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Example Sheet 8

Boundary layers

1. Consider the singularly perturbed second-order linear ODE for $u(y)$:

$$\epsilon u'' + u' = 1, \tag{1}$$

where $\epsilon > 0$ is small and fixed, with boundary conditions $u(0) = 0$, $u(1) = 2$.

(a) Show that the exact solution is

$$u(y) = y + \frac{1 - \mathrm{e}^{-y/\epsilon}}{1 - \mathrm{e}^{-1/\epsilon}}.$$

Sketch $u(y)$ carefully. Explain briefly, but carefully, why $u_{BL} = 1 - \mathrm{e}^{-y/\epsilon}$ is a 'good approximation' when $0 \le y < \epsilon$ and why $u_M = y + 1$ is a 'good approximation' to $u(y)$ when $\epsilon < y < 1$. In the rest of this question we will explore the meaning of the term 'good approximation'.

(b) Consider computing a series solution to (1) by writing $u(y) = u_0(y) + \epsilon u_1(y) + \cdots$. Write down the form of u_0 that satisfies the boundary condition $u(1) = 2$.

(c) Now rescale (1) by changing the independent variable to $Y = y/\epsilon$. Write down the rescaled differential equation for $\tilde{u}(Y) = u(y)$. Consider a series solution $\tilde{u}(Y) = \tilde{u}_0(Y) + \epsilon\tilde{u}_1(Y) + \cdots$ and show that the form of $\tilde{u}_0$ which satisfies the boundary condition $\tilde{u}(0) = 0$ is given by

$$\tilde{u}_0 = A(1 - \mathrm{e}^{-Y}).$$

Observe that the matching condition

$$\lim_{Y\to\infty} \tilde{u}_0 = \lim_{y\to 0} u_0. \tag{2}$$

enables the constant A to be determined, and find it.

2. (a) Show that the streamfunction $\psi(r, \theta)$ for steady two-dimensional flow of a viscous fluid satisfies the equation

$$-\frac{1}{r}\frac{\partial(\psi, \nabla^2\psi)}{\partial(r, \theta)} = \nu\nabla^4\psi \tag{3}$$

where $\partial(f, g)/\partial(x, y) \equiv \partial f/\partial x\ \partial g/\partial y - \partial f/\partial y\ \partial g/\partial x$ is the Jacobian of $f(x, y)$ and $g(x, y)$.

(b) Show that (3) admits solutions of the form $\psi(r, \theta) = \nu f(\theta)$ as long as

$$f'''' + 4f'' + 2f'f'' = 0.$$

Hence show that $F(\theta) = f'(\theta)$ is given implicitly by

$$\int \left(C_1 + C_2F - 4F^2 - \frac{2}{3}F^3\right)^{-1/2} dF = \theta + C_3,$$

where C_1, C_2, C_3 are constants.

3. Consider the steady 2D boundary layer equations near a rigid wall at $y = 0$:

$$
\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}, && (4)\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, && (5)
\end{aligned}
$$

subject to the boundary condition

$$
u \to U(x) \qquad \text{as} \qquad y/\delta \to \infty, \tag{6}
$$

where $\delta \propto \nu^{1/2}$ is a typical measure of the boundary layer thickness. Take the pressure gradient to be that driven by the free stream acceleration, i.e. set

$$
-\frac{1}{\rho}\frac{dp}{dx} = U\frac{dU}{dx}. \tag{7}
$$

(a) Consider a general similarity solution in the form

$$
\psi = F(x)f(\eta), \qquad \text{where} \qquad \eta = y/g(x)
$$

to (4) - (5). Show that the 'free stream' boundary condition (6) demands that F takes the form

$$
F(x) = cU(x)g(x),
$$

where c is a constant that we can (wlog) set to unity.

(b) By substituting into (4) - (5) show that f satisfies the ODE

$$
(f')^2 - \left(1 + \frac{U}{U'}\frac{g'}{g}\right)ff'' = 1 + \frac{\nu f'''}{g^2 U'} \tag{8}
$$

(note the use of primes to denote either d/dx or $d/d\eta$ as appropriate). Deduce that a similarity solution is possible, i.e. (8) is just an ODE for $f(\eta)$, if (and only if) either

$$
U(x) \propto (x - x_0)^m \qquad \text{or} \qquad U(x) \propto \mathrm{e}^{\alpha x},
$$

where x_0, m and α are constants.

(c) In the case $U(x) = Ax^m$, $A > 0$, show that $g(x) \propto x^{(1-m)/2}$. Hence demonstrate that by choosing

$$
g(x) = \left(\frac{2\nu}{(m+1)Ax^{m-1}}\right)^{1/2}
$$

we can reduce the ODE for f to the form

$$
f''' + ff'' + \frac{2m}{m+1}\left(1 - (f')^2\right) = 0.
$$

Explain briefly why appropriate boundary conditions for this third-order ODE are $f(0) = f'(0) = 0$ and $f'(\eta) \to 1$ as $\eta \to \infty$.

4. A thin two-dimensional jet of fluid emerges from a narrow slit in a wall at $x = 0$ into fluid in $x > 0$ which is at rest. Assuming that the velocity u varies much more rapidly across the jet than along it we may apply boundary layer theory, i.e.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{9}$$

taking the pressure gradient to be zero since the outer fluid velocity is zero, see (7). Suitable boundary conditions are that $u \to 0$ as we move away from the jet, and $\partial u/\partial y = 0$ at the centre $y = 0$ of the narrow slit (by symmetry).

(a) By integrating (9) across the jet, and performing an integration by parts, show that

$$M \equiv \int_{-\infty}^{\infty} u^2\, dy \tag{10}$$

is constant (that is, M is independent of x).

(b) Consider similarity solutions in the form

$$\psi = F(x)f(\eta), \qquad \text{where} \qquad \eta = y/g(x)$$

where, wlog, we choose f to satisfy

$$\int_{-\infty}^{\infty} \left[f'(\eta)\right]^2\, d\eta = \frac{2}{3}. \tag{11}$$

Show that

$$F(x) = \left(\frac{3M}{2}\right)^{1/2} (g(x))^{1/2}.$$

From the boundary layer equation (9) now show that $g(x) \propto x^{2/3}$.

(c) Show that the choice $g(x) = \left(\frac{2}{3M}\right)^{1/3} (3\nu x)^{2/3}$ reduces the boundary layer equation to the ODE

$$f''' + ff'' + (f')^2 = 0 \tag{12}$$

and that the appropriate boundary conditions are $f(0) = f''(0) = 0$ and $f'(\eta) \to 0$ as $\eta \to \infty$.

(d) Integrate (12) three times and deduce that $f(\eta) = 2A\tanh(A\eta)$ for some constant A which can then be determined using (11). Deduce that the velocity profile in the jet is

$$u = \frac{1}{2}\left(\frac{3M^2}{4\nu x}\right)^{1/3} \operatorname{sech}^2\left(\frac{\eta}{2}\right),$$

and sketch the velocity profile at two different downstream positions.