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Hopf bifurcation on a square superlattice

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Abstract

Techniques of equivariant bifurcation theory are used to study the Hopf bifurcation problem on a square lattice where the group $\Gamma = D_4 \ltimes T^2$ acts Γ-simply on $\mathbb{C}^8$. This enables the analysis of the stability of solutions found in a previous analysis (Silber and Knobloch 1991 *Nonlinearity* **4** 1063–106) of the Γ-simple representation on $\mathbb{C}^4$ to both solutions which are spatially periodic on a rhombic lattice, and to a countably infinite number of oscillatory 'superlattice' solutions. The normal form for the bifurcation is computed, and conditions for the stability of all 17 $\mathbb{C}$-axial branches are given.

Numerical investigations indicate that there exist open regions of coefficient space where the dynamics of the cubic order truncation of the normal form are chaotic. Chaotic dynamics have not previously been found in simpler Hopf bifurcation problems in normal form.

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1. Introduction

A common situation arising in chemical reactions, biological systems and physical experiments is the existence of a pattern-forming instability on the plane $\mathbb{R}^2$; a uniform, time-independent initial state loses stability to perturbations with a non-zero wavenumber as a physical parameter is varied. The homogeneous state is invariant under the group $E(2)$ of planar translations, reflections and rotations and hence the centre manifold $\mathbb{E}^c(\mathbb{R}^2)$ at the bifurcation point must contain Fourier modes of the critical wavenumber pointing in every horizontal direction. Thus $\mathbb{E}^c(\mathbb{R}^2)$ is infinite dimensional. Since many of the patterns that result from such an instability are (at least approximately) spatially periodic, a common approach to studying the formation of these spatially periodic patterns on $\mathbb{R}^2$ from a homogeneous $E(2)$-invariant state is to restrict attention to classes of solutions which are periodic in the plane with respect to a lattice $\mathcal{L}$. In this paper we will consider only square lattices. Such a restriction results in a reduction in the symmetry of the problem from the Euclidean group $E(2)$ to the compact group $D_4 \ltimes T^2$ and enables equivariant bifurcation theory to be applied.

Many previous studies of Hopf bifurcations on square lattices in $E(2)$-equivariant physical systems have relied on the theoretical results of Silber and Knobloch [5] for the eight-dimensional representation of $D_4 \ltimes T^2$ on $\mathbb{C}^4$. In this analysis the spacing of the lattice $\mathcal{L}$ equalled the wavelength of the perturbations considered. This does not have to be the case, and consideration of other lattices leads naturally to the analysis of the countably infinite number of 16-dimensional representations of $D_4 \ltimes T^2$ on $\mathbb{C}^8$. Consideration of these higher-dimensional representations enables a more general study of the stability of previously identified patterns. Moreover, 'superlattice' patterns appear as $\mathbb{C}$-axial branches in the bifurcation problem. Since the irreducible representations of $D_4 \ltimes T^2$ are either four or eight dimensional, this paper completes the analysis of branches of solutions for Hopf bifurcations in the presence of a Γ-simple action of $D_4 \ltimes T^2$.

In what follows we make extensive use of the previous group-theoretic results of [3], and assume familiarity with the previous results of [2, 5] for related Hopf bifurcations with symmetry. The paper is organized as follows. In section 2 the details of the bifurcation problem are fleshed out. Section 3 tabulates and summarizes the $\mathbb{C}$-axial solution branches and their isotropy subgroups. The normal form for the bifurcation problem is derived in section 4, and complete stability results for the $\mathbb{C}$-axial branches are calculated in section 5. Section 6 briefly indicates that it is possible for branches of stable chaotic dynamics to be created in the bifurcation.

2. Statement of the problem

We consider a set of smooth PDEs written in evolutionary form

$$\frac{\partial \boldsymbol{u}}{\partial t} = \mathcal{F}(\boldsymbol{u}, \mu) \tag{1}$$

which has a solution $\boldsymbol{u}(x, y, z, t) = 0$ for all values of the parameter μ. We assume that this uniform, time-independent state loses stability in an oscillatory bifurcation to perturbations with a non-zero wavenumber k_c as the bifurcation parameter μ increases through zero. The uniform state is invariant under the non-compact Euclidean group $E(2)$ of planar translations, rotations and reflections. The Euclidean group is a semi-direct product $E(2) = O(2) \ltimes \mathbb{R}^2$ of the group $O(2)$ of orthogonal transformations of the (x, y)-plane (rotations about the origin and reflections in lines containing the origin) and the group of planar translations $\mathbb{R}^2$. The group $E(2)$ acts in a natural way on the variables (x, y) and this induces an action on $\boldsymbol{u}$. In this paper we consider only the scalar representation of $E(2)$ and do not discuss the alternative pseudoscalar representation [10, 12]. In all that follows we suppress the dependence of $\boldsymbol{u}$ on the z-coordinate. A direct consequence of the $E(2)$ symmetry of the solution $\boldsymbol{u} = 0$ is that the centre manifold at the bifurcation point is infinite dimensional; it describes the continuous circle of wavevectors of length k_c. To reduce the problem to one with a finite-dimensional centre manifold we restrict the analysis to solutions which lie on the square lattice $\mathcal{L} = \{n\ell_1 + m\ell_2 : (n, m) \in \mathbb{Z}^2,\ \ell_1 = (2\pi, 0),\ \ell_2 = (0, 2\pi)\}$ in the (x, y)-plane. Thus $\boldsymbol{u}(x + 2\pi n, y + 2\pi m, z, t) = \boldsymbol{u}(x, y, z, t)$ for any pair of integers (n, m).

