3 Generation and diffusion of vorticity

3.1 The vorticity equation

We start from Navier–Stokes:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} \tag{1}$$

where we have not included a term describing a body force, since the usual ones that we consider will, if present, be conservative (i.e. $\mathbf{F} = \nabla V$) and therefore can be absorbed into the pressure term.

Recall the vector identity (one of those listed in the vector calculus appendix in the book by Acheson)

$$\mathbf{u}\times\boldsymbol{\omega} \equiv \frac{1}{2}\nabla(\mathbf{u}^2) - \mathbf{u}\cdot\nabla\mathbf{u}$$

which implies

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u}\times\boldsymbol{\omega} - \frac{1}{\rho}\nabla\left(p + \frac{1}{2}\rho\mathbf{u}^2\right) + \nu\nabla^2\mathbf{u}.$$

Then, taking the curl of this equation (and noting that ∇^2 commutes with taking the curl) we obtain the **vorticity equation**:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla\times(\mathbf{u}\times\boldsymbol{\omega}) + \nu\nabla^2\boldsymbol{\omega}. \tag{2}$$

A second very useful vector identity (also in the appendix to Acheson) is

$$\nabla\times(\mathbf{u}\times\boldsymbol{\omega}) \equiv \boldsymbol{\omega}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\boldsymbol{\omega} + \mathbf{u}(\nabla\cdot\boldsymbol{\omega}) - \boldsymbol{\omega}(\nabla\cdot\mathbf{u}).$$

Clearly the last two terms on the RHS are zero, from the definition of $\boldsymbol{\omega}$ and by incompressibility. So we have the equivalent forms of the vorticity equation

$$\begin{aligned} \frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} &= \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\omega} \\ \text{or} \qquad \frac{D\boldsymbol{\omega}}{Dt} &= \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\omega} \end{aligned}$$

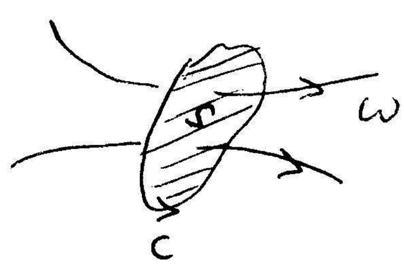
Remarks:

- when $\nu = 0$, vortex lines are 'frozen into' the fluid and the circulation κ is conserved for every material circuit (Kelvin's theorem, see earlier):

  $$\kappa = \oint_C \mathbf{u}\cdot\mathbf{dx} = \int_S \boldsymbol{\omega}\cdot\mathbf{n}\,dS$$

  is the flux of vorticity through the surface S.

- the term $\mathbf{u}\cdot\nabla\boldsymbol{\omega}$ represents 'advection of vorticity by the velocity field'
- the term $\boldsymbol{\omega}\cdot\nabla\mathbf{u}$ represents 'rotation and stretching of vortex lines', although this term really cannot be separated from the $\mathbf{u}\cdot\nabla\boldsymbol{\omega}$ term since they arrive together.



Two-dimensional flow

In the case of 2D flow the vorticity $\boldsymbol{\omega}$ reduces to a scalar quantity, closely related to the streamfunction $\psi(x, y, t)$, since

$$\begin{aligned} \mathbf{u} &= (\partial\psi/\partial y, -\partial\psi/\partial x, 0) \\ \Rightarrow \boldsymbol{\omega} &= (0, 0, -\nabla^2\psi) = (0, 0, \omega) \end{aligned}$$

defining the scalar ω to be the only non-zero (i.e. third) component of $\boldsymbol{\omega}$. For 2D flow we can compute that

$$\boldsymbol{\omega} \cdot \nabla\mathbf{u} = -\nabla^2\psi\frac{\partial}{\partial z}\mathbf{u} = 0$$

and also that

$$\mathbf{u} \cdot \nabla\boldsymbol{\omega} = (0, 0, \mathbf{u} \cdot \nabla\omega)$$

so the vorticity equation simplifies to the scalar equation

$$\frac{D\omega}{Dt} \equiv \frac{\partial\omega}{\partial t} + \mathbf{u} \cdot \nabla\omega = \nu\nabla^2\omega$$

where the $\mathbf{u} \cdot \nabla\omega$ term represents advection and the term on the RHS represents diffusion of vorticity. Therefore, this kind of PDE is called an ‘advection–diffusion’ equation. In this case the PDE is inherently nonlinear since $\mathbf{u}$ and ω are related.

