2 Flows without inertia

These notes are heavily based on lectures delivered by Keith Moffatt in DAMTP. Much of the material should be found in any sufficiently mathematical textbook on the subject. The books by Batchelor and Acheson are good references for the material in this and later sections of the course, and some precise references are given where appropriate.

A small number of proofs are included here, but they are non-examinable. All non-examinable material is indicated by being surrounded by brackets: [** $\cdots$ **].

2.1 Introduction

Recall, from previous courses, the momentum equation

$$\rho\frac{Du_i}{Dt} = \rho F_i + \frac{\partial}{\partial x_j}\sigma_{ij} \tag{1}$$

due to Cauchy, where u_i is the ith component of the fluid velocity, and σ_{ij} is the stress tensor. For a Newtonian fluid we propose that the stress tensor is linearly related to the velocity gradient tensor $\partial u_i/\partial x_j$. Assuming that the fluid is isotropic, and proposing the most general relation between σ_{ij} and $\partial u_i/\partial x_j$ we are lead to the constitutive relation for an incompressible Newtonian fluid:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij} \tag{2}$$

where μ is the shear viscosity of the fluid and p is the pressure. Inserting (2) into (1) leads to the Navier–Stokes equation:

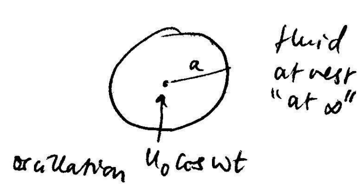
$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \rho\mathbf{F} - \nabla p + \mu\nabla^2\mathbf{u}\,. \tag{3}$$

When $\mu = 0$ this reverts to the Euler equation already seen, but the limit $\mu \to 0$ is not a straightforward one: immediately we can see that this involves ignoring the highest derivatives of $\mathbf{u}$ and therefore it is a singular perturbation.

2.2 Orders of magnitude of terms

Let L be a typical length scale for a given flow situation. This is not uniquely defined, but usually there are sensible candidates: consider for example flow past a cylinder of cross-sectional radius L, or a sphere of radius L, or flow in a channel of width L. Similarly, let U be a typical velocity for the flow, usually derived from that imposed at a boundary surface. Let T be a typical timescale (often it happens that T is of the same magnitude as L/U, but this is not always the case).

Example: Oscillating sphere



Consider a rigid sphere of radius a whose centre moves at a velocity $U_0\mathbf{e}_z\cos\omega t$, surrounded by fluid at rest at infinity. Appropriate choices are then $L = a$, $U = U_0$ and

$T = 2\pi/\omega$. Then, considering the orders of magnitude of the terms in (3) we have

$$\begin{aligned} \left|\frac{\partial \mathbf{u}}{\partial t}\right| &\sim \frac{U}{T} \\ |\mathbf{u}\cdot\nabla\mathbf{u}| &\sim \frac{U^2}{L} \\ \left|\nu\nabla^2\mathbf{u}\right| &\sim \frac{\nu U}{L^2} \end{aligned}$$

Now we can investigate the relative importance of these terms. Firstly,

$$\frac{|\mathbf{u}\cdot\nabla\mathbf{u}|}{|\nu\nabla^2\mathbf{u}|} \sim \frac{UL}{\nu} \tag{4}$$

defines the Reynolds number $Re \equiv UL/\nu$ which measures the relative importance of the nonlinear and viscous terms. Secondly,

$$\frac{\left|\frac{\partial \mathbf{u}}{\partial t}\right|}{|\nu\nabla^2\mathbf{u}|} \sim \frac{L^2}{\nu T} \tag{5}$$

defines the Stokes number $S \equiv L^2/(\nu T)$ which is the ratio of the unsteady to the viscous terms. Note that if $T \sim L/U$ then $S \sim UL/\nu$ and so S and Re are equivalent.

If $Re \ll 1$ and $S \ll 1$ then we should be able to neglect the material derivative term on the LHS of (3) which would yield the Stokes equations (in the absence of body forces):

$$0 = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} \tag{6}$$

$$0 = \nabla\cdot\mathbf{u} \tag{7}$$

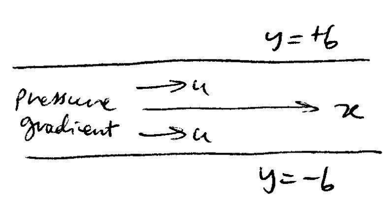
Solutions of the Stokes equations will be focus of this chapter.

2.3 Poiseuille Flow

[see Batchelor pages 181-183 for a discussion of several cases including this one.]

There is a special case in which the inertia terms are not approximately small, but identically zero. This arises for unidirectional flow where the velocity is a function only of a second coordinate.

Consider a steady pressure-driven flow between two planes $y = \pm b$. Assume that the velocity field is $\mathbf{u} = (u(y), 0, 0)$.



Then a simple calculation shows that both parts of $D\mathbf{u}/Dt$ are identically zero. Therefore such a flow must satisfy

$$0 = -\nabla p + \mu\nabla^2\mathbf{u}$$

i.e.

$$\frac{\partial p}{\partial x} = \mu\frac{d^2 u}{dy^2}, \qquad \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$$

So $p = p(x)$ but $\partial p/\partial x$ is a function of y which is a contradiction unless $\partial p/\partial x =$ constant $= -G$, say. Then

$$\frac{d^2u}{dy^2} = -G/\mu$$

and, requiring no slip on $y = \pm b$ (i.e. $u = 0$ on $y = \pm b$) implies

$$u(y) \quad = \quad -\frac{G}{2\mu}(y^2 - b^2)$$

which is a parabolic velocity profile. The total volume flux (per unit length in the z-direction is given by

$$Q \quad = \quad \int_{-b}^{b} u(y)dy = \frac{2Gb^3}{3\mu}.$$

In the case that the streamlines are curved rather than straight (as here) we can see that $\mathbf{u} \cdot \nabla \mathbf{u} \sim \kappa u^2$ where κ is the curvature of the streamlines. If the curvature is small in this case then we may still be able to use the Stokes equations as an approximation to the flow.

2.4 Properties of the Stokes Equations

[Batchelor, section 4.8, pages 227-228]

Consider a volume V containing fluid, bounded by a surface S on which the velocity field is prescribed: $\mathbf{u} = \mathbf{U}(\mathbf{x})$ on S.

Remark: Linearity.

The Stokes equations (6) - (7) are linear: if $\{\mathbf{u}_1, p_1\}$ and $\{\mathbf{u}_2, p_2\}$ are any two solutions, then $\{\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2, \lambda_1 p_1 + \lambda_2 p_2\}$ is also a solution (and satisfies the linearly combined boundary conditions on all boundaries).

Remark: Reversibility.

The Stokes equations are reversible. This is the case $\lambda_1 = -1$, $\lambda_2 = 0$ of the remark above, in fact. Reversing the boundary conditions implies that the flow is reversed everywhere and is still a solution.