We also define the dual lattice $\mathcal{L}^*$, spanned by integer combinations of the unit wavevectors $(1, 0)$ and $(0, 1)$. The dual lattice $\mathcal{L}^*$ (identical, apart from a scaling, to the real-space lattice $\mathcal{L}$ in this case) intersects the critical circle of radius k_c in only a finite number of points, generally only four or eight. Since these intersections correspond to modes which together span the centre manifold, the centre manifold is rendered finite dimensional and all other points in $\mathcal{L}^*$ are bounded away from the critical circle. The four-point intersection yields a Γ-simple representation of $D_4 \ltimes T^2 \times S^1$ on $\mathbb{C}^4$: this Hopf bifurcation was studied in detail

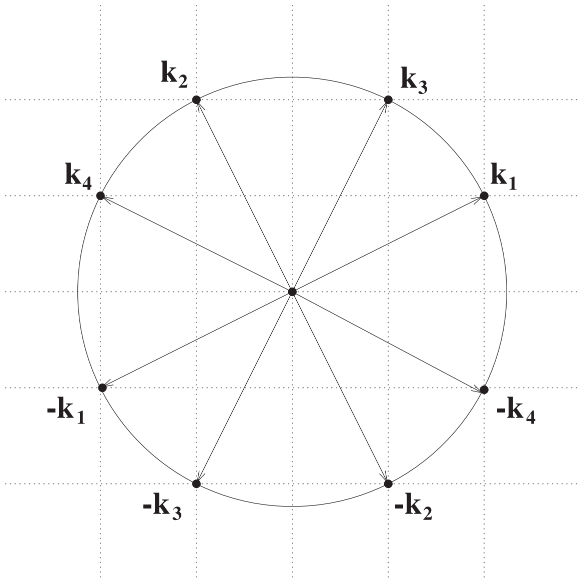

Figure 1. Superlattice planform for the case $\alpha = 2$, $\beta = 1$: the full circle indicates the circle of critical modes. This intersects the dual lattice $\mathcal{L}^*$ at eight points giving the wavevectors $\pm \boldsymbol{k}_1, \ldots, \pm \boldsymbol{k}_4$. Note that in this problem the lattice $\mathcal{L}$ and the dual lattice $\mathcal{L}^*$ are equal.

by Silber and Knobloch [5]. In this paper we study the eight-point intersection which yields an irreducible representation of $D_4 \ltimes T^2 \times S^1$ on $\mathbb{C}^8$. There are countably many of these larger irreducible representations which we distinguish by the integers α and β which define the periodic lattice. Figure 1 shows the circle of critical wavevectors and the intersections with the dual lattice $\mathcal{L}^*$ giving the eight critical wavevectors $\pm \boldsymbol{k}_1, \ldots, \pm \boldsymbol{k}_4$, defined to be $\boldsymbol{k}_1 = (\alpha, \beta)$, $\boldsymbol{k}_2 = (-\beta, \alpha)$, $\boldsymbol{k}_3 = (\beta, \alpha)$, $\boldsymbol{k}_4 = (-\alpha, \beta)$. Without loss of generality we have scaled lengths in the original partial differential equations (PDEs) (1) so that $k_c^2 = \alpha^2 + \beta^2$. Restricted to the lattice, we express the perturbations to the uniform solution $\boldsymbol{u} = 0$ in the form of a sum of these Fourier modes:

$$\hat{u} = \mathrm{Re}\left[z_1 \mathrm{e}^{\mathrm{i}(\alpha x + \beta y - \omega_0 t)} + z_2 \mathrm{e}^{\mathrm{i}(-\beta x + \alpha y - \omega_0 t)} + z_3 \mathrm{e}^{\mathrm{i}(-\alpha x - \beta y - \omega_0 t)} + z_4 \mathrm{e}^{\mathrm{i}(\beta x - \alpha y - \omega_0 t)} \right.$$
$$\left. + w_1 \mathrm{e}^{\mathrm{i}(\alpha x - \beta y - \omega_0 t)} + w_2 \mathrm{e}^{\mathrm{i}(\beta x + \alpha y - \omega_0 t)} + w_3 \mathrm{e}^{\mathrm{i}(-\alpha x + \beta y - \omega_0 t)} + w_4 \mathrm{e}^{\mathrm{i}(-\beta x - \alpha y - \omega_0 t)}\right]. \quad (2)$$

