Resonance bifurcations from robust homoclinic cycles

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Abstract. We present two calculations for a class of robust homoclinic cycles with symmetry $\mathbb{Z}_n \ltimes \mathbb{Z}_2^n$, for which the sufficient conditions for asymptotic stability given by Krupa and Melbourne are not optimal.

Firstly, we compute optimal conditions for asymptotic stability using transition matrix techniques which make explicit use of the geometry of the group action.

Secondly we consider a specific polynomial vector field that contains a robust heteroclinic cycle with this symmetry. Through an explicit computation of the global parts of the Poincaré map near the cycle we show that, generically, the resonance bifurcations from the cycles are supercritical: a unique branch of asymptotically stable period orbits emerges from the resonance bifurcation and exists for coefficient values where the cycle has lost stability.

This second calculation is of a novel kind: it is the first calculation that explicitly computes the criticality of a resonance bifurcation, and it answers a conjecture of Field and Swift in a particular limiting case. Moreover, we are able to obtain an asymptotically-correct analytic expression for the period of the bifurcating orbit, with no adjustable parameters, which has not previously been achieved. We show that the asymptotic analysis compares very favourably with numerical results.

1. Introduction

Heteroclinic orbits between saddle-like invariant sets are of great interest in dynamical systems since they generate many kinds of non-trivial behaviour, including intermittency and chaotic dynamics [14]. In continuous-time dynamical systems (i.e. ordinary differential equations) a non-transversal intersection may give rise to intermittent dynamics in the sense that trajectories approach neighbourhoods of the saddle-like invariant sets and spent a substantial amount of time there before moving rapidly to a neighbourhood of the next invariant set.

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It is well known that, while heteroclinic orbits in generic systems of nonlinear ordinary differential equations (ODEs) are of codimension at least one, in sets of ODEs containing invariant subspaces they can exist for open sets of parameter values, that is, they are codimension zero, and hence are referred to as 'robust' [14, 8]. Three situations in which such invariant subspaces can arise are (i) due to equivariance with respect to a symmetry group [15, 16], (ii) modelling assumptions such as the permanence of death in Lotka–Volterra-type models of population dynamics [17, 11, 12], and (iii) structural restrictions such as the coupled cell structures investigated recently by Aguiar et al. [1]. Detailed results on the existence of homoclinic cycles in particular families of equivariant vector fields have been given by many authors, for example [13, 2, 10].

A number of codimension-one bifurcations have been identified in which robust heteroclinic cycles are created or destroyed, or in which their stability changes. Issues of stability turn out to be more subtle than might be at first thought. In particular, weaker definitions of stability than asymptotic stability turn out to be useful, and indeed for some systems describe the generic case. The most prevalent of these weaker notions is 'essential asymptotic stability', introduced by Melbourne [18] who gave an example of an essentially asymptotically stable robust heteroclinic cycle. In this context we note the recent work by Driesse & Homburg [6] which discusses examples of essentially asymptotically stable homoclinic cycles which are produced in codimension-one bifurcations from asymptotically stable robust homoclinic cycles as a 'transverse' eigenvalue (which we shall define in section 2) crosses the imaginary axis.

In this paper we discuss a particularly simple class of homoclinic cycles in $\mathbb{R}^n$. Although necessarily restrictive, this class contains the essence of the Lotka–Volterra type examples discussed at length by Rabinovich and co-workers [20, 24, 21, 22] as models for neural decision-making processes. The class of homoclinic cycles that we consider can undergo both transverse and resonance bifurcations and we focus here on the resonant case, beginning by deriving necessary and sufficient conditions for asymptotic stability, improving on the general result given by Krupa & Melbourne [15, 16]. Such a resonance bifurcation is shown to yield a single periodic orbit which lies close to the (now unstable) homoclinic cycle. The stability and direction of bifurcation of this periodic orbit depend on coefficients in the Poincaré return map which come from the global maps, that is, those which describe how trajectories near the homoclinic cycle behave outside neighbourhoods of the equilibria on the cycle [3]. Usually these coefficients are impossible to compute analytically, but here we describe a particular example for which the calculation turns out to be tractable, asymptotically in the limit where the sum of the transverse eigenvalues is small. To the best of our knowledge no calculation along these lines has been attempted previously. The result of the calculation is to show that the resonance bifurcation for this class of simple homoclinic cycles is always supercritical, at least when the sum of the transverse eigenvalues is sufficiently small compared to the leading expanding and contracting eigenvalues.

Field & Swift [9] study in detail a particular example from the class of cycles we consider, in the case $n = 4$. They conjecture that the resonant homoclinic bifurcation

is always supercritical. Our results prove this conjecture for an open set of parameter values.

The outline of the paper is as follows. In section 2 we introduce our notation and the class of robust homoclinic cycles that we study. We state our result on asymptotic stability of such cycles (Theorem 1). Section 3 contains the proof of Theorem 1. In section 4 we present the calculation of the coefficients in the Poincaré return map. The calculations are reasonably straightforward to follow but become remarkably lengthy. Some more detailed parts of the calculations are relegated to the Appendix. In section 5 we use our return map calculations to provide an estimate of the period of the bifurcating periodic orbit. We also show that the agreement between the analytic leading-order estimate and numerical results is very good. We conclude in section 6.

2. Robust homoclinic cycles

We consider continuous time dynamical systems in the form of Γ-equivariant ODEs:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{1}$$

where $\Gamma \subset \mathbf{O}(n)$ is a finite Lie group. An equilibrium $\xi \in \mathbb{R}^n$ of (1) satisfies $f(\xi) = 0$. We begin by giving a number of definitions; these are all standard in the literature, see for example [14, 16].

Definition 1 $\phi_j(t)$ *is a* heteroclinic orbit *between two equilibria ξ_j and ξ_{j+1} of (1) if $\phi_j(t)$ is a solution of (1) which is backward asymptotic to ξ_j and forward asymptotic to ξ_{j+1}.*

Definition 2 *A* heteroclinic cycle *is an invariant set $X \subset \mathbb{R}^n$ consisting of the union of a set of equilibria $\{\xi_1, ..., \xi_m\}$ and orbits $\{\phi_1, ..., \phi_m\}$, where ϕ_j, $1 \leq j \leq m$, is a heteroclinic orbit between ξ_j and ξ_{j+1}; and $\xi_{m+1} \equiv \xi_1$. We require that $m \geq 2$.*

In the case $m = 1$, that is, $\xi_2 = \xi_1$, we say that ϕ_1 is a *homoclinic orbit* to ξ_1.

Definition 3 *A heteroclinic cycle is a* homoclinic cycle *if there exists $\gamma \in \Gamma$ such that $\gamma \xi_j = \xi_{j+1}$ for all $1 \leq j \leq m$.*

Definition 4 *For $x \in \mathbb{R}^n$ the* isotropy subgroup Σ_x *is*

$$\Sigma_x = \{\sigma \in \Gamma : \sigma x = x\}. \tag{2}$$

Definition 5 *For Σ an isotropy subgroup of Γ, the* fixed-point subspace Fix Σ *is*

$$\text{Fix } \Sigma = \{x \in \mathbb{R}^n : \sigma x = x \; \forall \sigma \in \Sigma\}. \tag{3}$$

Definition 6 *A heteroclinic cycle X is* robust *if for each j, $1 \leq j \leq m$, there exists a fixed-point subspace, $P_j = \text{Fix } \Sigma_j$ where $\Sigma_j \subset \Gamma$ and*

(i) ξ_j is a saddle and ξ_{j+1} is a sink for the flow restricted to P_j,

Table 1. Classification of eigenvalues. $P \ominus L$ denotes the orthogonal complement in P of the subspace L.

Eigenvalue class	Subspace
Radial (r)	$L_j \equiv P_{j-1} \cap P_j$
Contracting (c)	$V_j(c) = P_{j-1} \ominus L_j$
Expanding (e)	$V_j(e) = P_j \ominus L_j$
Transverse (s)	$V_j(s) = (P_{j-1} + P_j)^\perp$

(ii) there is a heteroclinic connection from ξ_j to ξ_{j+1} contained in P_j.

Importantly, robust heteroclinic cycles may occur as codimension-zero phenomena in systems with symmetry. That is, they may exist for open sets of parameter values. We define $L_j \equiv P_{j-1} \cap P_j$ and clearly $\xi_j \in L_j$. Following [15], the eigenvalues of the linearisation of $f(x)$ about each equilibrium can be classified according to the subspaces in which the eigenspaces lie, as shown in table 1.