Theorem 1 (Solutions to the Stokes Equations are unique) . *Suppose that* $\{\mathbf{u}^{(1)}, p^{(1)}\}$ *and* $\{\mathbf{u}^{(2)}, p^{(2)}\}$ *are two solutions satisfying the same boundary condition, i.e.* $\mathbf{u}^{(1)} = \mathbf{U}(\mathbf{x})$ *on* S *and* $\mathbf{u}^{(2)} = \mathbf{U}(\mathbf{x})$ *on* S. *Then* $\mathbf{u}^{(1)} = \mathbf{u}^{(2)}$ *and* $p^{(1)} - p^{(2)} = \text{const}$ *everywhere in* V.

[** **Proof:** Consider the velocity field $\tilde{\mathbf{u}} = \mathbf{u}^{(2)} - \mathbf{u}^{(1)}$, so that $\tilde{\mathbf{u}} = 0$ on S. Similarly, let $\tilde{p} = p^{(2)} - p^{(1)}$, $\tilde{e}_{ij} = e_{ij}^{(2)} - e_{ij}^{(1)}$ and $\tilde{\sigma}_{ij} = \sigma_{ij}^{(2)} - \sigma_{ij}^{(1)}$. Then from the consitutive law we have

$$\tilde{\sigma}_{ij} \quad = \quad \tilde{p}\delta_{ij} + 2\mu\tilde{e}_{ij}$$

and the Stokes equations imply $\partial/\partial x_j(\tilde{\sigma}_{ij}) = 0$. Now, consider

$$\begin{aligned}
2\mu \int_V \tilde{e}_{ij}\tilde{e}_{ij}\, dV &= \int_V \tilde{e}_{ij}\tilde{\sigma}_{ij}\, dV \qquad \text{since } \tilde{e}_{kk} = 0 \\
&= \int_V \frac{\partial}{\partial x_j}(\tilde{u}_i)\tilde{\sigma}_{ij}\, dV \qquad \text{using } \tilde{\sigma}_{ij} = \tilde{\sigma}_{ji} \\
&= -\int_V \tilde{u}_i \frac{\partial}{\partial x_j}(\tilde{\sigma}_{ij})\, dV \qquad \text{using } \tilde{\mathbf{u}} = 0 \text{ on } S \\
&= 0.
\end{aligned}$$

Hence $\tilde{e}_{ij} \equiv -$ in V, and so $e_{ij}^{(1)} = e_{ij}^{(2)}$.
Now, we can also compute, using $\tilde{\mathbf{u}} = 0$ on S, that

$$\int_V |\tilde{\boldsymbol{\omega}}|^2 \, dV \quad = \quad 2 \int_V \tilde{e}_{ij} \tilde{e}_{ij} \, dV$$

and therefore this quantity is also zero, implying that $\boldsymbol{\omega}^{(1)} \equiv \boldsymbol{\omega}^{(2)}$ everywhere in V.
Putting these two results together implies that $\partial / \partial x_j (u_i^{(1)} - u_i^{(2)}) = 0$ everywhere in V, and, since the velocity fields coincide on S they must therefore be equal everywhere in V. From this it follows also that the pressure fields can differ by at most a constant. □ **]

Theorem 2 (The Minimum Dissipation Theorem) . *Let* $\mathbf{u}^{(1)}$ *be the unique Stokes flow satisfying* $\mathbf{u}^{(1)} = \mathbf{U}(\mathbf{x})$ *on* S. *Let* $\mathbf{u}^{(2)}$ *be any kinematically possible flow, i.e. one that merely satisfies* $\nabla \cdot \mathbf{u}^{(2)} = 0$ *in* V *and* $\mathbf{u}^{(2)} = \mathbf{U}(\mathbf{x})$ *on* S. *Then*

$$\int_V e_{ij}^{(1)} e_{ij}^{(1)} dV \leq \int_V e_{ij}^{(2)} e_{ij}^{(2)} dV$$

with equality only if $\mathbf{u}^{(2)} \equiv \mathbf{u}^{(1)}$. *In other words, the Stokes flow has a smaller rate of viscous energy dissipation* Φ *than any other flow in* V *(recall the definition of* $\Phi = 2\mu \int_V e_{ij} e_{ij} dV$*).*

[** **Proof:** Consider the velocity field $\tilde{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2$ as before. Then we can compute that

$$2\mu \int_V \tilde{e}_{ij} e_{ij}^{(1)} \, dV = \int_V \tilde{e}_{ij} \sigma_{ij}^{(1)} \, dV = \int_V \frac{\partial \tilde{u}_j}{\partial x_i} \sigma_{ij}^{(1)} \, dV = - \int_V \tilde{u}_j \frac{\partial}{\partial x_i} \sigma_{ij}^{(1)} \, dV = 0$$

Hence

$$\begin{aligned} \int_V e_{ij}^{(2)} e_{ij}^{(2)} - e_{ij}^{(1)} e_{ij}^{(1)} \, dV \quad &= \quad \int_V \tilde{e}_{ij} \left(e_{ij}^{(2)} + e_{ij}^{(1)} \right) \, dV \\ &= \quad \int_V \tilde{e}_{ij} \tilde{e}_{ij} \, dV + 2 \int_V \tilde{e}_{ij} e_{ij}^{(1)} \, dV \geq 0, \end{aligned}$$

with equality if and only if $\tilde{e}_{ij} = 0$, i.e. if and only if $\mathbf{u}^{(1)} = \mathbf{u}^{(2)}$. □ **]

Theorem 3 (Unsteady Stokes flow relaxes monotonically to steady Stokes flow) . *Consider a time-dependent solution* $\mathbf{u}(\mathbf{x}, t)$ *to the unsteady Stokes equation*

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \mu \nabla^2 \mathbf{u}$$

subject to the steady boundary condition $\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x})$ *on* S. *Then the kinetic energy of the flow decreases monotonically to its minimum value which is attained when the flow becomes steady Stokes flow.*

[** **Proof:** Consider the rate of change in time of the viscous energy dissipation:

$$\begin{aligned} \frac{d}{dt} \int_V e_{ij} e_{ij} \, dV \quad &= \quad 2 \int_V e_{ij} \frac{\partial}{\partial t} e_{ij} \, dV \\ &= \quad \frac{1}{\mu} \int_V \sigma_{ij} \frac{\partial}{\partial t} e_{ij} \, dV \qquad \text{since } e_{ii} = 0 \; \forall \; t \\ &= \quad \frac{1}{\mu} \int_V \sigma_{ij} \frac{\partial}{\partial t} \left(\frac{\partial u_i}{\partial x_j} \right) \, dV \qquad \text{note that } \sigma_{ij} = \sigma_{ji} \\ &= \quad \frac{1}{\mu \rho} \int_V \sigma_{ij} \frac{\partial^2}{\partial x_j \partial x_k} \sigma_{ik} \, dV \end{aligned} \tag{8}$$

substituting in the Stokes equation again. Now suppose that we maintain the steady boundary condition $\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x})$ on S for all t, so that $\partial \mathbf{u}/\partial t = 0$ on S, i.e. $\partial/\partial x_k \sigma_{ik} = 0$ on S, then, integrating (8) by parts we obtain

$$\frac{d}{dt}\int_V e_{ij}e_{ij}\,dV = -\frac{1}{\mu\rho}\int_V \left(\frac{\partial}{\partial x_j}\sigma_{ij}\right)\left(\frac{\partial}{\partial x_k}\sigma_{ik}\right)\,dV \leq 0.$$

so that Φ decreases monotonically to its minimum value, obtained when $\partial\sigma_{ij}/\partial x_j \equiv 0$ in V. □ **]