One further manipulation is useful in 2D:

$$\mathbf{u} \cdot \nabla\omega = \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} \equiv -\frac{\partial(\psi, \omega)}{\partial(x, y)} = -J(\psi, \omega)$$

which is a Jacobian quantity: these are often written even more compactly as $J(\psi, \omega)$ where the order of the two arguments clearly matters. So a final way of writing the vorticity equation (in 2D) is

$$\frac{\partial\omega}{\partial t} - \frac{\partial(\psi, \omega)}{\partial(x, y)} = \nu\nabla^2\omega. \tag{3}$$

The inviscid limit $\nu = 0$

If $\nu = 0$ and the flow is steady, i.e. $\partial\omega/\partial t = 0$, then (3) reduces to

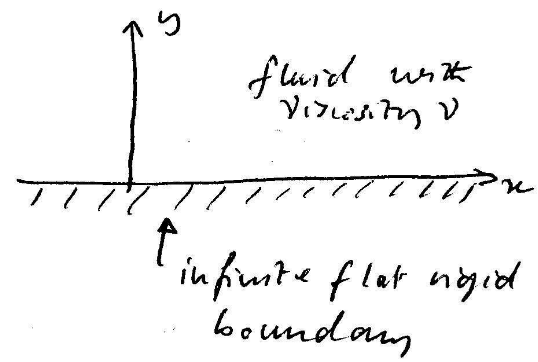
$$\frac{\partial(\psi, \omega)}{\partial(x, y)} = 0$$

which implies that there is a (possibly nonlinear) functional relationship $\omega = F(\psi)$ between ω and ψ. Then ψ satisfies the equation $\nabla^2\psi = -F(\psi)$.

In particular, ψ being constant implies ω is constant, so along streamlines we must have ω constant.

3.2 An impulsively started plate - the ‘Rayleigh problem’

Vorticity is often generated at rigid boundaries. To illustrate the generation and diffusion of vorticity in the simplest possible case, we consider a stationary fluid with a boundary that is impulsively instantaneously accelerated from rest to a constant velocity $\mathbf{U} = (U, 0, 0)$:



More precisely, suppose that for $t < 0$ the fluid and the plate at $y = 0$ are at rest. For $t \geq 0$ the plate moves with velocity $\mathbf{U} = (U, 0, 0)$. We compute the velocity field $\mathbf{u}$ for $y > 0$ and $t > 0$.

Physical intution suggests that the flow is only in the x-direction and is independent of x: $\mathbf{u} = (u(y,t), 0, 0)$ with boundary conditions

$$
\begin{aligned}
u(0,t) &= 0 && \text{for} \quad t < 0, \quad \text{and} \\
u(0,t) &= U && \text{for} \quad t \geq 0, \\
u(y,t) &\to 0 && \text{as} \quad y \to \infty, \quad \text{for all } t
\end{aligned}
$$

and initial condition $u(y, 0) = 0$ in $y > 0$.

Since we anticipate that streamlines are $y =$ constant and therefore are straight lines, we suppose that the inertia term is identically zero:

$$\mathbf{u} \cdot \nabla \mathbf{u} = u \frac{\partial}{\partial x} \mathbf{u}(y,t) = 0$$

so the Navier–Stokes equation (1), in components, becomes

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u, \\
0 &= -\frac{1}{\rho}\frac{\partial p}{\partial y}, \\
0 &= -\frac{1}{\rho}\frac{\partial p}{\partial z}.
\end{aligned}
$$

So we must have $p = p(x,t)$, then, since $p \to \partial_0$ (a constant) as $y \to \infty$, in fact it must be the case that $p = p_0$ for all x and t. So we are left with