The specific differential equations we consider in this paper are of the form:

$$\dot{x} = f(x) = \mu x + Q(x), \quad x \in \mathbb{R}^n \tag{4}$$

where $\mu \in \mathbb{R}_+$ (and so, following a rescaling of time, can be set equal to 1), $n \geq 3$, and $Q(x)$ is a Γ-equivariant polynomial, that is,

$$\gamma Q(x) = Q(\gamma x), \quad \gamma \in \Gamma,$$

which contains only nonlinear terms. The group Γ is of the form

$$\Gamma = \mathbb{Z}_n \ltimes \Delta_n, \tag{5}$$

where $\Delta_n \equiv \mathbb{Z}_2^n$ acts on $\mathbb{R}^n$ as n reflections κ_j, $j = 1, \ldots, n$:

$$\kappa_j(x_1, \ldots, x_j, \ldots, x_n) = (x_1, \ldots, -x_j, \ldots, x_n), \tag{6}$$

The actions of the reflections κ_j mean that every coordinate hyperplane is a fixed-point subspace, and hence invariant under (4). The $\mathbb{Z}_n$ subgroup of Γ is generated by a cyclic permutation element ρ which acts as:

$$\rho(x_1, \ldots, x_n) = (x_n, x_1, \ldots, x_{n-1}). \tag{7}$$

It is clearly sufficient to consider the dynamics restricted to the domain

$$\mathbb{R}_+^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1, \ldots, x_n \geq 0\}.$$

We now describe the robust homoclinic cycles X that we study in this paper. For now, we suppose that it is possible to construct a vector field with the following properties. In section 4 we give an explicit example of ODEs that contain such a cycle. We make the following assumptions on the ODEs (4):

(H1) There exist n equilibria ξ_j, $j = 1, \ldots n$, related by the symmetry element ρ:

$$\xi_{j+1} = \rho \xi_j.$$

each with isotropy exactly Δ_{n-l}, $(1 \leq l \leq n - 2)$, and therefore with l non-zero coordinates.

(H2) The unstable manifold of ξ_j, $W^u(\xi_j)$, is one-dimensional, and has isotropy Δ_{n-l-1}, that is, points on the manifold have $(l+1)$ non-zero coordinates.

Without loss of generality, we can choose the non-zero coordinates of ξ_1 to be $x_1, \ldots, x_l$, and the non-zero coordinates of $W^u(\xi_1)$ to be $x_1, \ldots, x_{l+1}$. Therefore ξ_1 and ξ_2 are contained in the subspace P_1, spanned by $x_1, \ldots, x_{l+1}$. We make the further assumption:

(H3) ξ_2 is the only sink in P_1 and the unstable manifold of ξ_1 is asymptotic to ξ_2.

Therefore the union of the ξ_j and their unstable manifolds forms a homoclinic cycle we label X.

From the action of ρ^{-1}, it is clear that P_n is spanned by $x_n, x_1, \ldots, x_l$. Using table 1 we deduce that $V_1(c)$ is the x_n axis and $V_1(e)$ is the x_{l+1} axis. The radial subspace of ξ_1 has l dimensions, and so there are $n-l-2$ transverse directions. Since ξ_j has a one-dimensional unstable manifold, all the transverse and radial eigenvalues are negative.

We label the contracting eigenvalue (that is, the eigenvalue in the direction spanned by $V_j(c)$ at ξ_j) $-c < 0$, and the expanding eigenvalue (the eigenvalue in the direction spanned by $V_j(e)$), $e > 0$. We label the transverse eigenvalues $-s_1, \ldots, -s_{n-l-2}$ where the eigenvector corresponding to the eigenvalue $-s_k$ at ξ_j is in the x_{l+j+k} direction, taking subscripts modulo n as appropriate.

Theorem 1 *The homoclinic cycle X is asymptotically stable if*

$$c + \sum_{j=1}^{n-l-2} s_j > e. \tag{8}$$

If $c + \sum_{j=1}^{n-l-2} s_j < e$ then the cycle is unstable.

When the stability of the cycle changes at $c + \sum_{j=1}^{n-l-2} s_j = e$, the cycle undergoes a *resonance bifurcation*. In this bifurcation a unique long-period periodic orbit bifurcates; the bifurcation may be supercritical (in which case the bifurcating periodic orbit is stable and exists for parameters for which X is unstable) or subcritical (and the bifurcating periodic orbit is unstable and exists for parameters for which X is stable). It turns out that the criticality of the resonance bifurcation is determined by the behaviour of trajectories in the 'global parts' of the flow near X (i.e. the parts of the flow away from neighbourhoods of the equilibria). In the second half of this paper we give an example of a system which contains a homoclinic cycle of the form described above, and we explicitly calculate these constants. For the example we give, we show that the bifurcation always occurs supercritically. The calculation also allows us to compute the period of the bifurcating stable periodic orbit.

3. Proof of theorem 1

This section is devoted to the proof of theorem 1. The proof is divided into three sections; first we construct return maps on Poincaré sections in the standard way by

dividing the flow into 'local' and 'global' parts. The return maps describe the behaviour of trajectories asymptotically close to X. We then relate these maps to transition matrices, and finally deduce results on the eigenvalues of the transition matrices which enable us to prove the theorem.

A complete and detailed discussion of the construction of local and global maps near X relies implicitly on the local linearisability of the ODEs near the equilibria. Linearisability is a well-known issue, discussed at length by many authors for lower-dimensional robust heteroclinic cycles, for example Field & Swift [9], Chossat & Lauterbach [4] and Driesse & Homburg [7]. In this work we are content to provide formal, asymptotic, results and we do not provide detailed proofs of the estimates we use. As Field & Swift [9] remark (below their 'Remark 5.3') the detailed estimates computed by these, and other, authors carry over naturally to higher-dimensional settings such as those considered here. If the eigenvalues at the equilibrium satisfy a finite number of non-resonance conditions, then Sternberg's linearisation theorem [23] guarantees that the vector field in a neighbourhood of an equilibrium can be transformed into its linearisation by a C^1 coordinate change. In cases where the vector field is not linearisable, our computations of the leading order forms of the local and global maps are modified, just as in Field & Swift [9], by the inclusion of a small parameter $\tilde{\epsilon}$ which can be made arbitrarily small by restricting to a sufficiently small neighbourhood of the robust heteroclinic cycle. Finally we note that other techniques, such as the use of averaged Lyapunov functions [12], are available to deduce conditions for stability of robust heteroclinic cycles: these avoid the resonance issues noted above which arise in the Poincaré map approach. Of course, the Poincaré map offers greater insight into the bifurcation behaviour associated with a change in stability which is why we pursue it here.

Finally in this subsection we remark that, although this result can be deduced from a theorem of Hofbauer and Sigmund [12] (see chapter 17, pages 220–232), we see substantial value in the proof presented here since it is more transparent for the situation at hand, and it makes the geometry of the invariant subspaces explicit.

3.1. Poincaré maps

We define Poincaré intersections about ξ_1 in the standard way:

$$H_1^{\text{in}} : \{x_n = h^2\}, \quad H_1^{\text{out}} : \{x_{l+1} = h^2\},$$

for some small $h > 0$. We consider an initial point on H_1^{in}:

$$x^{\text{in},1} = (v_1^{\text{in},1}, \ldots, v_l^{\text{in},1}, x_{l+1}^{\text{in},1}, \ldots, x_{n-1}^{\text{in},1}, h^2),$$

where the v_j are radial coordinates which are zero at ξ_1. The 'time of flight' for the trajectory to reach H_1^{out} is

$$T_1 = -\frac{1}{e} \log \left(\frac{x_{l+1}^{\text{in},1}}{h^2} \right). \tag{9}$$

Let the coordinates of the trajectory when it reaches H_1^{out} be

$$x^{\text{out},1} = (v_1^{\text{out},1}, \ldots, v_l^{\text{out},1}, x_{l+1}^{\text{out},1}, \ldots, x_n^{\text{out},1})$$

then using the linearised flow at ξ_1, we have

$$
\begin{aligned}
v_j^{\text{out},1} &= h^{-\frac{2r_j}{e}} v_j^{\text{in},1} \left(x_{l+1}^{\text{in},1} \right)^{\frac{r_j}{e}}, & j &= 1, \ldots, l \\
x_{l+1}^{\text{out},1} &= h^2, & & \\
x_j^{\text{out},1} &= h^{\frac{-2s_{j-l-1}}{e}} x_j^{\text{in},1} \left(x_{l+1}^{\text{in},1} \right)^{\frac{s_{j-l-1}}{e}}, & j &= l+2, \ldots, n-1 \\
x_n^{\text{out},1} &= h^{2-\frac{2c}{e}} \left(x_{l+1}^{\text{in},1} \right)^{\frac{c}{e}}.
\end{aligned}
\tag{10}
$$

This defines the local map near ξ_1.