2.5 2D Stokes flow

Recall that in 2D the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ may always be satisfied by introducing a streamfunction $\psi(x, y)$ such that

$$\mathbf{u} = \nabla \times (\mathbf{e}_z\psi) = (\partial\psi/\partial y, -\partial\psi/\partial x, 0).$$

Recall also that

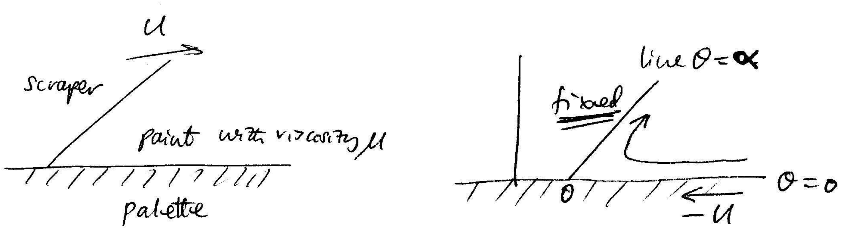
$$\boldsymbol{\omega} = (0, 0, -\nabla^2\psi).$$

Therefore, taking the curl of (6) we obtain

$$\nabla^2\boldsymbol{\omega} \equiv \nabla^2\nabla^2\psi = 0$$

which is the biharmonic equation, often written $\nabla^4\psi = 0$.

2.6 The paint-scraper problem



(a) (b)

(a) Paint scraper in the frame where the wall is stationary and the scraper moves at velocity U. (b) In the frame where the scraper in stationary and the wall moves at a velocity $-U$.

Choosing a frame of reference in which the scraper (the line $\theta = \alpha$) is steady and the wall (at $\theta = 0$) moves with velocity $-U$ in the x-direction we have to solve $\nabla^4\psi = 0$ in the wedge $0 \leq \theta \leq \alpha$ subject to the boundary conditions $u_r = -U$, $u_\theta = 0$ on $\theta = 0$ and $u_r = u_\theta = 0$ on the line $\theta = \alpha$.

From dimensional analysis we see that ψ has dimensions of length $\times$ velocity, and given that there is only one variable with each of these units available, i.e. r and U respectively, we propose a solution in the (variable-separable) form $\psi(r, \theta) = rUf(\theta)$ since θ is dimensionless. Now we reduce the biharmonic equation to an ODE for $f(\theta)$ and apply the boundary conditions.

First, it is convenient to introduce the notation $F(\theta) = f + f''$ since

$$\nabla^2\psi = \frac{U}{r}(f + f'') = \frac{U}{r}F(\theta)$$

So then

$$\nabla^4\psi = \frac{U}{r^3}(F + F'') = 0$$

which implies

$$F(\theta) \quad = \quad A' \cos\theta + B' \sin\theta$$

for some constants A' and B'. Then we find the general solution for f:

$$f(\theta) \quad = \quad A\cos\theta + B\sin\theta + C\theta\cos\theta + D\theta\sin\theta.$$

The radial velocity boundary condition $u_r = (1/r)\partial\psi/\partial\theta \equiv Uf'(\theta)$ on $\theta = 0, \alpha$ implies

$$f'(0) = -1, \qquad f'(\alpha) = 0.$$

Two further boundary conditions are given by the fact that we require the lines $\theta = 0, \alpha$ to be streamlines on which $\psi = 0$, since $\psi = 0$ at $r = 0$. Therefore

$$f(0) = 0, \qquad f(\alpha) = 0.$$

Applying these four boundary conditions determines the four constants of integration A, B, C, D:

$$f(\theta) \quad = \quad \frac{\alpha(\alpha-\theta)\sin\theta - \theta\sin(\alpha-\theta)\sin\alpha}{\sin^2\alpha - \alpha^2}.$$

It is instructive also to find the pressure distribution in the fluid. From integrating the Stokes equation (6) this is found to be

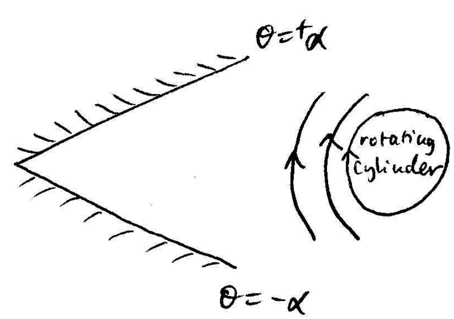
$$p \quad = \quad p_0 + \frac{2\mu U}{r}\left(\frac{\alpha\sin\theta + \sin\alpha\sin(\alpha-\theta)}{\alpha^2 - \sin^2\alpha}\right)$$

from which we see that $p \to \infty$ as $r \to 0$, so an infinite force is needed to keep the scraper in contact with the wall.

2.7 Flow in a corner generated by remote forcing

[Acheson section 7.3, pages 229-232]

In this section we will investigate the generation of steady flow in a corner, between rigid boundaries fixed to be at $\theta = \pm\alpha$. We suppose that there is some remote device, e.g. a rotating cylinder, that drives the flow, see sketch.



We ask for the structure of the flow near the corner, i.e. at small r.

Similar to the previous problem, we guess that there is a separable solution for the streamfunction, and take $\psi(r,\theta) = r^\lambda f(\theta)$ - taking the leading-order term in the r-dependence. Then, defining $F(\theta) = \lambda^2 f + f''$ in a manner also similar to the previous subsection, we have

$$\begin{aligned}\nabla^2\psi &= (\lambda^2 f + f'')r^{\lambda-2} \equiv r^{\lambda-2}F(\theta)\\ \Rightarrow \nabla^4\psi &= (F'' + (\lambda-2)^2F)r^{\lambda-4} = 0\\ \Rightarrow F(\theta) &= A'\cos(\lambda-2)\theta + B'\sin(\lambda-2)\theta\\ \Rightarrow f(\theta) &= A\cos\lambda\theta + B\sin\lambda\theta + C\cos(\lambda-2)\theta + D\sin(\lambda-2)\theta\end{aligned}$$

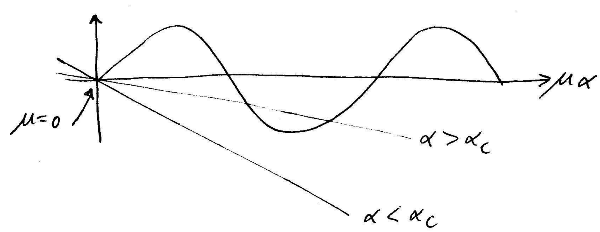
which has four linearly independent functions as long as $\lambda \neq 1$. Now we need to satisfy the boundary conditions $f(\pm\alpha) = 0$ and $f'(\pm\alpha) = 0$. Symmetry considerations imply that $u_r(r,\theta)$ is odd in θ and $u_\theta(r,\theta)$ is even in θ. Hence $\psi(r,\theta)$ must be even in θ, and so $B = D = 0$. Now we are left with the pair of equations

$$f(\alpha) = A\cos\lambda\alpha + C\cos(\lambda-2)\alpha = 0$$
$$f'(\alpha) = -\lambda A\sin\lambda\alpha - C(\lambda-2)\sin(\lambda-2)\alpha = 0$$

So, for a nonzero solution for A and C we see that we require

$$\sin 2\mu\alpha = -\mu\sin 2\alpha \tag{9}$$

where $\mu = \lambda - 1$. There is no obvious solution to this except $\lambda = 1$ which is degenerate (we ruled this case out earlier in the computation in fact). Graphically it is clear that real solutions to (9) exist as long as $\alpha > \alpha_c$ where the critical value $\alpha_c \approx 73°$, see figure.



