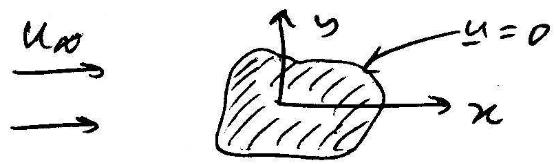
4 Flow at high Reynolds number

4.1 Introduction

In this chapter we will consider 'fast' steady 2D flows around rigid obstacles, with flow predominantly in the positive x direction, such that $\mathbf{u} \to \mathbf{u}_\infty$ (a constant) as $|x| \to \infty$:



Consider the vorticity equation

$$-\frac{\partial(\psi, \omega)}{\partial(x, y)} = \nu \nabla^2 \omega \qquad (1)$$

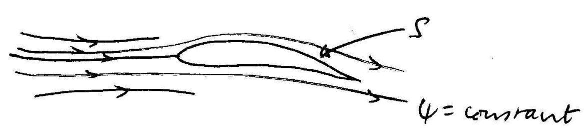
(where $\omega = -\nabla^2 \psi$ as usual, $\boldsymbol{\omega} = (0, 0, \omega)$ and $\mathbf{u} = (u, v, 0)$). Let L be a typical length-scale, usually introduced via the boundary conditions, and let U be a typical flow velocity. Then by 'fast' we mean the case $Re = UL/\nu \gg 1$. For low Re the vorticity equation is dominated by $\nu\nabla^2\omega$ as we saw in chapter 2; for high Re it is dominated by the left-hand side.

4.2 The Euler limit

If we simply put $\nu = 0$, i.e. $Re = \infty$, then we are left with

$$\frac{\partial(\psi, \omega)}{\partial(x, y)} = 0$$

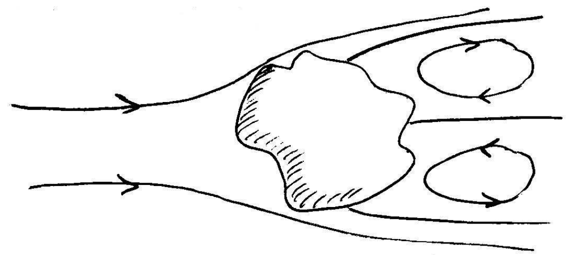
which implies $\nabla^2\psi = F(\psi)$ for some function $F(\psi) = -\omega$. So the vorticity is constant on streamlines $\psi =$ constant. If, for example, $\omega = 0$ at ∞ (which is what we usually assume) then we would have $\omega = 0$ on all streamlines which come in from infinity. For example, for an aerofoil shape



we would have $\omega = 0$ on every streamline, which implies $\omega = 0$ everywhere so the flow is irrotational and $\mathbf{u} = \nabla\phi$, i.e. this must be a potential flow which satisfies the traditional equation $\nabla^2\phi = 0$ from incompressibility, together with the standard boundary conditions $\mathbf{u} \cdot \mathbf{n} \equiv \mathbf{n} \cdot \nabla\phi = 0$ on the surface S and $\mathbf{u} \to \mathbf{u}_\infty$, i.e. $\phi \to \mathbf{u}_\infty x$ as $|x| \to \infty$. The solution to this problem is unique, but has the drawback that it generically will not satisfy the additional requirement of the no-slip condition on S: the tangential velocity on S is not zero.

The resolution of this problem is that in fact there must be a layer near the surface S in which viscous effects remain important, however large Re is, and in which layer the Euler limit is not valid. From previous examples we might guess that the thickness of this layer $\delta = O(\nu^{1/2})$.

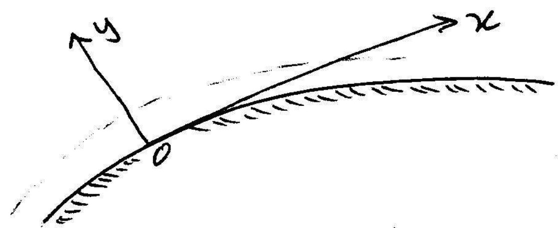
For many high Reynolds number flows, in particular those around 'bluff bodies' we find in practice that the flow is neither steady nor closely resembles the potential flow solution away from the obstacle. Often there are regions of closed streamlines and 'separation' occurs: new stagnation points appear on the boundary which divide the flow into different regions:



Then we may find $\omega \neq 0$ in regions of close streamlines (but $\omega = 0$ in regions connected to infinity still). In general such problems are extremely difficult to solve analytically (or numerically).

4.3 The Prandtl limit

Let O be a point on the surface S and take local Cartesian axes Ox tangential to the surface and Oy normal to the surface:



within the boundary layer at small y we expect that variations in ψ will be much more rapid in y (approximately across streamlines) than in x (approximately along streamlines):

$$\left|\frac{\partial^2\psi}{\partial y^2}\right| \gg \left|\frac{\partial^2\psi}{\partial x^2}\right|$$

(recalle that this kind of argument is reminiscent of lubrication theory).