We now compute the global part of the Poincaré map. The isotypic decomposition of $\mathbb{R}^n$ with respect to the isotropy of $W^u(\xi_1)$ is

$$\mathbb{R}^n = P_1 \oplus V_1(s_1) \oplus \ldots \oplus V_1(s_{n-l-2}) \oplus V_1(c)$$

where $V_1(s_j) = \langle (0, \ldots, 0, x_{l+j+1}, 0, \ldots, 0) \rangle$ is the eigenspace spanned by the eigenvalue s_j. Recall that $\dim P_1 = l+1$. The form of the global map from H_1^{out} to a point $x^{\text{in},2}$ on $H_2^{\text{in}} : \{x_1 = h^2\}$, up to linear terms, is therefore:

$$
\begin{pmatrix}
x_1^{\text{in},2} \\
v_2^{\text{in},2} \\
\vdots \\
v_{l+1}^{\text{in},2} \\
x_{l+2}^{\text{in},2} \\
\vdots \\
x_n^{\text{in},2}
\end{pmatrix}
=
\begin{pmatrix}
h^2 \\
w_2{}^\star \\
\vdots \\
w_{l+1}{}^\star \\
0 \\
\vdots \\
0
\end{pmatrix}
+
\begin{pmatrix}
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
c_{21} & \cdots & c_{2l} & c_{2l+1} & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
c_{l+11} & \cdots & c_{l+1l} & c_{l+1l+1} & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & c_{l+2l+2} & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & c_{nn}
\end{pmatrix}
\begin{pmatrix}
v_1^{\text{out},1} \\
\vdots \\
v_l^{\text{out},1} \\
h^2 \\
x_{l+2}^{\text{out},1} \\
\vdots \\
x_n^{\text{out},1}
\end{pmatrix}
\tag{11}
$$

where the c_{ij} are $O(1)$ constants which depend on the global flow. By the invariance of the coordinate planes, $c_{jj} > 0$ for $l+2 \le j \le n$. In the section 4, we find approximations for these c_{jj} for a specific set of ODEs. The $w_j{}^\star$ are $O(h^2)$ constants which define where the heteroclinic connection $W^u(\xi_1)$ hits the plane H_2^{in}. Note that $x_1^{\text{in},2} = h^2$ by definition, and the radial coordinates on H_2^{in} are $v_2^{\text{in},2}, \ldots, v_{l+1}^{\text{in},2}$.

We compose the local and global maps, and use the symmetry ρ^{-1} to map points on H_2^{in} onto H_1^{in}, to give a return map ϕ on H_1^{in}. That is, we associate the point $\rho^{-1} x^{\text{in},2}$ with a point on H_1^{in}, so

$$\phi(x^{\text{in},1}) = \rho^{-1}(x^{\text{in},2}),$$

or

$$\phi\begin{pmatrix} v_1^{in,1} \\ \vdots \\ v_l^{in,1} \\ x_{l+1}^{in,1} \\ x_{l+2}^{in,1} \\ \vdots \\ x_n^{in,1} \end{pmatrix} = \rho^{-1}\begin{pmatrix} x_1^{in,2} \\ v_2^{in,2} \\ \vdots \\ v_{l+1}^{in,2} \\ x_{l+2}^{in,2} \\ \vdots \\ x_n^{in,2} \end{pmatrix} = \begin{pmatrix} v_2^{in,2} \\ \vdots \\ v_{l+1}^{in,2} \\ x_{l+2}^{in,2} \\ \vdots \\ x_n^{in,2} \\ x_1^{in,2} \end{pmatrix}.$$

The radial (v_j) components are contracting, and do not affect the other components, so we only need consider an $n - l - 1$ dimensional map ψ that describes the dynamics of the flow near the homoclinic cycle:

$$\psi\begin{pmatrix} x_{l+1}^{in,1} \\ x_{l+2}^{in,1} \\ \vdots \\ x_{n-2}^{in,1} \\ x_{n-1}^{in,1} \end{pmatrix} = \begin{pmatrix} x_{l+2}^{in,2} \\ x_{l+3}^{in,2} \\ \vdots \\ x_{n-1}^{in,2} \\ x_n^{in,2} \end{pmatrix} = \begin{pmatrix} \tilde{C}_{l+2}x_{l+2}^{out,1} \\ \tilde{C}_{l+3}x_{l+3}^{out,1} \\ \vdots \\ \tilde{C}_{n-1}x_{n-1}^{out,1} \\ \tilde{C}_n x_n^{out,1} \end{pmatrix} = \begin{pmatrix} \tilde{C}_{l+2}h^{-\frac{2s_1}{e}}x_{l+2}^{in,1}\left(x_{l+1}^{in,1}\right)^{\frac{s_1}{e}} \\ \tilde{C}_{l+3}h^{-\frac{2s_2}{e}}x_{l+3}^{in,1}\left(x_{l+1}^{in,1}\right)^{\frac{s_2}{e}} \\ \vdots \\ \tilde{C}_{n-1}h^{-\frac{2s_{n-l-2}}{e}}x_{n-1}^{in,1}\left(x_{l+1}^{in,1}\right)^{\frac{s_{n-l-2}}{e}} \\ \tilde{C}_n h^{2(1-\frac{c}{e})}\left(x_{l+1}^{in,1}\right)^{\frac{c}{e}} \end{pmatrix} \quad (12)$$

where $\tilde{C}_j = c_{jj}$. For convenience, we relabel the coordinates and constants. We write $p = n - l - 1$,

$$y_j = x_{l+j}^{in}, \ j = 1, \ldots, p,$$

$$C_j = h^{-\frac{2s_j}{e}}\tilde{C}_{l+j+1}, \ \text{for } 1 \le j \le p-1, \quad C_p = h^{2(1-\frac{c}{e})}\tilde{C}_n,$$

and

$$a_j = \frac{s_j}{e}, \ \text{for } 1 \le j \le p-1, \quad a_p = \frac{c}{e}, \quad (13)$$

to reach:

$$\psi\begin{pmatrix} y_1 \\ \vdots \\ y_{p-1} \\ y_p \end{pmatrix} = \begin{pmatrix} C_1 y_2 y_1^{a_1} \\ \vdots \\ C_{p-1}y_p y_1^{a_{p-1}} \\ C_p y_1^{a_p} \end{pmatrix}. \quad (14)$$

The map ψ has a fixed point at $y_1 = \ldots = y_p = 0$ which corresponds to the homoclinic cycle in the flow. The map has a second fixed point, the y_1 coordinate of which satisfies

$$y_1 = \left(\prod_{j=1}^{p}C_j\right)(y_1)^\delta,$$

where

$$\delta = \sum_{j=1}^{p} a_j = \frac{c + \sum_{j=1}^{p-1} s_j}{e}.$$

Note that $\prod_{j=1}^{p} C_j = h^{2(1-\delta)} \prod_{j=l+2}^{n} \tilde{C}_j$. This fixed point may correspond to a periodic orbit in the flow, and we discuss the existence and stability of this fixed point further in section 4 below.

In the next section we use transition matrix methods to determine the stability of the zero fixed point of ψ and hence the stability of the homoclinic cycle in the flow.

3.2. Transition matrices

Let G be the set of mappings $g : \mathbb{R}^p \to \mathbb{R}^p$ that have at lowest order the form

$$g(x_1, \ldots, x_p) = (C_1 x_1^{\alpha_{11}} x_2^{\alpha_{12}} \cdots x_p^{\alpha_{1p}}, \ldots, C_p x_1^{\alpha_{p1}} \cdots x_p^{\alpha_{pp}})$$

for constants $\alpha_{ij} \geq 0$ and C_i non-zero, $1 \leq i, j \leq p$. G is clearly closed under composition. We define the *transition matrix* [16, 9] of g to be the $p \times p$ real matrix $M(g)$ with entries $[M(g)]_{ij} = \alpha_{ij}$. It is easily verified that if $g_1, g_2 \in G$, then

$$M(g_2 \circ g_1) = M(g_2) M(g_1).$$

Any $g \in G$ has a fixed point at $x_1 = \ldots = x_p = 0$. The zero fixed point of the map g will be stable if all the row sums of $M(g)^N$ diverge to $+\infty$ as $N \to \infty$. Conversely, if any of the row sums of $M(g)^N$ tends to 0, then the fixed point is unstable.

For the homoclinic cycles described above, the transition matrix corresponding to the Poincaré map (14) is:

$$M(\psi) = A_p = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{p-1} & 0 & \cdots & 0 & 1 \\ a_p & 0 & \cdots & 0 & 0 \end{pmatrix},$$

with the a_i as in (13). We now recall the Perron–Frobenius theorem, but first describe our notation. We write $M \cdot \mathbf{v}$ to be the matrix-vector product with components $(M \cdot \mathbf{v})_i = M_{ij} v_j$, for a matrix M with components M_{ij} and column vector $\mathbf{v}$ with components v_j. We similarly write $\mathbf{v}^\top \cdot M$ to be the usual vector-matrix product, $\mathbf{v}^\top \cdot \mathbf{u}$ to be the vector-vector inner product (for $\mathbf{u}$ also a column vector), and $\mathbf{v} \cdot \mathbf{u}^\top$ to be the vector-vector outer product. The row sums of A_p^N can be written as $A_p^N \cdot \mathbf{1}$ where $\mathbf{1}$ is the vector $(1, 1, \ldots, 1)^T$.