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{4}$$

which is the usual linear diffusion equation. By linearity, $u(y,t)$ must depend linearly on the driving velocity U, i.e. $u(y,t) = Uf(y,t;\nu)$ where the function f must be dimensionless. It follows that f is a function of dimensionless combinations of the remaining variables in the problem: y, t and ν. In terms of units we have $[y] = L$, $[t] = T$ and $[\nu] = L^2T^{-1}$ so there is only one dimensionless combination of these three variables: $y/\sqrt{\nu t}$. We define the new dimensionless variable η by

$$\eta = \frac{y}{2\sqrt{\nu t}}$$

where the factor 2 is for later convenience. Thus η is another example of a similarity variable. Let $u(y,t) = Uf(\eta)$. Then

$$\frac{\partial \eta}{\partial t} = -\frac{1}{2}\frac{\eta}{t} \qquad \text{and} \qquad \frac{\partial \eta}{\partial y} = \frac{1}{2\sqrt{\nu t}}$$

so we have

$$\frac{\partial u}{\partial t} = Uf'(\eta)\left(-\frac{1}{2}\frac{\eta}{t}\right) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = Uf''(\eta)\frac{1}{4\nu t}.$$

Substituting these into (4) we obtain

$$f'' + 2\eta f' = 0$$

which is an ODE that can be integrated once after dividing by f' to obtain

$$f' = C \exp\left(-\eta^2\right)$$

which implies

$$f(\eta) = C\int_0^\eta e^{-\xi^2}\, d\xi + D$$

where C and D are constants of integration. From the boundary and initial conditions we have that (being careful with limits)

- At any fixed $t > 0$, on $y = 0$ we have $u(0,t) = U$ which implies at $\eta = 0$ we have $f = 1$, hence $D = 1$.
- At $t = 0$, at any fixed $y > 0$ we have $u(y,0) = 0$ which implies that $\lim_{\eta\to\infty} f(\eta) = 0$, so

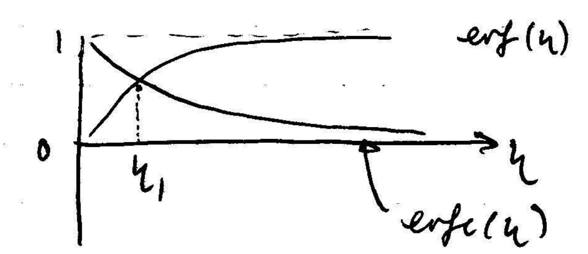
  $$C\int_0^\infty e^{-\xi^2}\, d\xi = -1$$

  hence $C = -2/\sqrt{\pi}$.

So the solution is

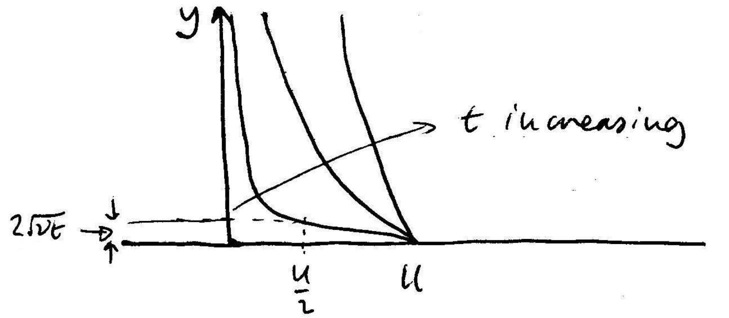
$$f(\eta) = 1 - \mathrm{erf}(\eta) \qquad \text{where} \qquad \mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\xi^2}\, d\xi$$

$\mathrm{erf}(\eta)$ is the 'error function' - the incomplete integral of the Gaussian function. Often $1 - \mathrm{erf}(\eta)$ is called $\mathrm{erfc}(\eta)$, the complementary error function:



The function $\mathrm{erfc}(\eta)$ has a characteristic shape, and the value of $\eta = \eta_*$ at which $\mathrm{erfc}(\eta) = \frac{1}{2}$ is $\eta_* \approx 0.48$. This gives a characteristic length for the diffusion process: $y = 2\sqrt{\nu t}\eta_* \approx 0.96\sqrt{\nu t}$.

