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Example Sheet 7

The generation and diffusion of vorticity

1. (a) Consider solutions of the form

$$\mathbf{u} = u_\theta(r,t)\mathbf{e}_\theta$$

to the Navier–Stokes equations in cylindrical polar coordinates. Explain why $\partial p/\partial\theta$ must be a function of r and t only, say $\partial p/\partial\theta = P(r,t)$ and hence deduce that $P(r,t)$ must be zero.

(b) Use the result of part (a) to show that the time-dependent line vortex flow

$$\mathbf{u} = \frac{\Gamma(r,t)}{2\pi r}\mathbf{e}_\theta$$

in cylindrical polar coordinates (r,θ,z) is an exact solution of the Navier–Stokes equations, provided that Γ obeys the equation

$$\frac{\partial\Gamma}{\partial t} = \nu\left(\frac{\partial^2\Gamma}{\partial r^2} - \frac{1}{r}\frac{\partial\Gamma}{\partial r}\right). \qquad (1)$$

(c) Show that $\eta = r/(\nu t)^{1/2}$ is the only dimensionless combination of r, t and ν and hence find a similarity solution to (1), using the initial condition $\Gamma(r,0) = \Gamma_0\delta(r)$ in which the vorticity is concentrated at $r = 0$ and is zero elsewhere, and the boundary condition $\Gamma(0,t) = 0$ in $t > 0$.

(d) Find u_θ and sketch u_θ as a function of r at different times.

2. A viscous flow is generated in $r > a$ by a circular porous cylinder of radius a which rotates with constand angular velocity Ω. Suction inside the cylinder generates a radial inflow so that $u_r = -U$ on $r = a$.

(a) Show that

$$u_r = -\frac{Ua}{r} \qquad \text{in} \qquad r \geq a,$$

and that

$$r^2\frac{d^2u_\theta}{dr^2} + (R+1)r\frac{du_\theta}{dr} + (R-1)u_\theta = 0, \qquad (2)$$

where $R = Ua/\nu$.

(b) Show that if $R < 2$ then there is only one solution of (2) that satisfies both the no-slip condition on $r = a$ and that has finite circulation $\Gamma = 2\pi r u_\theta$ as $r \to \infty$, but that if $R > 2$ then there are many such solutions.

3. Viscous fluid flows between two rigid boundaries $y = 0$ and $y = h$. The lower boundary moves in the x-direction with constant speed U and the upper boundary is at rest. The boundaries are porous and there is an imposed constant vertical flow with velocity $-V$ at each boundary, i.e. fluid is introduced at $y = h$ and removed by suction at $y = 0$.

(a) Show that the resulting steady flow is

$$u(y) = U\left(\frac{\mathrm{e}^{-Vy/\nu} - \mathrm{e}^{-Vh/\nu}}{1 - \mathrm{e}^{-Vh/\nu}}\right), \qquad v = -V.$$

(b) Compute the vorticity, and sketch the horizontal velocity profile $u(y)$ in the cases $V = 0$ and $Vh/\nu \gg 1$.

(c) Now consider the modified problem where the walls are rigid and stationary (but porous) and in the x-direction the flow is driven by a constant pressure gradient G $(= -\partial p/\partial x)$.