F(\chi), \tag{11}$$

where $f = \frac{1}{3}{<}\chi, \tilde{\varphi}{>}$. We do not know a priori that χ is the exterior derivative of a bounded form, so we cannot yet apply Proposition 3.4 to show that $\tilde{\varphi} + \chi$ is torsion-free.

3.2 Regularity

We first show by elliptic regularity that χ is smooth and uniformly decaying.

Proposition 3.6 *If s is sufficiently small then $\chi \in L^{14}_k(\Lambda^3)$ for all $k \geq 0$.*

Proof Since $F(\chi)$ depends only point-wise on χ and is of quadratic order we can write

$$*\,\mathrm{d}F(\chi) = P(\chi, \nabla\chi) + Q(\chi), \tag{12}$$

where $P(u, v)$ is linear in v and smooth of linear order in u, whilst $Q(u)$ is smooth of quadratic order in u for u small. We can then rephrase (11) as stating that $\beta = \chi$ is a solution of

$$\begin{aligned} \mathrm{d}^*\beta - P(\chi, \beta) - \mathrm{d}^*(f(\beta)\psi) &= \mathrm{d}^*\psi + Q(\chi), \\ \mathrm{d}\beta &= 0, \end{aligned} \tag{13}$$

where $f(\beta) = \frac{1}{3}{<}\beta, \tilde{\varphi}{>}$. The LHS is a linear partial differential operator acting on β. Its symbol depends on χ and ψ, but not on their derivatives. By taking s small we can ensure that χ and ψ are both small in the uniform norm (see (10) and hypothesis a in Proposition 3.5) so that the equation is elliptic.

Now suppose that χ has regularity L^{14}_k. Then so do the coefficients and the RHS of (13). Because $\beta = \chi \in L^{14}_1(\Lambda^3)$ is a solution of (13), standard interior estimates (see Morrey

[27, Theorems 6.2.5 and 6.2.6]) imply that it must have regularity L^{14}_{k+1} locally. Moreover, because the metric is asymptotically cylindrical the local bounds are actually uniform, so in fact χ is globally L^{14}_{k+1}. The result follows by induction on k. □

In the next result and in §3.3, we interchangeably consider χ on the cylindrical end $M_\infty = \mathbb{R}_+ \times X$ as a family of sections over X depending on a real parameter t.

Corollary 3.7 *If s is sufficiently small then on the cylindrical end of M the form χ decays, with all derivatives, uniformly on X as $t \to \infty$.*

Proof Because M is EAC, standard Sobolev embedding results imply that we can pick $r > 0$ such that M is covered by balls $B(x_i, r)$ with the following property:

$$\|\chi|_{B(x_i,r)}\|_{C^k} < C\|\chi|_{B(x_i,2r)}\|_{L^{14}_{k+1}},$$

where the constant $C > 0$ is independent of $x_i \in M$. If we ensure that each point of M is contained in no more than N of the balls $B(x_i, 2r)$ then

$$\sum_i \|\chi|_{B(x_i,r)}\|^{14}_{C^k} < NC^{14}\|\chi\|^{14}_{L^{14}_{k+1}}.$$

As the sum is convergent the terms tend to 0, i.e. the k-th derivatives of χ decay uniformly. □

3.3 Exponential decay

To complete the proof of Theorem 3.1 it remains to prove that the rate of decay of χ is exponential. Then $\chi = \mathrm{d}\eta$ for some exponentially asymptotically translation-invariant η by the Hodge decomposition Theorem 2.4, since χ is closed and L^2-orthogonal to the decaying harmonic forms $\mathcal{H}^3_+$. Proposition 3.4 then implies that $\tilde{\varphi} + \mathrm{d}\eta$ is torsion-free, so that $\mathrm{d}\eta$ has all the desired properties.

By hypothesis, $\tilde{\varphi}$ is exactly cylindrical on the cylindrical end $M_\infty = \{t \geq 0\}$ of M, and ψ is supported in the compact piece $M_0 = \{t \leq 0\}$. Thus on the cylindrical end the Eq. (11) for χ simplifies to

$$\mathrm{d}^*\chi = *\mathrm{d}F(\chi). \tag{14}$$

On the cylindrical end $t > 0$ we can write

$$\chi = \sigma + \mathrm{d}t \wedge \tau,$$
$$F(\chi) = \beta + \mathrm{d}t \wedge \gamma,$$

where $\tau \in \Omega^2(X)$, $\sigma, \gamma \in \Omega^3(X)$ and $\beta \in \Omega^4(X)$ are forms on the cross-section X depending on the parameter t. Let d_X denote the exterior derivative on X. Then the conditions $\mathrm{d}\chi = 0$ and (14) are equivalent to

$$d_X\sigma = 0, \tag{15a}$$
$$\tfrac{\partial}{\partial t}\sigma = d_X\tau, \tag{15b}$$
$$d_X{*}\tau = -d_X\beta, \tag{15c}$$
$$\tfrac{\partial}{\partial t}{*}\tau = -d_X{*}\sigma - \tfrac{\partial}{\partial t}\beta + d_X\gamma. \tag{15d}$$

(15b) implies that $\sigma(t_1) - \sigma(t_2)$ is exact for any $t_1, t_2 > 0$. Since the exact forms form a closed subspace of the space of 3-forms on X (in the L^2 norm) and $\sigma \to 0$ as $t \to \infty$ it follows that σ is exact for all $t > 0$. Similarly (15d) implies that $*\tau - \beta$ is exact for all $t > 0$. (The Eqs. (15a) and (15c) are thus redundant.) The path (σ, τ) is therefore constrained to lie in the space

$$\mathcal{F} = \{(\sigma, \tau) \in d_X L^2_1 \left(\Lambda^2 T^* X\right) \times L^2 \left(\Lambda^2 T^* X\right) : *\tau - \beta \text{ is exact}\}.$$

Remark 3.8 We have not assumed that χ is in L^1 on M.

β is a function of σ and τ, and it is of quadratic order. The implicit function theorem applies to show that if we replace $\mathcal{F}$ with a small neighbourhood of 0 then it is a Banach manifold with tangent space

$$T_0\mathcal{F} = B = d_X L^2_1 \left(\Lambda^2 T^* X\right) \times d^*_X L^2_1 \left(\Lambda^3 T^* X\right).$$

We can now interpret (15b) and (15d) as a flow on $\mathcal{F}$, or equivalently near the origin in B. By the chain rule we can write $\frac{\partial}{\partial t}\beta$ as

$$\tfrac{\partial}{\partial t}\beta = A_2\left(\tfrac{\partial}{\partial t}\tau\right) + A_3\left(\tfrac{\partial}{\partial t}\sigma\right) + \beta',$$

where A_m is a linear map from $\Lambda^m T^* X$ to $\Lambda^4 T^* X$, determined point-wise by σ and τ and of linear order, whilst β' is a 4-form determined point-wise by σ and τ and of quadratic order. In particular, for large t the norm of A_2 is small, and (15b) and (15d) are equivalent to

$$\begin{aligned} \tfrac{\partial}{\partial t}\sigma &= d_X\tau, \\ \tfrac{\partial}{\partial t}\tau &= (id + *A_2)^{-1}(d^*_X\sigma - *A_3 d_X\tau - *\beta' + *d_X\gamma). \end{aligned} \tag{16}$$

The origin is a stationary point for the flow, and the linearisation of the flow near the origin is given by the (unbounded) linear operator $L = \begin{pmatrix} 0 & d_X \\ d^*_X & 0 \end{pmatrix}$ on B. Because L is formally self-adjoint B has an orthonormal basis of eigenvectors. Also, L is injective on B, so B can be written as a direct sum of subspaces with positive and negative eigenvalues,

$$B = B_+ \oplus B_-.$$

Then $\{e^{\mp tL} : t \geq 0\}$ defines a continuous semi-group of bounded operators on $B_\pm$. If we let μ denote the smallest absolute value of the eigenvalues of L then $e^{t\mu}e^{\mp tL}$ is uniformly bounded on $B_\pm$ for $t \geq 0$, so the origin is a hyperbolic fixed point. By analogy with finite-dimensional flows, we expect that any solution of (15b) and (15d) approaching the origin must do so at an exponential rate.

A similar problem of exponential convergence for an infinite-dimensional flow is considered by Mrowka, Morgan and Ruberman [26, Lemma 5.4.1]. Their problem is more general in that the linearisation of their flow has non-trivial kernel, so that they need to consider convergence to a 'centre manifold' rather than to a well-behaved isolated fixed point. As a simple special case we can prove the L^2 exponential decay for χ.

Proposition 3.9 *Let $\delta > 0$ such that δ^2 is smaller than any positive eigenvalue of the Hodge Laplacian on X. Then χ is L^2_δ.*

Proof Identify $\mathcal{F}$ with a neighbourhood of the origin in the tangent space B, and let x be the path in B corresponding to (σ, τ) in $\mathcal{F}$. Then (16) transforms to a differential equation for x,

$$\frac{dx}{dt} = Lx + Q(x),$$

where L is the linearisation of (16) as above, and Q is the remaining quadratic part. Let $x = x_+ + x_-$ with $x_\pm \in B_\pm$. If, as before, μ denotes the smallest absolute value of the eigenvalues of L then

$$\|Lx_+\|_{L^2} \geq \mu\|x_+\|_{L^2}, \quad -\|Lx_-\|_{L^2} \leq -\mu\|x_-\|_{L^2}.$$

Applying the chain rule to the quadratic part of (16) gives

$$\|Q(x)\|_{L^2} < O(\|x\|_{L^2})\|x\|_{L^2_1} + O(\|x\|^2_{L^2}).$$

By corollary 3.7, x converges uniformly to 0 with all derivatives as $t \to \infty$. Therefore, for any fixed $k > 0$, we can find t_0 such that

$$\|Q(x)\|_{L^2} < k\|x\|_{L^2}$$

for any $t > t_0$. As μ^2 is an eigenvalue for the Hodge Laplacian on X we may fix k so that $\mu - 2k > \delta$.

We thus obtain that for $t > t_0$

$$\frac{d}{dt}\|x_+\|_{L^2} \geq \quad \mu\|x_+\|_{L^2} - k\|x\|_{L^2}, \tag{17a}$$

$$\frac{d}{dt}\|x_-\|_{L^2} \leq -\mu\|x_-\|_{L^2} + k\|x\|_{L^2}. \tag{17b}$$

In particular, $\|x_+\|_{L^2} - \|x_-\|_{L^2}$ is an increasing function of t. Because it converges to 0 as $t \to \infty$,

$$\|x_+\|_{L^2} \leq \|x_-\|_{L^2}$$

for all $t > t_0$. Substituting into (17b)

$$\frac{d}{dt}\|x_-\|_{L^2} \leq -\mu\|x_-\|_{L^2} + 2k\|x_-\|_{L^2},$$

so $\|x_-\|_{L^2}$ is of order $e^{(-\mu+2k)t}$. Hence so is $\|x\|_{L^2}$, so $e^{\delta t}\chi$ is L^2-integrable on M. □

Corollary 3.10 *χ decays exponentially with rate δ.*

Proof We prove by induction that χ is $L^2_{k,\delta}$ for all $k \geq 0$. Interior estimates for the elliptic operator $d + d^*$ on M imply that we can fix some $r > 0$ and cover the cylindrical part of M with open balls $U = B(x, r)$ such that

$$\|\chi\|_{L^2_{k+1}(U)} < C_1\left(\|\mathrm{d}\chi\|_{L^2_k(U)} + \|\mathrm{d}^*\chi\|_{L^2_k(U)}\right) + C_2\|\chi\|_{L^2(U)}.$$

The constants C_1 and C_2 depend on the local properties of the metric and the volume of U. Since M is EAC we can take the constants to be independent of U. Recall that on the cylinder $\mathrm{d}\chi = 0$ and $\mathrm{d}^*\chi = *\mathrm{d}F(\chi)$. In view of the chain rule expression (12) there is a constant $C_3 > 0$ such that

$$\|\mathrm{d}F(\chi)\|_{L^2_k(U)} < C_3\|\chi\|_{C^k(U)}\left(\|\nabla\chi\|_{L^2_k(U)} + \|\chi\|_{L^2_k(U)}\right).$$

As χ decays uniformly we can ensure that $\|\chi\|_{C^k(U)} < 1/2C_1C_3$ by taking U to be sufficiently far along the cylindrical end. Then

$$\|\chi\|_{L^2_{k+1}(U)} < \|\chi\|_{L^2_k(U)} + 2C_2\|\chi\|_{L^2(U)}.$$

Hence χ is $L^2_{k,\delta}$ for all $k \geq 0$. □

This completes the proof of Theorem 3.1.

4 Constructing an EAC G_2-manifold

We shall obtain examples of torsion-free EAC G_2-structures by modifying one of the compact 7-manifolds M with holonomy G_2 constructed by Joyce [16]. Our EAC G_2-manifolds will arise in pairs via a decomposition of a compact M into two compact manifolds identified along their common boundary, a 6-dimensional submanifold $X \subset M$,

$$M = M_{0,+} \cup_X M_{0,-}. \tag{18a}$$

A collar neighbourhood of the boundary of each $M_{0,\pm}$ is diffeomorphic to $I \times X$, for an interval $I \subset \mathbb{R}$. Define

$$M_\pm = M_{0,\pm} \cup_X (\mathbb{R}_+ \times X). \tag{18b}$$

It is on the manifolds $M_\pm$ with cylindrical ends that we shall construct EAC G_2-structures satisfying the hypotheses of Theorem 3.1, such that the resulting EAC G_2-manifolds have holonomy G_2. (Of course, $M_\pm$ is homeomorphic to the interior of $M_{0,\pm}$.)

4.1 Joyce's example of a compact irreducible G_2-manifold

In order to give examples of $M_\pm$ as above, we need to recall part of the construction of a relatively uncomplicated example of a compact G_2-manifold in [16, §12.2]. Consider the action on a torus T^7 by the group $\Gamma \cong \mathbb{Z}_2^3$ generated by

$$\begin{aligned}
\alpha &: (x_1, \dots, x_7) \mapsto (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7),\\
\beta &: (x_1, \dots, x_7) \mapsto (x_1, -x_2, -x_3, x_4, x_5, \tfrac{1}{2} - x_6, -x_7),\\
\gamma &: (x_1, \dots, x_7) \mapsto (-x_1, x_2, -x_3, x_4, \tfrac{1}{2} - x_5, x_6, \tfrac{1}{2} - x_7).
\end{aligned} \tag{19}$$

These maps preserve the standard flat G_2-structure on T^7 (cf. (1)), so T^7/Γ is a flat compact G_2-orbifold. It is simply-connected.