The velocity profile therefore evolves with increasing time by smoothing out the imposed step change in the velocity on the boundary:



and at a given time t the velocity of the fluid has reached $U/2$ at a distance $y = 2\sqrt{\nu t}\eta_*$ from the plate.

So far we have not discussed the vorticity of the flow. A simple calculation gives

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \left(0, 0, -\frac{\partial u}{\partial y}\right)$$

so let us define the usual scalar vorticity

$$\omega(y,t) = -\partial u/\partial y = -\frac{Uf'(\eta)}{2\sqrt{\nu t}} = \frac{U}{\sqrt{\pi \nu t}}\mathrm{e}^{-y^2/(4\nu t)}.$$

This expression shows that the vorticity distribution is initially δ-like and concentrated on the boundary $y = 0$ but then evolves into an increasingly flat Gaussian profile: it is clear that the vorticity enters the flow 'through' the boundary.

A further short computation of interest is to compute the total vorticity in the flow, for fixed $t > 0$:

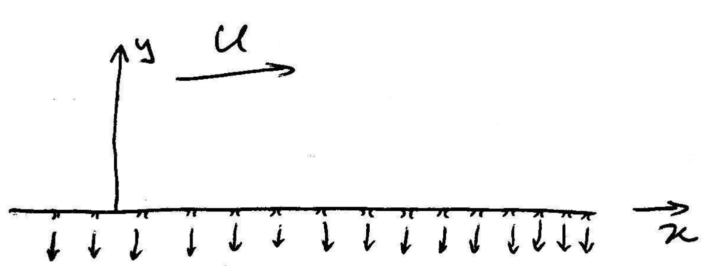
$$\int_0^\infty \omega \, dy = \int_0^\infty -\frac{\partial u}{\partial y} \, dy = [u]_0^\infty = U$$

So the vorticity of the flow changes discontinuously at $t = 0$ and then diffuses into the interior of the flow. The characteristic 'thickness' of the viscous boundary layer so generated is $\delta(t) \approx 0.96\sqrt{\nu t}$.

3.3 The suction boundary layer

In this section we solve a steady-state problem for which a boundary layer is clearly generated and maintained close to a boundary.

We consider flow over a porous boundary through which fluid flows by suction:



Suppose that we extract fluid at a constant velocity $(0, -V, 0)$ on the boundary $y = 0$, and that $\mathbf{u} \sim (U, 0, 0)$ at $y = \infty$. Again, the problem set-up implies that a solution exists that does not depend on x, so we look for a steady solution $\mathbf{u} = (u(y), v(y), 0)$.

Incompressibility $\nabla \cdot \mathbf{u} = 0$ implies that $dv/dy = 0$ throughout the domain, hence

$$v(y) = \text{constant} = -V$$

(since $v(0) = -V$). Now we consider (steady) Navier–Stokes:

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}$$

and the inertial term is

$$\mathbf{u} \cdot \nabla \mathbf{u} = \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) (u(y), -V, 0) = \left(-V \frac{du}{dy}, 0, 0 \right)$$

which gives, for the components of the Navier–Stokes equation:

$$\begin{aligned} -V \frac{du}{dy} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{d^2 u}{dy^2} \\ 0 &= -\frac{1}{\rho} \frac{\partial p}{\partial y} \\ 0 &= -\frac{1}{\rho} \frac{\partial p}{\partial z} \end{aligned}$$

So $p = p(x) = p_0$ constant by the same argument as in the previous subsection: the pressure is independent of y and constant at $y = \infty$.