As a result, $\nabla^2\psi \approx \partial^2\psi/\partial y^2 = \psi_{yy}$ introducing notation where subscripts mean partial derivatives. Then the vorticity equation (1) becomes

$$\begin{aligned} \psi_y\psi_{xyy} - \psi_x\psi_{yyy} &= \nu\psi_{yyyy} \\ \Rightarrow \psi_y\psi_{xy} - \psi_x\psi_{yy} &= \nu\psi_{yyy} + G(x) \end{aligned} \tag{2}$$

where $G(x)$ is a constant of integration after integrating w.r.t. y. Equation (2) is equivalent to writing

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = G(x) + \nu\frac{\partial^2 u}{\partial y^2} \tag{3}$$

which we will refer to as *the boundary layer equation.* It is the x-component of the Navier–Stokes equations with $\nabla^2 u$ replaced by $\partial^2 u/\partial y^2$, provided that we identify $G(x)$ with the pressure term $-\frac{1}{\rho}\frac{\partial p}{\partial x}$. Therefore, in the boundary-layer approximation we have that the pressure gradient is necessarily independent of y, meaing that it must be determined by conditions outside the boundary layer where $y \gg \delta$ (formally, for $y \to \infty$).

4.4 A formal derivation of the boundary layer equation

A less hand-waving derivation of (2) can be obtained by rescaling y and ψ appropriately and then taking the limit of small ν. Returning to the vorticity equation (1), let

$$\hat{y} = y/\nu^{1/2} \qquad \text{and} \qquad \hat{\psi} = \psi/\nu^{1/2} \tag{4}$$

so that $\hat{y}$ is $O(1)$ when $y = O(\nu^{1/2})$, i.e. small, and so $\partial\psi/\partial y = \partial\hat{\psi}/\partial\hat{y}$ remains $O(1)$.

Substituting this into the vorticity equation we obtain

$$-\frac{(\nu^{1/2})^2 \partial\left(\hat{\psi}, \left(\frac{\partial^2}{\partial x^2} + \frac{1}{\nu}\frac{\partial^2}{\partial \hat{y}^2}\right)\hat{\psi}\right)}{\nu^{1/2}\partial(x,\hat{y})} = \left(\frac{\partial^2}{\partial x^2} + \frac{1}{\nu}\frac{\partial^2}{\partial \hat{y}^2}\right)^2 \hat{\psi}.$$

Now we take the formal limit $\nu \to 0$ assuming that $\hat{\psi}$ and $\hat{y}$ etc remain $O(1)$ in the limit. We keep only the leading-order terms to find

$$-\frac{\partial(\hat{\psi}, \hat{\psi}_{\hat{y}\hat{y}})}{\partial(x,\hat{y})} = \hat{\psi}_{\hat{y}\hat{y}\hat{y}\hat{y}} \tag{5}$$

in which we note that ν has disappeared - this is an indication that we have found the correct rescaling. Equation (5) can be integratated once straight away to yield the rescaled version of (2):

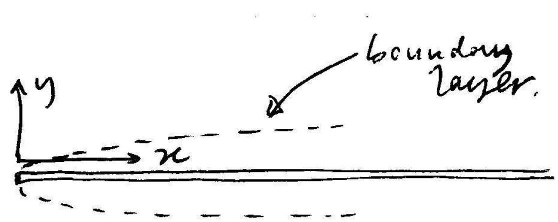
$$\begin{aligned} \hat{\psi}_{\hat{y}}\hat{\psi}_{x\hat{y}\hat{y}} - \hat{\psi}_x\hat{\psi}_{\hat{y}\hat{y}\hat{y}} &= \hat{\psi}_{\hat{y}\hat{y}\hat{y}\hat{y}} \\ \Rightarrow \hat{\psi}_{\hat{y}}\hat{\psi}_{x\hat{y}} - \hat{\psi}_x\hat{\psi}_{\hat{y}\hat{y}} &= \hat{\psi}_{\hat{y}\hat{y}\hat{y}} + \tilde{G}(x). \end{aligned}$$

Remark: In fact the rescaling (4) is the only rescaling that produces a sensible boundary-layer equation, in the sense that if we consider the more general rescaling

$$\hat{y} = y/\nu^q \qquad \text{ans} \qquad \hat{\psi} = \psi/\nu^q$$

so that $\partial\hat{\psi}/\partial\hat{y} = \partial\psi/\partial y$ remains unchanged, where $0 < q \leq 1$, then any choice of $q < \frac{1}{2}$ gives the Euler limit where we just ignore the viscous term, and any $q > \frac{1}{2}$ gives the Stokes equations limit where we ignore the inertial terms. The choice $q = \frac{1}{2}$, which gives the Prandtl boundary-layer limit, is the only other choice.