Definition 7 *A real non-negative $p \times p$ matrix M is* primitive *if there exists an $N > 0$ such that all entries of M^N are strictly positive.*

Theorem 2 (Perron–Frobenius) *If M is a $p \times p$ non-negative primitive matrix, then there exists a unique and simple positive eigenvalue λ_{pf} which is dominant in the sense that $|\lambda| < \lambda_{\mathrm{pf}}$ for all other eigenvalues λ of M. There exist right and left eigenvectors $\mathbf{u}, \mathbf{v} > \mathbf{0}$ such that $M \cdot \mathbf{u} = \lambda_{\mathrm{pf}} \mathbf{u}$ and $\mathbf{v}^{\top} \cdot M = \lambda_{\mathrm{pf}} \mathbf{v}^{\top}$. If $\mathbf{u}$ and $\mathbf{v}$ are normalised such that $\mathbf{v}^{\top} \cdot \mathbf{u} = 1$ then*

$$\lim_{N \to \infty} (\lambda_{\mathrm{pf}}^{-N} M^N) = T \equiv \mathbf{u} \cdot \mathbf{v}^{\top}.$$

By inspection, A_p is primitive, since $a_i > 0$ for all i, and so $(A_p)^p$ will have strictly positive entries.

We now consider matrices of the form A_p, with $a_i > 0$, and in the remainder of this section, prove the following lemma.

Lemma 1 *If $\sum_{j=1}^{p} a_j < 1$, then the dominant eigenvalue λ_{pf} of A_p satisfies $\lambda_{\mathrm{pf}} < 1$. If $\sum_{j=1}^{p} a_j > 1$ then $\lambda_{\mathrm{pf}} > 1$.*

Using the Perron–Frobenius theorem, we have

$$\lim_{N \to \infty} \lambda_{\mathrm{pf}}^{-N} A_p^N \cdot \mathbf{1} = T \cdot \mathbf{1} = \mathbf{u}' > \mathbf{0},$$

where $\mathbf{u}' = \left(\sum_j v_j \right) \mathbf{u}$, and $\mathbf{u}$ and $\mathbf{v}$ are the left and right eigenvectors of A_p. If $\lambda_{\mathrm{pf}} < 1$, then the row sums tend to zero, that is, $\lim_{N \to \infty} A_p^N \cdot \mathbf{1} = 0$, and so the zero fixed point of the map ψ is unstable. If $\lambda_{\mathrm{pf}} > 1$, then $A_p^N \cdot \mathbf{1}$ will diverge to $+\infty$, and the zero fixed point of the map will be stable. Since the stability of the fixed point in the map corresponds to the stability of the homoclinic cycle in the flow, substituting for the a_j from (13) completes the proof of theorem 1.

We now prove lemma 1.

Claim 1 *The eigenvalues λ of A_p satisfy*

$$\lambda^p - \left(a_1 \lambda^{p-1} + a_2 \lambda^{p-2} + \ldots + a_{p-1} \lambda + a_p \right) = 0. \tag{15}$$

Proof. We will show by induction (on p) that

$$D_p \equiv \det(A_p - \lambda I) = (-1)^p \left[\lambda^p - \left(a_1 \lambda^{p-1} + a_2 \lambda^{p-2} + \ldots + a_{p-1} \lambda + a_p \right) \right]. \tag{16}$$

Firstly, we have

$$D_1 = a_1 - \lambda = (-1)[\lambda - a_1].$$

Then

$$D_k = \begin{vmatrix} a_1 - \lambda & 1 & 0 & \cdots & 0 \\ a_2 & -\lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{k-1} & 0 & \cdots & -\lambda & 1 \\ a_k & 0 & \cdots & 0 & -\lambda \end{vmatrix} = -\lambda D_{k-1} + (-1)^{k-1} a_k.$$

Assuming (16) for $p = k - 1$ we have

$$D_k = -\lambda(-1)^{k-1} \left[\lambda^{k-1} - \left(a_1 \lambda^{k-2} + a_2 \lambda^{k-3} + \ldots + a_{k-2} \lambda + a_{k-1} \right) \right] + (-1)^{k-1} a_k$$
$$= (-1)^k \left[\lambda^k - \left(a_1 \lambda^{k-1} + a_2 \lambda^{k-2} + \ldots + a_{k-1} \lambda + a_k \right) \right].$$

Hence (16) is satisfied for all integer $p \geq 1$. $\square$

The roots of equation (15) are the eigenvalues of A_p. The Perron–Frobenius theorem states that there is at least one positive root, the next claim relates the size of the positive root(s) to the entries a_j of A_p.

Claim 2 *Let λ be a positive root of (15). If $\sum_{j=1}^{p} a_j < 1$ then $\lambda < 1$; if $\sum_{j=1}^{p} a_j > 1$ then $\lambda > 1$.*

Proof. Consider the case $\sum_{j=1}^{p} a_j < 1$, and suppose that $\lambda \geq 1$:

$$\lambda^p = a_1 \lambda^{p-1} + a_2 \lambda^{p-2} + \ldots + a_{p-1} \lambda + a_p \leq \lambda^{p-1} \sum_{j=1}^{p} a_j < \lambda^{p-1} \leq \lambda^p.$$

This is a contradiction and hence $\lambda < 1$. Similarly, for the case $\sum_{j=1}^{p} a_j > 1$, suppose that $\lambda \leq 1$, and then

$$\lambda^p = a_1 \lambda^{p-1} + a_2 \lambda^{p-2} + \ldots + a_{p-1} \lambda + a_p \geq \lambda^{p-1} \sum_{j=1}^{p} a_j > \lambda^{p-1} \geq \lambda^p,$$

again a contradiction, and hence $\lambda > 1$. $\square$

Claim 2 is true for all positive roots λ, therefore it is certainly true for the dominant eigenvalue λ_{pf}, completing the proof of lemma 1. $\square$

4. Bifurcation criticality computation

In this section we give an example set of ODEs which contain a homoclinic cycle of the type discussed in section 2. For these equations, we are able to explicitly compute the shape of the heteroclinic connections between the equilibria, and use this to calculate the unknown constants $\tilde{C}_j$ in the global part of the Poincaré map (12) derived in section 3.1. This allows us to compute the criticality of the resonance bifurcation and in addition, the period of the resulting bifurcating orbit.

4.1. System description

The following equations satisfy conditions **(H1)** - **(H3)** and are equivariant under the action of Γ given in (5), (6) and (7).

$$\dot{x}_j = x_j(1 - \sum_{i=1}^{n} x_i^2 - cx_{j+1}^2 - s_{n-3}x_{j+2}^2 - \cdots - s_1 x_{j-2}^2 + ex_{j-1}^2), \quad j = 1, \ldots, n \qquad (17)$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $c, e, s_k > 0$, $(k = 1, \ldots, n-3)$. In addition, in this example, $l = 1$, and $Q(x)$ is truncated at third order. We consider only $n \geq 4$. The case $n = 3$ is the example of Guckenheimer and Holmes [10] and it is well known that the system truncated at third order has a degenerate resonance bifurcation at $c = e$.

The system (17) has n equilibria on the coordinate axes we label ξ_j; the equilibrium ξ_j has coordinates $x_j = 1$, $x_k = 0$, $k \neq j$. Each equilibrium is hyperbolic, and the eigenvalues of the linearisation about ξ_j are -2 (in the radial, that is, x_j direction),

$-c$ (in the x_{j-1} direction), e (in x_{j+1} direction), and $-s_k$ in the x_{k+j+1} direction ($k = 1, \ldots, n - 3$). Note that a homoclinic cycle between the equilibria ξ_j exists if $-c$ and e are of opposite sign, but without loss of generality we consider only the case $c, e > 0$. Thus e is the expanding eigenvalue and c is the contracting eigenvalue. The $-s_j$ are transverse eigenvalues; we assume $-s_j < 0$. By theorem 1, the homoclinic cycle is asymptotically stable if $c + \sum_{j=1}^{n-3} s_j > e$. If $c + \sum_{j=1}^{n-3} s_j < e$ the cycle is unstable.

In section 3.1 we constructed a Poincaré map (12) on a section H_1^{in}, approximating the flow near the homoclinic cycle. This map has a non-trivial fixed point with x_2-coordinate given by

$$x_2 = h^2 \left(\prod_{j=l+2}^{n} \tilde{C}_j \right)^{\frac{1}{1-\delta}} \tag{18}$$

where the $\tilde{C}_j$ are defined in the map (12).

This point corresponds to a periodic orbit in the flow, branching from the resonant homoclinic bifurcation, only if it is small as $\delta \to 1$. Let $C = \left(\prod_{j=l+2}^{n} \tilde{C}_j \right)$. For $C < 1$ the fixed point is small if $\delta < 1$ (where the heteroclinic cycle is unstable), so the bifurcation is supercritical. Conversely for $C > 1$ the orbit exists in $\delta > 1$, and the bifurcation is subcritical. In the following, we write $A = \log C$. Thus the bifurcation is supercritical if $A < 0$.

4.2. The solution for the heteroclinic trajectory

We now compute an approximation for the form of the heteroclinic trajectory near the resonance bifurcation, under the assumption that the difference between the expanding and contracting eigenvalues, $e - c$, is small. We write $e = c + \beta$ and take $|\beta| \ll 1$. We consider the heteroclinic connection from ξ_1 to ξ_2 in the x_1-x_2 plane; all other connections are symmetry-related. The equations in the x_1-x_2 plane are:

$$\dot{x}_1 = x_1(1 - (x_1^2 + x_2^2) - cx_2^2),$$
$$\dot{x}_2 = x_2(1 - (x_1^2 + x_2^2) + ex_1^2).$$

For ease of computation, we make a change of variables, writing $u = x_1^2$, $v = x_2^2$:

$$\dot{u} = 2u(1 - (u + v) - cv) \tag{19}$$
$$\dot{v} = 2v(1 - (u + v) + eu) \tag{20}$$

The heteroclinic connection is a solution to (19) and (20) with boundary conditions $\{u = 0, v = 1\}$ and $\{v = 0, u = 1\}$. When $c = e$ ($\beta = 0$), there exists an exact solution $v = v_0(u) \equiv 1 - u$.

