Example Sheet 3

Bernoulli's equation for unsteady flow

1. A horizontal thin tube of uniform cross-section and length L is initially filled with fluid at rest. One end of the tube is closed with a plug. The other is connected to a large open water tank containing fluid whose free surface is at a height h above the tube. At $t = 0$ the plug is removed and water begins to flow. Neglecting changes in h, show that the outflow velocity u_{out} for $t > 0$ is approximately

$$u_{out}(t) \quad = \quad \sqrt{2gh}\tanh\left(\frac{t\sqrt{2gh}}{2L}\right).$$

Deduce that the time taken for the flow to accelerate to a fraction $(\mathrm{e}^2 - 1)/(\mathrm{e}^2 + 1) = 0.7616...$ of its limiting value is $2L/\sqrt{2gh}$. Verify that this time is roughly 3s for a garden hose $L = 9$m long supplied by a rainwater tank for which $h = 1.8$m.

2. A sphere is immersed in an infinite ocean of inviscid fluid of density ρ that is initially at rest. The radius of the sphere oscillates in time according to $R(t) = a + b\sin nt$, where a, b and n are positive constants and $b < a$. The fluid moves radially and the pressure at infinity P is constant. If $a \geq 5b$ show that the maximum pressure attained on the surface of the sphere is $P + \rho n^2 b(a - b)$.

Find the corresponding formula if $b < a \leq 5b$.

Vorticity and Kelvin's Circulation Theorem

3. Calculate the vorticity $\boldsymbol{\omega}$ of the velocity field

$$\begin{aligned} u &= -\alpha x - yrf(t) \\ v &= -\alpha y + xrf(t) \\ w &= 2\alpha z \end{aligned}$$

where $r^2 = x^2 + y^2$. Show that this describes an incompressible flow for any $f(t)$.

Assuming that the fluid is inviscid, find the general form of $f(t)$ which allows this velocity field to be a solution of the vorticity equation.

At time $t = 0$ the fluid on the curve $r = a_0$, $z = 0$ is marked with dye. Deduce how this curve evolves in time and verify explicitly that the circulation theorem holds in this case.

4. Consider the velocity field $\mathbf{u}$ given by $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{x}$, corresponding to uniform rotation with angular velocity Ω. Show that the vorticity $\boldsymbol{\omega} = 2\boldsymbol{\Omega}$. If the flow is two-dimensional $(u(x, y), v(x, y), 0)^T$, show that $\boldsymbol{\omega} = (0, 0, -\nabla^2\psi)^T$ where ψ is the streamfunction.

A long cylinder with an elliptical cross-section is filled with inviscid fluid. The cross-section has semi-major axes a and b. When $t < 0$ both the cylinder and the water within it rotate about the axis of the cylinder as a solid body, with uniform angular velocity $(0, 0, \Omega)^T$. Find the streamfunction and sketch the streamlines. Why can they intersect the boundary of the cylinder?

At $t = 0$ the cylinder is suddenly brought to rest. Assuming that the flow remains two-dimensional, find $\boldsymbol{\omega}$ for $t > 0$. Hence determine $\nabla^2\psi$ in $t > 0$ and verify that the flow can be described by

$$\psi = \frac{a^2 b^2 \Omega}{a^2 + b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

Is this solution unique? Sketch the streamlines for $t > 0$.