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Example Sheet 5

Suffix notation question

1. (a) Recall the definitions

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \text{and} \qquad \omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

Show that the vorticity tensor ω_{ij} is related to the vorticity vector $\boldsymbol{\omega}$ by

$$\omega_k = -\epsilon_{kpq}\,\omega_{pq},$$

and hence show that

$$\omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k.$$

(b)We have defined the rate of energy dissipation by viscosity to be

$$\Phi \;=\; 2\mu \int_V e_{ij}\, e_{ij}\, dV.$$

Show that an equivalent expression is given by

$$\Phi \;=\; \mu \int_V |\boldsymbol{\omega}|^2\, dV$$

where $\boldsymbol{\omega}$ is the vorticity vector.

Flows without inertia

2. Fluid flows steadily through a rigid cylindrical tube parallel to the z-axis with velocity

$$\mathbf{u} \;=\; (0, 0, w(x, y)),$$

in Cartersian co-ordinates, under a uniform pressure gradient $G = -\partial p/\partial z$.

(a) Show that the Navier–Stokes equations are satisfied provided

$$\nabla^2 w \;=\; -G/\mu,$$

and state the appropriate boundary condition for the flow.

(b) Suppose that the tube has an elliptical cross-section with semi-major axes a and b. Show that

$$w(x, y) \;=\; w_0\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

Find w_0, and also show that the volume flux Q (i.e. the volume of fluid passing any cross-section of the tube in unit time) is given by

$$Q \;=\; \frac{\pi a^3 b^3 G}{4(a^2 \;+ b^2)\mu}.$$

3. Viscous fluid is confined in the space $a < r < b$ between two infinitely long concentric coaxial rigid cylinders. The inner cylinder rotates with constant angular velocity Ω about its axis, and the outer cylinder is at rest.

(a) Consider steady solutions of the Navier–Stokes equation in the form $\mathbf{u} = (0, v(r), 0)$ in cylindrical polar co-ordinates $r, \theta, z)$. Derive the second-order linear ODE that $v(r)$ must satisfy, and state the relevant boundary conditions.

(b) Look for solutions to this ODE in the form $v(r) \propto r^\alpha$ and hence find $v(r)$.

(c) Show that the torque (per unit length in z) that must be applied to the inner cylinder to maintain the motion is

$$T \equiv -\int_0^{2\pi} 2\mu\, e_{r\theta}\, a^2 d\theta = \frac{4\pi\mu\Omega a^2 b^2}{b^2 - a^2}$$

where, for this flow, $e_{r\theta} = \frac{r}{2}\frac{d}{dr}\left(\frac{v}{r}\right)$.

4. Two infinite plates are hinged together at $r = 0$ and open slowly over time with angular velocities $\pm\Omega$, so that their positions are $\theta = \pm\Omega t$. The space $-\Omega t < \theta < \Omega t$, $0 < r < \infty$ between them is filled with very viscous fluid.

(a) Write down the boundary conditions satisfied by u_r and u_θ at $\theta = \pm\Omega t$ and hence write down the boundary conditions for the streamfunction $\psi(r, \theta, t)$.

(b) Discuss the assumptions behind the derivation of the (approximate) slow flow equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \psi = 0.$$

Solve this equation by examining possible solutions of the form $\psi = r^\alpha f(\theta)$, and derive the solution

$$\psi = -\frac{\Omega r^2}{2}\left(\frac{\sin 2\theta - 2\theta\cos 2\Omega t}{\sin 2\Omega t - 2\Omega t \cos 2\Omega t}\right) \qquad (1)$$

(c) Show that when $2\Omega t = \pi/2$ the (instantaneous) streamlines are rectangular hyperbolas.

(d) Use (1) to make rough order-of-magnitude estimates of the terms $\partial\mathbf{u}/\partial t$, $\mathbf{u}\cdot\nabla\mathbf{u}$ and $\nu\nabla^2\mathbf{u}$ and hence deduce that the slow flow approximation is only valid if

$$\frac{\Omega r^2}{\nu} \ll 1.$$

Discuss the validity of the solution (1) for small and large r.