MA50215 Specialist Reading Course
The Dynamics of Fluids

J.H.P. Dawes
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Example Sheet 6

Stokes flow in 3D

1. A rigid sphere of radius a rotates with angular velocity $\mathbf{\Omega}$ in a viscous fluid which is at rest at infinity. Show that the Stokes equations can be satisfied by a velocity field $\mathbf{u} = \nabla \times (f(r)\mathbf{\Omega})$ where $f(r)$ satisfies $\nabla^2 f = 0$.

   Hence find $f(r)$.

Thin films

2. Viscous fluid is contained between two planes $y = \pm b$. It undergoes a two-dimensional flow with stream function $\psi(x, y)$ generated by some kind of motion (e.g. a rotating cylinder) near $x = y = 0$. Find the form of the flow field for x large and positive by finding the general solution of $\nabla^4 \psi = 0$ in the form

$$\psi(x, y) = f(y)\mathrm{e}^{-kx}$$

   where $Re(k) > 0$ and $f(y)$ is an even function of y. Hence show that k is given implicitly as the solution of the equation

$$2kb + \sin 2kb = 0. \qquad (1)$$

   Show that (1) has no real nonzero solutions. Sketch the streamlines of the flow that arises.

3. A thin layer of viscous fluid flows under gravity down a plane inclined at an angle α to the vertical. Taking coordinates such that the origin O lies on the plane, the axis Ox is directed down the line of steepest slope, and the axis Oy is normal to the plane, the free surface of the fluid is given by $y = h(x, t)$ where $|\partial h/\partial x| \ll 1$. Assume that the pressure distribution in the liquid is hydrostatic.

   (a) Using the approximations appropriate to thin films (lubrication theory), show that the film height $h(x, t)$ evolves according to

$$\frac{\partial h}{\partial t} = \frac{g}{3\nu}\frac{\partial}{\partial x}\left[h^3\left(\sin\alpha\frac{\partial h}{\partial x} - \cos\alpha\right)\right].$$

   (b) Consider now the linearised form of this equation when $\alpha = \pi/2$ and the film height is nearly constant, i.e.

$$h = h_0 + \eta(x, t)\,, \qquad \text{where} \qquad |\eta| \ll h_0.$$

   Find the solution corresponding to the initial condition

$$\eta(x, 0) = \eta_0 \mathrm{e}^{-x^2/a^2}.$$

   (c) Discuss what happens in the case $\alpha = -\pi/2$, i.e. the fluid is on the underside of the plane.

4. A viscous fluid coats the outer surface of a cylinder of radius a which rotates with constant angular velocity Ω about its axis which is horizontal. The angle θ on the cylinder is measured from the horizontal on the rising side.

(a) Show that the volume flux (flow rate) per unit length $Q(\theta, t)$ is related to the film thickness $h(\theta, t)$ by

$$Q = \omega a h - \frac{g}{3\nu} h^3 \cos\theta.$$

Deduce an evolution equation for $h(\theta, t)$.

(b) Consider now the possibility of a steady state in which $Q =$ constant, $h = h(\theta)$. By considering the functions $F_1(h) = \Omega a h - g h^3/(3\nu)$ and $F_2 = \Omega a h + g h^3/(3\nu)$, show that a steady solution in which $h(\theta)$ is continuous and 2π-periodic exists only if

$$\Omega a > \left(\frac{9Q^2 g}{4\nu}\right)^{1/3}.$$