4.5 Uniform steady flow past a semi-infinite flat plate



We consider a flow $\mathbf{u} = (u, 0)$ (in Cartesian coordinates) in $y > 0$ above a rigid plate at $y = 0$, $x\ geq0$, such that $\mathbf{u} \to (u_\infty, 0)$ in $y \gg 1$. The Euler limit is trivial in this case: we would like to set $\mathbf{u} = (\mathbf{u}_\infty, 0)$ everywhere. Within the boundary layer we need to solve the boundary layer equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = G(x) + \nu\frac{\partial^2 u}{\partial y^2}.$$

$G(x)$ is determined by conditions outside the boundary layer where obviously $\partial p/\partial x = 0$ since $\mathbf{u} \to (\mathbf{u}_\infty, 0)$ constant. Hence $G(x) = 0$ for this flow. Using the Prandtl rescaling from the previous section the equation we wish to solve is (5) after integrating once and setting the constant of integration $G(x)$ to zero:

$$\hat{\psi}_{\hat{y}}\hat{\psi}_{x\hat{y}} - \hat{\psi}_{x}\hat{\psi}_{\hat{y}\hat{y}} = \hat{\psi}_{\hat{y}\hat{y}\hat{y}} \tag{6}$$

which does not contain ν, hence $\hat{\psi}$ can depend only on u_∞, x and $\hat{y}$. We now investigate similarity forms of solution to (6). First we write down the dimensions of each of these quantities:

$$[\hat{\psi}] = \left[\frac{\psi}{\nu^{1/2}}\right] = \left(L^2T^{-1}\right)^{1/2}, \qquad [u_\infty] = LT^{-1}$$
$$[x] = L \qquad [\hat{y}] = \left[\frac{y}{\nu^{1/2}}\right] = T^{1/2}$$

Suppose that $\eta = \hat{y}U_\infty^a x^b$ is a dimensionless combination of these variables (without loss of generality we take η to be linear in $\hat{y}$ since if η is dimensionless, so is η^q for any exponent q). Then equating powers of L and T we find

$$\begin{aligned} L &: \quad a + b = 0 \\ T &: \quad -a + \tfrac{1}{2} = 0 \end{aligned}$$

so the unique dimensionless combination is given by $a = 1/2$, $b = -1/2$. So we define

$$\eta = \hat{y}\left(\frac{u_\infty}{x}\right)^{1/2}.$$

Then, since the streamfunction $\hat{\psi}$ has units $LT^{-1/2}$ a similarity solution must take the form

$$\hat{\psi} = (u_\infty x)^{1/2} f(\eta) \tag{7}$$

for some dimensionless function f. We now substitute the similarity ansatz (7) into (6), computing the various derivatives carefully, noting that

$$\frac{\partial \eta}{\partial x} = -\frac{\hat{y}u_\infty^{1/2}}{2x^{3/2}}, \qquad \text{and} \qquad \frac{\partial \eta}{\partial \hat{y}} = \left(\frac{u_\infty}{x}\right)^{1/2}$$

to obtain the *Blasius Equation*

$$f''' + \frac{1}{2}ff'' = 0. \tag{8}$$

This nonlinear third-order ODE must be solved numerically. The appropriate boundary conditions can be easily seen by considering $u = \psi_y = \hat{\psi}_{\hat{y}} = u_\infty f'(\eta)$: we require $f'(\eta) \to 1$ as $\eta \to \infty$ to match to the Euler limit for the inviscid flow outside the boundary layer. In addition we require $f'(0) = 0$ since $\hat{y} = 0$ is a rigid boundary, and $f(0) = 0$ since $\hat{y} = 0$ is an impermeable boundary, and therefore it is a streamline.

Numerically, the easiest way to solve (8) is therefore a shooting method, varying the initial guess for $f''(0)$ to achieve the far-field boundary condition $f'(\eta) = 1$ as $\eta \to \infty$. We find the numerical value $f''(0) = 0.332...$.

4.6 Remarks on the boundary layer equation, and matching

The boundary layer equation (3)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = G(x) + \nu\frac{\partial^2 u}{\partial y^2}$$

closely resembles the x-component of the Navier–Stokes equation, except for the following important differences:

- $\nabla^2 u$ is approximated by $\partial^2 u/\partial y^2$
- the pressure term $-\frac{1}{\rho}\frac{\partial p}{\partial x}$ appears here as a function $G(x)$ of only the streamwise coordinate. Moreover, from the condition that we have uniform flow $U(x)$ above the boundary layer, we can deduce, considering the limit $\hat{y} \to \infty$, that $G(x) = U\frac{dU}{dx}$ since in this regime we expect the velocity field $\mathbf{u}$ to approach $(U(x), 0)$. This dependence of $G(x)$ on the free stream velocity $(U(x), 0)$ shows how the pressure term drives the boundary layer dynamics. Sometimes one says that the external pressure gradient is 'impressed upon the boundary layer'.

'Matching' refers to the asymptotic procedure by which a solution of the boundary layer equation (i.e. in the Prandtl limit, $\nu \to 0$ taken so that $y/\nu^{1/2}$ remains $O(1)$) is fitted to a solution of the Euler limit ($\nu \to 0$ taken so that y remains $O(1)$). We demand that these solutions agree in the sense that the x-velocity computed as one emerges upwards from the boundary layer the solution obtained agrees with that found by descending downwards from the free stream. More mathematically:

$$\lim_{\hat{y}\to\infty} \frac{\partial \hat{\psi}}{\partial \hat{y}} = \lim_{y \to 0} \frac{\partial \psi}{\partial y}.$$

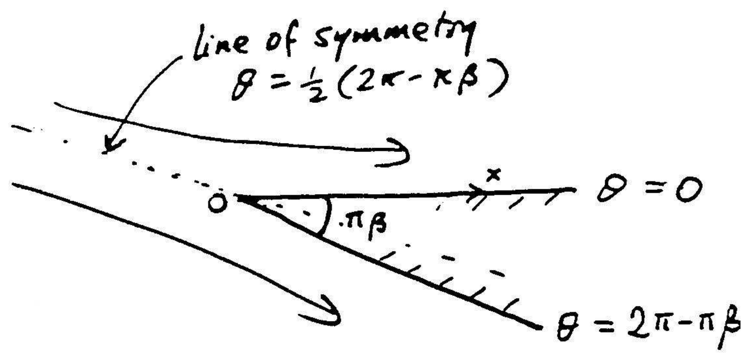
We will not say too much more about the matching process here - there is plenty more than can be said!