From the x-component of Navier–Stokes we have, integrating once,

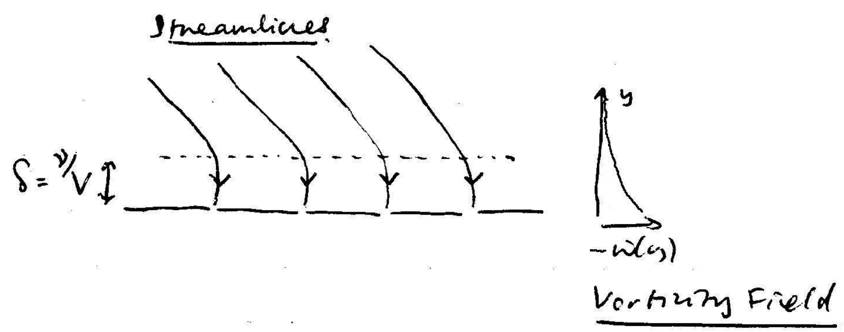
$$-Vu = \nu \frac{du}{dy} + \text{constant} = -VU \text{ at } y = \infty$$

since $u = U$ at $y = \infty$. Hence

$$u(y) = U \left(1 - \mathrm{e}^{-Vy/\nu} \right)$$

which satisfies the boundary conditions $u(0) = 0$ and $u(\infty) = U$.

Hence $-\omega(y) = (UV/\nu) \exp(-Vy/\nu)$:

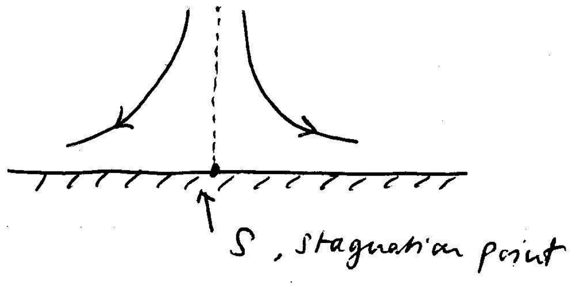


The interpretation is then that vorticity is generated at the boundary $y = 0$ and tries to diffuse away, but it is prevented from doing so by the applied suction. Notice the appearance of the dimensionless group Vy/ν in the solution.

3.4 Stagnation point boundary layer

In this subsection we return to consideration of the flow near a stagnation point at $x = y = 0$, for which the corresponding inviscid flow is straightforward: we would have the streamfunction $\psi = \alpha xy$ in suitable coordinates (suppose $\alpha > 0$), with the velocity field

$$u = \frac{\partial \psi}{\partial y} = \alpha x, \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x} = -\alpha y$$



Now we consider the case of a viscous fluid with a rigid boundary at $y = 0$, such that the flow far from the boundary is the inviscid one we thought of earlier: $\psi \sim \alpha xy$ as $y \to \infty$. Can we propose a solution to this modified problem in the form

$$\psi(x, y) = \alpha x f(y)?$$

Actually, no, on dimensional grounds - we would be forced back into the inviscid solution. The way around this is to introduce a new parameter which incorporates viscosity ν. Let us first consider the dimensions of the quantities we have introduced so far:

$$[\nu] = L^2T^{-1}, \qquad [\alpha] = [u/x] = T^{-1}, \qquad [x] = [y] = L$$

so we can define the new lengthscale

$$\delta = \left(\frac{\nu}{\alpha}\right)^{1/2} \tag{5}$$

which does incorporate viscosity. So, the combination $\eta = y/\delta$ is dimensionless, and a modification of the inviscid streamfunction which is dimensionally correct is to write

$$\psi = \alpha x \delta F(\eta)$$

where F is now a dimensionless function of a dimensionless argument. We can use this ansatz to investigate whether solutions of the vorticity equation of this form as possible. First we compute the velocity components

$$\begin{aligned} u &= \frac{\partial \psi}{\partial y} = \alpha x \delta F'(\eta)\frac{1}{\delta} = \alpha x F'(\eta) \\ v &= -\frac{\partial \psi}{\partial x} = -\alpha \delta F(\eta) \end{aligned}$$

from the form of u and v we deduce the boundary conditions for $F(\eta)$: we need $u \to \alpha x$ as $\eta \to \infty$ (i.e. $y \to \infty$), hence

