Stable quasiperiodic solutions in the Hopf bifurcation with $D_4 \ltimes T^2$ symmetry.

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Abstract

The Hopf bifurcation with $D_4 \ltimes T^2$ symmetry generically has open regions of the normal form coefficient space where all branches of periodic solutions bifurcate supercritically but none is stable [1]. In such regions we prove the existence of an attracting set near the origin. A new possibility for the attractor is a quasiperiodic solution branch related to Standing Cross Rolls (SCR). The new solution physically represents a planform we call *Drifting Standing Cross Rolls*. Unlike Standing Cross Rolls, this solution can be stable, as can a further triply-periodic solution. This explains the behaviour in the regions of coefficient space omitted by Silber and Knobloch [1] and completes their analysis.

Key words: quasiperiodic solutions, symmetric bifurcation theory
PACS Codes: 47.20 (bifurcation theory) ; 47.54 (pattern formation).

Patterns which are periodic on a square lattice are often seen in numerical and laboratory experiments of a fluid dynamical, chemical or biological nature. This has motivated theoretical analyses of solutions and their stability in the relevant equivariant (symmetric) bifurcation problems. In this letter we consider oscillatory patterns on a square lattice and extend the theoretical analysis of Silber and Knobloch [1] of the Hopf bifurcation with $D_4 \ltimes T^2$ symmetry. Five distinct periodic solutions are guaranteed to appear (by the Equivariant Hopf Theorem [2]) in such a Hopf bifurcation, but there are open regions of the coefficient space where none is stable. We prove a new result giving necessary and sufficient conditions for the existence of a stable solution branch. We highlight coefficient values at which this stable solution is a doubly or triply-periodic oscillation.

Section 1 introduces the amplitude equations for the Hopf bifurcation with $D_4 \ltimes T^2$ symmetry (the notation is the same as in [1]), and contains the statement and proof of the existence of an attractor. The Drifting Standing Cross Roll solution and its relevance to this result is discussed in section 2. Section 3 contains an example path through the normal form coefficient space which captures the associated bifurcation structure. Conclusions are drawn in section 4.

1 Existence of an attractor

Following [1] neutrally stable modes at the bifurcation point have the form

$$u = \mathrm{Re}(v_1 e^{ikx} + v_2 e^{iky} + w_1 e^{-ikx} + w_2 e^{-iky})$$

where x and y are the horizontal directions in the planform of the pattern. Requiring equivariance with respect to the group $D_4 \ltimes T^2$ leads us to amplitude equations for $(v_1, v_2, w_1, w_2) \in \mathbb{C}^4$:

$$\dot{v}_1 = (\nu + a|w_1|^2 + bN_1 + cN_2)v_1 + d\bar{w}_1 v_2 w_2 \tag{1}$$
$$\dot{v}_2 = (\nu + a|w_2|^2 + bN_2 + cN_1)v_2 + d\bar{w}_2 v_1 w_1 \tag{2}$$
$$\dot{w}_1 = (\nu + a|v_1|^2 + bN_1 + cN_2)w_1 + d\bar{v}_1 v_2 w_2 \tag{3}$$
$$\dot{w}_2 = (\nu + a|v_2|^2 + bN_2 + cN_1)w_2 + d\bar{v}_2 v_1 w_1 \tag{4}$$

where $N_1 = |v_1|^2 + |w_1|^2$, $N_2 = |v_2|^2 + |w_2|^2$ and $\nu = \lambda + i\sigma(\lambda)$ with λ and σ both real and $\sigma(0) = \omega_c$ the frequency of periodic solutions at the bifurcation point. The coefficients a, b, c and d will in general be complex; their real and imaginary parts are denoted by subscripts r and i. Truncating the amplitude equations at third order as we have done here is believed to be sufficient (generically) to determine the behaviour in a neighbourhood of the origin.

The amplitude equations (1) - (4) are also equivariant with respect to time translations which generate a group S^1. This is a symmetry which arises naturally in normal forms for Hopf bifurcation problems.

