THE NORMAL FORM FOR A $1:\sqrt{3}$ HOPF/STEADY-STATE MODE INTERACTION

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Abstract

The interaction of three-dimensional steady and oscillatory patterns on a hexagonal planar lattice is considered, when the ratio of the pattern lengthscales is $1 : \sqrt{3}$. The normal form for the mode interaction is derived from symmetry considerations for the simplest case; this is where the size of the imposed lattice is chosen to ensure that the relevant symmetry group, $D_6 \ltimes T^2$, acts by its fundamental representation on both the steady and oscillatory modes. This analysis is of interest for many pattern-forming systems because the wavenumbers involved in mode interactions of this kind are those selected naturally by the system in a spatially-extended domain.

AMS subject classification: 37C80, 37G05, 37G40, 37L10

1 Introduction

A common feature of many spatially-extended continuum systems is that they undergo pattern-forming instabilities of an initially uniform state. These often produce steady regular periodic patterns, for example stripes or hexagons. In some cases, though, the loss of stability is via a Hopf bifurcation and oscillatory phenomena such as standing or travelling waves are seen. Near the boundaries in parameter space which divide these two kinds of instability we expect some sort of interaction between steady and oscillating patterns. These interactions may well lead to complex dynamics. In this paper we consider such an interaction and derive, by symmetry arguments, the normal form for the behaviour close to the pattern-forming instability threshold when the system parameters are near the boundary which divides the regions of steady and oscillatory pattern-forming behaviour.

Much of the time this transition from steady instability to oscillatory instability takes place at a point where the preferred wavenumbers (for a spatially-extended plane layer) for the two kinds of instability are distinct; there is a jump in the preferred horizontal scale of the pattern. In this paper we examine the case where the ratio of the preferred wavenumbers is $\sqrt{3}$. Although this seems very restrictive, the normal form we derive is the simplest possible one for a Hopf/steady-state mode interaction on a hexagonal lattice and provides a starting point for consideration of more complex mode interactions at different wavenumber ratios. The frequency of the oscillatory instability remains bounded away from zero at the codimension-2 point; this fact distinguishes clearly between Takens-Bogdanov bifurcations [7] and the

Hopf/steady-state bifurcations considered here and previously in [2]. Since many physical systems form hexagonal, rather than square, patterns it is clearly of interest to frame problems which have hexagonal solutions. A clear recent example of a physical system where this analysis may be applicable is a two-layer convection problem where the effects of surface-tension are important [6].

2 Mode interaction

We consider a set of smooth PDEs

$$\frac{\partial \mathbf{u}}{\partial t} = \mathcal{F}(\mathbf{u}, \lambda_1, \lambda_2) \tag{1}$$

for the quantities $\mathbf{u}(\mathbf{x}, z, t)$ in the domain $(\mathbf{x}, z) \in \mathbb{R}^2 \times [0, 1]$ where $\mathcal{F}$ is a nonlinear function and λ_1 and λ_2 are (real) physical parameters for the system. We assume that there is a uniform, time-independent basic state $\mathbf{u} = 0$ which exists for all (λ_1, λ_2). The uniform state is invariant under the natural action of the group $E(2)$ of all translations, rotations and reflections of the plane. We further assume that at the point $\lambda_1 = \lambda_1^c$, $\lambda_2 = \lambda_2^c$ the uniform state is simultaneously unstable to perturbations at two distinct wavenumbers; for the smaller wavenumber the eigenvalues are imaginary and for the larger wavenumber they are zero. The marginal stability curves for the uniform state take the form shown in figure 1.

The $E(2)$ invariance of the basic state $\mathbf{u} = 0$ causes problems; as all horizontal directions are equivalent there are whole circles of critical wavevectors to consider. We would like to perform a centre manifold reduction to obtain a finite-dimensional set of ODEs describing the mode interaction. This is only possible if we introduce extra constraints.

