J.H.P. Dawes

Example Sheet 2

Streamlines, particle paths

1. A steady 2D 'pure straining' flow is given by $u = \alpha x$, $v = -\alpha y$ with $\alpha > 0$ and constant.
   (i) Find the equation for a general streamline of the flow, and sketch typical streamlines.
   (ii) At $t = 0$ the fluid particles on the curve $x^2 + y^2 = a^2$ are marked with dye without disturbing the flow. Find the equation for this material fluid curve for $t > 0$.
   (iii) Does the area within the curve change in time, or not? Why?

Streamfunctions and velocity potentials

2. A 2D flow is represented by a streamfunction $\psi(x, y)$ with $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Show that
   (i) the streamlines are given by $\psi = \text{const.}$,
   (ii) $|\mathbf{u}| = |\nabla\psi|$ so that the flow is faster where streamlines are closer together,
   (iii) the volume flux crossing any curve from a point $\mathbf{x}_0$ to a point $\mathbf{x}_1$ is given by $\psi(\mathbf{x}_1) - \psi(\mathbf{x}_0)$, *Hint: take* $\mathbf{n}\, ds = (dy, -dx)$
   (iv) $\psi = \text{const}$ on any *stationary* boundary.

3. Verify that the 2D flow given in Cartesian coordinates by

$$u = \frac{y - b}{(x - a)^2 + (y - b)^2}, \quad v = \frac{a - x}{(x - a^2 + (y - b)^2}$$

is incompressible. Find the streamfunction $\psi(x, y)$ such that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Sketch the streamlines.

4. A 2D irrotational flow occupies the half-space $y < 0$ and is given by the velocity potential $\phi = \mathrm{e}^{ky} \sin kx$, for $k > 0$. Show that the flow is incompressible. Calculate the velocity field $\mathbf{u}(x, y)$ and the streamfunction $\psi(x, y)$. Sketch the streamlines.

5. Verify that the 2D flow given in plane polar coordinates (r, θ) by

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \qquad u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

is incompressible. Find the streamfunction $\psi(r, \theta)$. Sketch the streamlines and describe the flow.
Recall that:

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \quad \text{and} \quad u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r}$$

6. Do the following velocity potentials describe incompressible flows or not?

   (a) $\phi = C(x^2 + y^2)$
   (b) $\phi = C(x^2 - y^2)$

   If so, determine the corresponding streamfunction. Sketch the streamlines in all cases, and the lines of constant potential ϕ.

7. Do the following streamfunctions describe irrotational flows or not?

   (a) $\psi = C(x^2 + y^2)$
   (b) $\psi = C(x^2 - y^2)$

   If so, determine the corresponding velocity potentials.
   Sketch the streamlines in all cases and the lines of constant potential ϕ wherever possible.

8. In spherical polar co-ordinates (r, θ, z) the velocity potential $\phi = -kr^2 P_2(\cos\theta)$ represents the axisymmetric flow produced by the confluence of two equal and opposite streams. Sketch the streamline pattern in a plane containing the axis of symmetry.

   An impermeable sphere of radius a is now placed in this flow, with its centre at the origin. How is the velocity potential modified? Determine the velocity and pressure over the surface of the sphere.