The analysis of Silber and Knobloch [1] shows the existence for all coefficient values of five periodic solution branches to (1) - (4). These have two-dimensional fixed point subspaces and hence maximal isotropy subgroups [2]. They are summarised in table 1, and will be referred to as *axial* branches. Other solution branches introduced later are still *primary* branches (meaning that they are created at the origin, when $\lambda = 0$ and exist for all values of λ on one side of the bifurcation point) even though they may be quasiperiodic. Due to the lack of quadratic terms in the normal form, there are no secondary bifurcations in (1) - (4) as we vary λ away from $\lambda = 0$.

Table 1
Axial branches in the Hopf bifurcation with $D_4 \ltimes T^2$ symmetry.

Fixed point subspace	Solution branch	Representative solution form	Amplitude
II	Travelling Rolls (TR)	$(\mathbf{z},0,0,0)$	$\lvert\mathbf{z}\rvert^2 = -\frac{\lambda}{b_r}$
III	Travelling Squares (TS)	$(\mathbf{z},\mathbf{z},0,0)$	$\lvert\mathbf{z}\rvert^2 = -\frac{\lambda}{a_r+2b_r}$
IV	Standing Rolls (SR)	$(\mathbf{z},0,\mathbf{z},0)$	$\lvert\mathbf{z}\rvert^2 = -\frac{\lambda}{b_r+c_r}$
V	Standing Squares (SS)	$(\mathbf{z},\mathbf{z},\mathbf{z},\mathbf{z})$	$\lvert\mathbf{z}\rvert^2 = -\frac{\lambda}{a_r+2b_r+2c_r+d_r}$
VI	Alternating Rolls (AR)	$(\mathbf{z},i\mathbf{z},\mathbf{z},i\mathbf{z})$	$\lvert\mathbf{z}\rvert^2 = -\frac{\lambda}{a_r+2b_r+2c_r-d_r}$

Writing $r_1{=}|v_1|$, $r_2{=}|v_2|$, $r_3{=}|w_1|$, $r_4{=}|w_2|$, and defining $\psi = \arg(v_1 w_1 \bar{v}_2 \bar{w}_2)$ we can derive equations for the evolution of the amplitude moduli and the phase variable ψ:

$$\dot{r_1} = r_1\Big[\lambda + a_r r_3^2 + b_r N_1 + c_r N_2\Big] + r_3 r_2 r_4 \mathrm{Re}(de^{i\psi}) \tag{5}$$
$$\dot{r_2} = r_2\Big[\lambda + a_r r_4^2 + b_r N_2 + c_r N_1\Big] + r_4 r_1 r_3 \mathrm{Re}(de^{-i\psi}) \tag{6}$$
$$\dot{r_3} = r_3\Big[\lambda + a_r r_1^2 + b_r N_1 + c_r N_2\Big] + r_1 r_2 r_4 \mathrm{Re}(de^{i\psi}) \tag{7}$$
$$\dot{r_4} = r_4\Big[\lambda + a_r r_2^2 + b_r N_2 + c_r N_1\Big] + r_2 r_1 r_3 \mathrm{Re}(de^{-i\psi}) \tag{8}$$
$$\dot{\psi} = r_1 r_3\Big(\frac{r_2}{r_4} + \frac{r_4}{r_2}\Big)\mathrm{Im}(de^{-i\psi}) - r_2 r_4\Big(\frac{r_1}{r_3} + \frac{r_3}{r_1}\Big)\mathrm{Im}(de^{i\psi}) + f_i(r_2^2 + r_4^2 - r_1^2 - r_3^2) \tag{9}$$

where $f = a + 2b - 2c$.

Theorem 1 *The amplitude equations for a Hopf bifurcation with $D_4 \ltimes T^2$ symmetry are given in equations (1)-(4). Assume that the following combinations of coefficients do not vanish: b_r, $a_r + 2b_r$, $b_r + c_r$, $a_r + b_r + 2c_r \pm d_r$, so that all axial branches bifurcate either subcritically or supercritically. Then for all $|\lambda| < 1$ there exists a compact set $\mathcal{U}$ – fixed independent of λ – with the following properties:*

P1: *The interior of $\mathcal{U}$ contains the origin.*

P2: *The interior of $\mathcal{U}$ contains all primary branches.*

P3: *$\mathcal{U}$ is forward invariant under the flow derived from the vector field (1) - (4):* $\phi_t(\mathcal{U}) \subset \mathcal{U} \quad \forall \quad t > 0.$

if and only if all axial branches bifurcate supercritically.