Recall that at resonance,

$$c + s = e,$$

where $s = \sum_{j=1}^{n-3} s_j$. Since $e = c + \beta$, at resonance we also have

$$s = \beta.$$

Therefore, close to resonance, s is $O(\beta)$, although it is important to note that it is not necessary that each s_j is individually $O(\beta)$. We will now look for approximate solutions to (19) and (20) when $\beta \neq 0$. Write

$$v = v_0(u) + \beta v_1(u) + O(\beta^2). \tag{21}$$

Expanding equations (19) and (20) in powers of β, we find at $O(\beta)$:

$$v_1'(u) - \frac{1}{cu(1-u)} v_1(u) = -\frac{1}{c},$$

which gives

$$v_1(u) = \frac{1}{c} \left(\frac{u}{1-u} \right)^{1/c} \int_u^1 \left(\frac{1-\xi}{\xi} \right)^{1/c} \mathrm{d}\xi. \tag{22}$$

In the following we also find it convenient to write the heteroclinic connection as a function $u(v)$, that is, we write the connection in the form

$$u(v) = 1 - v + \beta u_1(v) + O(\beta^2). \tag{23}$$

By rearranging (23) and substituting into (21) we find that $u_1(v) = v_1(1-v)$, that is

$$u_1(v) = \frac{1}{c} \left(\frac{1-v}{v} \right)^{1/c} \int_0^v \left(\frac{\eta}{1-\eta} \right)^{1/c} \mathrm{d}\eta. \tag{24}$$

4.3. Constants in the global map

We now use the approximated form of the homoclinic connection (21) to compute the constants $\tilde{C}_j$ in the map (12). From (12) we have

$$\tilde{C}_j = \frac{x_j^{\mathrm{in},2}}{x_j^{\mathrm{out},1}}, \quad j = 3, \ldots, n. \tag{25}$$

Consider a trajectory close to the x_1-x_2 plane, so that all other coordinates are small. Approximations to the x_3-,...,-x_n equations in (17) are therefore (ignoring higher order terms in the small coordinates):

$$\dot{x}_3 = x_3(1 - (x_1^2 + x_2^2) - s_1 x_1^2 + e x_2^2), \tag{26}$$
$$\dot{x}_j = x_j(1 - (x_1^2 + x_2^2) - s_{j-2} x_1^2 - s_{j-3} x_2^2), \quad j = 4, \ldots, n-1, \tag{27}$$
$$\dot{x}_n = x_n(1 - (x_1^2 + x_2^2) - c x_1^2 - s_{n-3} x_2^2). \tag{28}$$

We consider a trajectory which starts close to the heteroclinic cycle. This trajectory will approximately follow the heteroclinic trajectory in the x_1-x_2 plane, so we integrate along the trajectory, from H_1^{out} to H_2^{in}. So from (26) for instance, we find:

$$\int_{H_1^{\mathrm{out}}}^{H_2^{\mathrm{in}}} \frac{1}{x_3} \mathrm{d}x_3 = \int_{H_1^{\mathrm{out}}}^{H_2^{\mathrm{in}}} (1 - (x_1^2 + x_2^2) - s_1 x_1^2 + e x_2^2) \, \mathrm{d}t. \tag{29}$$

We can compute the left hand side of (29):

$$\int_{H_1^{\mathrm{out}}}^{H_2^{\mathrm{in}}} \frac{1}{x_3} \mathrm{d}x_3 = \log\left(x_3^{\mathrm{in},2} \right) - \log\left(x_3^{\mathrm{out},1} \right) = \log\left(\frac{x_3^{\mathrm{in},2}}{x_3^{\mathrm{out},1}} \right) = \log(\tilde{C}_3),$$

(where the last equality follows from (25)), and we find similar expressions for the other $\log(\tilde{C}_j)$ by integrating equations (27) and (28).

As noted at the end of section 4.1, the criticality of the bifurcation depends on the sign of A, where

$$A = \log C = \log \left(\prod_{j=3}^{n} \tilde{C}_j \right) = \sum_{j=3}^{n} \log \tilde{C}_j.$$

Summing the integrated forms of equations (26) to (28) we find, again writing $x_1^2 = u$, and $x_2^2 = v$,

$$A = \sum_{j=3}^{n} \log \tilde{C}_j = \int_{H_1^{\text{out}}}^{H_2^{\text{in}}} \left((n-2)(1-(u+v)) - cu - \sum_{j=1}^{n-3} s_j(u+v) + ev \right) dt$$

where again the integral is taken along the heteroclinic connection in the x_1-x_2 plane. Note that this expression does not depend independently on the s_j; it only depends on the sum $s = \sum_{j=1}^{n-3} s_j$.

We now rewrite the integral as an integral in u. Recall that the heteroclinic connection can be written as $v(u) = 1 - u + \beta v_1(u) + O(\beta^2)$, and $e = c + \beta$. From (19), we have

$$\dot{u} = -2cu(1-u) - 2\beta(c+1)uv_1(u) + O(\beta^2),$$

so

$$\frac{1}{\dot{u}} = \frac{-1}{2cu(1-u)} + \beta \frac{c+1}{2c^2} \frac{v_1}{u(1-u)^2} + O(\beta^2).$$

To determine the bounds on the integral, note that on H_1^{out}, $x_2^2 = h^2$, so $v = h$, and the u-coordinate of the heteroclinic connection is $u = 1 - h + \beta u_1(h) + O(\beta^2)$. On H_2^{in}, $u = h$.

Thus, expanding in powers of β (recall that $s = O(\beta)$) we find:

$$A = \int_{H_1^{\text{out}}}^{H_2^{\text{in}}} ((n-2)(1-(u+v(u))) - cu - s(u+v(u)) + ev) \frac{1}{\dot{u}} du$$

$$= \int_{1-h+\beta u_1(h)}^{h} \{-cu + c(1-u) - s + \beta[-(n-2-c)v_1(u) + (1-u)]\}$$

$$\times \left\{ -\frac{1}{2cu(1-u)} + \beta \left[\frac{c+1}{2c^2} \frac{v_1}{u(1-u)^2} \right] \right\} du + O(\beta^2),$$

which, after rearranging terms and tidying up, becomes

$$2A = \int_{1-h+\beta u_1(h)}^{h} \frac{u - (1-u)}{u(1-u)} du - \beta \int_{1-h}^{h} \frac{1}{cu} du + s \int_{1-h}^{h} \frac{1}{cu(1-u)} du$$

$$+ \beta \int_{1-h}^{h} \frac{(n-2-c)v_1(u)}{cu(1-u)} + (1-2u)\frac{c+1}{c} \frac{v_1(u)}{u(1-u)^2} du + O(\beta^2),$$

$$= \int_{1-h+\beta u_1(h)}^{h} \frac{1}{1-u} - \frac{1}{u} du - \frac{\beta}{c} \int_{1-h}^{h} \frac{1}{u} du + \frac{s}{c} \int_{1-h}^{h} \frac{1}{u} + \frac{1}{1-u} du$$

$$+\frac{\beta}{c}\int_{1-h}^{h}\frac{(n-2-c)(1-u)+(1-2u)(c+1)}{u(1-u)^2}v_1(u)\,\mathrm{d}u+O(\beta^2),$$

$$=\log\left(1-\frac{\beta u_1(h)}{h}\right)+\log\left(1+\frac{\beta u_1(h)}{1-h}\right)+\frac{2s-\beta}{c}(\log h-\log(1-h))$$

$$-\beta\frac{n-1}{c}\int_{h}^{1-h}\frac{v_1(u)}{u(1-u)}\mathrm{d}u+\beta\frac{c+1}{c}\int_{h}^{1-h}\frac{v_1(u)}{(1-u)^2}\,\mathrm{d}u+O(\beta^2). \qquad (30)$$

Since the functions u_1 and v_1 are (in principal) known through the integrals (24) and (22) the above expression determines the sign of A, and hence yields the criticality of the resonance bifurcation. The remainder of this section is devoted to the computation of A which is perhaps surprisingly algebraically complicated.

