Example Sheet 4

Complex potentials

1. (Acheson, page 125). Check (by careful differentiation!) the derivation of the first equality in equation (4.15), i.e. that the complex potential $\Phi = \phi + \mathrm{i}\psi$ satisfies

$$\frac{d\Phi}{dz} = \frac{\partial \phi}{\partial x} + \mathrm{i}\frac{\partial \psi}{\partial x}$$

2. Compute the potential ϕ and streamfunction ψ for the complex potentials

   (i) $\Phi = Uz\mathrm{e}^{-\mathrm{i}\alpha}$,
   (ii) $\Phi = \frac{1}{2}\beta z^2$,
   (iii) $\Phi = -\frac{\mathrm{i}\Gamma}{2\pi}\log(z - z_0)$ (line vortex),
   (iv) $\Phi = \frac{Q}{2\pi}\log(z - z_0)$ (line source).

   In each case sketch the streamlines.

   For (iii) show that the circulation $\oint_C \mathbf{u} \cdot \mathbf{dx}$ around *any* simple closed curve C containing z_0 is Γ. For (iv) show that the mass flux $\oint_C \mathbf{u} \cdot \mathbf{n}\, ds$ across *any* simple closed curve C containing z_0 is Q.

3. Consider the complex potential $\Phi = Az^\lambda$. Sketch streamlines and qualitatively describe the flows in the cases $\lambda = 3$, 2, 3/2, 1, 2/3 and 1/2.

4. Consider the complex potential

$$\tilde{\Phi}(z, z_0) = \frac{Q}{2\pi}\left(\log(z - z_0) - \log z\right)$$

   for $Q > 0$, which corresponds to a source at z_0 and a sink at the origin. Sketch streamlines of the flow.

   Now let $z_0 = r_0\mathrm{e}^{\mathrm{i}\theta_0}$ and let $Q = k/r_0$. Show that

$$\tilde{\Phi}(z, z_0) = \frac{k\mathrm{e}^{\mathrm{i}\theta_0}}{2\pi}\left(\frac{\log(z - z_0) - \log z}{z_0}\right).$$

   Compute the potential $\Phi(z) = \lim_{z_0 \to 0} \tilde{\Phi}(z, z_0)$ in which the source and sink merge. Sketch the streamlines of the complex potential $\Phi(z)$. [This flow is called a 'doublet' of strength k.]

5. Construct the complex potential for the flow in $x \geq 0$ produced by a line source at $(0, d)$ near an impermeable wall at $x = 0$. *Hint: by linearity we can consider superpositions of line sources.*

   Show that the pressure along $x = 0$ attains a minimum at $y = \pm d$ and increases monotonically to a constant p_∞ when $|y| > d$.

6. Describe the flow given by the complex potential $\Phi = U(a^2 + z^2)^{1/2}$. Hints: (i) taking the appropriate branch of the square root we can say that Φ is real on $z = \mathrm{i}y$ for $-a < y < a$; (ii) consider what streamlines must look like near $z \approx \pm\mathrm{i}a$, and for $|z|$ large.