MA6000M Topics in Applied Mathematics 2011-12 **J.H.P. Dawes**
Bifurcation Theory and Applications

Problem Sheet 1

Starred questions are not necessarily harder, just less central to subsequent course material. Send comments and queries to `J.H.P.Dawes@bath.ac.uk`.

1. Consider the linear system $\dot{x} = Ax$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $T = a + d$ and $D = ad - bc$. Indicate the regions in the (T, D)–plane where the origin is a stable node, stable focus, unstable focus, unstable node and saddle point. In which regions is the origin stable?

2. **Using the results of question 1**, sketch the phase portrait of the following linear systems in $\mathbb{R}^2$:

   (a) $\dot{x} = -2x - 3y$, $\dot{y} = 8x + 8y$;
   (b) $\dot{x} = x$, $\dot{y} = -3x - y$;
   (c) $\dot{x} = 7x - 2y$, $\dot{y} = 5x + 5y$;
   (d) $\dot{x} = 7x + 5y$, $\dot{y} = -10x - 7y$;
   (e) $\dot{x} = 2x + y$, $\dot{y} = 2y$;
   (f) $\dot{x} = 4x + 2y$, $\dot{y} = 2x + y$;
   (g) $\dot{x} = 8x + 3y$, $\dot{y} = -3x + 2y$;
   (h) $\dot{x} = -2x$, $\dot{y} = -2y$.

   For which systems is the origin a hyperbolic equilibrium and for which is it stable?

3. Find all equilibria of the system

$$\begin{aligned} \dot{x} &= 2x - x^3 - 3xy^2, \\ \dot{y} &= y - y^3 - x^2 y. \end{aligned}$$

   Find their stability types (stable node, saddle point etc) and (carefully) sketch the phase portrait, noting carefully what happens on and near the axes and any symmetries present.

4. (a) For the nonlinear oscillator $\ddot{x} + ax + bx^2 = 0$ show that $V = \frac{1}{2}p^2 + \frac{1}{2}ax^2 + \frac{1}{3}bx^3$ (where $p = \dot{x}$) is conserved along trajectories, i.e. $\frac{dV}{dt} = 0$. Sketch the level sets $V =$ constant in the (x, p)–plane assuming a, $b > 0$ and describe the different types of orbit in the system. Discuss stability (in both the Lyapunov and quasi-asymptotic senses) for all equilibrium points that you identify.

   (b) Now consider the system $\ddot{x} + k\dot{x} + ax + bx^2 = 0$. By computing $\frac{dV}{dt}$ show that when $k > 0$ no periodic orbits can exist. Compute the linear stability type of each equilibrium. Sketch the complete phase portrait and indicate the sets of initial conditions whose trajectories converge to each of the equilibria as $t \to \infty$.

5. Consider the ODEs

$$\dot{x} = -x - \frac{y}{\log\sqrt{x^2 + y^2}}, \qquad \dot{y} = -y + \frac{x}{\log\sqrt{x^2 + y^2}}.$$

   By changing to polar co-ordinates show that trajectories of the flow repeatedly circle around the origin, even though the linearised system at the origin is that of a stable node.

* 6. Solve explicitly the following differential equations for $x(t)$ and $y(t)$:

$$\begin{aligned} \dot{x} &= -2x + y^2, \\ \dot{y} &= -y. \end{aligned}$$

Eliminate t from the solution to obtain an equation $x = g(y)$ for the trajectory starting from (x_0, y_0) at $t = 0$. Hence show that there can be no C^2 conjugacy of solutions to trajectories of the linearised system, even though (by Hartman–Grobman) they are topologically (i.e. C^0) equivalent.

7. Show that the ODEs

$$\begin{aligned} \dot{x} &= a - (b+1)x + x^2 y, \\ \dot{y} &= bx - x^2 y. \end{aligned}$$

where a and b are positive parameters, have exactly one equilibrium point, and find it explicitly. Find the set of values of a and b at which there is a bifurcation and plot this bifurcation set (curve) in the (a, b)–plane.

8. Find the equilibrium points in the following 1D systems and determine their stability. Find the value(s) of μ at which bifurcations occur and sketch a bifurcation diagram in each case.

(a) $\dot{x} = x(\mu - x^2)(1 - \mu + \frac{1}{2}x^2)$;
(b) $\dot{x} = \mu - 2x^2 + x^4$;
(c) $\dot{x} = x(\mu + 2x^2 - x^4)$;
(d) $\dot{x} = x(x^2 - \mu)(x^2 + \mu^2 - 1)$.

9. Consider the behaviour of the two-parameter family of 1D systems $\dot{x} = \mu_1 + \mu_2 x + x^2$. Sketch in the (μ_1, μ_2) plane the sets of parameter values where the system has 0, 1 or 2 equilibria. **Use your sketch** to investigate the bifurcations which occur in the following one-parameter families:

(a) $\dot{x} = \mu + x + x^2$,
(b) $\dot{x} = 2\mu + \varepsilon - (\mu + 2)x + x^2$.

In each case sketch a bifurcation diagram showing the position and stability of the equilibria as μ varies. In (b) you should consider separately the three cases $\varepsilon = 0$, ε small and positive, and ε small and negative.

10. For each of the following maps, $x_{n+1} = F(x_n, \mu)$, describe the bifurcation that occurs at each given value of μ.

(a) $F(x, \mu) = (1 + \mu)x - x^2$ at $\mu = 0$,

(b) $F(x, \mu) = (1 + \mu)x - x^3$ at $\mu = 0$,

(c) $F(x, \mu) = \mu - x^2$ at $\mu = -\frac{1}{4}$ and $\mu = \frac{3}{4}$,

(d) $F(x, \mu) = \mu x - x^3$ at $\mu = -1$, $\mu = 1$ and $\mu = 2$.

Now consider the map $g(x, \mu) = F(F(x, \mu), \mu) \equiv F^2(x, \mu)$. Show that the trivial fixed point of g cannot undergo a period–doubling bifucation.

October 10, 2011