When $\alpha < \alpha_c$ we can continue the search by looking for complex solutions for $\lambda = p + \mathrm{i}q$. What does this mean physically? To begin with, linearity means that we are allowed to consider $\psi = \frac{1}{2}\left(r^\lambda f(\theta) + c.c.\right) = Re(r^\lambda f(\theta))$.

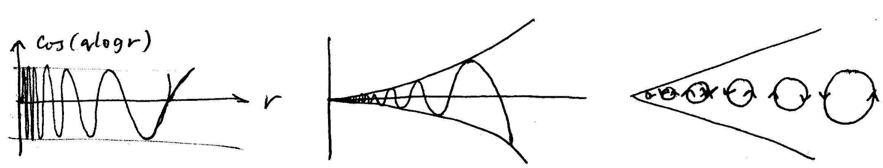
We have that

$$u_\theta = -\frac{\partial\psi}{\partial r} = -Re\left(\lambda r^{\lambda-1}f(\theta)\right)$$

Let $\lambda - 1 = p + \mathrm{i}q$, so that $r^\lambda = r^{p+\mathrm{i}q} = r^p\left(\cos(q\log r) + \mathrm{i}\sin(q\log r)\right)$. Then on $\theta = 0$ we have that

$$u_\theta|_{\theta=0} = Cr^p\cos(q\log r + \delta)$$

for some constants C and δ. Sketch (a) and (b) below indicate how this function oscillates infinitely often as $r \to 0$.



(a) (b) (c)

(a) Plot of $\cos(q\log r)$ against r. (b) Plot of $r^p\cos(q\log r)$ against r. (c) Sketch of sequence of eddies in the flow, rotating in alternate directions and (rapidly) decreasing in strength as $r \to 0$.

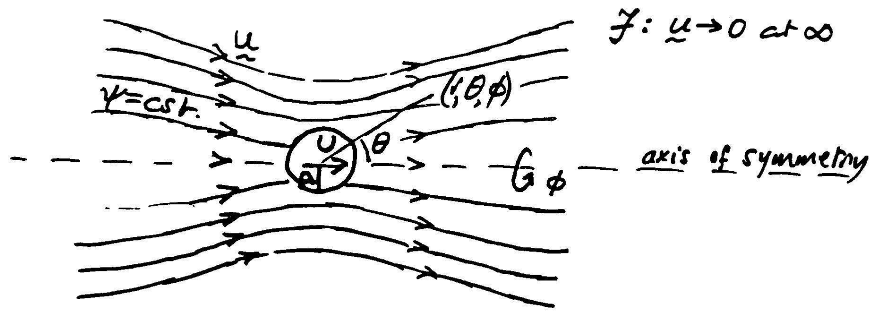
The sketch in (c) illustrates that the flow describes an infinite sequence of eddies as $r \to 0$. For example, in the case $2\alpha = \pi/6$ we can compute that $2\alpha p \approx 4.72$ and $2\alpha q \approx 2.20$. In this case the ratio of intensities of successive eddies as $r \to 0$ is $\approx 1/412$.

2.8 A sphere moving at constant speed

[**

[Acheson, section 7.2, pages 223- 226; Batchelor, section 4.9, pages 230 - 233]

This and the next section summarise a classic problem in Stokes Flow. The material is included because the result is widely quoted and used, but we won't rely on it later in the course. We take a fixed frame of reference $\mathcal{F}$ so that the fluid at infinity is at rest, and the sphere, of radius a is moving at a constant speed U (wlog in the z direction) and is instantaneously located at the origin $r = 0$:



We naturally make use of spherical polar coordinates (r, θ, ϕ) and introduce a 'Stokes streamfunction' $\psi(r, \theta)$. This is a like the streamfunction for 2D flow in Cartesian coordinates, but for axisymmetric flows in 3D. It is defined as the function ψ such that

$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}, \qquad \text{and} \qquad u_\theta = -\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r} \tag{10}$$

(and $u_\phi = 0$). It can be seen straightforwardly (Acheson section 5.5, page 173) that $D\psi/Dt \equiv \mathbf{u} \cdot \nabla\psi$ is zero, and hence ψ is constant on streamlines. The velocity field is computed equivalently from writing

$$\mathbf{u} = \nabla \times \left(0, 0, \frac{\psi}{r \sin\theta}\right). \tag{11}$$

Now we need to specify the problem mathematically: we first discuss appropriate boundary conditions and then the governing equation to solve.

Boundary conditions:

$$\begin{array}{rcc} \mathbf{u} \to 0 & \text{as} & r \to \infty \\ \left.\begin{array}{r} u_r = U\cos\theta \\ u_\theta = -U\sin\theta \end{array}\right\} & \text{on} & r = a \end{array}$$

i.e. on $r = a$ we require $\psi = \frac{1}{2}Ua^2 \sin^2\theta$ and $\partial\psi/\partial r = Ua\sin^2\theta$. From (10a) we see that we also require $\psi = o(r^2)$ as $r \to \infty$, i.e. $\psi/r^2 \to 0$ as $r \to \infty$ so that we guarantee that $|\mathbf{u}| \to 0$ as $r \to \infty$.

From (11) we find that

$$\boldsymbol{\omega} \quad = \quad \nabla \times \mathbf{u} = (0, 0, -\frac{1}{r\sin\theta}\mathbb{D}^2\psi$$

where

$$\mathbb{D}^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}$$

is known as the 'Stokes operator'. Then $\nabla^2\boldsymbol{\omega} = 0$ becomes

$$\nabla^2\boldsymbol{\omega} = \left(0, 0, -\frac{1}{r\sin\theta}\mathbb{D}^4\psi\right)$$

where $\mathbb{D}^4\psi \equiv \mathbb{D}^2(\mathbb{D}^2\psi)$. This is therefore the version of the biharmonic equation that we are required to solve in the axisymmetric case.