The fixed point set of each of α, β and γ consists of 16 copies of T^3 and these are all disjoint. $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$ and $\alpha\beta\gamma$ act freely on T^7. Furthermore, $\langle\beta, \gamma\rangle$ acts freely on the set of sixteen 3-tori fixed by α, so they map to 4 copies of T^3 in the singular locus of T^7/Γ. Similarly $\langle\alpha, \gamma\rangle$ and $\langle\alpha, \beta\rangle$ acts freely on the sixteen 3-tori fixed by β and γ, respectively. Thus the singular locus of T^7/Γ consists of 12 disjoint copies of T^3.

A neighbourhood of each component T^3 of the singular locus of T^7/Γ is diffeomorphic to $T^3 \times \mathbb{C}^2/\{\pm 1\}$. The blowup of $\mathbb{C}^2/\{\pm 1\}$ at the origin resolves the singularity giving a complex surface Y biholomorphic to $T^*\mathbb{C}P^1$, with the exceptional divisor corresponding to the zero section $\mathbb{C}P^1$. The canonical bundle of Y is trivial and Y has a family of asymptotically locally Euclidean (ALE) Ricci-flat Kähler (hyper-Kähler) metrics with holonomy $SU(2)$. These metrics may be defined via their Kähler forms $i\partial\bar\partial f_s$, in the complex structure on Y induced by from $T^*\mathbb{C}P^1$, where

$$f_s = \sqrt{r^4 + s^4} + 2s^2 \log s - s^2 \log(\sqrt{r^4 + s^4} + s^2), \quad r^2 = z_1\bar z_1 + z_2\bar z_2, \tag{20}$$

and z_1, z_2 are coordinates on $\mathbb{C}^2$ and $s > 0$ is a scale parameter. The forms $i\partial\bar\partial f_s$ admit a smooth extension over the exceptional divisor. Metrics induced by f_s in (20) are the well-known Eguchi–Hanson metrics [9,16, Chap. 7].

It is known (and easy to check) that for each $\lambda > 0$ the map $Y \to Y$ induced by $(z_1, z_2) \mapsto \lambda(z_1, z_2)$ pulls back $i\partial\bar\partial f_s$ to $i\lambda^2\partial\bar\partial f_{\lambda s}$. In particular, s is proportional to the diameter of the exceptional divisor on Y. Further, an important property of the Eguchi–Hanson metrics is that the injectivity radius is proportional to s whereas the uniform norm of the curvature is proportional to s^{-2}.

As discussed in §2.1, the product of an $SU(2)$-manifold and a flat 3-manifold has a 'natural' torsion-free G_2-structure (3). By replacing a neighbourhood of each singular T^3 in T^7/Γ by the product of T^3 and a neighbourhood $U \subset Y$ of the exceptional divisor in the Eguchi–Hanson space one obtains a compact smooth manifold M. Now f_s, for each $s > 0$, is asymptotic to r^2 as $r \to \infty$ and $i\partial\bar{\partial}r^2$ is the Kähler form of the flat Euclidean metric on $\mathbb{C}^2/\{\pm 1\}$. It is therefore possible to smoothly interpolate between the torsion-free G_2-structures on $T^3 \times U$ corresponding to the Eguchi–Hanson metrics and the flat G_2-structure on T^7/Γ away from a neighbourhood of the singular locus, using a cut-off function in the gluing region. In this way, one obtains, for each small $s > 0$, a closed stable 3-form, say $\varphi_s^{\mathrm{init}}$, on M, so that the induced G_2-structure is torsion-free, except in the gluing region. Altogether, according to [16, §11.5] the torsion $\varphi_s^{\mathrm{init}}$ is 'small' in the sense that $\mathrm{d}^*\varphi_s^{\mathrm{init}} = \mathrm{d}^*\psi_s$ for some 3-forms ψ_s satisfying

$$\|\psi_s\|_{L^2} < \lambda' s^4, \quad \|\psi_s\|_{C^0} < \lambda' s^{1/2}, \quad \|d^*\psi_s\|_{L^{14}} < \lambda', \tag{21}$$

for some constant λ' independent of s (cf. (6)). By [16, Theorem 11.6.1] (cf. Remark 3.2), there is a constant $\kappa_M > 0$, so that the G_2-structure $\varphi_s^{\mathrm{init}}$ can be perturbed into a torsion-free G_2-structure

$$\varphi_s = \varphi_s^{\mathrm{init}} + (\text{exact form}) \tag{22}$$

inducing a metric $g(\varphi_s)$ with holonomy G_2 on M whenever $0 < s < \kappa_M$.

We also recall from [16, §12.1] the technique for computing the Betti numbers of the resolution M. This will be needed later when we compute Betti numbers of the EAC G_2-manifolds $M_\pm$.

The cohomology of T^7/Γ is just the Γ-invariant part of the cohomology of T^7, so $b^2(T^7/\Gamma) = 0$ whilst $b^3(T^7/\Gamma) = 7$. For each of the 12 copies of T^3 in the singular locus we cut out a tubular neighbourhood, which deformation retracts to T^3, and glue in a piece of $T^3 \times Y$, which deformation retracts to $T^3 \times \mathbb{C}P^1$. Each of the operations increases the Betti numbers of M by the difference between the Betti numbers of $T^3 \times Y$ and T^3. This is justified using the long exact sequences for the cohomology of T^7/Γ relative to its singular locus and M relative to the resolving neighbourhoods. Hence

$$b^2(M) = 12 \cdot 1 = 12,$$
$$b^3(M) = 7 + 12 \cdot 3 = 43.$$

4.2 An EAC G_2-manifold

We can let the group Γ defined above act on $\mathbb{R} \times T^6$ instead of T^7, taking x_1 to be the coordinate on the $\mathbb{R}$-factor. Then $(\mathbb{R} \times T^6)/\Gamma$ is a flat G_2-orbifold with a single end. We want to resolve it to an EAC G_2-manifold.

The fixed point set of each of α and β in $\mathbb{R} \times T^6$ consists of 16 copies of $\mathbb{R} \times T^2$ and the fixed point set of γ consists of 8 copies of T^3. Resolving the singularities of $(\mathbb{R} \times T^6)/\Gamma$ arising from α, β by gluing in copies of $\mathbb{R} \times T^2 \times Y$ (along with resolving the T^3 singularities arising from γ as before) yields a smooth manifold M_+ with a single end (the cross-section X of M_+ is a resolution of T^6/Γ', where $\Gamma' \subset \Gamma$ is the subgroup generated by α and β). However, the G_2-structure defined by naively adapting the method of the last subsection would introduce torsion in a non-compact region, making it difficult to perturb to a torsion-free G_2-structure. To apply Theorem 3.1 we need to ensure that the G_2-structure is exactly cylindrical and torsion-free on the cylindrical end, so there may only be torsion in a compact region. We shall get round this problem by performing the resolution in two steps, and prove the following.

Theorem 4.1 *The manifold M_+ with cylindrical end and cross-section X, as defined in the beginning of this subsection, has an EAC metric with holonomy equal to G_2. The asymptotic limit metric on X has holonomy equal to $SU(3)$.*

Before giving the details of the proof of Theorem 4.1, let us change perspective slightly and explain how the latter 7-manifold M_+ arises in the setting (18), with M the compact 7-manifold discussed in §4.1. The image of a hypersurface $T^6 \subset T^7$ defined by $x_1 = \frac{1}{4}$ is a hypersurface orbifold X_0 which divides T^7/Γ into two open connected regions. In fact, X_0 is precisely T^6/Γ', as Γ' is the subgroup that acts trivially on the x_1 factor in T^7. Each component of $(T^7/\Gamma)\backslash X_0$ is the interior of a compact orbifold with boundary X_0 and we can attach product cylinders $\mathbb{R}_{>0} \times X_0$ to form orbifolds with a cylindrical end. One of these (the one containing the image of $x_1 = 0$) corresponds naturally to $(\mathbb{R} \times T^6)/\Gamma$.

Now, M_+ is well-defined as a resolution of singularities of this $(\mathbb{R} \times T^6)/\Gamma$ as described above and M_- is defined similarly by starting from the other component of $T^7/\Gamma\backslash X_0$.

Remark 4.2 In this particular example, the two EAC halves $M_\pm$ will be isometric, the isometry being induced from an involution on T^7/Γ,

$$(x_1, \ldots, x_7) \mapsto \left(x_1 + \tfrac{1}{2}, x_2, x_3, x_4, x_5, x_6, x_7\right),$$

which swaps the two components of $(T^7/\Gamma)\backslash X_0$. The restriction to X_0 induces an anti-holomorphic isometry on its resolution X.

We now state a technical result from which Theorem 4.1 will follow.

Proposition 4.3 *Let M be a smooth compact 7-manifold obtained by resolving singularities of T^7/Γ, as defined in §4.1. There exists a constant $\kappa' > 0$, such that for each s with $0 < s < \kappa'$, there is a closed stable $\tilde{\varphi}_s \in \Omega^3(M)$ with the following properties:*

(i) *There is a Calabi–Yau structure (Ω, ω) on a 6-manifold X and an interval $I = (-\varepsilon, \varepsilon)$ such that M has an open subset $N \cong X \times I$ with*

$$\tilde{\varphi}_s|_N = \Omega + \mathrm{d}t \wedge \omega, \tag{23}$$

and N retracts to X and the complement of N in M has exactly two connected components (diffeomorphic to the components of $M\backslash X$).

(ii) *There is a smooth 3-form ψ_s such that $\mathrm{d}^*\psi_s = \mathrm{d}^*\tilde{\varphi}_s$, satisfying the estimates (6), with $\lambda > 0$ independent of s.*

(iii) *ψ_s vanishes on N.*

(iv) *The 3-form $\tilde{\varphi}_s - \varphi_s^{init}$ is exact, where φ_s^{init} is the G_2-structure on M defined in §4.1.*

We can think of $\tilde{\varphi}_s$ as an 'intermediate' perturbation of φ_s^{init}. Instead of perturbing away all the torsion in one go, like in §4.1, we settle for eliminating the torsion from the neck region N, whilst keeping it controlled elsewhere. What we gain is that $\tilde{\varphi}_s$ is a product G_2-structure on N. We can therefore cut M into two halves along the hypersurface $X \times \{0\} \subset N$, and attach a copy of $X \times [0, \infty)$ to each half to form cylindrical-end manifolds $M_\pm$ with EAC G_2-structures $\tilde{\varphi}_{s,\pm} = \tilde{\varphi}_s|_{M_\pm}$. The properties (i)–(iii) achieved in Proposition 4.3 ensure that Theorem 3.1 then applies to each of $M_\pm$, giving $0 < \kappa \leq \kappa'$ such that $\tilde{\varphi}_{s,\pm}$ can be perturbed to torsion-free G_2-structures

$$\varphi_{s,\pm} = \tilde{\varphi}_{s,\pm} + \mathrm{d}\eta_{s,\pm},$$

whenever $0 < s < \kappa$.

The orbifold $(\mathbb{R} \times T^6)/\Gamma$ is simply-connected, and so is the resolution M_+. Therefore, any torsion-free G_2-structure on M_+ induces a metric with full holonomy G_2 by Corollary 2.2, which proves Theorem 4.1 assuming Proposition 4.3.

Remark 4.4 This construction of the EAC G_2-structures with small torsion is only superficially different from the description given before the statement of Theorem 4.1. That is, the choice of whether we cut the manifold in half and attach cylinders before or after resolving the singularities of the neck is not particularly important. The convenience of going with the latter choice in the proof is that it allows us to do most of the technical work on compact manifolds. Another advantage is that then it is better illuminated that we obtain a pair of torsion-free EAC G_2-manifolds whose asymptotic models are isomorphic. One can apply to this pair of G_2-structures the gluing theorem from [18, §5] and obtain a G_2-structure on the generalized connected sum of $M_\pm$ joined at their ends, giving a compact G_2-manifold with a long neck. This connected sum is, of course, diffeomorphic to the compact G_2-manifold M as obtained by resolving singularities T^7/Γ directly as in §4.1. Considering the G_2-metrics one may intuitively think of the EAC halves $M_\pm$ being obtained by 'pulling M apart'. This will be made more precise in §6, where the clause (iv) of Proposition 4.3 will be important.

4.3 Proof of Proposition 4.3

We find the desired cylindrical-neck G_2-structure $\tilde{\varphi}_s$ on the resolution M of T^7/Γ by performing the resolution in two stages. The group Γ preserves the product decomposition $T^7 = S^1 \times T^6$, where the S^1 factor corresponds to the x_1 coordinate. Let $\Gamma' \subset \Gamma$ be the subgroup generated by α and β; notice that Γ' acts on T^6 (and fixes the S^1-factor). Define $\Psi = \Gamma/\Gamma'$. Here is the strategy of our proof:

(1) Resolve the singularities of T^7/Γ' using Eguchi–Hanson hyper-Kähler spaces as described in §4.1 to form a compact manifold $M' \cong S^1 \times X^6$ equipped with a family of Ψ-invariant G_2-structures $\tilde{\varphi}'_s$ with small torsion. Perturb $\tilde{\varphi}'_s$ to a torsion-free Ψ-invariant product G_2-structure φ'_s on M'.

(2) The G_2-structure φ'_s is not flat near the fixed point set F of Ψ acting on M'. We perturb φ'_s by adding an exact 3-form supported near F, so that the resulting G_2-structure on M' interpolates between the flat structure near F and φ'_s away from F. The torsion introduced by the latter perturbation 3-form is controlled by estimates similar to (6). Furthermore, the interpolating G_2-structure is Ψ-invariant and descends to the orbifold M'/Ψ (see Fig. 1).

(3) Resolve the singularities of M'/Ψ, using the same Eguchi–Hanson hyper-Kähler structures as in the construction of $\varphi_s^{\mathrm{init}}$ in §4.1 (in particular, they have *the same scale parameter* s as in step 1) and construct the G_2-structure $\tilde{\varphi}_s$ on the compact manifold M. Finally, check that the difference $\tilde{\varphi}_s - \varphi_s^{\mathrm{init}}$ is essentially the exact form added in step 2.

Our first step is entirely analogous to the construction of $\varphi_s^{\mathrm{init}}$ outlined in §4.1, but this time we resolve the singularities of the orbifold $(S^1 \times T^6)/\Gamma'$ rather than T^7/Γ. This gives a compact 7-manifold M' with a family of closed S^1-invariant 3-forms, say $\tilde{\varphi}'_s$, inducing G_2-structures with small torsion in the sense of (21). Then, as noted in Remark 3.2, we can apply [16, Theorem 11.6.1] and obtain a $\kappa' > 0$, such that $\tilde{\varphi}'_s$ admits a perturbation to a torsion-free G_2-structure

$$\varphi'_s = \tilde{\varphi}'_s + \mathrm{d}\eta'_s, \tag{24}$$

for $0 < s < \kappa'$. The correction term satisfies

$$\|\mathrm{d}\eta_s\|_{L^2} < K's^4, \quad \|\mathrm{d}\eta_s\|_{C^0} < K's^{1/2}, \quad \|\nabla\mathrm{d}\eta_s\|_{L^{14}} < K', \tag{25}$$

with some constant K' independent of s [cf. (7)].