The importance of this link between the free stream and the boundary layer dynamics emerges as we now consider generalisations of the flow past a flat plate. Geometrically these will correspond to flow past a wedge and flow around a corner. Mathematically they can be treated very similarly, and we will derive a generalisation of the Blasius equation (8) that describes the boundary layer structure: this is the *Falkner–Skan equation.*

4.7 Flow past a wedge

In this section and the one following we present some of the details of the set-up: principally we need to recall the form of the potential flow that we expect far away from the boundaries. These details are very similar in the two cases, in fact. Then, in section 4.9 we turn to the details of the boundary layer flow. The Blasius equation for the velocity profile in the boundary layer generalises to the Falkner–Skan equation for these cases.

Consider flow in $0 < \theta < (2 - \beta)\pi$ past a wedge of angle $\pi\beta$:



First we need to determine the Euler limit of the flow, i.e. potential flow. It makes sense to assume potential flow since the flow arrives from far upstream where there is no mechanism to introduce vorticity into the flow. Then we are left solving $\nabla^2\phi = 0$, so that

$$\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta = \nabla\phi = \frac{\partial \phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\mathbf{e}_\theta$$

and the boundary conditions (impermeability) are

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0 \quad \text{on} \quad \theta = 0, \pi(2-\beta)/2 \quad \text{by symmetry.}$$

We try a solution (see an earlier example sheet)

$$\phi = Cr^{\lambda} \cos \lambda\theta$$

then computing $u_\theta = (1/r)\partial\phi/\partial\theta = -C\lambda r^{\lambda-1} \sin \lambda\theta$ we require $\sin \lambda\theta = 0$ on $\theta = \pi(2-\beta)/2$ and hence $\lambda = 2/(2-\beta)$. This solves the Euler problem for the potential flow outside the boundary layer.

We observe that on $\theta = 0$ the radial velocity

$$u_r = \frac{\partial \phi}{\partial r} = C\lambda r^{\lambda-1}$$

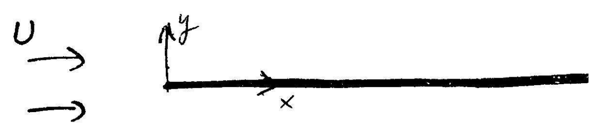
is exactly the x-velocity that we called u_infty previously. In this section we call this $u_\infty \equiv U(x) = Ax^m$ where $m = \lambda - 1 = \beta/(2-\beta)$ is a positive parameter.

We make two remarks. Firstly, the pressure gradient $-G(x) \equiv \partial p/\partial x$ in the x-direction can be computed from the x component of the Euler equation which reduces to

$$G(x) \equiv -\frac{\partial p}{\partial x} = U\frac{dU}{dx} = mA^2x^{2m-1}, \tag{9}$$

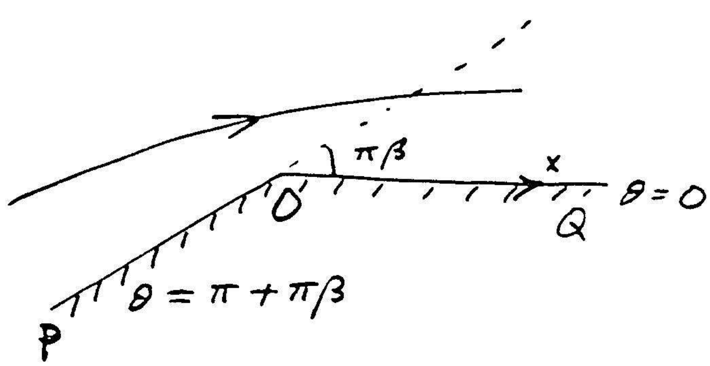
which is positive and therefore the pressure gradient $\partial p/\partial x$ is negative. In this case the potential flow is said to be 'favourable' and it shows that the flow continues to accelerate as it passes around the wedge.

Secondly, the special case $\beta = 0$ corresponds to flow past a semi-infinite flat plate: $m = 0$ and $U =$ constant so that the Euler flow is just a uniform stream. This is exactly the problem we looked at in section 4.5.