The space of all such perturbations $\hat{u}$ is (isomorphic to) a vector space $W \cong \mathbb{C}^8$ whose elements are vectors $\boldsymbol{w} = (z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4)$. This planform is shown graphically in figure 1. The planform inherits spatial symmetries $\Gamma = D_4 \ltimes T^2$ from the original $E(2)$-equivariance and, in normal form, also has a time-translation symmetry group S^1. The group D_4 is generated by a reflection in the x-axis, denoted by m_x, and a rotation of $\pi/2$ denoted by ρ. These act on the mode amplitudes $\boldsymbol{w} \in W \cong \mathbb{C}^8$ according to

$$m_x\colon \quad (x, y) \to (x, -y) \quad (3)$$

$$\boldsymbol{w} \to (w_1, w_4, w_3, w_2, z_1, z_4, z_3, z_2) \quad (4)$$

$$\rho\colon \quad (x, y) \to (y, -x) \quad (5)$$

$$\boldsymbol{w} \to (z_2, z_3, z_4, z_1, w_2, w_3, w_4, w_1). \quad (6)$$

The group $T^2 \times S^1$ of translations in the x and y directions and in time acts as follows

$$[(\xi,\eta),\phi]: \quad (x,y,t) \to (x+\xi, y+\eta, t+\phi) \tag{7}$$

$$\begin{aligned} \boldsymbol{w} \to (&z_1 e^{i(\alpha\xi+\beta\eta)}, z_2 e^{i(\alpha\eta-\beta\xi)}, z_3 e^{i(-\alpha\xi-\beta\eta)}, z_4 e^{i(\beta\xi-\alpha\eta)}, \\ &w_1 e^{i(\alpha\xi-\beta\eta)}, w_2 e^{i(\beta\xi+\alpha\eta)}, w_3 e^{i(-\alpha\eta+\beta\eta)}, w_4 e^{i(-\beta\xi-\alpha\eta)}) e^{-i\omega_0\phi} \end{aligned} \tag{8}$$

where $(\xi,\eta) \in T^2$ and $\phi \in S^1$. Following Dionne *et al* [1] and using exactly the same reasoning as they employed (namely to ensure that we have imposed the finest lattice which will support the set of critical modes) we require α and β to be relatively prime and not both odd. Without loss of generality we assume $\alpha > \beta \geqslant 1$. The representation of $D_4 \ltimes T^2 \times S^1$ defined by (4), (6) and (8) is Γ-simple since it is isomorphic to two copies of an absolutely irreducible representation acting independently on $(z_1+\bar{z}_3, z_2+\bar{z}_4, w_1+\bar{w}_3, w_2+\bar{w}_4)$ and $(i(z_1-\bar{z}_3), i(z_2-\bar{z}_4), i(w_1-\bar{w}_3), i(w_2-\bar{w}_4))$.

3. $\mathbb{C}$-axial branches and their isotropy subgroups

An isotropy subgroup $\Sigma_{\boldsymbol{w}}$ is the collection of symmetries γ which leave the point $\boldsymbol{w} \in W$ invariant: $\Sigma_{\boldsymbol{w}} = \{\gamma \in \Gamma \times S^1 : \gamma\boldsymbol{w} = \boldsymbol{w}\}$. An idea closely related to this is that of a fixed-point subspace of an isotropy subgroup: $\mathrm{Fix}(\Sigma) = \{\boldsymbol{w} \in W : \sigma\boldsymbol{w} = \boldsymbol{w}, \forall\sigma \in \Sigma\}$. For a (non-degenerate) Hopf bifurcation problem with symmetry group Γ the main result on the existence of branches of periodic solutions is the equivariant Hopf theorem [4, 9]: a branch of solutions with isotropy subgroup $\Sigma \subset \Gamma \times S^1$ is guaranteed for every isotropy subgroup that has dim $\mathrm{Fix}(\Sigma) = 2$. Such branches of solutions are called $\mathbb{C}$-*axial*.