$$F'(\eta) \to 1 \qquad \text{as} \qquad \eta \to \infty. \tag{6}$$

We also require $u = v = 0$ on $y = 0$, which implies

$$F(0) = F'(0) = 0. \tag{7}$$

Returning to the velocity components, we now compute the vorticity

$$\boldsymbol{\omega} = (0, 0, -\nabla^2\psi) = (0, 0, \omega)$$

where

$$\omega = -\frac{\alpha x \delta}{\delta^2}F''(\eta)$$

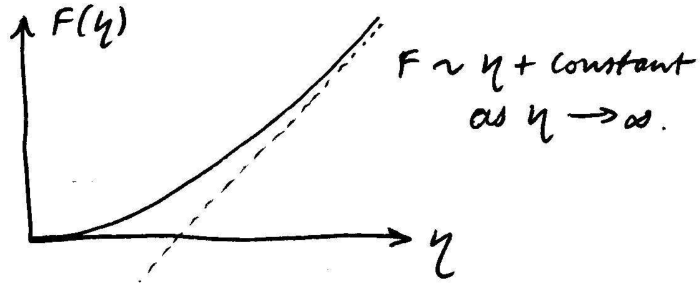
so the steady (scalar) vorticity equation $\mathbf{u}\cdot\nabla\omega = \nu\nabla^2\omega$ becomes

$$\begin{aligned}\left(\alpha x F'(\eta)\frac{\partial}{\partial x} - \alpha\delta F(\eta)\frac{\partial}{\partial y}\right)\left(-\frac{\alpha x}{\delta}F''(\eta)\right) &= \nu\left(-\frac{\alpha x}{\delta}\frac{1}{\delta^2}F''''(\eta)\right)\\ \Rightarrow -F'F'' + FF''' &= -F'''' \end{aligned} \tag{8}$$

after cancellations. This is a nonlinear ODE for $F(\eta)$. It can be integrated once, adding and subtracting $-F'F'' + F'F''$ to the LHS of (8). We then have

$$(F')^2 - FF'' - F''' = \text{constant}$$

and this constant is 1 by applying boundary condition (6). No analytical methods are available to integrate this third-order nonlinear ODE further, so we resort to numerical solutions which satisfy the remaining boundary conditions (7) and which therefore look roughly like



The interpretation is that vorticity, created at $y = 0$ via the rigid boundary, diffuses away from the boundary into $y > 0$ but it is advected back towards $y = 0$ by the y-component $\sim -\alpha y$ of the velocity field.

3.5 The bathtub vortex

Our last example again illustrates the existence of steady states in which the diffusion and advection of vorticity are balanced against each other. In the bathtub vortex we consider a superposition of a uniform strain flow

$$\mathbf{u}_1 = (-\alpha x, -\alpha y, 2\alpha z)$$

(where we take $\alpha > 0$), in Cartesian coordinates $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. This flow is axisymmetric and incompressible. In cylindrical polar coordinates (r, θ, z) we have

$$\mathbf{u}_1 = (-\alpha r, 0, 2\alpha z) \tag{9}$$

with respect to unit vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$. Note that this flow is irrotational: $\nabla\times\mathbf{u}_1 = 0$.

Now suppose that we superpose a swirling velocity field $\mathbf{u}_2 = (0, v(r), 0)$ so that the total velocity field $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$. Then the vorticity $\boldsymbol{\omega}$ is

$$\begin{aligned}\boldsymbol{\omega} &= \nabla\times(\mathbf{u}_1 + \mathbf{u}_2)\\ &= \nabla\times\mathbf{u}_2\\ &= \frac{1}{r}\begin{vmatrix}\mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ 0 & rv(r) & 0\end{vmatrix}\\ &= \frac{1}{r}\frac{\partial}{\partial r}(rv(r))\mathbf{e}_z \equiv \omega(r)\mathbf{e}_z\end{aligned}$$

showing that the vorticity is everywhere in the z-direction.