The constraints we introduce are that we require the bifurcating modes to be periodic with respect to a doubly-periodic lattice $\mathcal{L}$ in the plane, $\mathbf{u}(\mathbf{x}+$

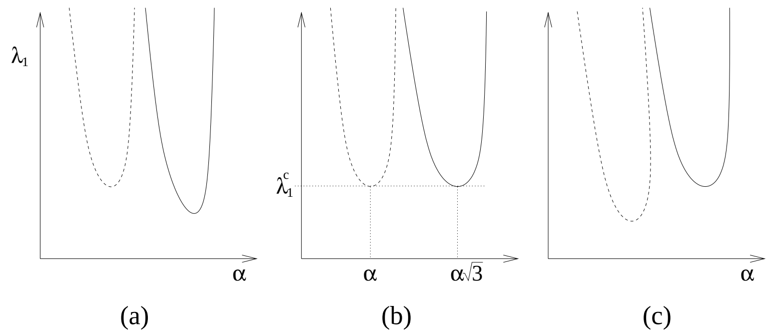

Figure 1: (a) Marginal stability curves (wavenumber α against λ_1) for the uniform state for $\lambda_2 < \lambda_2^c$. Below the curves the uniform state is stable to steady (solid line) and oscillatory (dashed line) perturbations. As λ_1 increases the first instability is a steady one at this value of λ_2. (b) $\lambda_2 = \lambda_2^c$. (c) $\lambda_2 > \lambda_2^c$; at this point the oscillatory instability occurs first as λ_1 increases.

$\ell, z, t) = \mathbf{u}(\mathbf{x}, z, t)$ for all $\ell \in \mathcal{L}$. For this problem we take the real-space lattice $\mathcal{L}$ to be

$$\mathcal{L} = \left\{ n\ell_1 + m\ell_2 : (n,m) \in \mathbb{Z}^2,\ \ell_1 = \frac{2\pi}{\alpha}\left(1, -\frac{1}{\sqrt{3}}\right),\ \ell_2 = \frac{2\pi}{\alpha}\left(0, \frac{2}{\sqrt{3}}\right) \right\} \tag{2}$$

and the corresponding dual lattice $\mathcal{L}^*$, defined by

$$\mathcal{L}^* = \left\{ n\mathbf{k}_1 + m\mathbf{k}_2 : (n,m) \in \mathbb{Z}^2,\ \mathbf{k}_1 = \alpha(1,0),\ \mathbf{k}_2 = \alpha\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \right\} \tag{3}$$

so that $\mathbf{k}_i \cdot \ell_j = 2\pi\delta_{ij}$. The factor α is the critical wavenumber for oscillatory perturbations, and could be removed by re-scaling lengths in the original PDEs. Having restricted the problem to a lattice, we have ensured that the dimension of the centre manifold of the bifurcation problem is finite, and have bounded away from zero the growth rates of all other modes not on the critical circle. The geometry of the lattice is shown in figure 2. The mode interaction involves six oscillatory modes $z_1, \ldots, z_6$ and three steady ones w_1, w_2, w_3.