Clearly, there is a diffeomorphism

$$M' \simeq S^1 \times X,$$

where X denotes a blowup of the complex orbifold T^6/Γ'. Since $\tilde{\varphi}'_s$ is S^1-invariant, so is φ'_s; in fact, more is true. The lemma below can be thought of as a simple version of the Cheeger–Gromoll line splitting theorem (cf. [8]) and ensures that φ'_s is a product G_2-structure determined by some Calabi–Yau structure on X and some diffeomorphism $M' \cong S^1 \times X$ (but not necessarily the same one as for $\tilde{\varphi}'_s$).

Lemma 4.5 (cf. Chan [7, p. 15]) *Let T^m be a torus and X a compact manifold with $b^1(X) = 0$. If g is a Ricci-flat metric on $T^n \times X$ that is invariant under translations of the torus factor then there is a function $f : X \to \mathbb{R}^n$ such that the graph diffeomorphism*

$$T^n \times X \to T^n \times X, \quad (t, x) \mapsto (t + f(x), x)$$

pulls g back to a product metric.

Sketch proof Let $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ be the coordinate vector fields on T^n and $\alpha_i = (\frac{\partial}{\partial x^i})^\flat$. Each $\frac{\partial}{\partial x^i}$ is a Killing vector field on a Ricci-flat manifold, so the 1-forms α_i are harmonic. Since $b^1(X) = 0$ the closed forms α_i are exact. Define $f : X \to \mathbb{R}^n$ by choosing f_i such that $\alpha_i = -df_i$. □

The following commutative diagram shows the relation between $M' \simeq S^1 \times X$ and M in the resolution of singularities and will be useful for keeping track of the construction of the desired $\tilde{\varphi}_s$ on M from G_2-structures $\tilde{\varphi}'_s$ and φ'_s on M'.

$$\begin{tikzcd}
 & M \arrow[d] \arrow[dd, dashed, bend left] \\
M' \arrow[r, "{[\Psi]}"] \arrow[d] & M'/\Psi \arrow[d] \\
S^1 \times (T^6/\Gamma') \arrow[r, "{[\Psi]}"] & T^7/\Gamma
\end{tikzcd} \tag{26}$$

Here, we used $[\Psi]$ to denote the quotient maps for the actions of $\Psi = \Gamma/\Gamma' \cong \mathbb{Z}_2$. The vertical arrows are the resolution maps (essentially blowups) locally modelled on $T^3 \times U \to T^3 \times (\mathbb{C}^2/\pm 1)$, with U a neighbourhood of the exceptional divisor in an Eguchi–Hanson space. Note that there is a unique way to lift the action of Ψ to M', so that the diagram (26) commutes. (One can further 'fill in' the top left corner of (26), the respective manifold being essentially the blowup of the fixed point set of Ψ in M', but we won't need that.)

The singular locus of M'/Ψ consists of 4 copies of T^3, corresponding to the fixed point set of γ, cf. §4.1. We can choose the resolutions in constructing $\tilde{\varphi}'_s$ so that it becomes Ψ-invariant, moreover, so that away from a neighbourhood S of the fixed point set of Ψ, $\tilde{\varphi}'_s$ is the pull-back of φ^{init}_s via $M' \backslash S \to M$. Then φ'_s is Ψ-invariant too, so both $\tilde{\varphi}'_s$ and φ'_s descend to well-defined G_2-structures on the quotient M'/Ψ. A neighbourhood of each T^3 component of the singular locus is homeomorphic to $T^3 \times (\mathbb{C}/\{\pm 1\})$. However, a consequence of our

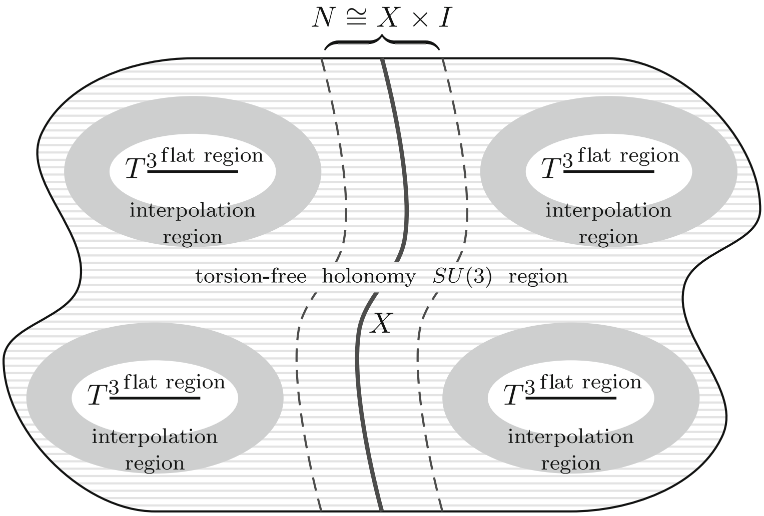

Fig. 1 An interpolating G_2-structure $\tilde{\varphi}'_s + \mathrm{d}(\eta' - \rho\chi)$ on the orbifold M'/Ψ

previous step is that the G_2-structure φ'_s on M'/Ψ is not necessarily flat near the singular locus. Therefore, we cannot immediately use Joyce's method discussed in §4.1, resolving the singularities of M'/Ψ by patching φ'_s with the product G_2-structure on $T^3 \times U$, in a way that keeps the torsion small.

On the other hand, the G_2-structure $\tilde{\varphi}'_s$ on M' *is* flat except near the resolved singularities. In particular, $\tilde{\varphi}'_s$ is flat near the fixed point set $F \subset M'$ of Ψ, since the elements of Γ have disjoint fixed point sets. We now wish to define on M', for $0 < s < \kappa'$, a closed Ψ-invariant G_2-structure with small torsion, by smoothly interpolating between the flat $\tilde{\varphi}'_s$ near F and the torsion-free φ'_s in a Ψ-invariant region $N' = \big((\frac{1}{4}-\varepsilon, \frac{1}{4}+\varepsilon) \cup (\frac{3}{4}-\varepsilon, \frac{3}{4}+\varepsilon)\big) \times X \subset (\mathbb{R}/\mathbb{Z}) \times X \simeq M'$, for some $0 < \varepsilon < \frac{1}{4}$. Note that although N' has two components, its image in the resolution M of M'/Ψ is connected and will be the cylindrical neck region N in the statement of Proposition 4.3. See Fig. 1. To achieve small torsion, we use a generalization of the classical Poincaré inequality.

Lemma 4.6 *Let F be a compact Riemannian manifold and I a bounded open interval. For any $n \geq 0, k \geq 0$ and $p \geq 1$ there is a constant $C_{n,p,k} > 0$, such that for every exact L^p_k m-form $\mathrm{d}\eta$ on the Riemannian product $S = F \times I^n$ there is an $(m-1)$-form χ with $\mathrm{d}\chi = \mathrm{d}\eta$ and*

$$\|\chi\|_{L^p_{k+1}} < C_{n,p,k}\|\mathrm{d}\eta\|_{L^p_k}. \tag{27}$$

Proof The proof is by induction on n. The result holds for $n = 0$ by standard Hodge theory and elliptic estimate for the Laplacian on compact F. For the inductive step, we show that if a manifold S satisfies the assertion of the lemma, then so does $S \times I$ with the product metric.

Let t denote the coordinate on I and S_t denote the hypersurface $S \times \{t\}$. We can write

$$\mathrm{d}\eta = \alpha + \mathrm{d}t \wedge \beta,$$

with α and β sections of the pull-back of $\Lambda^* T^* S$ to $S \times I$. Write $\alpha(t)$, $\beta(t)$ for the corresponding forms on S_t. Fix $t_0 \in I$ and let

$$\chi_1(t) = \int_{t_0}^{t} \beta(u)\mathrm{d}u.$$

Let ∇ denote the covariant derivative on $S \times I$, and consider χ_1 as a form on $S \times I$. For any $0 \le i \le k$ and $t \in I$

$$\begin{aligned}\|(\nabla^i \chi_1)(t)\|^p_{L^p(S_t)} &= \int_S \left\| \int_{t_0}^{t} (\nabla^i \beta)(u)\mathrm{d}u \right\|^p vol_S \\ &\le V^{p-1} \int_S \int_{t_0}^{t} \|(\nabla^i \beta)(u)\|^p \mathrm{d}u\, vol_S \le V^{p-1} \|\nabla^i \beta\|^p_{L^p(S\times I)},\end{aligned}$$

where V is the length of I. Hence

$$\|\nabla^i \chi_1\|^p_{L^p(S\times I)} \le \int_I \|(\nabla^i \chi_1)(u)\|^p_{L^p(S_u)} \mathrm{d}u \le V^p \|\nabla^i \beta\|^p_{L^p(S\times I)},$$

and

$$\|\chi_1\|_{L^p_k(S\times I)} \le V\|\mathrm{d}\eta\|_{L^p_k(S\times I)}.$$

$\mathrm{d}(\eta - \chi_1)$ has no $\mathrm{d}t$-component, so the $\mathrm{d}t$-component of $\mathrm{d}^2(\eta - \chi_1)$ is $\frac{\partial}{\partial t}\mathrm{d}(\eta - \chi_1) = 0$. Hence $d(\eta - \chi_1)$ is the pull-back to $S \times I$ of an exact form on S. By the inductive hypothesis there is a form χ_2 such that $\mathrm{d}\chi_2 = \mathrm{d}(\eta - \chi_1)$ and $\chi = \chi_1 + \chi_2$ satisfies (27) for some C independent of $\mathrm{d}\eta$. □

Let $S \cong F \times I^4$ be a tubular neighbourhood of F in M'. Applying Lemma 4.6 to $\mathrm{d}\eta'_s$ in (24), we obtain a 2-form χ_s on S such that

$$\mathrm{d}\chi_s = \mathrm{d}\eta'_s|_S$$

and χ_s satisfies the L^2 estimate

$$\|\chi_s\|_{L^2} < C_{4,2,0}\|\mathrm{d}\eta'_s|_S\|_{L^2} \le K_2 s^4 \tag{28a}$$

as well as the L^{14}_1 estimate

$$\|\chi_s\|_{L^{14}_1} < C_{4,14,0}\|\mathrm{d}\eta'_s|_S\|_{L^{14}} \le C_{4,14,0}\,\mathrm{vol}(S)^{1/14}\|\mathrm{d}\eta'_s|_S\|_{C^0} < K_{14}s^{1/2} \tag{28b}$$

with K_2, K_{14} independent of s. Here, we also used (25). We shall also need an estimate on the uniform norm of χ_s which is obtained from (28) and the following version of Sobolev embedding.

Theorem 4.7 **([16, Theorem G1])** *Let μ, ν and s be positive constants, and suppose M is a complete Riemannian 7-manifold, whose injectivity radius δ and Riemannian curvature R satisfy $\delta \ge \mu s$ and $\|R\|_{C^0} \le \nu s^{-2}$. Then there exists $C > 0$ depending only on μ and ν, such that if $\chi \in L^{14}_1(\Lambda^3) \cap L^2(\Lambda^3)$ then*

$$\|\chi\|_{C^0} \le C(s^{1/2}\|\nabla\chi\|_{L^{14}} + s^{-7/2}\|\chi\|_{L^2}).$$

We deduce that

$$\|\chi_s\|_{C^0} < C(K_2 s + K_{14}s^{1/2}) < \widetilde{C}s^{1/2}$$

as $s > 0$ varies in a bounded interval.

Let ρ be a cut-off function (not depending on s) which is 1 near F and 0 outside S. Then

$$\|\mathrm{d}(\rho\chi)\|_{L^2} < K''s^4, \ \|\mathrm{d}(\rho\chi)\|_{C^0} < K''s^{1/2}, \ \|\nabla\mathrm{d}(\rho\chi)\|_{L^{14}} < K'', \tag{29}$$

with K'' independent of s.

Remark 4.8 A key point in achieving the estimates (28) and (29) is that a tubular neighbourhood $S \cong F \times I^4$ does not meet the region affected by resolution of singularities in our first step. Therefore, the metric on S and the respective constants in (27) can be taken to be independent of s. See also Remark 5.2 below.

For each $0 < s < \kappa'$, $\tilde{\varphi}'_s + \mathrm{d}(\eta'_s - \rho\chi_s)$ is a closed G_2-structure which is flat near F. It is clear from the chain rule that it has small torsion in the sense of Theorem 3.1: there is a form ψ'_s such that $\mathrm{d}{*}\psi'_s = \mathrm{d}\Theta(\tilde{\varphi}'_s + \mathrm{d}(\eta'_s - \rho\chi_s))$, satisfying (6). (Here Θ denotes the non-linear mapping $\varphi \mapsto *_\varphi\varphi$; note that Θ depends only on the smooth structure and orientation on M'.) However, we need to take care to choose ψ'_s in such a way that it vanishes not only on the cylindrical region N', but also near F. Because F has dimension 3 any closed 4-form on the tubular neighbourhood S is exact. By Lemma 4.6 we can write

$$\left(\Theta\left(\tilde{\varphi}'_s + \mathrm{d}\eta'_s\right) - \Theta\left(\tilde{\varphi}'_s\right)\right)|_S = \mathrm{d}\chi'_s$$

for some 3-form χ'_s on S, so that $\mathrm{d}(\rho\chi'_s)$ satisfies estimates of the form (29). We can then take

$$\psi'_s = *\left(\Theta\left(\tilde{\varphi}'_s + \mathrm{d}\left(\eta'_s - \rho\chi_s\right)\right) - \Theta\left(\tilde{\varphi}'_s + \mathrm{d}\eta'_s\right) + \mathrm{d}\left(\rho\chi'_s\right)\right).$$

This is supported in S and vanishes near F and satisfies (6) for some $\lambda > 0$ (depending on the constants K' and K'' from (25) and (29), but *not* on s). We can ensure that all forms are Ψ-invariant, so ψ'_s descends to a small 3-form, still denoted by ψ'_s on the orbifold M'/Ψ. As this form is supported away from the singular locus, ψ'_s is also well-defined on the resolution M.