Now we consider what the form of $v(r)$ must be in order for this flow field to be a steady solution of the vorticity equation

$$0 = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}$$

where $\mathbf{u} = (-\alpha r, v(r), 2\alpha z)$, and $\boldsymbol{\omega} = (0, 0, \omega(r))$ in $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ coordinates. Then

$$\begin{aligned} (\mathbf{u}_1 + \mathbf{u}_2) \times \boldsymbol{\omega} &= (v\omega, \alpha r\omega, 0) \\ \Rightarrow \nabla \times ((\mathbf{u}_1 + \mathbf{u}_2) \times \boldsymbol{\omega}) &= \frac{1}{r} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ v\omega & \alpha r^2 \omega & 0 \end{vmatrix} \\ &= \frac{1}{r}\frac{\partial}{\partial r}(\alpha r^2 \omega)\mathbf{e}_z. \end{aligned}$$

Therefore the z-component of the vorticity equation is

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(\alpha r^2 \omega) + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right)$$

which integrates once immediately to give

$$\alpha r^2 \omega + \nu r \frac{\partial \omega}{\partial r} = C.$$

for some constant of integration C. Natural boundary conditions to apply in this case are that the vorticity field is localised around the orgin, i.e. $\omega \to 0$ and $\partial\omega/\partial r \to 0$ as $r \to \infty$. This implies $C = 0$, so

$$\frac{d\omega}{dr} + \frac{\alpha}{\nu} r\omega = 0$$

which can be solved by separation of variables:

$$\begin{aligned} \log \omega &= -\frac{\alpha r^2}{2\nu} + \text{constant} \\ \Rightarrow \omega(r) &= \omega_0 \mathrm{e}^{-\alpha r^2/(2\nu)} \end{aligned}$$

From the initial axisymmetric strain flow (9) we see that $[\alpha] = T^{-1}$ so $[\nu/\alpha] = L^2$ and hence we can define a lengthscale δ by $\delta^2 = 2\nu/\alpha$ to write $\omega(r)$ in the form

$$\omega(r) = \omega_0 \mathrm{e}^{-(r/\delta)^2}$$

which shows that the distribution of vorticity is Gaussian, with a characteristic width δ which is the unique lengthscale that can be formed as a ratio of the strengths of the outward diffusion of vorticity and the inward advection by the strain flow. We finish by computing the swirling velocity field $v(r)$ and the total circulation of the vortex.

Since

$$\begin{aligned} \omega(r) &= \frac{1}{r}\frac{\partial}{\partial r}(rv(r)) \\ \Rightarrow rv(r) &= \int_0^r \omega_0 \tilde{r} \mathrm{e}^{-(\tilde{r}/\delta)^2}\, d\tilde{r} \\ \Rightarrow v(r) &= \frac{\omega_0 \nu}{\alpha r}\left(1 - \mathrm{e}^{-\alpha r^2/(2\nu)}\right). \end{aligned}$$

Equivalently we can write this as

$$v(r) = \frac{\Gamma}{2\pi r}\left(1 - \mathrm{e}^{-\alpha r^2/(2\nu)}\right).$$

where $\Gamma = 2\pi\omega_0\nu/\alpha$ is the total circulation of the vortex: check that

$$\Gamma = \int_0^\infty \omega(r) 2\pi r \, dr.$$

Note that for small r we have $v(r) \approx \frac{1}{2}\omega_0 r$ and for large r we have $v(r) \approx \Gamma/(2\pi r)$.

As already mentioned, the interpretation is the balance between outward diffusion of vorticity and inward advection.

Remarks:

1. Although the velocity $v(r)$ goes to zero at the centre of the vortex, the pressure attains its minimum there: consider the $\mathbf{e}_r$ component of the Navier–Stokes equation:

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \text{other terms}$$

and

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\frac{v^2}{r}\mathbf{e}_r$$

which is the usual 'centrifugal force' term familiar from rigid body motion in a circle. So

$$\begin{aligned} \frac{\partial p}{\partial r} &= \frac{1}{r}\rho v^2 + \text{other terms} \\ \Rightarrow p &= p_0 + \int_0^r \frac{1}{r}\rho v^2 \, dr \\ \Rightarrow p(r) &= p_0 + \rho \frac{\omega_0^2}{8} r^2 + O(r^4) \end{aligned}$$

for small r.

2. As a result of the low pressure at the vortex centre, experimentally, vorticies in liquids can be visualised by introducing small air bubbles which therefore migrate towards the centres of the vortices (particles denser than the liquid would be 'thrown outwards').