4.8 Flow around a corner

In this case we are considering the flow in the region $0 < \theta < \pi(1+\beta)$:



As before we have a the solution $\phi = Cr^{\lambda} \cos \lambda\theta$ to $\nabla^2\phi = 0$ for the Euler flow. The boundary conditions

$$u_\theta = 0 \quad \text{on } \theta = 0 \text{ and } \theta = \pi(1+\beta)$$

imply that $\lambda = 1/(1+\beta)$. As before we calculate the radial velocity on $\theta = 0$: $u_r = C\lambda r^{\lambda-1}$ so that $U(x) = Ax^m$ where $m = \lambda - 1 = -\beta/(1+\beta)$ is negative for this case. Interestingly this implies singular behaviour: $U(x) \to \infty$ as $x \to 0$. We can also compute the pressure gradient from the Euler equation which gives (9) as before, but now $\partial p/\partial x$ is positive since $m < 0$. Therefore the flow is *decelerating* as x increases and the pressure gradient is said to be 'adverse'.

4.9 The Falkner–Skan equation

Both the flow past a wedge and the flow around a corner result in $U(x) = Ax^m$ as the 'outer boundary condition' for the boundary layer, i.e. the flow we should obtain in the limit $\hat{y} \to \infty$.

We now use dimensional analysis to propose a form for the streamfunction $\hat{\psi}(x, \hat{y})$ that satisfies both the boundary layer equation and the above outer boundary condition; $\partial\hat{\psi}/\partial\hat{y} \to U(x) = Ax^m$ as $\hat{y} \to \infty$.

First, note that $\hat{y}$ and $\hat{\psi}$ have dimensions as follows:

$$\begin{aligned}
[\hat{y}] \equiv \quad [y/\nu^{1/2}] \quad &= \frac{L}{(L^2T^{-1})^{1/2}} = T^{1/2} \\
\left[\hat{\psi}\right] \equiv \quad [\psi/\nu^{1/2}] \quad &= \frac{L^2T^{-1}}{(L^2T^{-1})^{1/2}} = (L^2T^{-1})^{1/2}
\end{aligned}$$

Since, in the Prandtl limit, the viscosity ν disappears from the boundary layer equation, the only dimensional parameter available to be combined with x and $\hat{y}$ to form a dimensionless combination is A. We observe that

$$\begin{aligned}
\left[Ax^{m+1}\right] \quad &= \quad \text{velocity} \ \times \ \text{length} = L^2T^{-1} \\
\left[Ax^{m-1}\right] \quad &= \quad \text{velocity} \ / \ \text{length} = T^{-1}
\end{aligned}$$

so the combination

$$\eta = \hat{y}(Ax^{m-1})^{1/2}$$

is clearly dimensionless (and it is the only distinct such combination). Then we are led to the following similarity form for $\hat{\psi}$:

$$\hat{\psi}(x, \hat{y}) = (Ax^{m+1})^{1/2} f(\eta)$$

which is equivalent to writing

$$\psi(x, y) = (\nu Ax^{m+1})^{1/2} f(\eta), \qquad \eta = y\left(\frac{A}{\nu x^{1-m}}\right)^{1/2}. \tag{10}$$

It is important to note that this argument for a similarity form of solution relies crucially on the form of $U(x)$ involving only a single dimensional parameter. If, for example, we had $U(x) = A_1x^{m_1} + A_2x^{m_2}$ then we would only be able to propose a solution in the form

$$\hat{\psi}(x, \hat{y}) = (A_1x^{m_1+1})^{1/2} f\left(\eta, \left(\frac{A_1}{A_2}\right)^{\frac{1}{m_1-m_2}} x\right)$$

which is far less helpful since now f is a function of two arguments and the problem will not produce an ODE for f.

We now proceed in the standard fashion, computing the derivatives of (10) with respect to x and y and substituting them into the boundary layer equation (3). We have that

$$\begin{aligned}
\frac{\partial \eta}{\partial y} &= \left(\frac{A}{\nu x^{1-m}}\right)^{1/2}, \\
\frac{\partial \eta}{\partial x} &= \frac{1}{2}(m-1)\eta/x, \\
u = \frac{\partial \psi}{\partial y} &= (\nu A x^{m+1})^{1/2} f'(\eta) \left(\frac{A}{\nu x^{1-m}}\right)^{1/2} = A x^m f'(\eta), \\
v = -\frac{\partial \psi}{\partial x} &= -\frac{1}{2}(\nu A x^{m-1})^{1/2} \left[(m+1)f + (m-1)\eta f'\right].
\end{aligned} \tag{11}$$

Note that (11) implies that we will need the boundary condition $f'(\eta) \to 1$ and $\eta \to \infty$. Similarly we can then compute that

$$\begin{aligned}
\frac{\partial u}{\partial x} &= A x^{m-1} \left[m f' + \frac{1}{2}(m-1)\eta f''\right] \\
\frac{\partial u}{\partial y} &= A x^m f''(\eta) \left(\frac{A}{\nu x^{1-m}}\right)^{1/2} \\
\frac{\partial^2 u}{\partial y^2} &= A x^m f'''(\eta) \frac{A}{\nu x^{1-m}} = \frac{A^2}{\nu} x^{2m-1} f'''(\eta).
\end{aligned}$$

Substituting these expressions into (3) and recalling that $G(x) \equiv U dU/dx = mA^2 x^{2m-1}$ we find that we can cancel a factor of $A^2 x^{2m-1}$ and we obtain

$$f'\left(m f' + \frac{1}{2}(m-1)\eta f''\right) - \frac{1}{2}\left[(m+1)f + (m-1)\eta f'\right] f'' = m + f'''.$$