For $0 < s < \kappa'$, the form $\tilde{\varphi}'_s + \mathrm{d}(\eta'_s - \rho\chi_s)$ descends to an orbifold G_2-structure on M'/Ψ with small torsion. By construction, it is a product G_2-structure on the image $N \cong I \times X \subset M'/\Psi$ of $N' \subset M'$. Its orbifold singularities are modelled on quotients of the flat G_2-structure, so the singularities can be resolved like in §4.1 to define a closed G_2-structure $\tilde{\varphi}_s$ on M. We make sure that the Eguchi–Hanson spaces used in this resolution have the same scale as those used for the resolution of the first-step singularities. The torsion introduced by the resolution is then small, in the sense that there is a smooth 3-form ψ''_s on M, supported near the pre-image F' of the singular locus, such that $\mathrm{d}^*\psi''_s = \mathrm{d}^*\tilde{\varphi}_s$ near F' and ψ''_s satisfies the estimates (6). Thus for each $0 < s < \kappa'$, $\tilde{\varphi}_s$ is a G_2-structure on M with small torsion (controlled by $\psi_s = \psi'_s + \psi''_s$) and N is a cylindrical neck region, so that $\tilde{\varphi}_s$ satisfies the claims (i)–(iii) of Proposition 4.3.

To prove the remaining claim iv we identify the difference between our $\tilde{\varphi}_s$ and the G_2-structure $\varphi_s^{\mathrm{init}}$ obtained (in §4.1) by resolving all the singularities of T^7/Γ in a single step. By construction in the previous paragraph, $\tilde{\varphi}_s - \varphi_s^{\mathrm{init}}$ vanishes on a neighbourhood of the pre-image in M of the singular locus of M'/Ψ (see (26)). Therefore, we may interchangeably consider $\tilde{\varphi}_s - \varphi_s^{\mathrm{init}}$ as a Ψ-invariant form on M' supported away from a neighbourhood S of the fixed point set of Ψ.

Now recall that $\tilde{\varphi}'_s$ is a Ψ-invariant form on M' and the restriction of $\tilde{\varphi}'_s$ agrees with the pull-back of $\varphi_s^{\mathrm{init}}$ to $M\backslash S$. On the other hand, the difference between the pull-back of $\tilde{\varphi}_s$ to $M\backslash S$ and $\tilde{\varphi}'_s|_{M'\backslash S}$ is $d(\eta'_s - \rho\chi_s)$. The 2-form $\eta'_s - \rho\chi_s$ is Ψ-invariant, as η'_s and χ_s are so.

As $\eta'_s - \rho\chi_s$ is also supported away from S, it is the pull-back via $M'\backslash S \to M$ of a well-defined 2-form, say ξ, on M. We find that $\tilde{\varphi}_s - \varphi_s^{\text{init}}$ is the exact form $d\xi$. This completes the proof of Proposition 4.3.

5 Further examples and applications

We now construct a few further examples of EAC G_2-manifolds with different types of cross-sections and discuss their topology. We also give examples of EAC coassociative submanifolds.

5.1 Topology of the example of §4

To study the topology of the EAC G_2-manifold M_+ we consider it as a resolution of $(T^6 \times \mathbb{R})/\Gamma$. As noted in §4.2, both the orbifold and its resolution are simply-connected.

Recall that we chose Γ' to be the stabiliser of the S^1 factor corresponding to the x_1 coordinate in (19), i.e. $\Gamma' = \langle \alpha, \beta \rangle$. The resolution of the intermediate quotient $S^1 \times T^6/\Gamma'$ is isomorphic to $S^1 \times X_{19}$, for a simply-connected Calabi–Yau 3-fold X_{19}. This X_{19} is then the cross-section of M_+.

We find that the Betti numbers of $\left(T^6 \times \mathbb{R}\right)/\Gamma$ are $b^2 = 0, b^3 = 4, b^4 = 3, b^5 = 0$. The singular locus in $\left(\mathbb{R} \times T^6\right)/\Gamma$ consists of 8 copies of $T^2 \times \mathbb{R}$ and 2 copies of T^3. Resolving the former adds 1, 2 and 1 to b^2, b^3 and b^4, respectively. Therefore

$$\begin{aligned}
b^2(M_+) &= 8 \cdot 1 + 2 \cdot 1 = 10,\\
b^3(M_+) &= 4 + 8 \cdot 2 + 2 \cdot 3 = 26,\\
b^4(M_+) &= 3 + 8 \cdot 1 + 2 \cdot 3 = 17,\\
b^5(M_+) &= 2 \cdot 1 = 2.
\end{aligned}$$

We can also compute the Betti numbers of the cross-section X_{19}, and find that $b^2(X_{19}) = 19$, $b^3(X_{19}) = 40$. Therefore its Hodge numbers are

$$h^{1,1}(X_{19}) = h^{1,2}(X_{19}) = 19.$$

Remark 5.1 The Calabi–Yau 3-fold X_{19} can be obtained in a slightly different way. Blowing up the singularities of $T^6/\langle\alpha\rangle$ gives a product of a Kummer K3 surface and an elliptic curve $\mathcal{E} \cong T^2$. The map β descends to a holomorphic involution of K3 $\times\, \mathcal{E}$, still denoted by β. The restriction $\beta|_{\mathcal{E}}$ induced by -1 on $\mathbb{C}$ has 4 fixed points in $\mathcal{E}$ and $(\beta|_{\mathrm{K3}})^*$ multiplies the holomorphic (2,0)-forms on the K3 surface by -1. The 3-fold X_{19} is then the blowup of the orbifold $(K3 \times \mathcal{E})/\langle\beta\rangle$ at its singular locus. Calabi–Yau 3-folds obtained from K3 $\times\, \mathcal{E}$ and an involution β with the above properties were studied by Borcea [4] and Voisin [32] in connection with mirror symmetry, and are sometimes called *Borcea–Voisin manifolds*.

According to [29, Proposition 3.5], the dimension of the moduli space of torsion-free EAC G_2-structures on $M_\pm$ can be written in terms of Betti numbers as

$$b^4(M_\pm) + \tfrac{1}{2}b^3(X) - b^1(M_\pm) - 1, \tag{30}$$

so in this example we find that the moduli space has dimension 36.

5.2 Two more EAC G_2-manifolds

Let us consider some variations of the example in the previous subsection in order to get examples of different topological types. Especially, we want to show that an EAC manifold with holonomy exactly G_2 may have a cross-section X whose holonomy is a proper subgroup of $SU(3)$. Here and below by holonomy of a cross-section we mean 'holonomy at infinity', corresponding to the Calabi–Yau structure on X defined by the asymptotic limit of G_2-structure along the cylindrical end (cf. §2.2).

When we let the group Γ from (19) act on $\mathbb{R} \times T^6$ in the previous subsection, we could have taken the $\mathbb{R}$-factor to correspond to a coordinate on T^7 other than x_1. In the geometric interpretation of Remark 4.4 this means pulling the compact G_2-manifold M apart along a hypersurface defined by $x_i = \text{const}$ rather than $x_1 = \text{const}$. Pulling apart M in the x_2 or x_4 direction we get essentially the same pair of 7-manifolds $M_\pm$ as for the x_1 direction in §4.2. We just need to use $\langle \gamma, \alpha \rangle$ or $\langle \beta, \gamma \rangle$ as Γ' to define the intermediate resolution.

If we pull apart along the x_3 direction we get a slightly different geometry and new examples. The subgroup of Γ acting trivially on the x_3 factor is $\Gamma' = \langle \alpha, \beta\gamma \rangle$, which only contains one non-identity element with fixed points. The cross-section of the neck is a resolution X_{11} of T^6/Γ'. It is a non-singular quotient of $T^2 \times K3$ by an involution that acts as -1 on the T^2 factor, so the first Betti number $b^1(X_{11})$ vanishes, but the holonomy of X_{11} is $\mathbb{Z}_2 \ltimes SU(2)$. The EAC G_2-manifolds $M_\pm$ are however simply-connected with a single cylindrical end. Thus, by Corollary 2.2, these are examples of irreducible EAC G_2-manifolds with locally reducible cross-section. These are not homeomorphic to the example in §4.2 as the cross-section X_{19} of the latter example is simply-connected, whereas X_{11} is not. We can also compute the Betti numbers of $M_\pm$.

In the present case, the singular locus in each half is 4 copies of T^3 and 4 copies of $T^2 \times \mathbb{R}$. The Betti numbers are therefore

$$\begin{aligned}
b^2(M_\pm) &= 4 \cdot 1 + 4 \cdot 1 = 8,\\
b^3(M_\pm) &= 4 + 4 \cdot 2 + 4 \cdot 3 = 24,\\
b^4(M_\pm) &= 3 + 4 \cdot 1 + 4 \cdot 3 = 19,\\
b^5(M_\pm) &= 4 \cdot 1 = 4.
\end{aligned}$$

The Hodge numbers of $X_{11} = (T^2 \times K3)/\mathbb{Z}_2$ are

$$h^{1,1}(X_{11}) = h^{1,2}(X_{11}) = 11.$$

By the formula (30) the moduli space of torsion-free EAC G_2-structures on $M_\pm$ has dimension 31.

It is also possible to pull apart M in the x_5, x_6 or x_7 directions. In all three cases, the resulting EAC G_2-manifolds have $b^1(M_\pm) = 1$, so do not have full holonomy G_2. In §7, we shall consider the case of x_5 in greater detail, and relate $M_\pm$ to quasiprojective complex 3-folds with holonomy $SU(3)$ and to a 'connected-sum construction' of compact irreducible G_2-manifolds [18,19]. The case x_6 is similar, but the case x_7 is qualitatively different in that the cross-section X is not $T^2 \times K3$ but a non-singular quotient of T^6.

In order to find an example of an EAC manifold with holonomy G_2 whose cross-section at infinity is flat, we replace Γ with the group Γ_1 generated by

$$\begin{aligned}
\alpha &: (x_1, \ldots, x_7) \mapsto (\ \ x_1, \ \ x_2, x_3, -x_4, -x_5, -x_6, -x_7)\,,\\
\beta &: (x_1, \ldots, x_7) \mapsto \left(\ \ x_1, -x_2, -x_3, x_4, x_5, \tfrac{1}{2} - x_6, -x_7\right),\\
\gamma_1 &: (x_1, \ldots, x_7) \mapsto \left(-x_1, \ \ x_2, \tfrac{1}{2} - x_3, x_4, \tfrac{1}{2} - x_5, x_6, -x_7\right).
\end{aligned} \tag{31}$$

The orbifold T^7/Γ_1 can be resolved in the same way as T^7/Γ, and the resulting compact G_2-manifold M_1 has the same Betti numbers as M. Pulling M_1 apart in the x_7 direction gives an EAC manifold with holonomy exactly G_2 whose cross-section is the non-singular quotient of T^6 by $\Gamma' = \langle \alpha\beta, \beta\gamma_1 \rangle \cong \mathbb{Z}_2^2$. In particular, the cross-section is flat (in this case, there is no need for any intermediate resolution in the construction of the EAC G_2-structure). The manifold has Betti numbers

$$\begin{aligned} b^2(M_+) &= 6 \cdot 1 = 6, \\ b^3(M_+) &= 4 + 6 \cdot 3 = 22, \\ b^4(M_+) &= 3 + 6 \cdot 3 = 20, \\ b^5(M_+) &= 6 \cdot 1 = 6. \end{aligned}$$

The cross-section has $b^1(T^6/\mathbb{Z}_2^2) = 0$ (this is in any case a necessary condition for the EAC manifolds $M_\pm$ to have full holonomy G_2, by [29, Proposition 5.16] and Theorem 2.1) and

$$h^{1,1}\left(T^6/\mathbb{Z}_2^2\right) = h^{1,2}\left(T^6/\mathbb{Z}_2^2\right) = 3.$$

The moduli space of torsion-free EAC G_2-structures on $M_\pm$ has dimension 23.

Remark 5.2 Looking carefully, the argument for pulling apart a compact G_2-manifold obtained by resolving T^7/Γ (provided a method for resolving its singularities with small torsion) relies on two properties of the group Γ. The first is that Γ preserves a product decomposition $T^7 = S^1 \times T^6$, with some elements acting as reflections on the S^1 factor. The other is that, in order to apply Lemma 4.6, the fixed point sets of elements of the subgroup Γ' acting trivially on the S^1 factor must not intersect fixed point sets of the remaining elements (cf. Remark 4.8).

In [16], Joyce gives a number of examples of suitable groups Γ, where such fixed point sets of elements are pair-wise disjoint. Most of them preserve a product decomposition, so can be pulled apart (possibly in more than one way) giving further examples of EAC G_2-manifolds.

More generally, a method is proposed in [16, p. 304] for constructing G_2-structures with small torsion on a resolution of singularities of $S^1 \times X^6/(-1, a)$, where X^6 is a Calabi–Yau 3-fold and a is an anti-holomorphic involution on X^6. As discussed in §2.1, the Calabi–Yau structure of X is completely determined by two closed forms, the real part Ω of a non-vanishing holomorphic (3, 0)-form and the Kähler form ω. Then $a^*\omega = -\omega$ and without loss of generality $a^*\Omega = \Omega$. The product torsion-free G_2-structure $\Omega + dt \wedge \omega$ as in (2) is well-defined on $S^1 \times X$ and invariant under $(-1, a)$, thus descends to a well-defined G_2-structure on the quotient. The singular locus of $S^1 \times X^6/(-1, a)$ is of the form $\{0, \frac{1}{2}\} \times L$, where $L \subset X$ is the fixed point set of a, necessarily a real 3-dimensional submanifold of X (more precisely, L is special Lagrangian).

A resolution of singularities of $(S^1 \times X)/(-1, a)$ should be locally modelled on $\mathbb{R}^3 \times Y$, where Y is an Eguchi–Hanson space. It is explained in [16, p. 304] that to get a well-defined G_2-structure (initially with small torsion) on the resolution one would need to make a choice of smooth family of ALE hyper-Kähler metrics on Y.

Assuming such choice, one could equally well define EAC G_2-structures with small torsion on $(\mathbb{R} \times X)/(-1, a)$, and use Theorem 3.1 to obtain EAC manifolds with holonomy G_2.

5.3 EAC coassociative submanifolds

Let M be a 7-manifold with a G_2-structure given by a 3-form φ. A *coassociative submanifold* $C \subset M$ is a 4-dimensional submanifold such that $\varphi|_C = 0$. It is not difficult to check that then the 4-form $*_\varphi\varphi$ never vanishes on C, thus every coassociative submanifold is necessarily orientable.

If a G_2-structure φ is torsion-free then $d*_\varphi\varphi = 0$ and the 4-form $*_\varphi\varphi$ is a *calibration* on M as defined by Harvey and Lawson [11]. In this case, coassociative submanifolds (considered with appropriate orientation) are precisely the submanifolds calibrated by $*_\varphi\varphi$, in particular, every coassociative submanifold of a G_2-manifold is a minimal submanifold [11, Theorem II.4.2]. Our definition of coassociative submanifold is not the same as in *op.cit.* but is equivalent to it via [11, Proposition IV.4.5 & Theorem IV.4.6].