4.4. Expansion of integrals

There are two types of terms in A which we cannot yet express explicitly: the terms involving $u_1(h)$ and the integrals involving $v_1(u)$. We will compute both of these by calculating a power series expansion for u_1. The two integrals in A which we need to compute are:

$$\int_{h}^{1-h}\frac{v_1(u)}{u(1-u)}\mathrm{d}u\equiv\int_{h}^{1-h}\frac{u_1(w)}{w(1-w)}\mathrm{d}w, \qquad (31)$$

and

$$\int_{h}^{1-h}\frac{v_1(u)}{(1-u)^2}\,\mathrm{d}u\equiv\int_{h}^{1-h}\frac{u_1(w)}{w^2}\,\mathrm{d}w. \qquad (32)$$

For convenience, we write $q=1/c$, then

$$u_1(w)=q\left(\frac{1-w}{w}\right)^q\int_{0}^{w}\left(\frac{\xi}{1-\xi}\right)^q\mathrm{d}\xi.$$

In the integrals (31) and (32) we only need to know $u_1(w)$ for $w<1$, so we can write the factors $(1-\xi)^{-q}$ and $(1-w)^q$ as power series. In the following, $\binom{q}{k}$ is a generalised binomial coefficient, that is,

$$\binom{q}{k}\equiv\frac{\Gamma(q+1)}{\Gamma(k+1)\Gamma(q-k+1)},$$

where $\Gamma(q)$ is the usual Gamma function. Substituting the expansions into $u_1(w)$ gives:

$$u_1(w)=qw^{-q}\sum_{k=0}^{\infty}\binom{q}{k}(-w)^k\int_{0}^{w}\xi^q\sum_{k=0}^{\infty}\binom{-q}{k}(-\xi)^k\mathrm{d}\xi,$$

$$=qw^{-q}\left(\sum_{k=0}^{\infty}(-1)^k\binom{q}{k}w^k\right)\left(\sum_{k=0}^{\infty}(-1)^k\binom{-q}{k}\int_{0}^{w}\xi^{q+k}\mathrm{d}\xi\right),$$

$$=qw^{-q}\left(\sum_{k=0}^{\infty}(-1)^k\binom{q}{k}w^k\right)\left(\sum_{k=0}^{\infty}(-1)^k\binom{-q}{k}\frac{1}{q+k+1}w^{q+k+1}\right),$$

$$=qw\sum_{n=0}^{\infty}c_nw^n \qquad (33)$$

where

$$c_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{q}{n-k} (-1)^k \binom{-q}{k} \frac{1}{q+k+1},$$

$$= \sum_{k=0}^{n} \binom{q}{n-k} \binom{q+k-1}{k} \frac{(-1)^{n-k}}{q+k+1} \tag{34}$$

Lemma 2 *The coefficient c_n in the expression for $u_1(w)$ given in (33), can be written as*

$$c_n = -\frac{q}{q+1} \frac{1}{\binom{n+q+1}{n}} \equiv -\frac{q\Gamma(n+1)\Gamma(q+1)}{\Gamma(q+n+2)}, \quad n \geq 1 \tag{35}$$

$$c_0 = \frac{1}{q+1}. \tag{36}$$

Proof. See appendix.

We now evaluate the integrals (31) and (32). We first consider (32), and writing $u_1(w) = qw \sum_{k=1}^{\infty} c_k w^k + qwc_0$ we find:

$$\int_h^{1-h} \frac{u_1(w)}{w^2} dw = q \int_h^{1-h} c_0 \frac{1}{w} + \sum_{k=1}^{\infty} c_k w^{k-1} dw,$$

$$= qc_0 [\log w]_h^{1-h} + q \sum_{k=1}^{\infty} \left[\frac{c_k}{k} w^k\right]_h^{1-h},$$

$$= qc_0(-\log h) + q \sum_{k=1}^{\infty} \frac{c_k}{k} + O(h).$$

In the appendix we show that

$$\sum_{k=1}^{\infty} \frac{c_k}{k} = -\frac{q}{(q+1)^2},$$

and so, using this result together with (36), we have

$$\int_h^{1-h} \frac{u_1(w)}{w^2} dw = -\frac{q}{q+1} \log h - \frac{q^2}{(q+1)^2} + O(h). \tag{37}$$

Now we consider (31). We expand both $u_1(w)$ and $(1-w)^{-1}$ to find:

$$\int_h^{1-h} \frac{u_1(w)}{w(1-w)} dw = q \int_h^{1-h} \left(\sum_{k=0}^{\infty} w^k\right) \left(\sum_{k=0}^{\infty} c_k w^k\right) dw,$$

$$= q \int_h^{1-h} \sum_{k=0}^{\infty} d_k w^k dw,$$

$$= q \left[\sum_{k=0}^{\infty} \frac{d_k}{k+1} w^{k+1}\right]_h^{1-h},$$

$$= q \sum_{k=0}^{\infty} \frac{d_k}{k+1} + O(h),$$

where we have defined $d_k = \sum_{j=0}^{k} c_j$. We evaluate d_k in the appendix to find:

$$\sum_{k=0}^{\infty} \frac{d_k}{k+1} = \frac{1}{q}.$$

Hence

$$\int_{h}^{1-h} \frac{u_1(w)}{w(1-w)} \mathrm{d}w = 1 + O(h). \tag{38}$$

Finally, we use our expansion of u_1 to find $u_1(h) = qh \sum_{k=0}^{\infty} c_k h^k$, and so

$$\log\left(1 - \frac{\beta u_1(h)}{h}\right) = -\beta q c_0 + O(\beta h) = -\beta \frac{q}{1+q} + O(\beta h) \tag{39}$$

and

$$\log\left(1 - \frac{\beta u_1(h)}{1-h}\right) = O(\beta h). \tag{40}$$

Therefore, substituting (37), (38), (39) and (40) into (30) we have

$$2A = -\beta \frac{q}{q+1} + q(2s - \beta)\log h - \beta q(n-1) + \beta(q+1)\left(\frac{q}{q+1}(-\log h) - \frac{q^2}{(q+1)^2}\right)$$
$$+ O(\beta h) + O(\beta^2),$$
$$= 2q(s - \beta)\log h + \beta\left(-\frac{q}{q+1} - nq + q - \frac{q^2}{q+1}\right) + O(\beta h) + O(\beta^2),$$

which simplifies dramatically to give

$$A = q(s - \beta)\log h - \frac{\beta n q}{2} + O(\beta h) + O(\beta^2).$$

By definition, as we approach the bifurcation point $s \to \beta$, and so, evaluating A at the bifurcation point we obtain to leading order

$$A \to -\frac{\beta n q}{2} < 0, \tag{41}$$

and hence the bifurcation is found to be supercritical for all values of $n \geq 4$ and $q = 1/c$. Note that although our calculation only computes A to leading order in β, since the criticality only depends on the sign of A, the bifurcation will be supercritical whenever β is sufficiently small.

5. Bifurcating periodic orbits

In this section we discuss the evolution of the unique stable periodic orbit produced in the resonance bifurcation for the system considered in section 4.

First we use the calculation of the return map to derive an approximate expression for the period of the orbit. We emphasise that the analytic result contains no adjustable parameters. Referring back to the calculation of the return map, combining equations (9) for the time spent in a neighbourhood of an equilibrium on the cycle with (18) which gives the leading-order relation between the location of the fixed point of the Poincaré map and the coefficients C_j, we obtain the following expression for

the period P of the periodic orbit, as usual neglecting the time spent moving between neighbourhoods of the equilibria on the cycle:

$$P \equiv nT = -\frac{n}{e} \log \left(C^{1/(1-\delta)} \right),$$

where $\delta = (c+s)/e$ and $C = \prod_{j=l+2}^{n} \tilde{C}_j$ as before. Substituting $\log C \equiv A = -n\beta q/2$ where $\beta = e - c > 0$ and $q = 1/c$ this expression simplifies to yield

$$P = \frac{n}{e} \frac{n\beta q}{2(1-\delta)} = \frac{n^2(e-c)}{2c(e-c-s)}. \tag{42}$$

This expression has been derived near the resonance bifurcation (i.e. $|e - c - s| \ll 1$), in the case that $|e - c| \ll 1$ and therefore is expected to be asymptotically correct in the limits of small s and small $e - c$.

We compare the analytic result (42) with numerical integrations of (17). Our numerical integrations are carried out with a standard 4th order Runge–Kutta scheme, with the return times $P_1, P_2, \ldots$ to the cross-section $x_1 = 0.2$ computed by linearly interpolating points on the trajectory that lie on either side of the cross-section. Since trajectories may be assumed to converge exponentially to the periodic orbit (discounting the possibility that it is nonhyperbolic) a highly accurate extrapolation of the true period of the orbit may be obtained by employing the *ansatz* $P_k = P + b_0 e^{-b_1 k}$ and eliminating the coefficients (which are supposed to be constant) b_0 and b_1 by considering a set of three return times P_k, P_{k+m} and P_{k+2m}. This leads to the extrapolation formula

$$P = \frac{P_{k+2m} P_k - P_{k+m}^2}{P_{k+2m} - 2P_{k+m} + P_k}, \tag{43}$$

which we find to give excellent results, even when the computed return times P_j used in (43) are far from the true (ultimate) period P.