The group action defined by (4), (6) and (8) on the vector space $W \cong \mathbb{C}^8$ defines the equivariant bifurcation problem. The general, and entirely group-theoretic, analysis of Dionne *et al* [3] proves the existence of exactly four branches of $\mathbb{C}$-axial periodic solutions which have translation-free isotropy subgroups. There are 13 other branches of periodic solutions which have two-dimensional fixed-point subspaces and hence are guaranteed to exist by the equivariant Hopf theorem. However, none of the 13 other branches have translation-free isotropy subgroups: they are all spatially periodic on a finer lattice than that on which the problem has been posed. To examine these non-translation-free $\mathbb{C}$-axial branches we define the following three subspaces of W:

$$\mathcal{S}_1 = \{\boldsymbol{w} \in W : \boldsymbol{w} = (z_1, z_2, z_3, z_4, 0, 0, 0, 0)\} \tag{9}$$

$$\mathcal{S}_2 = \{\boldsymbol{w} \in W : \boldsymbol{w} = (z_1, 0, z_3, 0, w_1, 0, w_3, 0)\} \tag{10}$$

$$\mathcal{S}_3 = \{\boldsymbol{w} \in W : \boldsymbol{w} = (z_1, 0, z_3, 0, 0, w_2, 0, w_4)\}. \tag{11}$$

That all $\mathbb{C}$-axial solution branches with non-translation-free isotropy subgroups must be contained in one of these subspaces is clear because if a fixed-point subspace contains points where more than four mode amplitudes are non-zero, the corresponding isotropy subgroup can contain no non-trivial spatial translations and hence the fixed-point subspace must be that of a translation-free branch. All possibilities (up to conjugacy) for subspaces with four or fewer non-zero amplitudes are covered by $\mathcal{S}_1, \ldots, \mathcal{S}_3$. The dynamics within $\mathcal{S}_1 \cong \mathbb{C}^4$ corresponds exactly to the analysis of the eight-dimensional representation of Γ by Silber and Knobloch [5]. The dynamics within $\mathcal{S}_2$ and $\mathcal{S}_3$ correspond to Hopf bifurcations on two distinct rhombic lattices. The Hopf bifurcation on a rhombic lattice has been previously studied by Silber *et al* [2] and we quote their results.

Table 1. Subspace $\mathcal{S}_1$: $\mathbb{C}$-axial branches with $\dim \mathrm{Fix}(\Sigma) = 2$. The last column defines the isotropy subgroups in terms of their generators. Groups isomorphic to a circle group S^1 are denoted S^1_* (where $*$ is some label to distinguish different groups), except for $SO(2)$. Cyclic groups of order q are similarly denoted $\mathbb{Z}^*_q$. Discrete spatial translation groups are denoted $S_{t,*}$. For each solution, all variables omitted from the description of the fixed-point subspace are set to zero.

	Solution branch, (abbreviation) and fixed-point subspace	Isotropy subgroup	Group definitions and generators
1	Travelling rolls (TR)	$SO(2) \times S^1_{TR}$	$SO(2) = \left\langle \left[\left(-\frac{\phi}{\alpha}, \frac{\phi}{\beta} \right), 0 \right] \right\rangle$
	z_1		$S^1_{TR} = \left\langle \left[\left(-\frac{\phi}{\alpha}, 0 \right), \phi \right] \right\rangle$
2	Standing rolls (SR)	$\mathbb{Z}^c_2 \times \mathbb{Z}^{\rho^2}_2 \times SO(2)$	$\mathbb{Z}^c_2 = \left\langle \left[\left(\frac{\pi(\alpha+\beta)}{\alpha^2+\beta^2}, \frac{\pi(\beta-\alpha)}{\alpha^2+\beta^2} \right), \pi \right] \right\rangle$
	$z_1 = z_3$		$\mathbb{Z}^{\rho^2}_2 = \langle \rho^2 \rangle$
3	Travelling squares (TS)	$S^1_{TS} \times S_{t,1}$	$S^1_{TS} = \left\langle \left[\left(\frac{\beta-\alpha}{\alpha^2+\beta^2}\phi, -\frac{(\alpha+\beta)}{\alpha^2+\beta^2}\phi \right), \phi \right] \right\rangle$
	$z_1 = z_2$		$S_{t,1} = \left\langle \left[\left(\frac{2\pi\alpha}{\alpha^2+\beta^2}, \frac{2\pi\beta}{\alpha^2+\beta^2} \right), 0 \right] \right\rangle$
4	Standing squares (SS) $z_1 = z_2 = z_3 = z_4$	$\mathbb{Z}^{SS}_4 \times \mathbb{Z}^c_2 \times S_{t,1}$	$\mathbb{Z}^{SS}_4 = \langle \rho \rangle$
5	Alternating rolls (AR)	$\mathbb{Z}^{AR}_4 \times \mathbb{Z}^c_2 \times S_{t,1}$	$\mathbb{Z}^{AR}_4 = \left\langle \left[\left(-\frac{\pi\alpha}{\alpha^2+\beta^2}, -\frac{\pi\beta}{\alpha^2+\beta^2} \right), \frac{\pi}{2} \right] \circ \rho \right\rangle$
	$z_1 = z_2 = \mathrm{i}z_3 = \mathrm{i}z_4$		

Table 2. Subspace $\mathcal{S}_2$: $\mathbb{C}$-axial branches with $\dim \mathrm{Fix}(\Sigma) = 2$. The last column defines the isotropy subgroups in terms of their generators. D^*_2 denotes a dihedral group of order 4, where $*$ is a distinguishing label for each group. See the caption to table 1 for other labelling conventions. For each solution, all variables omitted from the description of the fixed-point subspace are set to zero.