One way of producing examples of coassociative submanifolds is provided by the following.

Proposition 5.3 ([16, Proposition 10.8.5]) *Let $\sigma : M \to M$ be an involution such that $\sigma^*\varphi = -\varphi$. Then each connected component of the fixed point set of σ is either a coassociative 4-fold or a single point.*

Any σ as in the hypothesis of Proposition 5.3 is called an *anti-G_2 involution*. It is necessarily an isometry of M.

Let M^7 be the compact G_2-manifold discussed in §4.1 and φ its torsion-free G_2-structure. We shall consider two examples of anti-G_2 involution taken from [16, §12.6] which extend to well-defined anti-G_2 involutions of EAC G_2-manifolds constructed in §5.2.

Example 5.4 Define an orientation-reversing isometry of T^7 as in [16, Example 12.6.4].

$$\sigma : (x_1, \dots, x_7) \mapsto \left(\tfrac{1}{2} - x_1, x_2, x_3, x_4, x_5, \tfrac{1}{2} - x_6, \tfrac{1}{2} - x_7\right). \tag{32}$$

Then σ commutes with the action of Γ defined by (19) and pulls back φ_0 to $-\varphi_0$. When the singularities of T^7/Γ are resolved to form the compact G_2-manifold M one can ensure that σ lifts to an anti-G_2 involution of (M, φ). The fixed point set of σ in M consists of 16 isolated points and one copy of T^4, which is a coassociative submanifold of M.

We can also consider σ in (32) as an involution of $T^6 \times \mathbb{R}$. Provided that the $\mathbb{R}$ factor corresponds to the x_2, x_3 or x_4 coordinate this again commutes with the action of Γ. When we pull apart M in the x_2, x_3 or x_4 direction the resulting irreducible EAC G_2-manifolds $M_\pm$ are resolutions of $(T^6 \times \mathbb{R})/\Gamma$, so σ lifts to an anti-G_2 involution of $M_\pm$. The fixed point set in each half $M_\pm$ consists of 8 isolated points and one 4-manifold $C_\pm \cong T^3 \times \mathbb{R}$, which is an asymptotically cylindrical coassociative submanifold of $M_\pm$ (in the obvious coordinates for the cylindrical end of $M_\pm$, $C_\pm$ is a product submanifold).

Example 5.5 Here is another orientation-reversing isometry of T^7 taken from [16, Example 12.6.4].

$$\sigma : (x_1, \dots, x_7) \mapsto \left(\tfrac{1}{2} - x_1, \tfrac{1}{2} - x_2, \tfrac{1}{2} - x_3, x_4, x_5, x_6, x_7\right).$$

Its fixed point set in T^7/Γ consists of 16 isolated points and two copies of $T^4/\{\pm 1\}$. Again, σ lifts to an anti-G_2 involution of (M, φ) and the corresponding coassociative submanifolds in M are now, respectively, two copies of the usual Kummer resolution of $T^4/\{\pm 1\}$, diffeomorphic to a $K3$ surface.

If we pull apart M in the x_4 direction then σ again defines anti-G_2 involutions of the resulting irreducible EAC G_2-manifolds $M_\pm$. In each half the fixed point set has two 4-dimensional components, which are resolutions of $(T^3 \times \mathbb{R})/\{\pm 1\}$. These are asymptotically cylindrical coassociative submanifolds of M. Topologically, they are 'halves' of a $K3$ surface: attaching two copies by identifying their boundaries T^3 'at infinity' via an orientation-reversing diffeomorphism one obtains a closed 4-manifold diffeomorphic to $K3$.

Compact coassociative submanifolds have a well-behaved deformation theory. For any coassociative submanifold $C \subset M$, the normal bundle of C is isomorphic to the bundle $\Lambda^2_+ T^*C$ of self-dual 2-forms. McLean [24, Theorem 4.5] shows that the nearby coassociative deformations of a closed coassociative submanifold C is a smooth manifold of dimension $b^2_+(C)$ (see also [17, Theorem 2.5]).

Joyce and Salur prove an EAC analogue of McLean's result. Denote by $H^2_0(C, \mathbb{R}) \subseteq H^2(C, \mathbb{R})$ the subspace of cohomology classes represented by compactly supported 2-forms. Equivalently, $H^2_0(C, \mathbb{R})$ is the image of the natural 'inclusion homomorphism' of the cohomology with with compact support $H^2_c(M, \mathbb{R}) \to H^2(M, \mathbb{R})$.

Proposition 5.6 **([17])** *Let M^7 be an EAC G_2-manifold with cross-section X^6 and $C \subset M$ an EAC coassociative submanifold asymptotic to $\mathbb{R}_+ \times L$, for a 3-dimensional submanifold $L \subset X$. Then the space of nearby coassociative deformations of C asymptotic to $\mathbb{R}_+ \times L$ is a smooth manifold of finite dimension $b^2_{0,+}(C)$, which is the dimension of a maximal positive subspace for the intersection form on $H^2_0(C, \mathbb{R})$.*

For $T^3 \times \mathbb{R}$ or the half-$K3$-surface this quantity vanishes. Indeed, $H^i_0(T^3 \times \mathbb{R}) = 0$ for all i. The half-K3-surface can be regarded as a quotient of $T^3 \times \mathbb{R}$ blown up at some $\mathbb{C}^2/\{\pm 1\}$ singularities, so the only contribution to H^2_0 comes from the exceptional $\mathbb{C}P^1$ divisors, which have negative self-intersection. Thus the coassociative submanifolds in example 5.4 and 5.5 are rigid if their 'boundary L at infinity' is kept fixed.

6 Pulling apart G_2-manifolds

In §4 and §5 we constructed pairs of asymptotically cylindrical G_2-manifolds $(M_\pm, \varphi_{s,\pm})$. They were obtained from a decomposition (18) of compact G_2-manifolds (M, φ_s) taken from [16] which are resolutions of T^7/Γ. In this section we show how our construction of $(M_\pm, \varphi_{s,\pm})$ can be regarded as an inverse operation to a gluing construction in [18] that forms compact G_2-manifolds from a 'matching' pair of EAC G_2-manifolds. It is easy to see that joining the manifolds $M_\pm$ at their cylindrical ends yields a manifold diffeomorphic to M, but we shall prove a stronger statement that there is a continuous path of torsion-free G_2-structures connecting φ_s to the glued G_2-structures. In other words, pulling the compact G_2-manifold (M, φ_s) apart into EAC halves and gluing them back together again gives a G_2-structure that is deformation-equivalent to the original φ_s.

We begin by describing the gluing construction of compact G_2-manifolds from a matching pair of EAC G_2-manifolds. Let $(M_\pm, \varphi_\pm)$ be some EAC G_2-manifolds with cross-sections $X_\pm$. The restrictions of the EAC torsion-free G_2-structures $\varphi_\pm$ to the cylindrical ends $[0, \infty) \times X_\pm \subset M_\pm$ have the asymptotic form

$$\varphi_\pm|_{[0,\infty)\times X_\pm} = \varphi_{\pm,cyl} + d\eta_\pm,$$

where each

$$\varphi_{\pm,cyl} = \Omega_\pm + dt \wedge \omega_\pm$$

is a product cylindrical G_2-structure induced by a Calabi–Yau structure on X and each 2-form $\eta_\pm$ decays with all derivatives at an exponential rate as $t \to \infty$

$$\|\nabla^r \eta_\pm\|_{\{t\}\times X_\pm} < C_r e^{\lambda t}.$$

We say that $\varphi_\pm$ is a matching pair of EAC G_2-structures if there is an orientation-reversing diffeomorphism $F : X_+ \to X_-$ satisfying

$$F^*(\Omega_-) = \Omega_+, \qquad F^*(\omega_-) = -\omega_+. \tag{33}$$

For each sufficiently large $L > 0$, the 3-form

$$\tilde{\varphi}_\pm(L) = \varphi_\pm - \mathrm{d}(\alpha(t-L)\eta_\pm)$$

induces a well-defined G_2-structure. Here, we used $\alpha(t)$ to denote a smooth cut-off function, $0 \le \alpha(t) \le 1, \alpha(t) = 0$ for $t \le 0$ and $\alpha(t) = 1$ for $t \ge 1$. For $L > 1$, denote $M_\pm(L) = M_\pm \setminus ((L+1,\infty) \times X_\pm)$. A *generalized connected sum* of $M_\pm$ may be defined as

$$M(L) = M_+(L) \cup_F M_-(L)$$

identifying the collar neighbourhoods of the boundaries of $M(L)$ via $(t,x) \in [L, L+1] \times X_+ \to (2L+1-t, F(x)) \in [L, L+1] \times X_-$. The 3-forms $\tilde{\varphi}_\pm(L)$ agree on the 'gluing region' $[L, L+1] \times X_\pm$ and together define a closed G_2 3-form $\varphi(L)$ on $M(L)$. It is not difficult to check that the co-differential of this form, relative to the metric $g(\varphi(L))$ satisfies

$$\|\mathrm{d}{*_{\varphi(L)}}\varphi(L)\|_{L^p_k(M(L))} < C_{p,k}e^{\lambda L},$$

but need not vanish as the derivatives of the cut-off function introduce 'error terms'. Thus the G_2-structure $\varphi(L)$ has 'small' torsion on M, but need not be torsion-free.

For each L, the $M(L)$ is diffeomorphic, as a smooth manifold, to a fixed compact 7-manifold M, but the metrics $g(\varphi(L))$ have diameter asymptotic to $2L$, as $L \to \infty$.

Theorem 6.1 **([18, §5])** *Let a compact 7-manifold $M(L)$ and a G_2 3-form $\varphi(L) \in \Omega^3_+(M(L))$ be a generalized connected sum of a pair of EAC G_2-manifolds $(M_\pm, \varphi_\pm)$ with G_2-structures satisfying (33).*

Then there exists an $L_0 > 1$ and for each $L > L_0$ a 2-form η_L on M, so that the G_2-structure on M induced by $\varphi(L) + \mathrm{d}\eta_L$ is torsion-free. Furthermore, the form η_L may be chosen to satisfy $\|\eta_L\|_{L^p_k(M(L))} < C_{p,k}e^{-\delta L}$, for some positive constants $C_{p,k}$, δ independent of L.

The above is a variant of the 'gluing theorem' for solutions of nonlinear elliptic PDEs on generalized connected sums [20], adapted to (8). The proof uses a lower bound for the linearisation of (8) on M with carefully chosen weighted Sobolev norms and an application of the inverse mapping theorem in Banach spaces.

Definition 6.2 For a matching pair of torsion-free G_2-structures and $L > L_0$, let

$$\Phi(\varphi_+, \varphi_-, L) = \varphi(L) + \mathrm{d}\eta_L$$

be the G_2-structure on M defined in Theorem 6.1.

The family of G_2-metrics induced by $\Phi(\varphi_+, \varphi_-, L)$ may be thought of as stretching the neck of a generalized connected sum, defined by the decomposition of compact 7-manifold M along a hypersurface X. The pair of EAC G_2-manifolds $(M_\pm, \varphi_\pm)$ may be identified as a boundary point of the moduli space for G_2-structures on M corresponding to the limit of the path $\Phi(\varphi_+, \varphi_-, L)$, as $L \to \infty$ (see [30, §5] for more precise details).

Now we return to consider the pairs $(M_\pm, \varphi_\pm)$ of EAC G_2-manifolds constructed in §4.2 and §5. It follows from the decomposition (18) that $\varphi_\pm$ is a matching pair of EAC

G_2-structures in the sense of (33). The generalized connected sum of $M_\pm$ is clearly diffeomorphic to M in the left-hand side of (18a), so by Theorem 6.1 we obtain a family of torsion-free G_2-structures $\Phi(\varphi_+, \varphi_-, L) \in \Omega^3_+(M)$. On the other hand, in this case we can construct on M another path $\phi(L)$ of torsion-free G_2-structures, with the same asymptotic properties as $L \to \infty$, using the G_2-structure $\tilde{\varphi}_s$ defined in Proposition 4.3. Recall from (23) that $\tilde{\varphi}_s$ restricts to a product torsion-free G_2-structure on $N \subset M$, which is a finite cylindrical domain $N \cong (-\varepsilon, \varepsilon) \times X$. For each $L \geq 0$, using a diffeomorphism f_L between intervals in $\mathbb{R}$

$$t \in (-\varepsilon, \varepsilon) \to t_L = f_L(t) \in (-\varepsilon - L, \varepsilon + L), \tag{34}$$

we define a new G_2-structure $\tilde{\varphi}_s(L)$ on M so that $\tilde{\varphi}_s(L)|_N = \Omega + \mathrm{d}t_L \wedge \omega$ and $\tilde{\varphi}_s(L)$ coincides with $\tilde{\varphi}_s$ away from N. It is easy to see that the resulting family of metrics $g(\tilde{\varphi}_s(L))$ may be informally described as 'stretching' the neck region N in the Riemannian manifold $(M, g(\tilde{\varphi}_s))$. The change (34) of the cylindrical coordinate on N amounts to the diameter of $(M, g(\tilde{\varphi}_s))$ being increased by $2L$.

For each $L \geq 0$, the G_2-structure $\tilde{\varphi}_s(L)$ satisfies the same estimates on the torsion as $\tilde{\varphi}_s$ (this follows from the argument of §4.3). Therefore, the same method as in the case of $\tilde{\varphi}_s$ applies to show that $\tilde{\varphi}_s(L)$ can be perturbed to a torsion-free G_2-structure $\phi_s(L) = \tilde{\varphi}_s(L) +$ (exact form) [16, §11.6 and §12.2].

There is no obvious reason for the G_2-structures $\phi(L)$ to be isomorphic to $\Phi(\varphi_+, \varphi_-, L)$, but we show that the two families are 'asymptotic' to each other in the following sense.

Theorem 6.3 *Let M^7 be a compact manifold with a closed G_2-structure $\tilde{\varphi}_s$, such that the assertions i–iii of Proposition 4.3 hold, for each sufficiently small s. Assume that s is sufficiently small and define the path $\phi_s(L)$ as above. Let $\tilde{\varphi}_{s,\pm}$ be EAC G_2-structures on the manifolds $M_\pm$ with cylindrical ends defined after Proposition 4.3 and $\varphi_{s,\pm}$ the torsion-free perturbations of $\tilde{\varphi}_{s,\pm}$ within their cohomology class defined by Theorem 3.1.*

Then for every sufficiently large L, there are some matching deformations $\varphi'_{s,\pm} = \varphi'_{s,\pm}(L)$ of $\varphi_{s,\pm}$, satisfying $\|\varphi'_{s,\pm} - \varphi_{s,\pm}\| < C_1 L^{-1}$, and a real ε_L, satisfying $|\varepsilon_L| < C_2$, with $C_1, C_2 > 0$ independent of L, so that the G_2-structure $\phi_s(L)$ is isomorphic to $\Phi(\varphi'_{s,+}, \varphi'_{s,-}, L + \varepsilon_L)$.