Remark 1: There is a topological conjugacy between the flow for $\lambda = \lambda_+ > 1$ and the flow for $\lambda = 1$ (and similarly for $\lambda = \lambda_- < -1$ and $\lambda = -1$). So if the conditions on $\mathcal{U}$ are satisfied for $|\lambda| = 1$, then using the conjugacy (equivalent to a rescaling of time and the amplitudes $v_1, \ldots, w_2$) we can find a (larger) compact set with the properties P1 - P3 for any value of λ.

Remark 2: Since the bifurcation at $\lambda = 0$ is a purely local one it cannot affect the dynamics on $\partial\mathcal{U}$ (the boundary of $\mathcal{U}$). Moreover, since $\mathcal{U}$ contains all objects created in the bifurcation at $\lambda = 0$ (by property P2), the dynamics on $\partial\mathcal{U}$ are unaffected by the local bifurcation. In particular, trajectories crossing $\partial\mathcal{U}$ transversely for one value of λ will continue to do so for all $\lambda \in [-1, 1]$.

Remark 3: Condition P3 guarantees the existence of a stable branch but it does not have to be unique.

Proof: ($\Rightarrow$) We prove the contrapositive. Say an axial branch with isotropy subgroup Σ bifurcates subcritically. For $\lambda = -1$, restricting the flow to Fix(Σ)$\cong \mathbb{C}$ (which is a flow-invariant subspace) we have only 2 invariant sets: the origin and the periodic orbit P which is unstable within Fix(Σ). $\mathcal{U}$ must contain P to satisfy P2. As the periodic orbit is unstable, trajectories starting in the connected component of Fix(Σ)$\backslash P$ not containing the origin will grow unboundedly.

Since Fix(Σ)$\backslash P$ intersects $\mathcal{U}$ there are points which leave $\mathcal{U}$ under the flow, hence property P3 cannot be satisfied. So the existence of a subcritical axial branch means we cannot find a $\mathcal{U}$ satisfying P1 - P3. □

Proof: ($\Leftarrow$) To establish the existence of $\mathcal{U}$ in $\lambda > 0$ it is enough to show that $\mathcal{R}^2 = \Sigma r_i^2$ satisfies an equation of the form

$$\frac{1}{2}\frac{d(\mathcal{R}^2)}{dt} \leq \lambda \mathcal{R}^2 + T(r_1, \ldots, r_4) \tag{10}$$

where T contains only terms of order 4 in the r_i and T is strictly negative everywhere on the surface of a sphere $\mathcal{R}^2 = C$. Then, at large values of $\mathcal{R}^2$ the quartic terms will dominate, and trajectories starting on the surface of the sphere must flow inside it (and hence the sphere is flow-invariant). This has already been done for the case $a_r < 0$ by Silber and Knobloch [1, section 8]. Lemma 2 below proves the result for $a_r \geq 0$. □

Extremal points exist for each case that corresponds to an axial branch. The condition for the value of T at the extreme point to be negative (a linear constraint on the coefficients) is the same as the condition that the branch bifurcate supercritically. So when all axial branches bifurcate supercritically, a set $\mathcal{U}$ with properties P1 - P3 must exist, and hence there must be at least one stable branch of solutions. By remark 2 this attracting neighbourhood exists for all λ.

Lemma 2 *Let $\mathcal{R}^2 = \Sigma r_i^2$, then the evolution equation for $\mathcal{R}^2$ is*

$$\frac{1}{2}\frac{d(\mathcal{R}^2)}{dt} = \lambda \mathcal{R}^2 + Q \leq \lambda \mathcal{R}^2 + T \tag{11}$$

where

$$Q = 2a_r(r_1^2 r_3^2 + r_2^2 r_4^2) + b_r \mathcal{R}^4 + 2(c_r - b_r)N_1 N_2 + 4d_r \cos\psi r_1 r_2 r_3 r_4 \tag{12}$$
$$T = 2a_r(r_1^2 r_3^2 + r_2^2 r_4^2) + b_r \mathcal{R}^4 + 2(c_r - b_r)N_1 N_2 + 4|d_r| r_1 r_3 r_2 r_4 \tag{13}$$

and the subscript r means 'the real part of'. $\mathcal{L} = T + 2\mu(\Sigma r_i^2 - C)$ is the Lagrangian for the problem 'minimise T subject to $\Sigma r_i^2 = C > 0$'. μ is the Lagrange multiplier (the factor 2 is purely for convenience). Assume $a_r \geq 0$ in (1)-(4). If the values of T at all its extremal points given by

$$\frac{\partial \mathcal{L}}{\partial r_i} = 0 \qquad \text{for} \quad \text{all} \quad i = 1 \ldots 4 \tag{14}$$

are strictly negative then there exists a compact set $\mathcal{U}$ satisfying P1 - P3.