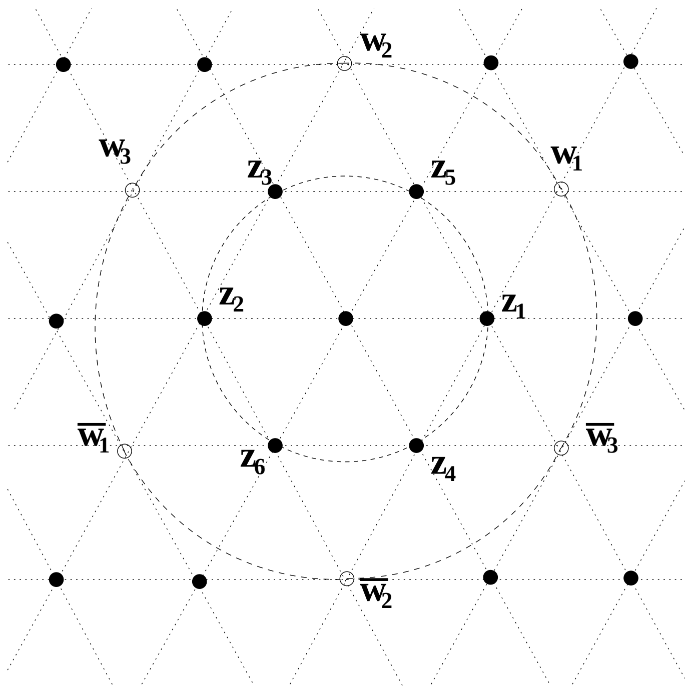

Figure 2: The dual lattice $\mathcal{L}^*$ for the $1:\sqrt{3}$ Hopf/steady-state mode interaction on a hexagonal lattice. The two dashed circles indicate the circles of critical wavevectors: they have a radius ratio of $\sqrt{3}$. The oscillatory modes are $z_1, \ldots, z_6$ and the steady modes are $w_1, \ldots, w_3$.

The symmetry group of the problem has also been reduced, from $E(2)$ to $\Gamma = D_6 \ltimes T^2$. The group D_6 is the holohedry of the lattice (the symmetry elements which leave the lattice invariant) and the two-torus of translations appears because there is no pre-determined spatial origin for the lattice. In normal form the amplitude equations for the modes $z_1, \ldots, z_6, w_1, w_2, w_3$ are also invariant under a normal form symmetry corresponding to a circle group S^1 of time translations. Hence the full symmetry group of the problem is $\Gamma \times S^1$.

3 Group action and linear theory

After restricting to the lattice, perturbations to the basic state take the form

$$\begin{aligned}\mathbf{u} \;=\;& Re[z_1 e^{\mathrm{i}(\alpha x-\omega t)} + z_2 e^{-\mathrm{i}(\alpha x+\omega t)} + z_3 e^{\mathrm{i}(-\frac{\alpha}{2}x+\frac{\alpha\sqrt{3}}{2}y-\omega t)} + z_4 e^{\mathrm{i}(\frac{\alpha}{2}x-\frac{\alpha\sqrt{3}}{2}y-\omega t)}\\ &+z_5 e^{\mathrm{i}(\frac{\alpha}{2}x+\frac{\alpha\sqrt{3}}{2}y-\omega t)} + z_6 e^{\mathrm{i}(-\frac{\alpha}{2}x-\frac{\alpha\sqrt{3}}{2}y-\omega t)} + w_1 e^{\mathrm{i}(\frac{3\alpha}{2}x+\frac{\alpha\sqrt{3}}{2}y)}\\ &+w_2 e^{\mathrm{i}\alpha\sqrt{3}y} + w_3 e^{\mathrm{i}(-\frac{3\alpha}{2}x+\frac{\alpha\sqrt{3}}{2}y)}]\mathbf{F}(z)\end{aligned} \tag{4}$$

where $\mathbf{F}(z)$ forms the vertical structure of the solution and ω is the frequency of the Hopf bifurcation. The space of perturbations $\mathbf{w} = (z_1, z_2, z_3, z_4, z_5, z_6, w_1, w_2, w_3)$ is then isomorphic to the vector space $W \cong \mathbb{C}^9$. The action of $\Gamma \times S^1$ on the mode amplitudes is inherited from its natural action on the plane $\mathbb{R}^2$; the action of D_6 is generated by the reflection $m_x : (x, y) \to (x, -y)$ and the rotation anticlockwise through an angle of $\pi/3$, denoted ρ:

$$m_x : \quad (z_1, z_2, z_3, z_4, z_5, z_6) \to (z_1, z_2, z_6, z_5, z_4, z_3) \tag{5}$$

$$(w_1, w_2, w_3) \to (\bar{w}_3, \bar{w}_2, \bar{w}_1) \tag{6}$$

$$\rho : \quad (z_1, z_2, z_3, z_4, z_5, z_6) \to (z_5, z_6, z_2, z_1, z_3, z_4) \tag{7}$$