When (M, φ_s) is an example of compact G_2-manifold discussed in §4.1 and §5 we shall deduce from the proof of Theorem 6.3 a further result which will be used in §7.

Theorem 6.4 *Let (M, φ_s) and $(M_\pm, \varphi_{s,\pm})$ be the G_2-manifolds defined in §4.1 and §4.2 or in §5. There is, for every sufficiently small $s > 0$, a continuous path of torsion-free G_2-structures on M connecting φ_s and $\Phi(\varphi_{s,+}, \varphi_{s,-}, L)$, whenever L is sufficiently large in the sense of Theorem 6.1.*

As we shall see, a closed G_2-structure $\tilde{\varphi}_s$ will be required once again in the argument of Theorem 6.4 and the clause iv of Proposition 4.3 will be important.

In order to prove Theorems 6.3 and 6.4 we need to recall some results concerning the moduli of torsion-free G_2-structures.

6.1 The moduli space of torsion-free G_2-structures

Let M be a compact G_2-manifold, $\mathcal{X}$ the space of torsion-free G_2-structures on M and $\mathcal{D}$ the group of diffeomorphisms of M isotopic to the identity. The group $\mathcal{D}$ acts on $\mathcal{X}$, and the quotient $\mathcal{M} = \mathcal{X}/\mathcal{D}$ is the *moduli space of torsion-free G_2-structures*. Since torsion-free G_2-structures are represented by closed forms there is a well-defined projection $\mathcal{M} \to H^3(M, \mathbb{R})$ via the de Rham cohomology.

One way to extend the definition of $\mathcal{M}$ to an EAC G_2-manifold M, with G_2-structure $\check{\varphi}$ say, is to set $\mathcal{X}$ to be the space of EAC torsion-free G_2-structures on M exponentially asymptotic to $\check{\varphi}$ along the cylindrical end. The group $\mathcal{D}$ is now taken to be the group of diffeomorphisms of M isotopic to the identity and on the cylindrical end exponentially asymptotic to the identity map. Then $\mathcal{M} = \mathcal{X}/\mathcal{D}$ is the *moduli space of torsion-free G_2-structures asymptotic to a fixed cylindrical G_2-structure*. It can be shown that for every φ exponentially asymptotic to $\check{\varphi}$ the de Rham cohomology class $[\varphi - \check{\varphi}]$ can be represented by a compactly supported closed 3-form on M. (More generally, one can define a moduli space for G_2-structures on M whose asymptotic model is allowed to vary, see [29] for the details.)

Theorem 6.5 (i) *Let M be a compact 7-manifold admitting torsion-free G_2-structures. Then the moduli space $\mathcal{M}$ of torsion-free G_2-structures on M is a smooth manifold, and the map*

$$\pi : \varphi\mathcal{D} \in \mathcal{M} \to [\varphi] \in H^3(M, \mathbb{R})$$

is a local diffeomorphism.

(ii) *Let $(M, \check{\varphi})$ be an EAC G_2-manifold. Then the moduli space $\mathcal{M}$ of torsion-free G_2-structures on M asymptotic to $\check{\varphi}$ is a smooth manifold, and the map to affine subspace*

$$\pi : \varphi\mathcal{D} \in \mathcal{M} \to [\varphi] \in [\check{\varphi}] + H_0^3(M, \mathbb{R}) \subset H^3(M, \mathbb{R})$$

is a local diffeomorphism. Here $H_0^3(M, \mathbb{R}) \subset H^3(M, \mathbb{R})$ denotes the subspace of cohomology classes represented by compactly supported closed 3-forms.

The clause (i) is proved in [16, Theorem 10.4.4] and (ii) in [29, Theorem 3.2 and Corollary 3.7].

The torsion-free G_2-structures discussed in this article are obtained as a perturbation of some closed stable 3-forms $\tilde{\varphi}_s$ by adding a 'small' exact form. In particular, a G_2-structure induced by $\tilde{\varphi}_s$ necessarily has small torsion. Our next result shows that two closed stable 3-forms, which are in the same de Rham cohomology class and have small torsion, will define the same point in $\mathcal{M}$ whenever their difference is also small.

Proposition 6.6 *Suppose that a 7-manifold M is either compact or has a cylindrical end. For $i = 0, 1$ let $\tilde{\varphi}_i$ be a closed stable 3-form defining a G_2-structure and a metric $\tilde{g}_i = g(\tilde{\varphi}_i)$ and Hodge star $*_i$ on M. If M has a cylindrical end, suppose further that $\tilde{\varphi}_i$ are EAC G_2-structures and that $\tilde{\varphi}_0 - \tilde{\varphi}_1$ decays to zero with all derivatives along the end.*

Let ψ_i be smooth 3-forms such that $d{*_i}\psi_i = d{*_i}\tilde{\varphi}_i$ and suppose that each $(\tilde{\varphi}_i, \psi_i)$ satisfies the hypotheses a–c in Theorem 3.1, relative to the metric $\tilde{g}_i$. Let φ_i be the torsion-free G_2-structures defined by Theorem 3.1 using $(\tilde{\varphi}_i, \psi_i)$.

Finally suppose that the 3-form $\tilde{\varphi}_0 - \tilde{\varphi}_1$ is exact and

$$\|\tilde{\varphi}_0 - \tilde{\varphi}_1\|_{L^2} < \lambda s^4, \quad \|\tilde{\varphi}_0 - \tilde{\varphi}_1\|_{C^0} < \lambda s^{1/2}, \quad \|\tilde{\varphi}_0 - \tilde{\varphi}_1\|_{L^{14}_1} < \lambda,$$

where the norms are defined using the metric $\tilde{g}_0$.

Then for each sufficiently small $s > 0$, the torsion-free G_2-structures φ_i are isomorphic and define the same point in $\mathcal{M}$.

Recall from Remark 3.2 that in the case when M is compact the statement of Theorem 3.1 recovers [16, Theorem 11.6.1].

Proof Let $\tilde{\varphi}_1 - \tilde{\varphi}_0 = \mathrm{d}\eta$, $\eta \in \Omega^2(M)$ and set $\tilde{\varphi}_u = \tilde{\varphi}_0 + u\,\mathrm{d}\eta$, for $u \in [0, 1]$. If $0 < s < s_0$ for a sufficiently small $s_0 > 0$ independent of the choice of $\tilde{\varphi}_j$ then $\tilde{\varphi}_u$ induces a well-defined path of G_2-structures on M. Define a path of 3-forms

$$\psi'_u = \tilde{\varphi}_u + *_u\left((1-u) *_0 (\psi_0 - \tilde{\varphi}_0) + u *_1 (\psi_1 - \tilde{\varphi}_1)\right),$$

where $*_u$ is the Hodge star of the metric defined by $\tilde{\varphi}_u$. Then $\psi'_0 = \phi_0$ and $\psi'_1 = \psi_1$ and $\mathrm{d}*_u\psi_u = \mathrm{d}*_u\varphi_u$, for each $u \in [0, 1]$.

By our hypothesis, $(\tilde{\varphi}_u, \psi'_u)$ satisfy for $u = 0$ and $u = 1$, all the estimates required in Theorem 3.1. The left-hand sides of these estimates depend continuously on u. Therefore, by choosing a smaller $s_0 > 0$ if necessary we obtain that the estimates on $(\tilde{\varphi}_u, \phi'_u)$ are satisfied for every $u \in [0, 1]$ and Theorem 3.1 produces a path of torsion-free G_2-structures φ_u, connecting the given φ_i, $i = 0, 1$. A standard argument verifies that φ_u is continuous in u.

By the construction, the de Rham cohomology class of $\tilde{\varphi}_u$ is independent of $u \in [0, 1]$. By Theorem 6.5, the path in the moduli space $\mathcal{M}$ defined by φ_u must be locally constant. It follows that φ_0 and φ_1 define the same point in $\mathcal{M}$ and the respective G_2-structures are isomorphic. □

6.2 Deformations and gluing. Proof of Theorems 6.3 and 6.4

We require one more ingredient for proving Theorem 6.3. The second author [30] shows that any small torsion-free deformation of $\Phi(\varphi_+, \varphi_-, L)$ is, up to an isomorphism, obtainable by gluing some small deformations of $\varphi_\pm$. More important to the present discussion is the following local description from the proof of that result.

There are pre-moduli spaces $\mathcal{R}_\pm$ of EAC torsion-free G_2-structures near $\varphi_\pm$, i.e. a submanifold of the space of EAC G_2-structures, which is homeomorphic to a neighbourhood of $\varphi_\pm$ in the moduli space of EAC G_2-structures on $M_\pm$. The subspace $\mathcal{R}_y \subseteq \mathcal{R}_+ \times \mathcal{R}_-$ of matching pairs is a submanifold. The connected-sum construction gives a well-defined map Φ from $\mathcal{R}_y \times (L_1, \infty)$ (for $L_1 > 0$ sufficiently large) to the moduli space $\mathcal{M}$ of torsion-free G_2-structures on M. It is best studied in terms of the composition with the local diffeomorphism $\mathcal{M} \to H^3(M)$,

$$\Phi_H : \mathcal{R}_y \times (L_1, \infty) \to H^3(M).$$

Topologically $M = M_+ \cup M_-$. Consider the Mayer–Vietoris sequence

$$\cdots \longrightarrow H^{m-1}(X) \xrightarrow{\delta} H^m(M) \xrightarrow{i_+^* \oplus i_-^*} H^m(M_+) \oplus H^m(M_-) \xrightarrow{j_+^* - j_-^*} H^m(X) \longrightarrow \cdots, \tag{35}$$

where $j_\pm : X \to M_\pm$ is the inclusion of the cross-section and $i_\pm : M_\pm \to M$ is the inclusion in the union (these maps are naturally defined up to isotopy).

The cohomology class of the glued G_2-structure satisfies $i_\pm^* \Phi_H(\varphi_+, \varphi_-, L) = [\varphi_\pm]$. Also $\frac{\partial}{\partial L}\Phi_H(\varphi_+, \varphi_-, L) = 2\delta([\omega])$, where ω denotes the Kähler form of the Calabi–Yau structure on X defined by the common asymptotic limit of $\varphi_\pm$. Thus, if we let $\mathcal{R}'_y$ be the submanifold

$$\mathcal{R}'_y = \{(\psi_+, \psi_-) \in \mathcal{R}_y : i_\pm^* \psi_\pm = i_\pm^* \varphi_\pm\},$$

then the restriction of Φ_H to $\mathcal{R}'_y \times (L_1, \infty)$ takes values in the affine subspace $K = [\varphi] + \delta(H^2(X))$, and can be written as

$$\Phi_H : \mathcal{R}'_y \times (L_1, \infty) \to K, \quad (\varphi'_+, \varphi'_-, L) \mapsto F(\varphi'_+, \varphi'_-) + 2L\delta([\omega']), \tag{36}$$

where ω' is the Kähler form of the common boundary value of $(\varphi'_+, \varphi'_-) \in \mathcal{R}_y$ and $F : \mathcal{R}'_y \to K$ is smooth. It is explained in [30, §5] that the image of $\mathcal{R}'_y \to \delta(H^2(X))$, $(\varphi'_+, \varphi'_-) \mapsto \delta([\omega'])$ is a submanifold transverse to the radial direction, so that (36) is diffeomorphism onto its image, which contains an open affine cone in K (if L_1 is large enough).

Proof of Theorem 6.3 Recall that the torsion-free G_2-structures $\phi_s(L)$ are obtained by perturbing the closed G_2-structures $\tilde{\varphi}_s(L)$ with small torsion, which are in turn defined by stretching the cylindrical neck $X \times I$ of $\tilde{\varphi}_s$ by a length $2L$. Their cohomology classes are $[\phi_s(L)] = [\tilde{\varphi}_s(L)] = [\varphi_s] + 2L\delta([\omega])$, where ω is the Kähler form on X, so the image of the path $\phi_s(L)$ in $H^3(M)$ is an affine line with slope $2\delta([\omega])$.

We also defined torsion-free EAC G_2-structures $\varphi_{s,\pm}$ on $M_\pm$ by perturbing the G_2-structures $\tilde{\varphi}_{s,\pm}$ obtained from $\tilde{\varphi}_s$ via decomposition (18) of M. The gluing Theorem 6.1 applied to $\varphi_{s,+}$ and $\varphi_{s,-}$ defines a path $\Phi(\varphi_{s,+}, \varphi_{s,-}, L)$ of torsion-free G_2-structures on M. The restrictions satisfy $i^*_\pm[\Phi(\varphi_{s,+}, \varphi_{s,-}, L)] = i^*_\pm[\varphi_s]$, so the image of the path in $H^3(M)$ lies in the affine space $K = [\varphi_s] + \delta(H^2(X))$. This is an affine line with the same slope $2\delta([\omega])$.

Our aim is to show that for every large L there is a small deformation $(\varphi'_{s,+}(L), \varphi'_{s,-}(L))$ of $(\varphi_{s,+}, \varphi_{s,-})$ and $L + \varepsilon_L$ at a bounded distance from L, so that $\phi_s(L)$ is isomorphic to $\Phi(\varphi'_{s,+}(L), \varphi'_{s,-}(L), L + \varepsilon_L)$. We prove this by appealing to to Proposition 6.6, showing first that we can find a small deformation such that the glued $\Phi(\varphi'_{s,+}(L), \varphi'_{s,-}(L), L + \varepsilon_L)$ has the same cohomology class as $\phi_s(L)$, and then checking that the gluing is close to $\tilde{\varphi}_s(L)$ in the relevant norms.

The difference between the cohomology classes $[\phi_s(L)]$ and $[\Phi(\varphi_{s,+}, \varphi_{s,-}, L)]$ is independent of L. Therefore, for each sufficiently large L, there is an $L + \varepsilon_L$ of bounded distance to L and a matching pair $(\varphi'_{s,+}(L), \varphi'_{s,-}(L)) \in \mathcal{R}'_y$, such that $\phi_s(L)$ is cohomologous to the glued G_2-structure $\Phi(\varphi'_{s,+}(L), \varphi'_{s,-}(L), L + \varepsilon_L)$. In fact, because the RHS of (36) is dominated by the $2L\delta([\omega])$ term for large L, the distance between $(\varphi'_{s,+}(L), \varphi'_{s,-}(L))$ and $(\varphi_{s,+}, \varphi_{s,-})$ is of order $1/L$, as $L \to \infty$, measured in the C^1 norm (since $\mathcal{R}_y$ has finite dimension all sensible norms are Lipschitz equivalent). Hence the difference between $\Phi(\varphi_{s,+}, \varphi_{s,-}, L)$ and $\Phi(\varphi'_{s,+}(L), \varphi'_{s,-}(L), L)$ is of order $1/L$ in C^0 norm. As the volume growth is of order L it follows also that the difference is of order $L^{-1/2}$ in L^2-norm, and order $L^{-13/14}$ in L^{14}_1-norm.