Proof: By an exhaustive analysis of the possible extremal points of (14) in the region

$\mathcal{S} = \{(r_1, r_2, r_3, r_4) \in \mathbb{R}_+^4 : \Sigma r_i^2 = C, \quad r_i > 0 \ \ \forall\ i\}.$

This will be split up into different cases, numbered II to XIII to correspond to the different isotropy subgroups of the original Hopf bifurcation with $D_4 \ltimes T^2$ symmetry, see Silber and Knobloch [1, table 1] and tables 1 and 2. From (14) we require

$$b_r r_1^3 + (a_r + b_r) r_1 r_3^2 + c_r r_1 (r_2^2 + r_4^2) + |d_r| r_2 r_3 r_4 + \mu r_1 = 0 \tag{15}$$

Table 2

All cases of possible extreme points for $\mathcal{L}$. Each case is numbered according to the corresponding fixed point subspace for the original Hopf bifurcation problem as given in [1].

Fixed point subspace	Conditions on variables	Number of zero variables
II	$r_1 \neq 0,\ r_2 = r_3 = r_4 = 0$	3
III	$r_1 = r_2 \neq 0,\ r_3 = r_4 = 0$	2
IV	$r_1 = r_3 \neq 0,\ r_2 = r_4 = 0$	2
V/VI	$r_1 = r_2 = r_3 = r_4 \neq 0$	0
VII	$r_1 = r_3 \neq r_2 = r_4 \neq 0$	0
VIII	$r_1 \neq r_2 \neq 0,\ r_3 = r_4 = 0$	2
IX	$r_1 \neq r_3 \neq 0,\ r_2 = r_4 = 0$	2
X/XI	$r_1 = r_2 \neq r_3 = r_4 \neq 0$	0
XII	$r_1 = r_3 \neq r_2 \neq r_4 \neq 0$	0
XIII	$r_1 \neq r_2 \neq r_3 \neq r_4 \neq 0$	0

$$b_r r_2^3 + (a_r + b_r) r_2 r_4^2 + c_r r_2 (r_1^2 + r_3^2) + |d_r| r_1 r_3 r_4 + \mu r_2 = 0 \quad (16)$$
$$b_r r_3^3 + (a_r + b_r) r_3 r_1^2 + c_r r_3 (r_2^2 + r_4^2) + |d_r| r_1 r_2 r_4 + \mu r_3 = 0 \quad (17)$$
$$b_r r_4^3 + (a_r + b_r) r_4 r_2^2 + c_r r_4 (r_1^2 + r_3^2) + |d_r| r_1 r_2 r_3 + \mu r_4 = 0 \quad (18)$$

Finding these extremal points is equivalent to finding the fixed points for a clearly related 'gradient' steady bifurcation problem with the Lagrange multiplier playing the role of the bifurcation parameter.

From the form of the last cubic term on the left hand sides in (15)-(18) there can be no extrema with exactly one variable zero. For each case which corresponds to an isotropy subgroup with a two-dimensional fixed point subspace of the original $D_4 \ltimes T^2$-symmetric problem we give the name of the axial branch in brackets. Having eliminated the phase variable ψ there are two cases where different isotropy subgroups for the original Hopf bifurcation problem give the same solution form here. Our definition of ψ differs by a factor of 2 from that in Silber and Knobloch [1].

Case II (Travelling Rolls - TR) $r_1 \neq 0$, $r_2 = r_3 = r_4 = 0$. There is a solution of (15)-(18) given by $\hat{r}_1^2 = -\mu/b_r$. This is an extremal point. From (13) we see that $b_r < 0$ is a sufficient condition to make $T(\hat{r}_1, 0, 0, 0) < 0$. This condition is exactly the one that ensures the TR branch for the original Hopf bifurcation problem bifurcates supercritically.