	Solution branch (abbreviation) and fixed-point subspace	Isotropy subgroup	Group definitions and generators
6	Travelling rhombs 1 (TRh_1)	$\mathbb{Z}^x_2 \times S^1_{TRh1} \times S_{t,2}$	$\mathbb{Z}^x_2 = \langle m_x \rangle$
	$z_1 = w_1$		$S^1_{TRh1} = \left\langle \left[\left(-\frac{\phi}{\alpha}, 0 \right), \phi \right] \right\rangle \equiv S^1_{TR}$ $S_{t,2} = \left\langle \left[\left(\frac{\pi}{\alpha}, \frac{\pi}{\beta} \right), 0 \right] \right\rangle$
7	Travelling rhombs 2 (TRh_2)	$\mathbb{Z}^y_2 \times S^1_{TRh2} \times S_{t,2}$	$\mathbb{Z}^y_2 = \langle m_y \rangle$
	$z_1 = w_3$		$S^1_{TRh2} = \left\langle \left[\left(0, -\frac{\phi}{\beta} \right), \phi \right] \right\rangle$
10	Standing rectangles 1 (SRec_1) $z_1 = z_3 = w_1 = w_3$	$D^{x,y}_2 \times S_{t,2}$	$D^{x,y}_2 = \langle m_x, m_y \rangle$
12	Alternating rectangles 1 (ARec_1)	$\tilde{D}^x_2 \times S_{t,2}$	$\tilde{D}^x_2 = \left\langle \rho^2, \left[\left(-\frac{\pi}{2\alpha}, -\frac{\pi}{2\beta} \right), \frac{\pi}{2} \right] \circ m_x \right\rangle$
	$\mathrm{i}z_1 = \mathrm{i}z_3 = w_1 = w_3$		

Since there are five $\mathbb{C}$-axial branches contained in $\mathcal{S}_1$ (see table 1), a further four contained in each of $\mathcal{S}_2$ and $\mathcal{S}_3$ (tables 2 and 3) and four 'superlattice' branches, there are exactly 17 branches guaranteed to exist by the equivariant Hopf theorem. Further branches of periodic solutions are possible (for example, the standing cross-rolls solution in $\mathcal{S}_1$); these have submaximal isotropy subgroups. The translation-free branches identified by [3] are listed in table 5. In [3] the four branches standing super-squares, standing anti-squares, alternating

Table 3. Subspace $\mathcal{S}_3$: $\mathbb{C}$-axial branches with dim Fix$(\Sigma) = 2$. The last column defines the isotropy subgroups in terms of their generators. See the captions to tables 1 and 2 for group labelling conventions. For each solution, all variables omitted from the description of the fixed-point subspace are set to zero.

	Solution branch (abbreviation) and fixed-point subspace	Isotropy subgroup	Group definitions and generators
8	Travelling rhombs 3 (TRh$_3$) $z_1 = w_2$	$\mathbb{Z}_2^d \times S^1_{TRh3} \times S_{t,3}$	$\mathbb{Z}_2^d = \langle m_d \rangle$ $S^1_{TRh3} = \left\langle \left[\left(-\frac{\phi}{\alpha+\beta}, -\frac{\phi}{\alpha+\beta} \right), \phi \right] \right\rangle$ $S_{t,3} = \left\langle \left[\left(\frac{2\pi\alpha}{\alpha^2-\beta^2}, -\frac{2\pi\beta}{\alpha^2-\beta^2} \right), 0 \right] \right\rangle$
9	Travelling rhombs 4 (TRh$_4$) $z_1 = w_4$	$\mathbb{Z}_2^{d'} \times S^1_{TRh4} \times S_{t,3}$	$\mathbb{Z}_2^{d'} = \langle m_{d'} \rangle$ $S^1_{TRh4} = \left\langle \left[\left(\frac{\phi}{\beta-\alpha}, -\frac{\phi}{\alpha-\beta} \right), \phi \right] \right\rangle$
11	Standing rectangles 2 (SRec$_2$) $z_1 = z_3 = w_2 = w_4$	$D_2^{d,d'} \times S_{t,3}$	$D_2^{d,d'} = \langle m_d,\ m_{d'} \rangle$
13	Alternating rectangles 2 (ARec$_2$) $\mathrm{i}z_1 = \mathrm{i}z_3 = w_2 = w_4$	$\tilde{D}_2^d \times S_{t,3}$	$\tilde{D}_2^d = \left\langle \rho^2, \left[\left(-\frac{\pi\alpha}{\alpha^2-\beta^2}, \frac{\pi\beta}{\alpha^2-\beta^2} \right), \frac{\pi}{2} \right] \circ m_d \right\rangle$

super-squares and alternating anti-squares were denoted by $S2_{\alpha,\beta}$, $S4_{\alpha,\beta}$, $S1_{\alpha,\beta}$ and $S3_{\alpha,\beta}$, respectively. For alternating super-squares and alternating anti-squares the exact form of the fixed-point subspace depends on the parity of α and β (either α or β is odd, but not both or else a finer lattice which supports these solution branches can be found) but the isotropy subgroup itself does not.