$$(w_1, w_2, w_3) \to (w_2, w_3, \bar{w}_1) \tag{8}$$

The action of the translation group $T^2 \times S^1$ is given by

$$[(\xi, \eta), \phi] : \quad (x, y, t) \to (x + \xi/\alpha, y + \eta/\alpha, t + \phi/\omega) \tag{9}$$

$$\begin{aligned}(z_1, z_2, z_3, z_4, z_5, z_6) \to\;& (z_1 e^{\mathrm{i}(\xi-\phi)}, z_2 e^{-\mathrm{i}(\xi+\phi)}, z_3 e^{\mathrm{i}(-\frac{\xi}{2}+\frac{\eta\sqrt{3}}{2}-\phi)},\\ & z_4 e^{\mathrm{i}(\frac{\xi}{2}-\frac{\eta\sqrt{3}}{2}-\phi)}, z_5 e^{\mathrm{i}(\frac{\xi}{2}+\frac{\eta\sqrt{3}}{2}-\phi)}, z_6 e^{\mathrm{i}(-\frac{\xi}{2}-\frac{\eta\sqrt{3}}{2}-\phi)})\end{aligned} \tag{10}$$

$$(w_1, w_2, w_3) \to (w_1 e^{\mathrm{i}(\frac{3\xi}{2}+\frac{\eta\sqrt{3}}{2})}, w_2 e^{\mathrm{i}\eta\sqrt{3}}, w_3 e^{\mathrm{i}(-\frac{3\xi}{2}+\frac{\eta\sqrt{3}}{2})}) \tag{11}$$

We require the normal form $\dot{\mathbf{w}} = \mathbf{f}(\mathbf{w}, \mu_1(\lambda_1, \lambda_2), \mu_2(\lambda_1, \lambda_2))$ to be $\Gamma \times S^1$-equivariant, i.e. $\mathbf{f}(\gamma\mathbf{w}, \mu_1, \mu_2) = \gamma\mathbf{f}(\mathbf{w}, \mu_1, \mu_2)$ for all $\gamma \in \Gamma \times S^1$. The Taylor series expansion of $\mathbf{f}$ up to terms of degree n can be made to commute with the S^1 action by applying near-identity transformations to remove terms order by order which do not commute with the S^1 action. In this paper we are interested only in computing the cubic truncation of $\mathbf{f}$ and assume that the cubic truncation has the normal form symmetry. We ignore questions concerning the influence of higher-order terms, in particular the 'tail' of terms of degree higher than n which may not be $\Gamma \times S^1$-equivariant, but only Γ-equivariant. Assuming $\mathbf{f}$ is a smooth function, we may write it as

$$\mathbf{f}(\mathbf{w}) = \sum_{j=1}^{n} \mathbf{g}_j(\mathbf{w}, \mu_1, \mu_2)\mathbf{h}_j(I_1, \ldots, I_K) \tag{12}$$

where the terms $\mathbf{g}_1(\mathbf{w}), \ldots, \mathbf{g}_n(\mathbf{w})$ are $\Gamma \times S^1$-equivariant and the $\mathbf{h}_j$ terms are polynomials in the K distinct $\Gamma \times S^1$ invariants $I_1, \ldots, I_K$.

We first consider the linearisation of the amplitude equations:

$$\dot{z}_j = (\mu_1(\lambda_1, \lambda_2) + \mathrm{i}\omega_0(\mu_1))z_j, \tag{13}$$

$$\dot{w}_k = \mu_2(\lambda_1, \lambda_2)w_k \tag{14}$$

for $1 \leq j \leq 6$ and $1 \leq k \leq 3$, where $\omega_0(\mu_1)$ is the nonlinear correction to the frequency of oscillations of the Hopf bifurcation; the physical oscillation frequency is $\omega + \omega_0(\mu_1)$. The linearised equations must take this form because the action of Γ on (w_1, w_2, w_3) is absolutely irreducible and the action on $(z_1, z_2, z_3, z_4, z_5, z_6)$ is isomorphic to two copies of the same absolutely irreducible representation [8], and hence is Γ-simple. The (real) bifurcation parameters μ_1 and μ_2 depend in some way on the physical parameters λ_1, λ_2 such that $(\mu_1, \mu_2) = (0, 0)$ at $(\lambda_1, \lambda_2) = (\lambda_1^c, \lambda_2^c)$ and there is a locally invertible co-ordinate transformation:

$$\det \left. \begin{pmatrix} \partial\mu_1/\partial\lambda_1 & \partial\mu_1/\partial\lambda_2 \\ \partial\mu_2/\partial\lambda_1 & \partial\mu_2/\partial\lambda_2 \end{pmatrix} \right|_{\lambda_1^c, \lambda_2^c} \neq 0 \tag{15}$$

4 $T^2 \times S^1$ invariants and the normal form

We now compute all invariants up to degree 4 since generically the behaviour near the mode interaction is determined by the third-order truncation of the normal form. Clearly there are nine invariants of degree 2; $|z_1|^2, \ldots, |z_6|^2, |w_1|^2, |w_2|^2, |w_3|^2$. After removing degree 2 invariants, a general invariant has the form $I = z_1^m z_2^n z_3^p z_4^q z_5^r z_6^s w_1^u w_2^v w_3^w$ where we adopt the usual convention that $z_j^l \equiv \bar{z}_j^{|l|}$ if $l < 0$. Requiring invariance under the $T^2 \times S^1$ action (10) - (11) leads to the following conditions

$$
\begin{aligned}
m - n - \frac{p}{2} + \frac{q}{2} + \frac{r}{2} - \frac{s}{2} + \frac{3u}{2} - \frac{3w}{2} &= 0, && (16)\\
\frac{p}{2} - \frac{q}{2} + \frac{r}{2} - \frac{s}{2} + \frac{u}{2} + v + \frac{w}{2} &= 0, && (17)\\
m + n + p + q + r + s &= 0, && (18)\\
|m| + |n| + |p| + |q| + |r| + |s| + |u| + |v| + |w| &\leq 4. && (19)
\end{aligned}
$$

This last condition restricts the search to those invariants of degree 4 or less. There are three distinct types of invariant: ones which contain only the steady modes w_j, ones which contain only oscillatory modes z_j and those which couple the two. All invariants in the first case occur in the steady bifurcation problem on a hexagonal lattice [1, 5]. Similarly, the oscillatory bifurcation problem was examined by Roberts, Swift & Wagner [8]. In these problems the following invariants were found, and are also invariants for the mode interaction considered here:

$$w_1 w_3 \bar{w}_2, \qquad (20)$$

$$z_1 z_2 \bar{z}_3 \bar{z}_4, \qquad z_1 z_2 \bar{z}_5 \bar{z}_6, \qquad \bar{z}_3 \bar{z}_4 z_5 z_6. \qquad (21)$$

Now we turn to considering invariants involving both w_j and z_k modes. We first restrict attention to computing all those involving z_1, and then apply the interchange symmetries in the group D_6 to find the whole 'group orbit' of invariants of that type.

From (18) we see that any invariant must involve an even number of the oscillatory modes; if this number is zero, the invariant must be (20) and if this number is four the resulting invariant must be one of (21). We need only now consider invariants containing exactly two oscillatory modes, one of which is (without loss of generality) z_1. All other invariants are then obtained by applying the D_6 interchange symmetries to the invariants we find. There are (up to conjugacy) three possibilities; (i) $m = -n = 1$, $p = q = r = s = 0$; (ii) $m = -p = 1$, $n = q = r = s = 0$; (iii) $m = -q = 1$, $n = p = r = s = 0$.

Case (i) implies

$$4 + 3(u - w) = 0, \qquad u + 2v + w = 0 \tag{22}$$

which has no integer solutions satisfying $|u| + |v| + |w| \le 2$.