Case III (Travelling Squares - TS) $r_1 = r_2 \neq 0$, $r_3 = r_4 = 0$. There is a solution of (15)-(18) given by $\hat{r}_1^2 = \hat{r}_2^2 = -\mu/(b_r + c_r)$. This is an extremal point. From (13) we see that $b_r + c_r < 0$ is a sufficient condition to make $T(\hat{r}_1, \hat{r}_2, 0, 0) < 0$. Similarly, this condition ensures that the TS branch for (1)-(4) bifurcates supercritically.

Case IV (Standing Rolls - SR) $r_1 = r_3 \neq 0$, $r_2 = r_4 = 0$. There is a solution of (15)-(18) given by $\hat{r}_1^2 = \hat{r}_3^2 = -\mu/(a_r + 2b_r)$. This is an extremal point. From (13) we see that $a_r + 2b_r < 0$ is a sufficient condition to make $T(\hat{r}_1, 0, \hat{r}_3, 0) < 0$. This condition ensures that the SR branch for (1)-(4) bifurcates supercritically.

Cases V/VI (Standing Squares / Alternating Rolls - SS / AR) $r_1 = r_2 = r_3 = r_4 \neq 0$. There is a solution of (15)-(18) given by $\hat{r}_1^2 = \hat{r}_2^2 = \hat{r}_3^2 = \hat{r}_4^2 = -\mu/(a_r + 2b_r + 2c_r + |d_r|)$. This is an extremal point. From (13) we see that $a_r + 2b_r + 2c_r + |d_r| < 0$ is a sufficient condition to make $T(\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4) < 0$. This condition ensures that the SS and AR branches for (1)-(4) bifurcate supercritically.

Cases VII – XI From the first two equations (15) and (16) we can deduce a contradiction: variables which were assumed to be different must be equal. We show case VII as an example: $r_1 = r_3 \neq 0$, $r_2 = r_4 \neq 0$. So

$$(15) \Rightarrow a_r r_1^2 + 2b_r r_1^2 + 2c_r r_2^2 + |d_r| r_2^2 + \mu = 0$$
$$(16) \Rightarrow a_r r_2^2 + 2b_r r_2^2 + 2c_r r_1^2 + |d_r| r_1^2 + \mu = 0$$

which implies $r_1^2 = r_2^2$ or $a_r + 2b_r - 2c_r - |d_r| = 0$. If the first condition holds, we have a contradiction since r_1 and r_2 were assumed to be different (we are in Case III otherwise). If the second condition holds (a degenerate situation) then from manipulation of (13) we can deduce that

$$T(r_1, r_2, 0, 0) = (a_r + 2b_r)(r_1^2 + r_2^2)^2 \tag{19}$$

which implies that the condition derived in Case IV is sufficient to keep $T < 0$. Cases VIII – XI are exactly similar, and in each case the degenerate situation is taken care of by one of the inequalities from Cases II – V/VI.

Case XII $r_1 \neq r_3 \neq 0$, $r_2 = r_4 \neq 0$. So

$$(15) r_3 - (17) r_1 \Rightarrow a_r r_1 r_3 + |d_r| r_2^2 = 0 \tag{20}$$

which has no solutions unless $a_r = d_r = 0$. If this is the case, then from (13) we find that sufficient conditions for T to be negative at any possible extremal point are $b_r < 0$ and $b_r + c_r < 0$. These conditions are already required by the analysis of Cases II and III.

Case XIII $r_1 \neq 0$, $r_2 \neq 0$, $r_3 \neq 0$, $r_4 \neq 0$.

$$(15)r_3 - (17)r_1 \Rightarrow a_r r_1 r_3 + |d_r| r_2 r_4 = 0 \tag{21}$$

which, as in case XII, has no solutions unless $a_r = d_r = 0$. If this is the case, the same argument as in Case XII shows that the necessary conditions for T to be negative on $\mathcal{S}$ are those given by Cases II and III again.

In general there might be points on the boundary of the region $\mathcal{S}$ where T is defined which are not extrema but which have larger values of T than all extreme points. We can exclude this possibility here as we have included all parts of the domain boundary in cases II-IV, VIII and IX. □

We conclude that an attractor is guaranteed when all axial branches bifurcate supercritically. In particular there must be an attractor in the 'blackened regions' omitted by Silber and Knobloch [1]. We now discuss previously unnoticed possibilities for stable branches of solutions in this region. We cannot rule out the coexistence of other stable (possibly complicated) types of behaviour in these regions.