We look for solutions in the separable form

$$\psi(r,\theta) = f(r)\sin^2\theta$$

and find that

$$\begin{aligned}\mathbb{D}^2\psi &\equiv \left(f'' - \frac{2f}{r^2}\right)\sin^2\theta \\ &= F(r)\sin^2\theta,\end{aligned}$$

which is convenient, because then

$$\mathbb{D}^4\psi = \left(F'' - \frac{2F}{r^2}\right)\sin^2\theta = 0.$$

We see that the equation $F'' - 2F/r^2 = 0$ is homogeneous so we expect solutions to be in the form $F(r) = r^\lambda$. Substituting this and solving we obtains $\lambda = -1$ and $\lambda = 2$ as linearly independent solutions. Therefore

$$F(r) = \tilde{A}r^2 + \tilde{C}/r$$

which implies

$$f'' - \frac{2f}{r^2} = \tilde{A}r^2 + \tilde{C}/r$$

and therefore

$$f(r) = Ar^4 + Br^2 + Cr + D/r.$$

The boundary conditions then imply that $A = B = 0$ and we have $f(a) = Ua^2/2$, $f'(a) = Ua$ which yields two linear equations for the coefficients C and D. These have the unique solution

$$C = \frac{3}{4}Ua, \qquad D = -\frac{1}{4}Ua^3.$$

And so, finally we have the solution

$$\psi(r,\theta) \quad = \quad \left(Cr + \frac{D}{r}\right)\sin^2\theta = \frac{1}{4}Ua^2\left(\frac{3r}{a} - \frac{a}{r}\right)\sin^2\theta. \tag{12}$$

The term $\propto r$ dominates for large r and is known as the 'Stokeslet' term. The term $\propto 1/r$ does not contribute to the vorticity of the flow, and is a 'dipole' contribution. The 'dipole' term decays much more rapidly at large r.

It is now also easy to compute the pressure field around the sphere. Since the vorticity $\boldsymbol{\omega} = (0, 0, \omega(r,\theta))$ is given by

$$\omega = -\frac{\mathbb{D}^2 v\psi}{r\sin\theta} = \frac{2C}{r^2}\sin\theta$$

we can use the Stokes equation in the form

$$-\frac{\partial p}{\partial r} = \mu(\nabla \times \boldsymbol{\omega})_r = \frac{4C\mu}{r^3}\cos\theta$$

and integrating this we obtain

$$p = p_\infty + \frac{2C\mu}{r^2}\cos\theta \tag{13}$$

2.9 Computation of the drag force on a sphere

Recall that the force $\mathbf{F}$ on the spherical surface is given by

$$F_i = \int_S \sigma_{ij} n_j \, dS$$

which, using the divergence theorem on a large but finite volume of fluid confined between the sphere S and a large sphere S_∞ 'at infinity', is equal to the expression

$$F_i = \int_{S_\infty} \sigma_{ij} n_j \, dS$$

because $\partial/\partial x_j(\sigma_{ij}) \equiv 0$ in V. Then, as $r \to \infty$, the contribution from the dipole term vanishes and we need only integrate the 'Stokelet' terms:

$$\mathbf{u} \sim \frac{C}{r}(2\cos\theta, -\sin\theta, 0)$$

So that the force (acting in the $-z$ direction) is given by

$$F = |\mathbf{F}| = -\int_{S_\infty} (\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta \, dS. \tag{14}$$

We now need to substitute in for the components of the stress tensor, and then for the pressure from (13):

$$\begin{aligned}
\sigma_{rr} &= -p + 2\mu\frac{\partial u_r}{\partial r} = -p_\infty - \frac{6C\mu}{r^2}\cos\theta \\
\sigma_{r\theta} &= 2\mu e_{r\theta} \equiv \mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) = 0.
\end{aligned}$$

We obtain

$$F = 6C\mu \int_0^\pi \cos^2\theta \frac{1}{r^2} \overbrace{2\pi} \underbrace{r^2 \sin\theta \, d\theta}$$

where the $\overbrace{}$ is from the ϕ integral and the $\underbrace{}$ is the usual element of surface area in spherical polar coordinates. So

$$F = 8\pi\mu C = 6\pi\mu a U \tag{15}$$

i.e. the drag force on the sphere is $\mathbf{F} = -6\pi a\mathbf{U}$. This is sometimes known as Stokes Law. In fact we could have deduced all of it from dimensional analysis apart from the constant 6π. So the integration work above is all in aid of computing that coefficient of 6π.

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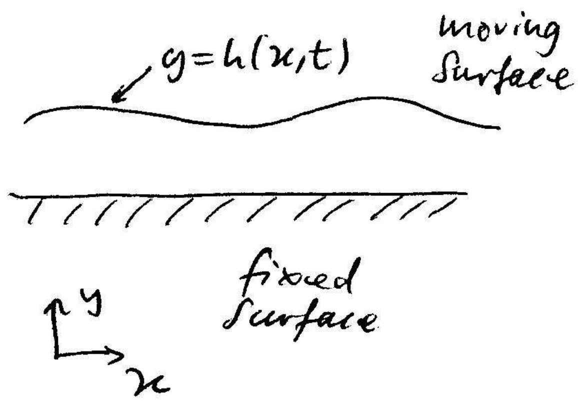
2.10 Flow in a Thin Layer

As previously in this chapter, we continue to assume that inertia is negligible. We consider fluid in a 2D thin layer $0 < y < h(x,t)$:



Let h_0 be a typical thickness (for example, an average value of $h(x,t)$). Let L be the scale of variation of $h(x,t)$ in the x-direction. For example, define

$$L \sim \min \left| \frac{h}{\partial h/\partial x} \right| .$$

We now make the additional fundamental assumption that $h_0 \ll L$. This assumption defines what we mean by 'a think layer'. Therefore we have

$$\left| \frac{\partial \mathbf{u}}{\partial y} \right| \gg \left| \frac{\partial \mathbf{u}}{\partial x} \right|$$

since the LHS is $O(U/h_0)$ and the RHS is $O(U/L)$. Since we know that

$$\mathbf{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0 \right) = (u, v, 0)$$

we must have

$$\left| \frac{\partial \psi}{\partial y} \right| \gg \left| \frac{\partial \psi}{\partial x} \right| .$$

As we would usually solve $\nabla^4 \psi = 0$ this approximation reduces the biharmonic equation to

$$\frac{\partial^4 \psi}{\partial y^4} = 0$$

(there are small cross-terms here that we ignore). We can integrate this once to obtain

$$\frac{\partial^3 \psi}{\partial y^3} \equiv \frac{\partial^2 u}{\partial y^2} = -G(x,t)/\mu$$

where $G(x,t)$ is some function of x and t. Therefore we must identify $G(x,t)$ with $\partial p/\partial x$, a function only of x and t but not of y. So we can integrate twice more with the boundary conditions $u = 0$ on $y = 0$ (a rigid boundary) and $u = U$ on $y = h(x,t)$ - a prescribed velocity. This yields the 'locally parabolic' profile

$$\mu u(x,y,t) = -\frac{1}{2} y(y-h) G(x,t) + \frac{Uy}{h}.$$

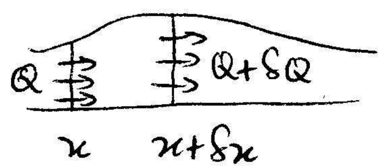
The volume flux of fluid (per unit length in the third dimension) is given by

$$Q(x,t) = \int_0^h u(x,y,t)\,dy = \frac{G(x,t)}{2\mu}\frac{h^3}{6} + \frac{Uh}{2}$$

so the relation between $Q(x,t)$ and $h(x,t)$ is roughly cubic. Incompressibility supplies the further condition that

$$\frac{\partial Q}{\partial x} = -\frac{\partial h}{\partial t}$$

which can be seen immediately from considering the flow between two points x and $x+\delta x$:



This in turn implies that

$$\begin{aligned}\frac{\partial h}{\partial t} = -\frac{\partial Q}{\partial x} &= -\frac{\partial}{\partial x}\left[\frac{G(x,t)}{12\mu}h^3 + \frac{Uh}{2}\right] \\ &= \frac{1}{12\mu}\frac{\partial}{\partial x}\left[h^3\frac{\partial p}{\partial x}\right] - \frac{1}{2}\frac{\partial}{\partial x}(Uh).\end{aligned}$$

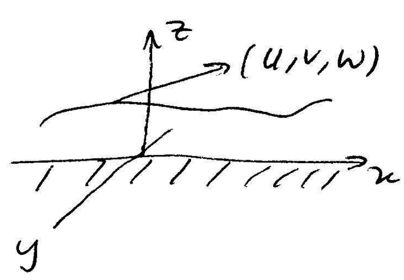
For a given $h(x,t)$, this equation determines $p(x,t)$ (we hope that that this equation comes with boundary conditions for p at two locations $x = x_0$, $x = x_1$.

Notes: [

] (i) There is a 3D generalisation of this 'thin film equation' which is

$$\frac{\partial h}{\partial t} = \frac{1}{12\mu}\nabla\cdot(h^3\nabla p) - \frac{1}{2}\nabla\cdot(\mathbf{u}h)$$

where $\nabla \equiv (\partial/\partial x, \partial/\partial y, 0)$ and $\mathbf{u} = (u, v, 0)$, i.e. there is no velocity in the z-direction.



(ii) We can justify the neglect of inertial forces *a posteori* as follows. Let U_0 be a typical velocity, then

$$|\mathbf{u}\cdot\nabla\mathbf{u}| \sim \frac{U_0^2}{L}$$

$$|\nu\nabla^2\mathbf{u}| \sim \left|\nu\frac{\partial^2}{\partial z^2}\mathbf{u}\right| \sim \nu\frac{U_0}{h_0^2}$$

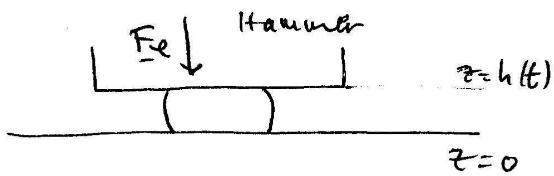
so the ratio of these is

$$\frac{|\mathbf{u}\cdot\nabla\mathbf{u}|}{|\nu\nabla^2\mathbf{u}|} = \frac{U_0}{\nu L}h_0^2 = Re\,\varepsilon$$

where $Re = U_0 h_0/\nu$ is a Reynolds number and $\varepsilon = h_0/L \ll 1$ is the slow scale of variation in the x-direction. Then we can safely neglect inertia if the product $Re\,\varepsilon \ll 1$.

Example 1. A thrust bearing.

Consider a hammer driven by an external force $\mathbf{F}_e$ onto an axisymmetric drop of honey on a rigid table at $z = 0$:



How can we determine the height $h(t)$ of the fluid drop?

Set $u = v = 0$ and $w = -dh/dt$ on $z = h(t)$. Then, using axisymmetry, and therefore plane polars (r, θ):

$$\frac{12\mu}{h^3}\frac{dh}{dt} = \nabla^2 p \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) \tag{16}$$

While the fluid remains entirely within the gap (which we assume) we suppose it occupies a region $0 \le r \le a(t)$, giving a (constant) fluid volume of $V_0 = \pi a^2 h$. Note that we really require $a \gg h$ to use 'lubrication theory'. Note that the pressure $p(r,t) = p_a$ atmospheric pressure at $r = a$. We are also neglecting surface tension forces at $r = a$. Integrating (16) w.r.t. r twice, we obtain

$$\begin{aligned} r\frac{\partial p}{\partial r} &= \frac{12\mu}{h^3}\frac{dh}{dt}\frac{r^2}{2} + C(t) \\ \Rightarrow \frac{\partial p}{\partial r} &= \frac{12\mu}{h^3}\frac{dh}{dt}\frac{r}{2} + \frac{C(t)}{r} \\ \Rightarrow p &= \frac{6\mu}{h^3}\frac{dh}{dt}\frac{r^2}{2} + C(t)\log r + D(t). \end{aligned}$$

Since p is finite at $r = 0$ we must have $C(t) \equiv 0$. Using atmospheric pressure at $r = a$ we can determine $D(t)$ and hence we have

$$p = p_a + \frac{3\mu}{h}\frac{dh}{dt}(r^2 - a^2). \tag{17}$$

Having computed the pressure in the fluid we can now use this expression to compute the excess force F exerted by the fluid on the hammer:

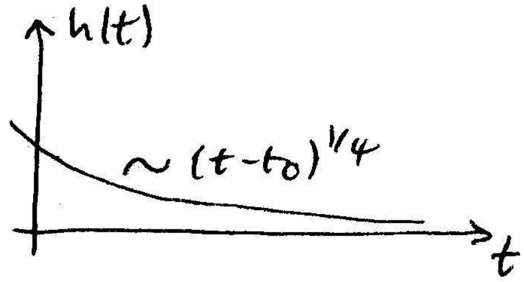
$$\begin{aligned} F &= \int_0^a p(r,t) - p_a\, 2\pi r\, dr \\ &= \frac{3\mu}{h^3}\frac{dh}{dt}2\pi\int_0^a r^3 - a^2 r\, dr \\ &= -\frac{3\mu\pi}{2h^3}\frac{dh}{dt}a^4 \end{aligned} \tag{18}$$

where we note that $dh/dt < 0$.

Now, if we neglect the inertia of the hammer (i.e. it is just sitting on top of the fluid drop) then $F = F_e$, constant. In this case we may integrate (18) again, after substituting for a^4 using the fact that $a^4 = \left(\frac{V_0}{\pi h}\right)^2$. This gives

$$h(t) = \left(\frac{3\mu V_0^2}{8\pi F_e}\right)^{1/4} \frac{1}{(t-t_0)^{1/4}}$$

where t_0 is a constant of integration. Thus the height of the drop decreases very very slowly $\sim (t-t_0)^{-1/4}$ as $t \to \infty$.