Case (ii) implies

$$w - u = 1, \qquad u + 2v + w = 1 \tag{23}$$

which has two independent solutions; $w = 1$, $u = v = 0$ and $u = -1$, $v = 1$, $w = 0$. These give rise to invariants $z_1\bar{z}_3 w_3$ and $z_1\bar{z}_3\bar{w}_1 w_2$.

Case (iii) implies

$$1 + 3(u - w) = 0, \qquad u + 2v + w = -1 \tag{24}$$

which again has no integer solutions.

The invariants found in case (ii) yield new $T^2 \times S^1$-invariants after the elements of the holohedry D_6 are applied to them. Using this fact, and applying m_x (the only element of D_6 which fixes z_1) to the invariants we deduce the other invariants containing z_1: $z_1\bar{z}_6\bar{w}_1$ and $z_1\bar{z}_6\bar{w}_2 w_3$. A complete list of invariants can now be compiled by applying the D_6 action to the invariants found so far. The equivariants E_j for the equation $\dot{z}_1 = f_1(\mathbf{w}, \mu_1, \mu_2)$ are given by $E_j = I_j/\bar{z}_1$ for all invariants I_j which contain $\bar{z}_1$.

From these computations, the amplitude equation $\dot{z}_1 = f_1(\mathbf{w}, \mu_1, \mu_2)$ is found to take the following form when truncated at cubic order

$$\begin{aligned}
\dot{z}_1 &= z_1 \left[\mu_1 + \mathrm{i}\omega_0 + a_1|z_1|^2 + a_2|z_2|^2 + a_3(|z_3|^2 + |z_6|^2) + a_4(|z_4|^2 + |z_5|^2)\right. \\
&\quad \left.+ a_5(|w_1|^2 + |w_3|^2) + a_6|w_2|^2\right] + b_1(z_3\bar{w}_3 + z_6 w_1) + b_2\bar{z}_2(z_3z_4 + z_5z_6) \\
&\quad + b_3(z_3w_1\bar{w}_2 + z_6w_2\bar{w}_3) \qquad (25)
\end{aligned}$$

where the complex coefficients $a_1, \ldots, a_6, b_1, b_2, b_3$ are formally functions of λ_1 and λ_2 but for the purpose of stability calculations we use the values evaluated at the bifurcation point $(\lambda_1^c, \lambda_2^c)$. The equations for $z_2, \ldots, z_6$ are obtained by applying the interchange (D_6) symmetries to (25). The corresponding equation for w_1 is

$$\begin{aligned}
\dot{w}_1 &= w_1 \left[\mu_2 + c_1|w_1|^2 + c_2(|w_2|^2 + |w_3|^2) + c_3(|z_1|^2 + |z_5|^2)\right. \\
&\quad \left.+ \bar{c}_3(|z_2|^2 + |z_6|^2)\right] + d_1 w_2\bar{w}_3 + d_2(z_1\bar{z}_6 + \bar{z}_2z_5) \\
&\quad + d_3(\bar{z}_2z_4w_2 + z_3\bar{z}_6\bar{w}_3) + \bar{d}_3(z_1\bar{z}_3w_2 + \bar{z}_4z_5\bar{w}_3) \qquad (26)
\end{aligned}$$

where the coefficients c_1, c_2, d_1 and d_2 are constrained to be real, but c_3, c_4 and d_3 are in general complex. These constraints on the coefficients can be easily checked by requiring equivariance under the transformations $\rho \circ m_x$ which fixes w_1 and $\rho^4 \circ m_x$ which maps $w_1 \to \bar{w}_1$. The amplitude equations for w_2 and w_3 are obtained by applying the rotations ρ and ρ^2 to (26).

This completes the computation of the normal form for the simplest case of the $1 : \sqrt{3}$ Hopf/steady-state mode interaction. More complicated amplitude equations can be derived when the lattice is chosen so that one or both critical circles intersect it in twelve points rather than six. These higher-dimensional problems enable the stability of solutions to a wider class of perturbations to be considered, and would also enable the analysis of interactions between steady and oscillatory superlattice patterns [4, 3].

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