2 Drifting Standing Cross Rolls

The normal form for a $D_4 \ltimes T^2$ Hopf bifurcation has one periodic branch of solutions with submaximal isotropy - *Standing Cross Rolls* (SCR). The SCR solution is of the form $v_1 = w_1$, $v_2 = w_2$. It does not exist for all coefficient values, and when it does exist, it is always unstable [1, section 6] and [3] in the cubic truncation considered here. The solution amplitudes and the value of the phase ψ_{scr} are given by

$$\cos \psi_{scr} = \frac{\mathrm{Re}(\bar{d} f)}{|d|^2} \tag{22}$$

$$\frac{r_1^2}{r_2^2} = \frac{\mathrm{Im}(\bar{d} f) - |d|^2 \sin \psi_{scr}}{\mathrm{Im}(\bar{d} f) + |d|^2 \sin \psi_{scr}} \tag{23}$$

The DSCR solution is of the form $\psi = const$, $r_1 \neq r_3 \neq r_2 = r_4$. As the moduli of three of the amplitudes are unequal, they cannot all oscillate at the same frequency: the solution must be quasiperiodic. The frequencies are constrained by the fact that $\dot{\psi} = 0$, so there are only two independent frequencies in the solution, not three. By linearising around the SCR solution we can locate the pitchfork bifurcation creating DSCR solutions from SCR solutions as the normal form coefficients are varied. We term this kind of bifurcation a *coefficient bifurcation*: let

$$|v_1| = r_1 + u \qquad |v_2| = |w_2| = r_2$$
$$|w_1| = r_1 - u \qquad \psi = \psi_{scr} + \phi$$

so that when $u = 0$ this is an SCR solution. After substitution into (5)-(9) and linearising, the equation for $\dot{\phi}$ decouples. From the $\dot{u}$ equation there is a bifurcation when

$$\lambda + (2b_r - a_r)r_1^2 + (2c_r + d_r \cos\psi_{scr} - d_i \sin\psi_{scr})r_2^2 = 0 \tag{24}$$

The new DSCR solution is not contained in the *SCR subspace* (defined by $v_1 = w_1$ and $v_2 = w_2$) which may be partly why it has been previously overlooked. The mode amplitudes in a DSCR solution can be determined from the normal form (1)-(4). Define

$$s_1 = \frac{r_1}{r_2} \qquad s_2 = \frac{r_3}{r_2} \tag{25}$$

then, taking the real parts of the amplitude equations (5)-(7) we get:

$$\frac{\lambda}{r_2^2} + b_r s_1^2 + (a_r + b_r)s_2^2 + 2c_r + \frac{s_2}{s_1}\mathrm{Re}(de^{i\psi}) = 0 \tag{26}$$
$$\frac{\lambda}{r_2^2} + a_r + 2b_r + c_r(s_1^2 + s_2^2) + s_1 s_2 \mathrm{Re}(de^{-i\psi}) = 0 \tag{27}$$
$$\frac{\lambda}{r_2^2} + b_r s_2^2 + (a_r + b_r)s_1^2 + 2c_r + \frac{s_1}{s_2}\mathrm{Re}(de^{i\psi}) = 0 \tag{28}$$

From the equation for $\dot{\psi}$ (9):

$$f_i(2 - s_1^2 - s_2^2) + 2s_1 s_2 \mathrm{Im}(de^{-i\psi}) - \mathrm{Im}(de^{i\psi})\Big[\frac{s_1}{s_2} + \frac{s_2}{s_1}\Big] = 0 \tag{29}$$

which yields an equation for the product $s_1 s_2$:

$$s_1 s_2 = -\frac{\mathrm{Re}(de^{i\psi})}{a_r} \tag{30}$$

From this and the other equations we can find two independent relations for $s_1^2 + s_2^2$:

$$s_1^2 + s_2^2 = \frac{\mathrm{Re}(de^{i\psi})\mathrm{Re}(de^{-i\psi}) - f_r a_r}{a_r b_r c_r} = \frac{2\mathrm{Re}(de^{i\psi})\mathrm{Im}(de^{-i\psi}) - 2 f_i a_r}{a_r[\frac{\mathrm{Im}(de^{i\psi})}{\mathrm{Re}(de^{i\psi})} a_r - f_i]} \quad (31)$$

which leads to a cubic equation for $\tan\psi$. Having determined ψ we can now use equations (30) and (31) to determine s_1 and s_2.

