MA6000M: Topics in Applied Mathematics 2010 - 11

Bifurcation Theory and Applications

J.H.P. Dawes

This course will introduce ideas and methods from nonlinear dynamics which are widely and routinely used to understand models of a wide range of physical systems, for example fluid flows, population dynamics, chemical reactions and coupled oscillators. The 'dynamical systems viewpoint' is to concentrate on features of the dynamics that are independent of the coordinate system, for example the long-term behaviour that the system 'settles down to'.
The first half of the course will be concerned with the qualitative behaviour of solutions to nonlinear ordinary differential equations, with an emphasis on structural changes in response to variations in parameters (bifurcation theory). There will be a brief discussion of the generation of complicated dynamics.

The second half of the course will extend these ideas to the case of symmetric systems. Symmetry naturally arises either from physical constraints, or from modelling assumptions. The existence of symmetry in the differential equations is, however, not 'generic'. This change in genericity comes with additional structure that provides ways of understanding the notion of a 'typical' bifurcation in the presence of symmetry, and typical dynamical behaviours. Basic ideas from group theory and representation theory will be developed as needed to describe the action of the group on the phase space of the dynamical system. Local bifurcations will be discussed in some detail, after which a variety of directions are possible including applications to pattern formation in two and three dimensions, heteroclinic cycling and coupled cell systems.

Any time remaining will be used to discuss problems of current interest in the field, for example (and these are given only as examples) the extension of ODE methods to pattern-forming PDEs such as the Swift–Hohenberg equation, the generation of localised states near subcritical Turing instabilities, and the development of bifurcation theory for locally-symmetric systems.

There will be a number of problem sheets and problem classes.

An over-ambitious course outline:

Part I

- Introduction (through which we will go quite fast): phase space and the qualitative description of solutions to ODEs. Topological equivalence, hyperbolicity and structural stability of flows. Stable and unstable manifolds. Codimension–one local bifurcations in flows and maps. Centre manifolds. Reduction to normal forms; normal form symmetries.
- Global bifurcations: Lorenz and Shil'nikov mechanisms. Codimension-two bifurcations: degenerate Hopf, Takens–Bogdanov.

Part II

- Groups and their irreducible representations. Fixed point subspaces and computation of their dimension. Isotropy subgroups. Steady-state bifurcations: the Equivariant Branching Lemma.
- Stability. Isotypic decomposition. Computing invariants and equivariants.
- Spatio-temporal symmetry. The Equivariant Hopf Theorem.
- Steady-state planar pattern formation. Square and hexagonal lattices. Scalar and pseudoscalar cases.
- Coupled cell systems. Robust heteroclinic cycles.

Further topics (time permitting):

- Pattern-forming PDEs. Multiple-scales reduction to the Ginzburg–Landau equation. Eckhaus instability.
- Localised states and the Hamiltonian-Hopf bifurcation.
- Groupoid symmetry. Takens–Bogdanov as a codimension-one bifurcation.

Desirable previous knowledge

Some previous exposure to nonlinear dynamics would be extremely advantageous. Only very very elementary ideas from group theory will be assumed.

Introductory reading

1. P. Glendinning, *Stability, Instability and Chaos.* Cambridge University Press, 1994.

   Clear, readable textbook based on lecture notes for the undergraduate courses on nonlinear dynamics at Cambridge.

2. M. Hirsch & S. Smale, *Differential Equations, Dynamical Systems and Linear Algebra.* Academic Press, New York, 1974.

3. J.D. Crawford, Introduction to bifurcation theory. *Rev. Mod. Phys.* **63**, 1991.

   Covers all the necessary background with more of the analytical details. Includes brief notes on bifurcations with symmetry towards the end.

4. S.H. Strogatz, Nonlinear Dynamics and Chaos. Perseus Books, Cambridge, MA.

Reading to complement course material

No single book covers both halves of the course together. In the references below, I've tried to indicate which half a book deals mostly with.

1. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer, **second or subsequent printing** 1986.

   An excellent general reference; dense and rewarding reading. (First half)

2. Y.A. Kuznetsov, *Elements of Applied Bifurcation Theory.* Second edition, Springer, 1998.

   Quite good for the first half of the course.

3. S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* Springer 1991.

   One of many general introductions to the subject. (First half)

4. R.B. Hoyle, *Pattern Formation: An Introduction to Methods.* CUP 2006.

   Based on notes for a course on very similar material, but stylistically a little different. (Second half)

5. M. Golubitsky, I.N. Stewart & D.G. Schaeffer, *Singularities and Groups in Bifurcation Theory. Volume II.* Springer, Applied Mathematical Sciences Series **69**, 1988.

   An intricate and careful development of equivariant bifurcation theory. Dense reading. (Second half)

6. M. Golubitsky and I.N. Stewart, *The Symmetry Perspective.* Progress in Mathematics, Volume 200. Birkhäuser, 2002.

   A more recent and straightforward account of equivariant bifurcation theory that develops applications and examples in parallel with the theory. (Second half)

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