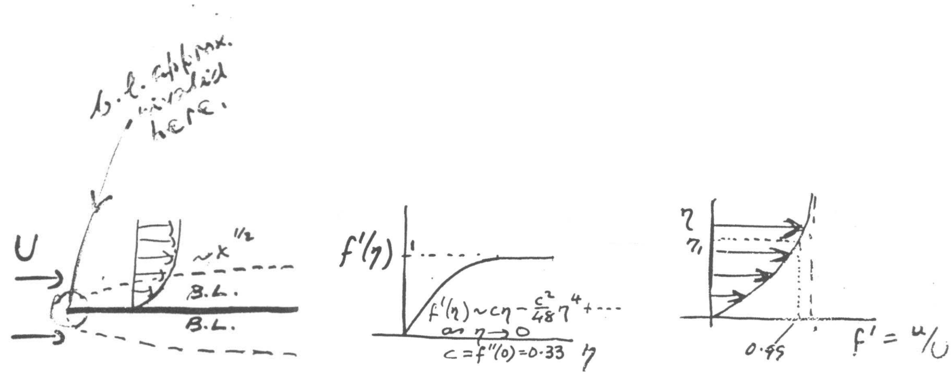
We observe that the two terms involving factors of $(m-1)$ cancel and we are left with the ODE

$$f''' + \frac{1}{2}(m+1) f f'' + m(1 - (f')^2) = 0 \tag{12}$$

which is the *Falkner–Skan equation*. It clearly reduces to (8) in the case $m = 0$. Three boundary conditions are required, and the natural boundary conditions are to require $f = f' = 0$ on $\eta = 0$ so that there is no slip at the boundary $x = 0$. The third boundary condition, as we have already seen, is $f'(\eta) \to 1$ as $\eta \to \infty$ to match to the outer boundary condition where the flow velocity is $U(x)$.

4.9.1 $m = 0$: flat plate

As already remarked, the case $m = 0$ corresponds to the Blasius boundary layer. In this case the similarity variable is just $\eta = y(U/(\nu x))^{1/2}$ and the boundary layer ansatz is valid away from the tip of the flat plate at $x = 0$, see figure (a) below:



(a) (b) (c)

(a) Boundary layer approximations are invalid near $x = 0$. (b) Sketch of the solution (obtained numerically) to the Blasius equation. (c) Velocity profile and definition of η_1.

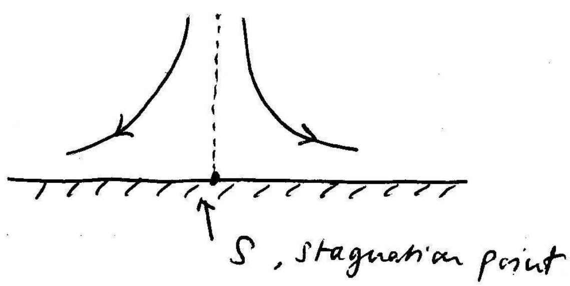
We can define a boundary layer thickness $\delta(x)$ by looking for the value of y at which the velocity u reaches a fixed proportion of the free stream value U. For example, let η_1 be the value at which $u/U \equiv f'(\eta_1) = 0.99$. Then we have

$$\delta(x) = \eta_1 \left(\frac{\nu x}{U}\right)^{1/2}$$

and we see that the thickness of the boundary layer increases parabolically with downstream distance x as sketched in (a) above.

4.9.2 $m = 1$: stagnation point flow

This case was discussed previously, in chapter 3 (section 3.4).



The free stream flow is given by $U = \alpha x$ and the boundary layer equation is in fact exact in this case, so the solution of the boundary layer equation gives an exact solution of the Navier–Stokes equations.

4.9.3 $0 < m < 1$: flow past a wedge

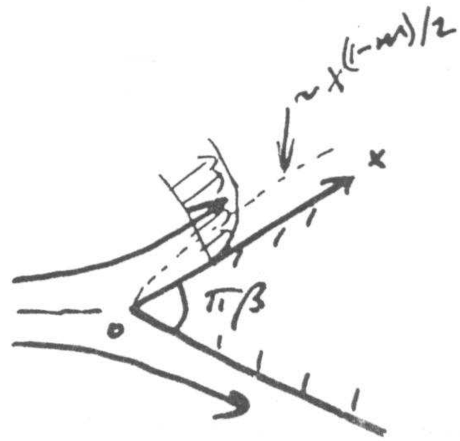
The wedge has interior angle $\pi\beta$ where $m = \beta/(2-\beta)$, or, equivalently, $\beta = 2m/(1+m)$. Then the similarity variable