3.1. Hidden symmetries

In the computations of the stability of these periodic solutions in section 5 we take into account various hidden symmetries of solution branches [4, 11]. Hidden symmetries are elements of $E(2)$ which are not elements of Γ but which nevertheless restrict the normal form equations and the Jacobian evaluated in the corresponding fixed-point subspace. Only the $\mathbb{C}$-axial branches in subspace $\mathcal{S}_1$ are affected by these calculations. A symmetry operation $\tilde{\gamma} \notin \Gamma$ is a *hidden symmetry* if there is a fixed-point subspace Fix(Σ) such that $U \equiv \mathrm{Fix}(\Sigma) \cap \tilde{\gamma}(\mathrm{Fix}(\Sigma)) \neq \emptyset$ but $U \neq \mathrm{Fix}(\Sigma)$. Then $\tilde{\gamma} : U \to U$ is a symmetry on U and any Γ-equivariant function $\boldsymbol{f}$ must in addition satisfy $\boldsymbol{f}(\tilde{\gamma}\boldsymbol{u}) = \tilde{\gamma}\boldsymbol{f}(\boldsymbol{u})$ for all $\boldsymbol{u} \in U$. Specifically, we define the hidden symmetry R to be the anticlockwise rotation by an angle $\theta = 2\tan^{-1}(\beta/\alpha)$. Then $m_x \circ R^{-1}$ acts only on $\mathcal{S}_1$:

$$m_x \circ R^{-1} : (z_1, z_2, z_3, z_4, 0, 0, 0, 0) \to (z_1, z_4, z_3, z_2, 0, 0, 0, 0). \tag{12}$$

In terms of the definition of a hidden symmetry given above, let Σ be the translation group $S_{t,1}$ generated by the translation $[(\xi, \eta), \phi] = [(\frac{2\pi\alpha}{\alpha^2+\beta^2}, \frac{2\pi\beta}{\alpha^2+\beta^2}), 0]$. Then Fix$(\Sigma) = \mathcal{S}_1$, and $U = \mathrm{Fix}(\Sigma) \cap m_x \circ R^{-1}\mathrm{Fix}(\Sigma) = \{(z_1, z_2, z_3, z_2, 0, 0, 0, 0) \cong \mathbb{C}^3\}$. See section 3.2 of [1] for a discussion of hidden symmetries in related steady-state bifurcation problems. The five $\mathbb{C}$-axial branches in $\mathcal{S}_1$ are periodic on a lattice $\hat{\mathcal{L}}$ which is finer than $\mathcal{L}$; by considering these branches to be periodic with respect to $\hat{\mathcal{L}}$ we can compute their full symmetry groups which include these hidden symmetries. These larger symmetry groups are given in table 4.

Table 4. Subspace $\mathcal{S}_1$: $\mathbb{C}$-axial branches with dim Fix$(\Sigma) = 2$ and their complete symmetry groups, including hidden symmetries. Refer to table 1 for group labelling conventions and the omitted group definitions.

	Solution branch	Symmetry group	Group definitions
1	Travelling rolls	$O(2) \times S^1_{TR}$	$O(2) = \left\langle m_x \circ R^{-1}, \left[\left(-\frac{\phi}{\alpha}, \frac{\phi}{\beta}\right), 0\right]\right\rangle$
2	Standing rolls	$\mathbb{Z}_2^c \times \mathbb{Z}_2^{\rho^2} \times O(2)$	
3	Travelling squares	$S^1_{TS} \times S_{t,1} \times \mathbb{Z}_2^{\mathrm{d}R}$	$\mathbb{Z}_2^{\mathrm{d}R} = \langle m_d \circ R^{-1} \rangle$
4	Standing squares	$D_4^{SS} \times \mathbb{Z}_2^c \times S_{t,1}$	$D_4^{SS} = \langle \rho, m_x \circ R^{-1} \rangle$
5	Alternating rolls	$D_4^{AR} \times \mathbb{Z}_2^c \times S_{t,1}$	$D_4^{AR} = \left\langle \left[\left(-\frac{\pi\alpha}{\alpha^2+\beta^2}, -\frac{\pi\beta}{\alpha^2+\beta^2}\right), \frac{\pi}{2}\right] \circ \rho, m_x \circ R^{-1} \right\rangle$

Table 5. $\mathbb{C}$-axial branches with translation-free isotropy subgroups. The groups denoted by D_4^* are dihedral groups of order 8. All other group labelling conventions are given in the captions to tables 1 and 2.