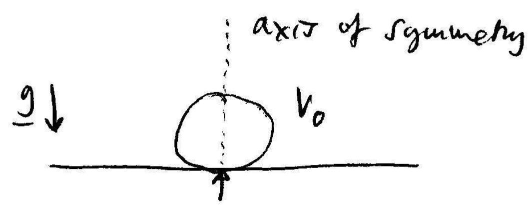


The hammer, of course (why?) never makes contact with the table.

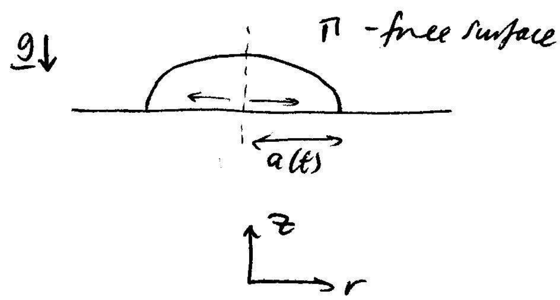
Example 2. Gravitational spreading.

In this example we consider how a drop of viscous liquid spreads out axisymmetrically on a horizontal surface under gravity. This is motivated, for example, by the spreading of hot lava from a volcano.

The drop lands on the table as a sphere, say (so that the flow is axisymmetric):



...and then 'very quickly' becomes a 'thin film' occupying the region $0 \le r \le a(t)$:



In contrast to Example 1, here the pressure is not constant in the layer, it is hydrostatic:

$$p = p_a + \rho g(h - z) \tag{19}$$

where the free surface Γ is given by $z = h(r,t)$. We now have two expressions for $\partial p/\partial r$: one from differentiating (19) and one from the thin film approximation of the Stokes equations:

$$\frac{\partial p}{\partial r} = \rho g \frac{\partial h}{\partial r}, \qquad \text{and} \qquad \frac{\partial p}{\partial r} = \mu \frac{\partial^2 u}{\partial z^2} \tag{20}$$

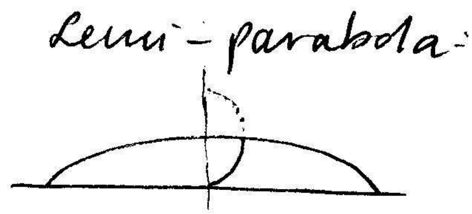
since $\mathbf{u} = (u(r,z,t),0,0)$. For this case, since the upper boundary is stress-free, the appropriate boundary conditions for the velocity are

$$\begin{aligned} u = 0 \qquad &\text{on} \qquad z = 0, \\ \frac{\partial u}{\partial z} = 0 \qquad &\text{on} \qquad z = h. \end{aligned}$$

Combining the two equations in (20) and integrating twice w.r.t. z we obtain

$$u(r,z,t) \quad = \quad \frac{\rho g}{2\mu}\frac{\partial h}{\partial r} z(z-2h)$$

which is a 'semi-parabolic' profile for the velocity:



The outward radial flux of material is straightforwardly computed to be

$$Q \quad = \quad 2\pi r \int_0^h u\,dz = -\frac{2\pi\rho g r}{3\mu} h^3 \frac{\partial h}{\partial r} \tag{21}$$

...and now we can use the final ingredient which is conservation of mass:

$$2\pi r \frac{\partial h}{\partial t} = -\frac{\partial Q}{\partial r}$$

and, combining this with the expression (21) we obtain the evolution equation for $h(r,t)$:

$$\frac{\partial h}{\partial t} \quad = \quad \frac{g}{3\nu}\frac{1}{r}\frac{\partial}{\partial r}\left(r h^3 \frac{\partial h}{\partial r}\right). \tag{22}$$

This equation looks rather like a diffusion equation for h, but with a nonlinearity due to the factor of h^3. Just like the usual linear heat equation, it has a similarity form of solution, that is, we look for a solution in the form

$$h(r,t) = t^{-\alpha} H(\eta), \qquad \text{where} \qquad \eta = r/t^{\beta}$$

where the exponents α and β need to be determined. We use the evolution equation (22) and the (constant) volume condition:

$$V = \int_0^{a(t)} h(r,t) 2\pi r\,dr = \text{constant} \tag{23}$$

to determine α and β, as follows.

We see that the derivatives of the similarity variable η w.r.t. r and t are given by

$$\frac{\partial \eta}{\partial r} = t^{-\beta}, \qquad \text{and} \qquad \frac{\partial \eta}{\partial t} = -\frac{\beta\eta}{t}.$$

Substituting these into (22) we obtain

$$-\left(\alpha H + \beta\eta\frac{dH}{d\eta}\right)t^{-\alpha-1} = \frac{g}{3\nu\eta}t^{-4\alpha-2\beta}\frac{d}{d\eta}\left(\eta H^3\frac{dH}{d\eta}\right), \tag{24}$$

which must hold for all t, implying

$$\begin{aligned} \alpha + 1 &= 4\alpha + 2\beta \\ \Rightarrow 3\alpha + 2\beta &= 1 \end{aligned} \tag{25}$$

From the constant volume condition we have that

$$V = \int_0^{a/t^\beta} t^{-\alpha}H2\pi t^{2\beta}\eta\,d\eta$$

must be independent of t, hence

$$2\beta - \alpha = 0. \tag{26}$$

Solving the two equations (25) and (26) we find the unique solution $\alpha = 1/4$, $\beta = 1/8$. Therefore $a(t) \propto t^{1/8}$.

Computing the shape $H(\eta)$ of the drop

We can now recast the evolution equation (24) and the constant-volume condition (23) in terms of the similarity variable η to solve for the shape of the droplet as it spreads out. From volume conservation we introduce A as the value of η at which we hit the boundary of the drop, i.e. the height falls to zero: $H(A) = 0$. We have

$$V = 2\pi\int_0^A H(\eta)\eta\,d\eta \tag{27}$$

and hence $A = a/t^{1/8}$. The evolution equation (24) becomes

$$-\left(\frac{1}{4}H + \frac{1}{8}\eta\frac{dH}{d\eta}\right) = \frac{g}{3\nu\eta}\frac{d}{d\eta}\left(\eta H^3\frac{dH}{d\eta}\right)$$

which is a nonlinear ODE for $H(\eta)$. It integrates once in the form given, multiplying through by a factor of η, to yield

$$\eta H^3\frac{dH}{d\eta} + \frac{3\nu}{8g}\eta^2 H = \text{constant}$$

but in fact this constant of integration must be zero since H and $dH/d\eta$ are both finite (in fact for a smooth solution we would expect $dh/d\eta = 0$) at $\eta = 0$. So we can cancel a factor of ηH and obtain

$$H^2\frac{dH}{d\eta} + \frac{3\nu}{8g}\eta = 0$$

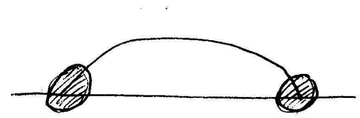
which can be integrated w.r.t. η to yield

$$\frac{1}{3}H^3 + \frac{3\nu}{8g}\frac{1}{2}\eta^2 = \text{constant} = \frac{3\nu}{16g}A^2$$

since A is defined to be the value of η at which $H(\eta) = 0$. Therefore the shape of the drop is

$$H(\eta) = \frac{9\nu}{16g}\left(A^2 - \eta^2\right)^{1/3}.$$

lubrication theory is not valid
in the shaded areas as
gradient of fluid not small
(infinite at join).