Figure 1 confirms the accuracy of the approximation (42), showing the period P as a function of $e - c - s$ for four different values of s in the case $n = 4$ (where there is only a single transverse eigenvalue). The excellent agreement for $s = 0.01$ (the solid curve and diamond symbols) and $s = 0.1$ (dashed curve and triangle symbols) shows that when $0 < s \ll c$ the approximate expression (42) is extremely accurate. Figure 2 confirms the accuracy in the limit $0 < s \ll c$ when c is varied at fixed s; the data for $c = 1$ are most accurately approximated by (42). Finally, figure 3 illustrates the dependence of P on n as being $P \sim n^2$ - not a scaling that one might intuitively have proposed. For this figure we have for convenience taken $s = 0.1$ for $n = 4, 5, 6$ and achieved this by setting $s_j = s/(n-3)$ for all $1 \le j \le n-3$. We remark that the accuracy of (42) does not appear to vary significantly with either increasing or decreasing n.

Further numerical investigations in the case that $s > 0$ but allowing one, or more, of the s_j to be negative, so that the transverse eigenvalues, $-s_j$ are positive, (i.e. when $e - c - s < 0$, the homoclinic cycle is only essentially asymptotically stable rather than asymptotically stable) show that a periodic orbit bifurcates at $e - c - s = 0$ and has a period that is extremely close to that for the $s_j = s/(n-3)$ case as long as s remains much less than e. For example, in the case $n = 5$, $s = s_1 + s_2 = 0.1$, $c = 0.5$, $e = 0.7$
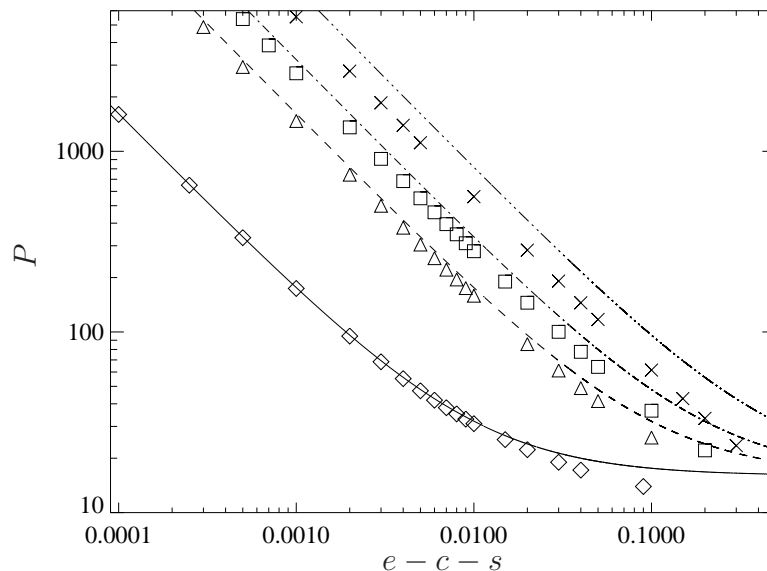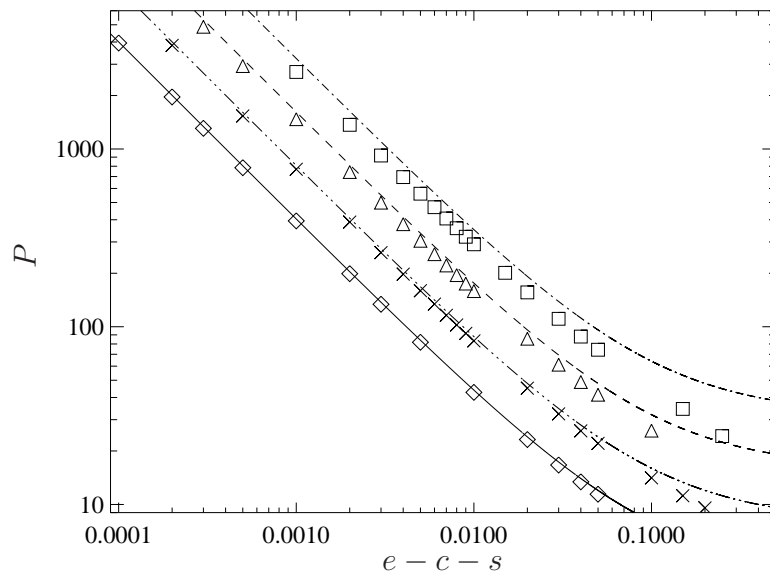
Figure 1. Orbit period P plotted against the eigenvalue combination $e - c - s$ for the periodic orbit created in the resonance bifurcation at $e - c - s = 0$. Parameters are $n = 4$, $c = 0.5$. Calculations are carried out by varying e for fixed values of s. Numerical data and the approximation (42) are shown for four values of s, from bottom to top: $s = 0.01$ ($\Diamond$, solid line); $s = 0.1$ ($\triangle$, dashed line); $s = 0.2$ ($\square$, dash-dotted line); $s = 0.5$ ($\times$, dash-triple-dotted line). The approximation is excellent for small s but systematically overestimates the period for larger s.

we observed that $P = 41.169$ for $s_1 = s_2 = 0.05$. On increasing s_1 (and correspondingly decreasing s_2 so that $s_1 + s_2 = 0.1$ always) we observed numerically that the period of the periodic orbit remained within $P = 41.17 \pm 0.04$ for $0.05 \leq s_1 \leq 0.65$ before reducing rapidly for larger s_1: $P = 35.4$ at $s_1 = 0.655$ and $P = 30.15$ at $s_1 = 0.66$. It seems reasonable to expect that the shape of the periodic orbit near the equilibria changes substantially when transverse eigenvalues become of the same order as the expanding eigenvalue e. We leave a detailed investigation of the dynamics to be the subject of future work.

6. Discussion

In this paper we have discussed resonance bifurcations in a class of vector fields containing robust homoclinic cycles that connect equilibria on a single group orbit, each lying in an l-dimensional hyperplane in $\mathbb{R}^n$.

First, using the well-known Poincaré return map technique and the construction of transition matrices, we proved a general stability result for this class of homoclinic cycles in section 3. We then discussed the resonant bifurcation that occurs when $c + s = e$, where s is the sum of the relevant transverse eigenvalues. Although our approach is asymptotic rather than rigorous, we anticipate that the methods of Driesse & Homburg
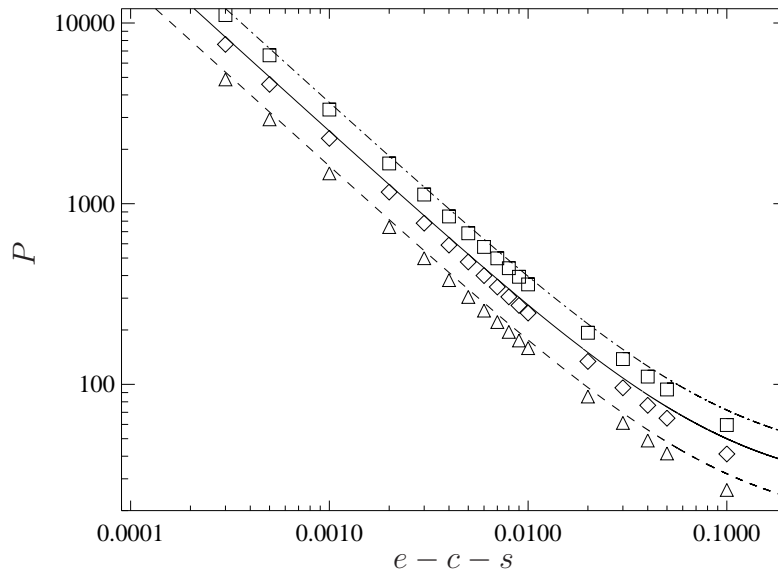
Figure 2. Orbit period P plotted against the eigenvalue combination $e - c - s$ for the periodic orbit created in the resonance bifurcation at $e - c - s = 0$. Calculations are carried out by varying e for fixed values of c. Parameters are $n = 4$, $s = 0.1$. Numerical data and the approximation (42) are shown for four values of c, from bottom to top: $c = 2$ ($\diamond$, solid line); $c = 1$ ($\times$, dash-triple-dotted line); $c = 0.5$ ($\triangle$, dashed line); $c = 0.25$ ($\square$, dash-dotted line). The approximation is excellent for large c but systematically overestimates the period for smaller c. Note that the data for $c = 0.5$ ($\triangle$, dashed line) is shown with the same symbols and line style in figures 1 and 3.

[7], or Field & Swift [9] could be applied to this example by straightforward extensions of their results on lower-dimensional robust homoclinic cycles.

Second, for a specific polynomial vector field in which the equilibria lie on the axes in $\mathbb{R}^n$ it is possible to make substantial further progress and compute the period of the bifurcating orbit by integrating along trajectories to explicitly compute the global parts of the return map. To the best of our knowledge, this is the first calculation of its kind.

The calculation rests on the assumption that the sum s of the relevant transverse eigenvalues is small compared to the leading stable and unstable eigenvalues (c and e respectively). The details of the calculation become algebraically rather complicated, but only involve computing a number of integrals; this is achieved through power series expansions and summations. Unlike most return map calculations where the global parts of the map introduce undetermined coefficients whose magnitudes are rarely known, our results contain no free parameters and are seen to agree well with numerical computations of the period of the bifurcating orbit.