	Solution branch (abbreviation)	Fixed-point subspace, isotropy subgroup and generators
14	Standing super-squares (SSS)	$z_1 = z_2 = z_3 = z_4 = w_1 = w_2 = w_3 = w_4$ $D_4^{SSS} \times \mathbb{Z}_2^{\pi}$ $\quad D_4^{SSS} = \langle \rho, m_x \rangle$ $\mathbb{Z}_2^{\pi} = \langle [(\pi, \pi), \pi] \rangle$
15	Standing anti-squares (SAS)	$-z_1 = -z_2 = -z_3 = -z_4 = w_1 = w_2 = w_3 = w_4$ $D_4^{SAS} \times \mathbb{Z}_2^{\pi}$ $\quad D_4^{SAS} = \langle \rho, [(\pi, \pi), 0] \circ m_x \rangle$
16	Alternating super-squares (ASS)	α odd: $z_1 = z_3 = w_1 = w_3 = -\mathrm{i}z_2 = -\mathrm{i}z_4 = -\mathrm{i}w_2 = -\mathrm{i}w_4$ β odd: $z_1 = z_3 = w_1 = w_3 = \mathrm{i}z_2 = \mathrm{i}z_4 = \mathrm{i}w_2 = \mathrm{i}w_4$ $D_4^{ASS} \times \mathbb{Z}_2^{\pi}$ $\quad D_4^{ASS} = \left\langle \left[(\pi, 0), \frac{\pi}{2}\right] \circ \rho, m_x \right\rangle$
17	Alternating anti-squares (AAS)	α odd: $z_1 = z_3 = -w_1 = -w_3 = \mathrm{i}z_2 = \mathrm{i}z_4 = -\mathrm{i}w_2 = -\mathrm{i}w_4$ β odd: $z_1 = z_3 = -w_1 = -w_3 = -\mathrm{i}z_2 = -\mathrm{i}z_4 = \mathrm{i}w_2 = \mathrm{i}w_4$ $D_4^{AAS} \times \mathbb{Z}_2^{\pi}$ $\quad D_4^{AAS} = \left\langle \left[(0, \pi), \frac{\pi}{2}\right] \circ \rho, [(\pi, \pi), 0] \circ m_x \right\rangle$

3.2. The isotropy lattice

The calculation of the four-dimensional fixed-point subspaces is a natural way to proceed after all $\mathbb{C}$-axial branches are known. This information is essential to investigate possible heteroclinic cycles between different periodic solutions. Although the analysis of heteroclinic cycles has not been pursued in detail, the calculation of the four-dimensional fixed-point subspaces is still important: it is of interest for calculating the relative stability of two periodic solutions which lie within the same four-dimensional subspace, and such stability calculations should agree with the stability results computed from the Jacobian matrix *via* isotypic decomposition. To determine the dynamics within a four-dimensional subspace Fix(Σ) we calculate the normalizer $\mathcal{N}(\Sigma) = \{\gamma \in \Gamma \times S^1 : \gamma^{-1}\Sigma\gamma = \Sigma\}$ of each corresponding isotropy subgroup Σ. The dynamics within Fix(Σ) is $\mathcal{N}(\Sigma)/\Sigma$-equivariant (note that Σ acts trivially on Fix(Σ) by definition), but may also be restricted by the form of the normal form equations restricted to Fix(Σ). Table 6 lists the 25 four-dimensional fixed-point subspaces and their normalizers and table 7

Table 6. Four-dimensional fixed-point subspaces. Note that the $\mathbb{C}$-axial branches are numbered 1–17. For each subspace, all variables omitted from the second column are set to zero and unconstrained variables are independent. A group isomorphic to the action of the normalizer is given in the last column: the elements generating isomorphic normalizers are different for different four-dimensional subspaces.