$$\eta = y\left(\frac{A}{\nu x^{1-m}}\right)^{1/2}$$

so the boundary layer thickness, defined as in subsection 4.9.1, is given by

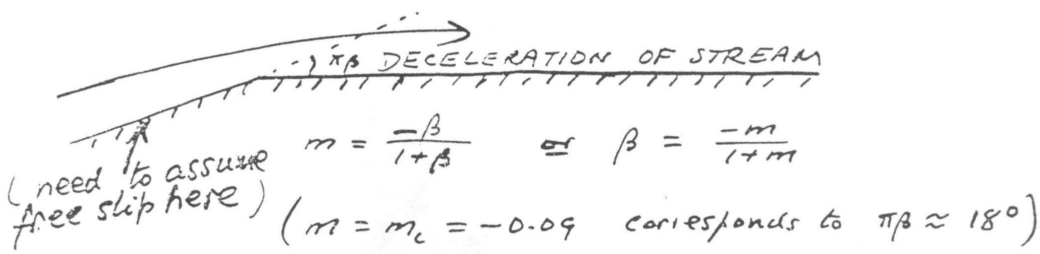
$$\delta(x) = \eta_1 \left(\frac{\nu x^{1-m}}{A} \right)^{1/2} \tag{13}$$

i.e. its width grows more slowly than that parabolic profile for the flat plate. We conclude that the acceleration of the free stream $U(x)$ inhibits the growth of the boundary layer:



4.9.4 $m_c < m < 0$: flow around a slight corner - I

If m is negative but greater than $m_c \approx -0.09$ then (numerically) solutions of the Falkner–Skan equation exist for which $0 < f'(\eta) < 1$ for all $0 < \eta < \infty$, i.e. $f'(\eta)$ remains positive and so the flow is always in the positive x direction:

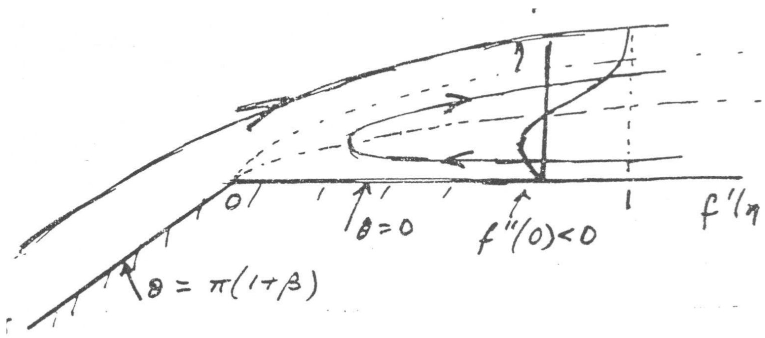


One slightly problematic issue with our solution method is that the section of rigid boundary in $x < 0$ we did assume to be free slip - we assumed that there was no boundary layer here and that the boundary layer only started at $x = 0$. Note that $m_c = -0.09$ corresponds to $\pi\beta \approx 18°$.

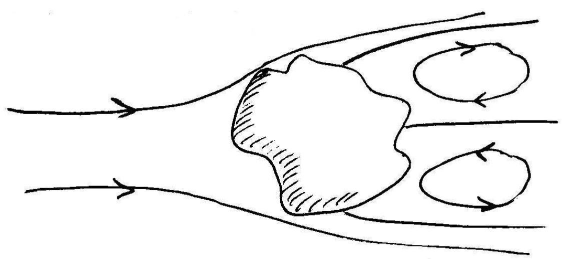
Since the boundary layer thickness $\delta(x) \sim (x^{1-m})^{1/2}$ and $m < 0$ we see that the boundary layer grows more rapidly with increasing x than in the parabolic case: this is due to the deceleration of the free stream $U(x)$ and so this deceleration promotes the growth of the boundary layer.

4.9.5 $m < m_c$: flow around a larger corner - II

If $m < m_c$, i.e. $\pi\beta$ greater than around 18°, then the boundary layer profiles obtained by numerical solution of the Falkner–Skan equation all have the property that $f''(0) < 0$, i.e. f is negative over a range of values of η. This indicates that in fact the flow direction is *reversed* near the wall:



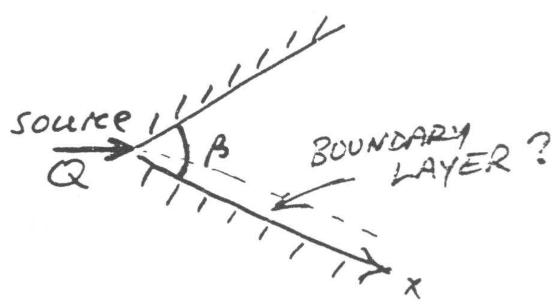
This strange flow configuration suggests than in real flows one would in fact observe the phenomenon of *flow separation* in which a new stagnation point appears on the rigid boundary, dividing the fluid into two regions separated by the streamline that emerges from the stagnation point. Regions of closed streamlines may well result from this, as in this sketch:



4.10 Flow in a diverging channel

We conclude this chapter with two sections highlighting that the existence and uniqueness of solutions to the boundary layer equation should not be taken for granted. Caution is necessary due to the nonlinear nature of the equation.

In this section we consider the flow produced by injecting fluid into the interior of a wedge of angle β from its point:



This is the case $m = -1$ where the Euler limit might reasonably be thought to be the radial flow $u_r = Q/(\beta r)$ due to a line source of strength Q at $r0$. Taking x to be a coordinate along the lower plane we then have $U(x) = A/x$ where $A = Q/\beta$ and so the pressure term driving the outer flow is

$$U\frac{dU}{dx} = -\frac{A^2}{x^3}$$

indicating a very rapid deceleration of the flow away from $x = 0$ (a rather singular point!).