Now $\phi_s(L)$ and $\Phi(\varphi'_{s,+}(L), \varphi'_{s,-}(L), L + \varepsilon_L(L))$ are both torsion-free perturbations of $\tilde{\varphi}_s(L)$ within its cohomology class, so we can try and use Proposition 6.6 to show that they are diffeomorphic. For large L, the difference between $\Phi(\varphi'_{s,+}(L), \varphi'_{s,-}(L), L + \varepsilon_L)$ and $\tilde{\varphi}_s(L)$ is dominated by the difference between $\tilde{\varphi}_{s,\pm}$ and $\varphi_{s,\pm}$, which is estimated in terms of s in (7). Therefore if s is sufficiently small then for all sufficiently large L the estimates required to apply Proposition 6.6 are satisfied, and

$$\Phi\left(\varphi'_{s,+}(L), \varphi'_{s,-}(L), L + \varepsilon_L\right) \cong \phi_s(L).$$

This completes the proof of Theorem 6.3. □

Proof of Theorem 6.4 We know from the argument of Theorem 6.3 and the preceding remarks that the pair $\varphi'_{s,+}(L), \varphi'_{s,-}(L)$, for each $L > L_1$, is contained in the pre-moduli space $\mathcal{R}'_y$ which we may assume connected. As discussed earlier in this subsection, the map $\Phi(\varphi_+, \varphi_-, L)$ induces a continuous function from $\mathcal{R}'_y \times (L_1, \infty)$ to the G_2 moduli space for M. We find that, for $L > L_1$, the torsion-free G_2-structure $\Phi(\varphi_{s,+}, \varphi_{s,-}, L)$ is a deformation of $\Phi(\varphi'_{s,+}(L), \varphi'_{s,-}(L), L)$.

By Theorem 6.3, we may further replace $\Phi(\varphi'_{s,+}(L), \varphi'_{s,-}(L), L)$, with the torsion-free G_2-structure $\phi_s(L - \varepsilon_L)$, assuming sufficiently large L. We saw above that the cohomology class $[\phi_s(L)]$ depends continuously on L and it is not difficult to check, using Theorem 6.5(i) that the forms $\phi_s(L)$ define a continuous path in the G_2 moduli space for M. Thus, we may further replace $\phi_s(L - \varepsilon_L)$ by the torsion-free G_2-structure $\phi_s(0)$. We claim that the latter is isomorphic to the G_2-structure φ_s.

By definition just before Theorem 6.3, $\phi_s(0)$ is a perturbation of G_2 3-form $\tilde{\varphi}_s(0) = \tilde{\varphi}_s$ given by Proposition 4.3 and $\phi_s(0) - \tilde{\varphi}_s$ is exact. On the other hand, recall from (22) that φ_s is a perturbation of G_2 3-form φ_s^{init} by an exact form. The latter two exact forms may be assumed 'small' in the sense of (25) by choosing a small s. Furthermore, $\tilde{\varphi}_s - \varphi_s^{\text{init}}$ is exact by Proposition 4.3 (iv) and the argument of §4.3 ((25) and (29)) again shows that $\tilde{\varphi}_s - \varphi_s^{\text{init}}$ is small. Proposition 6.6 now ensures that φ_s and $\phi_s(0)$ are diffeomorphic, for every sufficiently small s. □

7 Connected sums of EAC G_2-manifolds

We now revisit the orbifold T^7/Γ discussed in §4.1 but this time we shall split T^7/Γ into two connected components, $\hat{M}_{0,\pm}$ say, along a different orbifold hypersurface $\hat{X}_0$ which is the image of the 6-torus $\hat{T}^6 = \{x_5 \equiv 1/8 \mod \mathbb{Z}\} \subset T^7$. (As before, x_k modulo $\mathbb{Z}$ denote the standard coordinates on T^7 induced from $\mathbb{R}^7$.) As remarked in §5.2, this choice does not produce an irreducible EAC G_2-manifold but is interesting for its relation to the compact G_2-manifolds and EAC Calabi–Yau 3-folds constructed in [18,19].

More precisely, we shall show that the corresponding EAC G_2-manifolds $\hat{M}_\pm$ are of the form $S^1 \times W$, where W is a known complex 3-fold obtained by the algebraic methods of [19] with an EAC Calabi–Yau structure coming from a result in [18]. Application of Theorem 6.4 then shows that the G_2-structure on M constructed in [16] by resolution of singularities of T^7/Γ is a deformation of the G_2-structure obtainable from [18] by regarding M as generalized connected sum of EAC G_2-manifolds $\hat{M}_\pm$.

7.1 A G_2-manifold with holonomy $SU(3)$.

Recall that the singular locus of T^7/Γ consists of 12 disjoint copies of T^3, the union of 3 subsets of 4 copies of T^3 corresponding to the fixed point set of, respectively, the involutions α, β, γ defined in (19). Each of the 4 copies of T^3 in the singular locus of T^7/Γ, arising from the fixed points of β, intersects $\hat{X}_0$ in a 2-torus. The other 8 copies of T^3 in the singular locus do not meet $\hat{X}_0$. Let $\hat{M}_{0,+}$ denote the connected component of $(T^7/\Gamma)\backslash\hat{X}_0$ containing the image of $\{x_5 = 0\}$. Then $\hat{M}_{0,+}$ contains all the 3-tori coming from the fixed point set of α, whereas those coming from γ are in the image of $\{x_5 = \frac{1}{4}\}$ and contained in $\hat{M}_{0,-}$.

It is easy to see that $\hat{X}_0 = \hat{T}^6/\langle\beta\rangle \cong (T^4/\pm 1) \times T^2$ and that the orbifolds $\hat{M}_{0,\pm}$ are diffeomorphic, via the involution of T^7/Γ induced by the map

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto \left(x_1, x_2, x_3, x_4, x_5 + \tfrac{1}{4}, x_6, x_7\right). \tag{37}$$

The above is quite similar to the discussion in §4.1 and §4.2. In particular, it can be shown that the map (37) induces an isometry of the EAC G_2-manifolds $\hat{M}_\pm$ constructed from $\hat{M}_{0,\pm}$ (compare Remark 4.2).

Notice also that the pre-image of $\hat{M}_{0,+}$ in $T^7/\langle\alpha, \beta\rangle$ consists of two connected components and γ maps one of these diffeomorphically onto the other. In light of this, we can identify $\hat{M}_{0,+} \cong \left(\{|x_5| < \frac{1}{8}\} \times T^6\right)/\langle\alpha,\beta\rangle$, and disregard γ when restricting attention to $\hat{M}_{0,+}$. Replacing the interval $[-\frac{1}{8},\frac{1}{8}]$ by a copy of $\mathbb{R}$, with the coordinate still denoted by x_5, is equivalent to attaching a cylindrical end to $\hat{M}_{0,+}$. We have a diffeomorphism

$$\hat{M}_{0,+} \cong \left(\left(\mathbb{R}_{x_5} \times T^5\right)/\langle\alpha,\beta\rangle\right) \times S^1_{x_1}. \tag{38}$$

We see at once that the resolution of singularities of $\hat{M}_{0,+}$ amounts to resolving a 6-dimensional orbifold. We shall relate the latter resolution to blowing up *complex orbifolds*. Identify $\mathbb{R}^7 \cong \mathbb{R} \times \mathbb{C}^3$ using a real coordinate and three complex coordinates,

$$\theta = x_1, \quad z_1 = x_5 + ix_4, \quad z_2 = x_2 + ix_3, \quad z_3 = x_6 + ix_7. \tag{39}$$

In these coordinates, the involutions α, β are *holomorphic in z_k*

$$\alpha(\theta, z_1, z_2, z_3) = (\theta, -z_1, z_2, -z_3), \quad \beta(\theta, z_1, z_2, z_3) = \left(\theta, z_1, -z_2, \tfrac{1}{2} - z_3\right).$$

For the first step of the procedure explained in §4.2, we consider $\mathbb{R}_{x_5} \times \hat{T}^6/\langle\beta\rangle$. It is well-known that the resolution of singularities of $T^4/\pm 1$ using Eguchi–Hanson spaces (see p.232) produces a Kummer K3 surface, Y say. The Kummer construction defines on Y a one-parameter family of torsion-free $SU(2)$-structures, i.e. Ricci-flat Kähler structures, with a limit corresponding to the flat hyper-Kähler structure on $T^4/\pm 1$ induced from the Euclidean $\mathbb{R}^4$ [21]. Cf. (20); the parameter, still denoted by $s > 0$, is proportional to the diameter of the exceptional divisors on Y. We thus obtain $S^1_{x_1} \times S^1_{x_4} \times \mathbb{R}_{x_5} \times Y$ with a product torsion-free G_2-structure induced by a Kummer hyper-Kähler structure on Y (cf. (3)).

The Kummer construction can be performed α-equivariantly, so that α induces an involution on Y, say ρ_α, which preserves the $SU(2)$-structure. The quotient (38) takes the form $S^1_\theta \times Z_0$, where $Z_0 = (\mathbb{R}_{x_5} \times S^1_{x_4} \times Y)/\langle\alpha\rangle$ is a well-defined complex orbifold. Noting that $\mathbb{R}_{x_5} \times S^1_{x_4}$ is biholomorphic to $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, we can extend α holomorphically to an involution of $Y \times \mathbb{C}P^1$ (identifying $\mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$). The restriction of α to $\mathbb{C}P^1$ may be written as $\zeta \mapsto 1/\zeta$, where $\zeta = \exp(2\pi i z_1)$; it maps 0 and ∞ to each other and fixes precisely two points ± 1, both in the image of $\mathbb{R}_{x_5} \times S^1_{x_4}$ (the circle S^1_θ does not yet concern us). We can write Z_0 and its compactification Z as

$$Z_0 = \left(Y \times \mathbb{C}^\times\right)/\langle\alpha\rangle, \qquad Z = \left(Y \times \mathbb{C}P^1\right)/\langle\alpha\rangle \tag{40}$$

and it is not difficult to check that Z_0 is the complement in Z of an *anticanonical divisor D* biholomorphic to the K3 surface Y. The quotient of $\mathbb{C}P^1$ by the involution $\alpha|_{\mathbb{C}P^1}$ is biholomorphic to $\mathbb{C}P^1$, and we shall still denote the images of the fixed points by ± 1. It follows that the second projection on $Y \times \mathbb{C}P^1$ descends to a holomorphic map

$$p : Z_0 \to \mathbb{C}P^1 \tag{41}$$

with fibres biholomorphic to Y, except that the two fibres over ± 1 are biholomorphic to the quotients $Y/\langle\rho_\alpha\rangle$.

Denote by $\kappa_I, \kappa_J, \kappa_K$ a triple of closed 2-forms encoding the ρ_α-invariant $SU(2)$-structure on Y. Here, κ_I is the Kähler form of the Ricci-flat Kähler metric and $\kappa_J + i\kappa_K$ is a nowhere-vanishing holomorphic (2,0)-form, sometimes called a 'holomorphic symplectic form', which is unique up to a constant complex factor. We shall always require $\kappa_I^2 = \kappa_J^2 = \kappa_K^2$.

Observe that necessarily $\rho_\alpha^*(\kappa_J + i\kappa_K) = -\kappa_J - i\kappa_K$, so ρ_α acts by -1 on $H^{2,0}(Y)$. The latter makes Y into a K3 surface with 'non-symplectic involution' in the sense of [1], see also [28]. A general property of this class of K3 surfaces is that the sublattice of $H^2(Y, \mathbb{Z})$

fixed by ρ_α^* has signature $(1, t_-)$, so we must have $\rho_\alpha^*(\kappa_I) = \kappa_I$, because a Ricci-flat Kähler metric on Y is uniquely determined by the cohomology class of its Kähler form.

In order to compute some topological invariants later, we shall need some algebraic invariants of non-symplectic involutions, taken from [1]. One invariant is defined as the rank r of the sublattice L_ρ of the Picard lattice of Y fixed by ρ_α. It can be shown L_ρ has a natural embedding into its dual lattice L_ρ^* and the quotient has the form $L_\rho^*/L_\rho \cong (\mathbb{Z}_2)^a$. The integer a is another invariant that we shall need.

We determine the values of r, a in the present example from the classification of K3 surfaces with non-symplectic involution in [28], which includes a description of the fixed point set of ρ. Since the fixed point set of α has 4 components and the induced involution on $\mathbb{C}P^1$ fixes 2 points, we must have that the ρ_α fixes precisely two disjoint complex curves and each of these has genus 1. In this situation, there is only one possibility $r = 10$, $a = 8$ allowed by the classification of fixed point sets, [28, §4] or [1, §6.3].

A neighbourhood of each singular point in Z_0 is diffeomorphic to $(\mathbb{C}^2/\pm 1) \times \mathbb{C}$ and the singularities of $S^1_\theta \times Z_0$ may be resolved in an S^1_θ-invariant way by gluing in an Eguchi–Hanson space, similarly to several instances discussed in §4 and §5. The two-step procedure of §4 now produces a 7-manifold $\hat{M}_+ = S^1 \times W$ with an S^1-invariant, product G_2-structure having 'small' torsion. The torsion-free G_2-structure on $\hat{M}_+$ obtained by Theorem 3.1 is necessarily of product type (2) induced by an EAC Calabi–Yau structure on W.

We shall now show that after slightly changing some details of the method of §4 the same torsion-free G_2-structure on $\hat{M}_+$ can be recovered, up to an isomorphism, by constructing an EAC Calabi–Yau structure on W using the method of [18,19]. Recall from §2.1 that a Calabi–Yau structure on a 6-manifold may be determined by the complex structure (or, equivalently, the real part of a non-vanishing holomorphic 3-form) and the Kähler form.

The manifold W has a 'natural' complex structure defined by blowing up the singular locus of the complex orbifold Z_0. This is an instance of a general construction of quasiprojective complex 3-folds with trivial canonical bundle from $K3$ surfaces with non-symplectic involution.