Further, the drop size A can be directly related to the volume V since

$$V = 2\pi \int_0^A \frac{9\nu}{16g} \left(A^2 - \eta^2\right)^{1/3} \eta \, d\eta$$
$$\Rightarrow V = \frac{3\pi}{4} \left(\frac{9\nu}{16g}\right)^{1/3} A^{8/3}$$

or equivalently, in dimensional units, we have

$$a(t) = CV^{1/3} \left(\frac{t}{T}\right)^{1/8} \tag{28}$$

where $T = \nu/(gV^{1/3})$ is the characteristic viscous timescale for the spreading under gravity, and C is a pure number that we have computed via the thin film theory:

$$C = \left(\frac{2^{10}}{3^5\pi^3}\right)^{1/8} = 0.77921...$$

The $1/8^{th}$ power law relation (28) for the increase in the horizontal size of the drop over time has been verified experimentally (Huppert, 1982).

2.11 The Hele-Shaw Cell

This is another classic thin-film case, with a surprising link to the potential flows studied earlier in the course.

A Hele-Shaw cell consists of a pair of glass plates fixed at a small constant separation ($h \approx 1$mm) with fluid trapped between them:

we assume that there is a locally parabolic velocity profile between the rigid upper and lower boundaries.

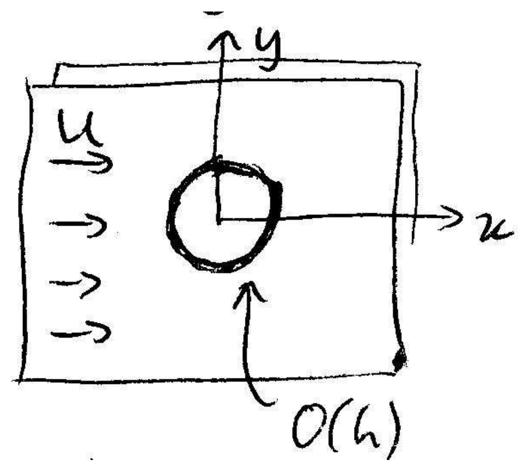
We then examine the thin film flow around objects placed in the cell. If we assume that the scale of the obstacles L is much greater than the plate separation h then the thin film equations are valid.

Recall that the volume flux $\mathbf{Q} = -\frac{1}{12\mu}h^3\nabla p$ in the thin film. Therefore, since the film height h is constant, the average velocity is given by

$$\bar{\mathbf{u}}(x,y) = \frac{1}{h}\mathbf{Q} = -\frac{1}{12\mu}h^2\nabla p = \nabla\phi(x,y)$$

where $\phi = -h^2/(12\mu)p$ is a 2D potential! So the averaged flow is potential flow, at least as long as $L \gg h$.

For example, flow at velocity U past a circular cylinder $r = a$ in a Hele-Shaw cell should look like



with $\phi \sim Ux = Ur\cos\theta$ as $r \to \infty$, and (impermeability) $\bar{\mathbf{u}} \cdot \mathbf{n} = 0$ on $r = a$, hence $\partial\phi/\partial r = 0$ on $r = a$. Incompressibility $\nabla \cdot \bar{\mathbf{u}} = 0$ implies $\nabla^2\phi = 0$ and we have the general solution

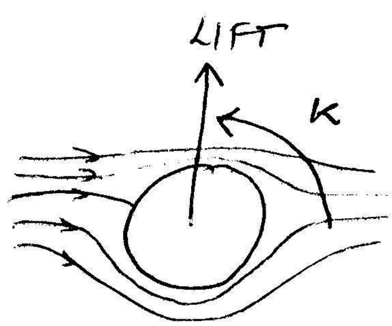
$$\phi(r, \theta) = U\left(r + \frac{a^2}{r}\right)\cos\theta,$$

which we saw earlier in the course, in chapter 1. However, it should be noted that we cannot satisfy the no-slip boundary condition on $r = a$:

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}|_{r=a} = -2U\sin\theta \neq 0$$

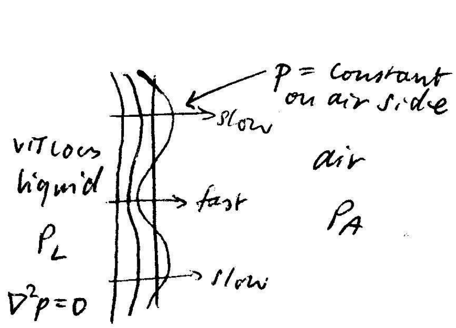
so there must be a small region, of width $O(h)$, around the cylinder in which the thin-film approximations are not valid. This is an example of a boundary layer, and we will see more of these later in the course.

A further limitation of the Hele-Shaw cell is that we cannot sustain flows with non-zero circulation κ. Mathematically, such flows contained additional contributions to the potential ϕ such as $\phi = \kappa\theta/(2\pi)$ which is not single-valued (but this did not matter in the section on potential flow since the only quantities with physical meaning were the derivatives of ϕ). In this context we have $\phi = -h^2/(12\mu)p$ and so ϕ is proportional to the pressure field and has a physical significance. So we cannot have solutions where ϕ has a contribution $\sim \theta$. If we attempt to generate circulation by, for example, rotating the cylinder, then the circulation is killed off by viscous effects within the boundary layer around the cylinder, leaving the flow outside this with zero circulation.

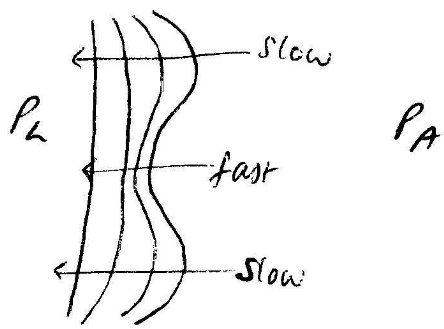


Two-fluid flow in the Hele-Shaw cell

When a more viscous fluid advances, displacing a less viscous one, the flat interface between them is stable to small disturbances. If, however, the more viscous fluid is receding, then the flat interface is unstable and a ‘fingering instability’ occurs.



(a)



(b)

In (a) the sketch shows viscous liquid with a pressure field $p_L(x, y)$ on the left and air, within which the pressure $p_A < p_L$ is constant, on the right. The lines show contours of constant p_L, with the rightmost one being the liquid-air interface at which $p_L = p_A$. Since $\bar{\mathbf{u}} \propto \nabla p$ in the liquid, the middle section, where the contours are closer together, movs faster to the right than the top and bottom sections and so the flat interface is stabilised (the sinusoidal perturbation decays).

In (b) we suppose that $p_A > p_L$ and the air is pushing the liquid back towards the left. The fast motion in the liquid is still in the centre and so the sinusoidal perturbation is amplified and the interface is no longer flat, but contains a (large) deformation in the centre.