3 Stability of DSCR

Near the coefficient bifurcation the DSCR solution must be unstable, as SCR solutions are always unstable in this problem. However, as the normal form coefficients are varied away from the bifurcation point (keeping the usual bifurcation parameter λ fixed and positive) it can be stabilised by a triply-periodic orbit in a 'Hopf bifurcation' as the coefficients are varied. The location of this bifurcation is algebraically too complicated to determine. This sequence of events has been observed along a path in coefficient space near a Hopf/steady-state mode interaction in three-dimensional magnetoconvection [4], using the continuation package AUTO [5]. In this example all coefficients in the normal form are functions of one free parameter ζ which lies between 0 and 1. Figure 1

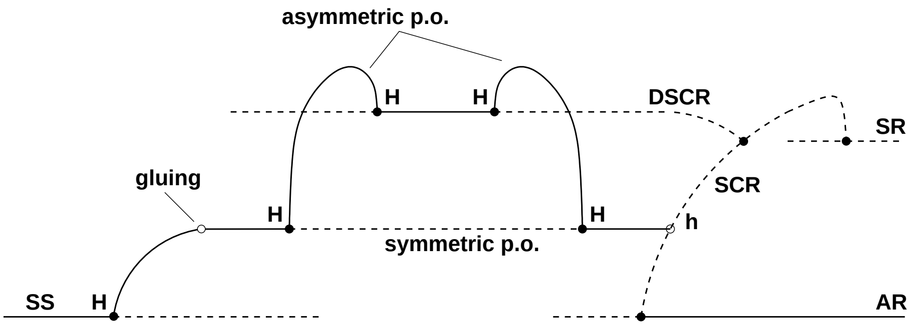

Fig. 1. *Sketch bifurcation diagram as ζ increases. H - Hopf bifurcation in the reduced system (5)-(9), h - homoclinic bifurcation. Solid lines - stable solutions, dashed lines - unstable solutions. The quasiperiodic solution branches only exist stably in certain open regions of the coefficient space. The bifurcations occur on the boundaries of these regions, as the particular path chosen here crosses from one region to another.*

shows how the stable primary branch of solutions created in the Hopf bifurcation changes as the normal form coefficients are varied along this path. The vertical scaling of the bifurcation diagram is purely schematic. The symmetric periodic orbit in figure 1 lies in the SCR subspace, as do the Standing Squares (SS), Alternating Rolls (AR) and Standing Rolls (SR) solutions. The symmetric periodic orbit undergoes a Hopf bifurcation to a triply-periodic orbit which lies outside the SCR subspace (labelled asymmetric p.o. on figure 1, and shown in figure 2). This triply-periodic orbit is labelled *asymmetric* because points on the orbit no longer satisfy both $|v_1| = |w_1|$ and $|v_2| = |w_2|$. The asymmetric orbit undergoes a reverse Hopf bifurcation as the normal form coefficients are varied to yield a DSCR solution, shown in figure 3.