Proposition 7.1 ([19, §4]) *Suppose that ρ is a non-symplectic involution of a K3 surface Y with invariants r, a and with a non-empty set of fixed points. Suppose that τ is a holomorphic involution of $\mathbb{C}P^1$ fixing precisely two points. Let $\overline{W}$ be the blowup of the singular locus of $(Y \times \mathbb{C}P^1)/(\rho, \tau)$ and let $D \subset \overline{W}$ be the pre-image of $Y \times \{p\}$, for some $p \in \mathbb{C}P^1$ with $\tau(p) \neq p$.*

Then both $\overline{W}$ and $W = \overline{W} \backslash D$ are non-singular and simply-connected and D is an anti-canonical divisor (biholomorphic to Y) in $\overline{W}$ with the normal bundle of D holomorphically trivial. Also, $b^2(\overline{W}) = 3 + 2r - a$ and $b^3(\overline{W}) = 44 - 2r - 2a$ and the pull-back map $\iota : H^2(\overline{W}, \mathbb{R}) \to H^2(D, \mathbb{R})$ induced by the embedding has rank r.

In particular, W admits nowhere-vanishing holomorphic (3, 0)-forms. An example of such form is obtained by starting on $\mathbb{R}_{x_5} \times S^1_{x_4} \times Y$ with the wedge product of $\mathrm{d}\zeta/\zeta = \mathrm{d}z_1 = \mathrm{d}x_5 + i\mathrm{d}x_4$ and the 'obvious' pull-back of a holomorphic symplectic form on Y. This (3, 0)-form is α-invariant and descends to Z_0. Denote its pull-back via the blowup $W \to Z_0$ by $\Omega' + i\Omega''$. This form is well-defined and may be alternatively obtained using the following resolution of singularities commutative diagram

$$\begin{tikzcd} \widetilde{W} \arrow[r] \arrow[d] & W \arrow[d] \\ Y \times \mathbb{C}^\times \arrow[r] & Z_0 \end{tikzcd}$$

where $\widetilde{W} \to Y \times \mathbb{C}^\times$ is the blowup of the fixed point set of α and $\widetilde{W} \to W$ is the quotient map for the involution of $\widetilde{W}$ induced by α.

We next construct a suitable Kähler form on W. The form $i\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \kappa_I$ defines an α-invariant Ricci-flat Kähler metric on $Y \times \mathbb{C}^\times$. Pulling back to W similarly to above, we obtain a 2-form ω_0 which is a well-defined Kähler form away from the exceptional divisor E on W. The exceptional divisors on Y arising from the Kummer construction induce divisors on Z_0, by taking a product with $\mathbb{C}^\times$ and dividing out by α. The proper transform of these defines a divisor, F say, on W. By choosing the parameter s in the Kummer construction sufficiently small we achieve that the curvature of ω_0 is small away from a tubular neighbourhood of F. Note that F does not meet E because the fixed point sets of α and β do not meet (see §4.1). We can choose disjoint tubular neighbourhoods of E and of F. Then on the intersection of a tubular neighbourhood V of E with the domain of ω_0 the metric ω_0 is close to flat whenever s is sufficiently small.

On the other hand, by taking a product of the Eguchi–Hanson metric (20) (with the same value of s) and the standard Kähler metric on an open domain in $\mathbb{C}$ we obtain a Kähler form ω_{EH} which is defined near E. With an appropriate choice of V, we can smoothly interpolate between the Kähler potentials of ω_0 and ω_{EH} to obtain a closed real $(1, 1)$-form ω_s, so that $\omega_s^3 \neq 0$ and ω_s is a well-defined Kähler form on W. An argument similar to that in §4.3 shows we can perform this construction of ω_s without introducing any more torsion of the corresponding G_2-structure than we would if ω_0 was actually flat. That is, the closed S^1-invariant G_2-structure $\varphi'_{W,s} = \Omega' + \mathrm{d}\theta \wedge \omega_s$ on the 7-manifold $\hat{M}_+$ has 'small' torsion in the sense of Proposition 4.3. Here we take $\psi = \psi_s = \Theta(\varphi'_{W,s}) - \mathrm{d}\theta \wedge \hat{\Omega}'' - \frac{1}{2}\omega_s \wedge \omega_s$. Then Theorem 3.1 produces an S^1-invariant torsion-free EAC G_2-structure $\varphi_{W,s} + \mathrm{d}\eta_s$ on $S^1 \times W$ determined by an EAC Calabi–Yau structure on W [cf. (3)]. Remark that the starting G_2-structure with small torsion and the choice of ψ may differ by a 'small amount' from those described in §4, but the resulting torsion-free G_2-structures are isomorphic by Proposition 6.6.

The latter EAC Calabi–Yau structure is asymptotic on the end $\mathbb{R}_{>0} \times S^1 \times Y$ of W to the product Calabi–Yau structure corresponding to the hyper-Kähler structure on Y and is obtained by the following 'non-compact version of the Calabi conjecture'.

Theorem 7.2 ([18, §3]) *Let $(\overline{W}, \overline{\omega})$ be a simply-connected complex 3-fold and suppose that a K3 surface $D \subset \overline{W}$ is an anticanonical divisor with the normal bundle of D holomorphically trivial and $W = \overline{W} \backslash D$ simply-connected. Let $\kappa_I, \kappa_J, \kappa_K$ be a triple of closed 2-forms inducing a Calabi–Yau structure on D, as above.*

Suppose that $\tilde{\omega}$ is a Kähler form on W which is asymptotically cylindrical in the following sense. There is a meromorphic function z on $\overline{W}$ vanishing to order one precisely on D. On the region $\{0 < |z| < \varepsilon\}$, for some $\varepsilon > 0$, ω has the asymptotic form

$$\kappa_I + \mathrm{d}t \wedge \mathrm{d}\theta + \mathrm{d}\tilde{\psi}$$

where $\exp(-t - i\theta) = z$ and a 1-form $\tilde{\psi}$ is exponentially decaying with all derivatives as $t \to \infty$.

Then W admits a asymptotically cylindrical Ricci-flat Kähler metric with Kähler form ω and a nowhere-vanishing holomorphic $(3, 0)$-form $\Omega' + i\Omega''$ such that

$$\omega = \tilde{\omega} + i\partial\bar{\partial}\psi_\infty$$

and Ω on the region $\{0 < |z| < \varepsilon\}$ has the asymptotic form

$$(\kappa_J + i\kappa_K) \wedge (\mathrm{d}t + i\mathrm{d}\theta) + \mathrm{d}\Psi_\infty,$$

where ψ_∞, Ψ_∞ are exponentially decaying with all derivatives as $t \to \infty$.

In the present example, we have $\tilde{\psi} = 0$ by construction. The function ψ_∞ is unique by [18, Proposition 3.11]. The uniqueness of $d\Psi_\infty$ follows from the uniqueness, up to a constant factor, of a non-vanishing holomorphic 3-form on W with a simple pole along $D = \overline{W}\backslash W$. Thus the G_2-structure obtained by application of Theorem 3.1 to the cylindrical end manifold $\hat{M}_+ = S^1 \times W$ with G_2-structure $\varphi'_{W,s}$ is unique and may be recovered from a blowup of complex orbifold and the Calabi–Yau analysis.

The Betti numbers for our example of $\hat{M}_+$ may be determined from those of $\overline{W}$ using Proposition 7.1 as we know that $r = 10$, $a = 8$. We obtain

$$b^3(\overline{W}) = 44 - 20 - 16 = 8 \quad \text{and} \quad b^2(\overline{W}) = 3 + 20 - 8 = 15,$$

and then, using the Mayer–Vietoris exact sequence for $\overline{W} = W \cup D$ similarly to [18, §8] and [19, §2],

$$b^2(W) = b^2(\overline{W}) - 1 = 14 \quad \text{and} \quad b^3(W) = b^3(\overline{W}) + 22 - b^2(W) + \dim \operatorname{Ker} \iota = 20,$$

using also the rank-nullity for ι. Therefore,

$$b^2\left(\hat{M}_+\right) = 14 \quad \text{and} \quad b^3\left(\hat{M}_+\right) = 34$$

by the Künneth formula.

The Betti numbers of W and $\hat{M}_+$ can also be recovered using the method explained at the end of §5.1.

7.2 The connected-sum construction of compact irreducible G_2-manifolds revisited

Everything that we said in the previous subsection about $\hat{M}_{0,+}$ and $\hat{M}_+$ can be repeated, with a change of notation, for $\hat{M}_{0,-}$ and $\hat{M}_-$. In particular $\hat{M}_- = W \times S^1$ with a product EAC G_2-structure. However, the roles of α and γ are swapped for $\hat{M}_{0,-}$ and the choice of identification $\mathbb{R}^7 = \mathbb{R}_\theta \times \mathbb{C}^3$ has to be revised too.

For $\hat{M}_{0,-}$, we set

$$\begin{aligned} \theta = x_4, \quad w_1 = x_5 + i x_1, \quad w_2 = x_2 + i x_6, \\ w_3 = x_7 + i x_3, \end{aligned} \tag{42}$$

so that

$$\begin{aligned} \beta(\theta, w_1, w_2, w_3) &= \left(\theta, w_1, \tfrac{i}{2} - w_2, -w_3\right), \\ \gamma(\theta, w_1, w_2, w_3) &= \left(\theta, \tfrac{1}{2} - w_1, w_2, \tfrac{1}{2} - w_3\right). \end{aligned}$$

We are interested in the image in $\hat{X}_0$ of the 4-torus corresponding to x_2, x_3, x_6, x_7. Writing

$$\begin{aligned} \kappa_1^0 &= dx_2 \wedge dx_3 + dx_6 \wedge dx_7, \quad \kappa_2^0 = dx_2 \wedge dx_6 + dx_7 \wedge dx_3, \\ \kappa_3^0 &= dx_2 \wedge dx_7 + dx_3 \wedge dx_6, \end{aligned}$$

we see that with respect to the complex structure on $\mathbb{R}^4_{x_2,x_3,x_6,x_7}$ defined by z_2, z_3 in (39) the Euclidean metric is Kähler with Kähler form κ_1^0 and a (2, 0)-form $\kappa_2^0 + i\kappa_3^0$. With respect to the complex structure of w_2, w_3 the Kähler form is κ_2^0 and a (2, 0)-form is $\kappa_1^0 - i\kappa_3^0$.

It follows by the symmetry of even permutations of x_2, x_3, x_6, x_7 and the equivariant properties of the Kummer construction that a similar statement holds for a triple of 2-forms, say $\kappa_I, \kappa_J, \kappa_K$ defining the hyper-Kähler structure on the resolution Y of $T^4_{x_2,x_3,x_6,x_7}/\langle\beta\rangle$.

In other words, the two Kummer K3 surfaces defined by using z- and w-coordinates correspond to choices of two anticommuting integrable complex structures say I and J coming from the hyper-Kähler structure on Y. The $\kappa_I, \kappa_J, \kappa_K$ are the Kähler forms corresponding, respectively, to $I, J, K = IJ$.

Recall from §2.1 that the product G_2-structure on a cylinder $\mathbb{R}_t \times S^1_{\theta_+} \times S^1_{\theta_-} \times D$ corresponding to a hyper-Kähler structure on D is induced by the 3-form

$$\varphi_D = \mathrm{d}\theta_+ \wedge \mathrm{d}\theta_- \wedge \mathrm{d}t + \mathrm{d}\theta_+ \wedge \kappa_I + \mathrm{d}\theta_- \wedge \kappa_J + \mathrm{d}t \wedge \kappa_K. \tag{43}$$

Here, $\theta_+ = x_1, \theta_- = x_4$, corresponding to (39),(42) and $x_5 = t$. The formula (43) is preserved by the transformation

$$\theta_+ \mapsto \theta_-, \quad \theta_- \mapsto \theta_+, \quad t \mapsto -t, \qquad \kappa_I \mapsto \kappa_J, \quad \kappa_J \mapsto \kappa_I, \quad \kappa_K \mapsto -\kappa_K.$$

Notice that the transformation of κ's corresponds precisely to changing the complex structure on Y from I to J (the latter is sometimes called a 'hyper-Kähler rotation'). It follows that we have an instance of a generalized connected sum of EAC G_2-manifolds discussed in the beginning of §6. In fact, more is true.

We can identify, in the present case, the isomorphism between the asymptotic models of EAC G_2 3-forms on the cylindrical ends of $\hat{M}_\pm \cong S^1_\pm \times W_\pm$. (Here $W_\pm$ are copies of W defined in the previous subsection and $\pm$ refers to using, respectively, the notation (39) or (42).) On the $D \cong Y$ factor the identification is an isometry with a change of complex structure, as discussed above. The $\pm x_5$ is the parameter along cylindrical end of $\hat{M}_\pm$, respectively. Finally, the S^1_+-factor with coordinate x_1 is identified with a circle around the K3 divisor in $\overline{W}_-$, whereas the S^1_- factor with coordinate x_4 corresponds to a circle around the K3 divisor in $\overline{W}_+$.

The matching described above between the asymptotic models of EAC G_2-manifolds $\hat{M}_\pm$ is precisely of the type studied in [18]. In particular, the gluing Theorem 6.1 constructs an irreducible torsion-free G_2-structure on M regarded as the generalized connected sum of the pair $\hat{M}_\pm$ defined above, with product EAC G_2-structures induced by the EAC Calabi–Yau structures on $W_\pm$ in the sense of Theorem 7.2.

The 'glued' G_2-metrics on M obtainable by Theorem 6.1 are of the type described in [19, Theorem 5.3]. When W_1, W_2 are constructed from a pair of K3 surfaces with non-symplectic involution with invariants r_j, a_j and with $d_j = \dim \operatorname{Ker} \iota_j$ as defined in Proposition 7.1, the resulting compact G_2-manifold M has

$$b^2(M) = d_1 + d_2 + \dim \left(\iota_1(H^2(W_+, \mathbb{R})) \cap \iota_2(H^2(W_-, \mathbb{R}))\right).$$

Recall that we have $b^2(M) = 12$ and $d_1 = d_2 = 4$, whence the last dimension in the right-hand side is 4. The examples explicitly discussed in [19] all have the latter intersection zero-dimensional, thus M is a new example for the construction given there.

By Theorem 6.4 and the work in §7.1, the glued torsion-free G_2-structure on M obtainable as in [18,19] is a continuous deformation of a torsion-free G_2-structure given by resolving singularities of T^7/Γ according to [16, §11]. Therefore, the moduli space for torsion-free G_2-structures on M has a connected component with boundary points corresponding to two types of degenerations of G_2-metrics: (1) those arising by pulling M apart into a pair of EAC G_2-manifolds and (2) those developing orbifold singularities but staying compact with volume and diameter bounded. To our knowledge, M is the first example of a compact irreducible G_2-manifold obtainable, up to deformation, both by the method of [16] and by the method of [18].

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