It is of interest that the calculation shows that for this class of systems the resonance bifurcation is always supercritical. It would be of interest to investigate extending the present work to find a correspondingly simple class of examples where the criticality of the bifurcation depended non-trivially on parameters. For instance, in [19] we give an

Figure 3. Orbit period P plotted against the eigenvalue combination $e - c - s$ for the periodic orbit created in the resonance bifurcation at $e - c - s = 0$. Calculations are carried out by varying e for fixed values of $s = 0.1$ and $c = 0.5$, keeping all the s_j equal as discussed in the text. Numerical data and the approximation (42) are shown for three values of n, from bottom to top: $n = 4$ ($\triangle$, dashed line); $n = 5$ ($\Diamond$, solid line); $n = 6$ ($\Box$, dash-dotted line). No trend in increasing or decreasing accuracy is observed as n varies. Note that the data for $n = 4$ ($\triangle$, dashed line) is shown with the same symbols and line style in figures 1 and 2.

example of a cycle which has a resonance bifurcation that switches from subcritical to supercritical as parameters vary. It would also be interesting to see to what extent the analysis here could be applied to more complicated heteroclinic cycles in Lotka–Volterra systems, such as those studied numerically by Rabinovich and co-workers [20, 24, 21, 22] or systems of more complicated kinds, such as those subjected to time-periodic forcing [5], where the Poincaré map approach has been found, asymptotically at least, to provide useful insight into the dynamics.

Appendix

In this appendix we provide the proof of lemma 2, and the computation of the sums $\sum_{k=1}^{\infty} \frac{c_k}{k}$ and $\sum_{k=0}^{\infty} \frac{d_k}{k+1}$ which are used in section 4.4 in the computation of A.

Proof of lemma 2 It is trivial to check that $c_0 = 1/(q+1)$. For $n \geq 1$, let

$$f(n) = \sum_{k=0}^{n} F(n, k)$$

where

$$F(n,k) = \frac{1+q}{q}\frac{(-1)^{n-k+1}}{q+k+1}\binom{q+k-1}{k}\binom{q}{n-k}\binom{n+q+1}{n},$$

that is, $f(n)$ is the right hand side of (34) divided by the right hand side of (35). We will show that $f(n)$ is independent of n, and specifically, that $f(n) = 1$, which proves the lemma.

Note that $F(n,k) = 0$ for $k \geq n+1$. Further, let

$$R(n,k) = \frac{-k(1+k+q)(k-n+q)(k-1-n-nq)}{qn(1+n)^2(1-k+n)}$$

for $k \neq n+1$, and define $G(n,k) = R(n,k)F(n,k)$. Note that $G(n,n+1)$ is well defined and finite, and $G(n,k) = 0$ for $k \geq n+2$. It can be verified that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

Summing over all k gives

$$\sum_{k=0}^{\infty} F(n+1,k) - \sum_{k=0}^{\infty} F(n,k) = \sum_{k=0}^{\infty} G(n,k+1) - \sum_{k=0}^{\infty} G(n,k), \tag{A.1}$$

and using the fact that $F(n,k) = 0$ when $k \geq n+1$ and $G(n,k) = 0$ when $k \geq n+2$ gives

$$\sum_{k=0}^{n+1} F(n+1,k) - \sum_{k=0}^{n} F(n,k) = \sum_{k=0}^{n} G(n,k+1) - \sum_{k=1}^{n+1} G(n,k) - G(n,0) \tag{A.2}$$

$$f(n+1) - f(n) = -G(n,0) = 0 \tag{A.3}$$

so $f(n)$ is independent of n. It is simple to check that $f(1) = 1$, completing the proof. $\square$

The following lemma is used a number of times in what follows:

Lemma 3
$$\sum_{j=0}^{k} \frac{\Gamma(x+1)\Gamma(j+1)}{\Gamma(x+j+1)} = \frac{x}{x-1}\left(1 - \frac{\Gamma(x)\Gamma(k+2)}{\Gamma(x+k+1)}\right) \tag{A.4}$$

Proof. We use induction on k. It is simple to check that (A.4) holds for $k = 1$. Then assuming (A.4) is true for some $k > 1$, we find for $k+1$,

$$\sum_{j=0}^{k+1} \frac{\Gamma(x+1)\Gamma(j+1)}{\Gamma(x+j+1)} = \sum_{j=0}^{k} \frac{\Gamma(x+1)\Gamma(j+1)}{\Gamma(x+j+1)} + \frac{\Gamma(x+1)\Gamma(k+2)}{\Gamma(x+k+2)}$$

$$= \frac{x}{x-1}\left(1 - \frac{\Gamma(x)\Gamma(k+2)}{\Gamma(x+k+1)}\right) + \frac{x\Gamma(x)\Gamma(k+2)}{(x+k+1)\Gamma(x+k+1)}$$

$$= \frac{x}{x-1}\left(1 - \frac{\Gamma(x)\Gamma(k+2)}{\Gamma(x+k+1)}\left(\frac{k+2}{x+k+1}\right)\right)$$

$$= \frac{x}{x-1}\left(1 - \frac{\Gamma(x)\Gamma(k+3)}{\Gamma(x+k+2)}\right)$$

$\square$

Corollary 1 *For $x > 1$,*

$$\sum_{j=0}^{\infty} \frac{\Gamma(x+1)\Gamma(j+1)}{\Gamma(x+j+1)} = \frac{x}{x-1}$$

Proof. The required result is equivalent to showing that

$$\lim_{k \to \infty} \frac{\Gamma(x)\Gamma(k+3)}{\Gamma(x+k+2)} = 0.$$

To show this we write the combination of Gamma functions as a Beta function:

$$\frac{\Gamma(x)\Gamma(k+3)}{\Gamma(x+k+2)} = (k+2)B(k+2, x) = (x-1)B(k+3, x-1), \tag{A.5}$$

where we define the usual Beta function

$$B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, dt = \int_0^\infty \frac{s^{q-1}}{(1+s)^{p+q}}\, ds, \tag{A.6}$$

making the change of variable $1 + s = t^{-1}$, see [25]. The last equality in (A.5) can be derived in a straightforward manner by integrating by parts the first integral expression in (A.6) for $B(p, q)$. Then we have

$$(x-1)B(k+3, x-1) = \int_0^\infty \frac{(x-1)s^{x-2}}{(1+s)^{x+k+2}}\, ds < \int_0^\infty (x-1)(1+s)^{-k-4}\, ds,$$

$$< -\left. \frac{(x-1)(1+s)^{-k-3}}{k+3} \right|_{s=0}^{\infty} = \frac{x-1}{k+3},$$

which clearly tends to zero as $k \to \infty$ for any fixed $x > 1$. $\qquad\square$

We are now able to compute the following two sums which are used in section 4.4. Firstly, consider

$$\sum_{k=1}^{\infty} \frac{c_k}{k} = -q \sum_{k=1}^{\infty} \frac{\Gamma(k+1)\Gamma(q+1)}{k\Gamma(q+k+2)},$$

$$= -\frac{q}{(q+2)(q+1)} \sum_{k=1}^{\infty} \frac{\Gamma(k)\Gamma(q+3)}{\Gamma(q+k+2)} = -\frac{q}{(q+2)(q+1)} \sum_{k=0}^{\infty} \frac{\Gamma(k+1)\Gamma(q+3)}{\Gamma(q+k+3)},$$

$$= -\frac{q}{(q+2)(q+1)} \frac{q+2}{q+1} = -\frac{q}{(q+1)^2}.$$

Now consider

$$d_k = \sum_{j=0}^{k} c_j,$$

$$= c_0 - q \sum_{j=1}^{k} \frac{\Gamma(j+1)\Gamma(q+1)}{\Gamma(q+j+2)},$$

$$= \frac{1}{1+q} + \frac{q\Gamma(q+1)}{\Gamma(q+2)} - q \sum_{j=0}^{k} \frac{\Gamma(j+1)\Gamma(q+1)}{\Gamma(q+j+2)},$$

$$= \frac{1}{1+q} + \frac{q}{1+q} - \frac{q}{q+1} \sum_{j=0}^{k} \frac{\Gamma(j+1)\Gamma(q+2)}{\Gamma(q+j+2)},$$

$$= 1 - \frac{q}{(q+1)} \frac{(q+1)}{q} \left(1 - \frac{\Gamma(q+1)\Gamma(k+2)}{\Gamma(q+k+2)} \right),$$
$$= \frac{\Gamma(q+1)\Gamma(k+2)}{\Gamma(q+k+2)},$$

so that the second sum can be computed to be

$$\sum_{k=0}^{\infty} \frac{d_k}{k+1} = \sum_{k=0}^{\infty} \frac{\Gamma(q+1)\Gamma(k+1)}{\Gamma(q+k+2)},$$
$$= \frac{1}{q+1} \sum_{k=0}^{\infty} \frac{\Gamma(q+2)\Gamma(k+1)}{\Gamma(q+k+2)},$$
$$= \frac{1}{q+1} \frac{q+1}{q} = \frac{1}{q}.$$

Acknowledgements

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