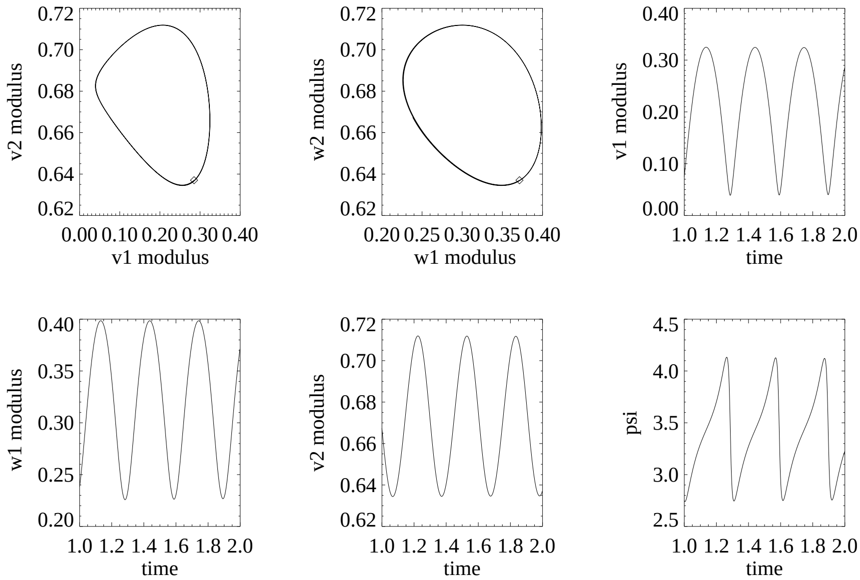

Fig. 2. *A solution trajectory of (1)-(4) showing the stable asymmetric triply-periodic orbit. Coefficient values are:* $a = 4.671 - 4.131i$, $b = -8.983 - 4.812i$, $c = -3.898 - 33.451i$, $d = -12.361 - 40.979i$.

4 Conclusion

Topologically, there must be some attractor for the amplitude equations (1)-(4) in $\lambda > 0$ when there exists a sphere containing the origin which is flow-invariant in forward time. In this letter we have shown that this condition is equivalent to ensuring that the five axial branches bifurcate supercritically. The criteria for existence of this sphere reduce to a small number of linear constraints on the normal form coefficients. The lack of quadratic terms in the amplitude equations, and the consequent structure of $\mathcal{L}$ allows us to show that the constraints for the existence of the flow-invariant sphere must be linear constraints on the normal form coefficients. Each of the axial branches gives rise to such a constraint.

We also demonstrate the existence of stable doubly and triply-periodic solution branches for the $D_4 \ltimes T^2$ Hopf bifurcation problem, which have not been noted before. In their analysis, Silber and Knobloch [1] investigated possible heteroclinic cycles between various periodic solutions. As they remark, their

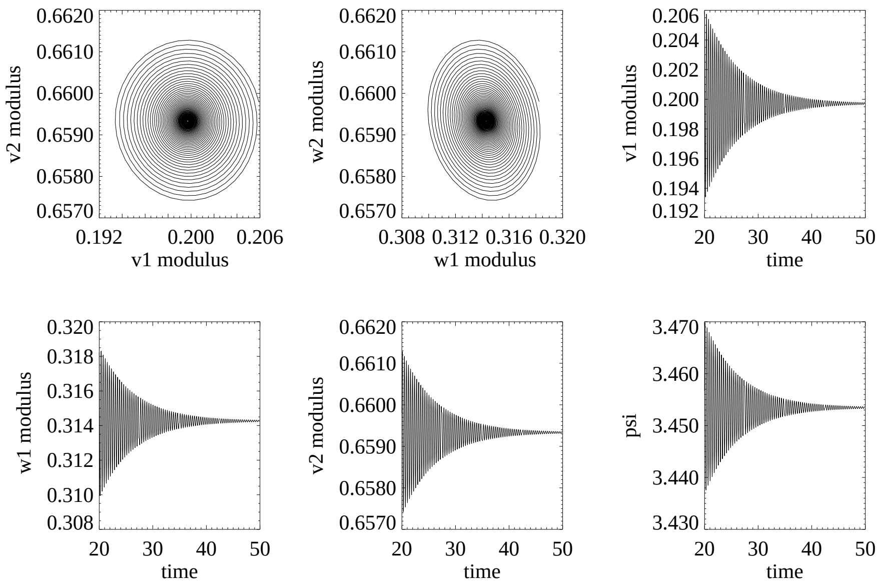

Fig. 3. *A solution trajectory of (1)-(4) converging to the stable DSCR solution (which appears as a fixed point in these plots). Coefficient values are:* $a = 3.497 - 5.548i$, $b = -8.630 - 5.449i$, $c = -4.375 - 34.069i$, $d = -13.036 - 42.079i$.

proposed heteroclinic cycle is not stable, so there must necessarily be more solution branches in this region of coefficient space.

It seems likely that similar straightforward, but careful, arguments can be applied to many other bifurcation problems, both steady and oscillatory. Further work is in progress.

Acknowledgements

I have benefited from useful discussions with Michael Proctor and Alastair Rucklidge. I also wish to thank two anonymous referees for a large number of helpful comments which have greatly improved the format and clarity of this paper. This work was funded by the EPSRC.

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