The Falkner–Skan equation for $m = -1$ is

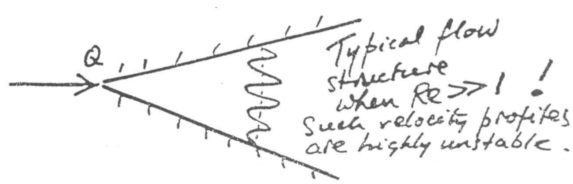
$$\begin{aligned} f''' + (f')^2 - 1 &= 0 \\ \Rightarrow f''f''' + f''(f')^2 - f'' &= 0 \\ \Rightarrow \frac{1}{2}(f'')^2 + \frac{1}{3}(f')^3 - f' &= \text{const} = -\frac{2}{3} \end{aligned}$$

since $f'(\eta) \to 1$ as $\eta \to \infty$. Hence, evaluating this last equation at $\eta = 0$ we have

$$\frac{1}{2}[f''(0)]^2 = -\frac{2}{3}$$

because $f'(0) = 0$ (the no-slip boundary condition). The LHS of this equation is clearly positive and the RHS is negative which is a contradiction. So **no solution of the boundary layer equation is possible in this case.** We conclude that the assumed Euler flow for the outer solution to the problem cannot be correct: the associated deceleration of the free stream $U(x) = A/x$ is too rapid to accomodate a boundary layer.

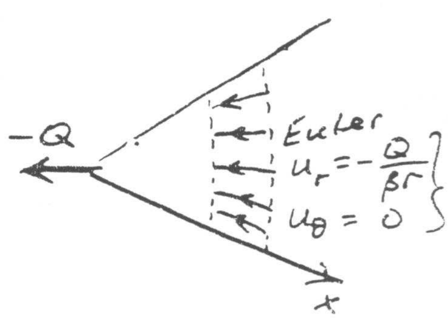
The typical flow structure found in Navier–Stokes for thie problem does not correspond to our supposed 'free stream + boundary layer' approach:



the number of oscillations in the radial velocity profile increases as $Re \to \infty$ and viscous effects remain important throughout the domain rather than being confined to near the boundaries.

4.11 Flow in a converging channel

Finally we consider the effect of substituting the source Q by a sink $-Q < 0$ in this problem of flow in the interior of a wedge.



Since $Q > 0$ we use it to define the similarity variable and streamfunction as before:

$$\eta = \left(\frac{Q}{\nu\beta}\right)^{1/2} \frac{y}{x}, \qquad \psi = \left(\frac{\nu Q}{\beta}\right)^{1/2} f(\eta)$$

so we obtain the Falkner–Skan equation with $m = -1$ as previously:

$$f''' + (f')^2 - 1 = 0$$

but now the outer boundary condition is $f'(\eta) \to -1$ as $\eta \to \infty$. The other boundary conditions are of course unchanged: $f(0) = f'(0) = 0$. We multiply by f'' as previously and integrate, using the outer boundary condition to find the constant of integration as before. Writing $F(\eta) = f'(\eta)$ we then have

$$\begin{aligned} \frac{1}{2}(F')^2 + \frac{1}{3}F^3 - F &= +\frac{2}{3} \\ \Rightarrow (F')^2 &= \frac{2}{3}(1+F)^2(2-F) \end{aligned}$$

which, happily and unexpectedly, can be integrated again as follows:

$$I \equiv \int \frac{dF}{(1+F)\sqrt{2-F}} = \pm\sqrt{\frac{2}{3}} \int d\eta = \pm\sqrt{\frac{2}{3}}\eta + mathrmconst.$$

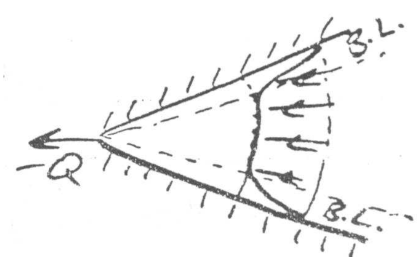
Now let $2 - F = 3\tanh^2\theta$ so that $1 + F = 3(1 - \tanh^2\theta) = 3\mathrm{sech}^2\theta$. Then $dF/d\theta = -6\tanh\theta\mathrm{sech}^2\theta$ so that

$$I = \int -\frac{6}{3\sqrt{3}} d\theta = -\frac{2}{\sqrt{3}}\theta$$

so

$$F(\eta) = 2 - 3\tanh^2\left(\frac{\eta}{\sqrt{2}} + C\right)$$

and the boundary condition $F(0) \equiv f'(0) = 0$ implies that the constant C satisfies $\tanh C = \pm\sqrt{2/3}$, i.e. $C = \pm 3.04...$ and there are two possible solutions for the boundary layer. The intuitively more likely solution corresponds to $C = +3.04...$ and gives a velocity profile that is monotonic in the boundary layer and looks like



The case $C = -3.04...$ has reversed flow in the boundary layer.

We conclude that life, or at any rate fluid mechanics, is full of surprises.