Subspace number	Fixed-point subspace	Structure of $\mathcal{N}(\Sigma)/\Sigma$
18	z_1, z_3	$O(2) \times S^1$
19	z_1, z_2	$O(2) \times S^1$
20	$z_1 = z_2, z_3 = z_4$	$O(2) \times S^1$
21	$\mathrm{i}z_1 = z_2, \mathrm{i}z_3 = z_4$	$O(2) \times S^1$
22	$z_1 = z_3, z_2 = z_4$	$D_4 \times S^1$
23	z_1, w_1	$O(2) \times S^1$
24	z_1, w_3	$O(2) \times S^1$
25	z_1, w_2	$O(2) \times S^1$
26	z_1, w_4	$O(2) \times S^1$
27	$z_1 = w_1, z_3 = w_3$	$O(2) \times S^1$
28	$z_1 = w_3, z_3 = w_1$	$O(2) \times S^1$
29	$z_1 = z_3, w_1 = w_3$	$D_4 \times S^1$
30	$\mathrm{i}z_1 = w_1, \mathrm{i}z_3 = w_3$	$O(2) \times S^1$
31	$\mathrm{i}z_1 = w_3, \mathrm{i}z_3 = w_1$	$O(2) \times S^1$
32	$z_1 = w_2, z_3 = w_4$	$O(2) \times S^1$
33	$z_1 = w_4, z_3 = w_2$	$O(2) \times S^1$
34	$z_1 = z_3, w_2 = w_4$	$D_4 \times S^1$
35	$\mathrm{i}z_1 = w_2, \mathrm{i}z_3 = w_4$	$O(2) \times S^1$
36	$\mathrm{i}z_1 = w_4, \mathrm{i}z_3 = w_2$	$O(2) \times S^1$
37	$z_1 = z_2 = z_3 = z_4, w_1 = w_2 = w_3 = w_4$	$\mathbb{Z}_2^2 \times S^1$
38	$z_1 = z_3 = w_1 = w_3, z_2 = w_2 = z_4 = w_4$	$D_4 \times S^1$
39	$z_1 = z_3 = w_2 = w_4, z_2 = z_4 = w_1 = w_3$	$\mathbb{Z}_2^2 \times S^1$
40	$z_1 = z_3 = -w_1 = -w_3, z_2 = z_4 = -w_2 = -w_4$	$D_4 \times S^1$
41	$z_1 = z_3 = -w_2 = -w_4, z_2 = z_4 = -w_1 = -w_3$	$\mathbb{Z}_2^2 \times S^1$
42	$z_1 = z_3 = \mathrm{i}z_2 = \mathrm{i}z_4, w_1 = w_3 = \mathrm{i}w_2 = \mathrm{i}w_4$	$\mathbb{Z}_2^2 \times S^1$

summarizes the structure of inclusions of $\mathbb{C}$-axial branches within four-dimensional fixed-point subspaces.

4. Derivation of the normal form

To determine conditions for the stability of the periodic solutions guaranteed by the equivariant Hopf theorem we first derive the form of the amplitude equations on the centre manifold. These amplitude equations are restricted by their equivariance with respect to the action of Γ defined by (4), (6) and (8):

$$\dot{\boldsymbol{w}} = \boldsymbol{f}(\boldsymbol{w}, \mu), \qquad \boldsymbol{f}(\gamma \boldsymbol{w}, \mu) = \gamma \boldsymbol{f}(\boldsymbol{w}, \mu) \qquad \forall\, \gamma \in \Gamma \times S^1. \tag{13}$$

Assuming $\boldsymbol{f}$ is a smooth function, it can be written as a sum of terms

$$\boldsymbol{f}(\boldsymbol{w}, \mu) = \sum_{j=1}^{n} \boldsymbol{g}_j(\boldsymbol{w}, \mu) h_j(\boldsymbol{w}) \tag{14}$$

where the terms $\boldsymbol{g}_1(\boldsymbol{w}, \mu), \ldots, \boldsymbol{g}_n(\boldsymbol{w}, \mu)$ are $\Gamma \times S^1$-equivariant and the $h_j(\boldsymbol{w})$ terms are polynomials in the invariants under $\Gamma \times S^1$. So we are required to find all low-order invariants

Table 7. The inclusions of $\mathbb{C}$-axial branches in four-dimensional fixed-point subspaces. This information is sufficient to construct the isotropy lattice.

Subspace number	$\mathbb{C}$-axial branch	Contained in these subspaces of dimension 4
1	TR	18, 19, 23, 24, 25, 26
2	SR	18, 22, 29, 34
3	TS	19, 20, 21
4	SS	20, 22, 37
5	AR	21, 22, 42
6	TRh_1	23, 27, 30
7	TRh_2	24, 28, 31
8	TRh_3	25, 32, 35
9	TRh_4	26, 33, 36
10	SRec_1	27, 28, 29, 38, 40
11	SRec_2	32, 33, 34, 39, 41
12	ARec_1	29, 30, 31
13	ARec_2	34, 35, 36
14	SSS	37, 38, 39
15	SAS	37, 40, 41
16	ASS	38, 42
17	AAS	40, 42

under the action of $\Gamma \times S^1$. The action of the interchange symmetries (4) and (6) determine the form of $f_2, \ldots, f_8$ once the form of the first component of $\boldsymbol{f}$ is found. Hence we will determine the form of f_1, the evolution equation for z_1; the equation $\dot{z}_1 = f_1(\boldsymbol{w}, \mu)$ is required to be equivariant with respect to the $T^2 \times S^1$ action defined by (8).

4.1. $T^2 \times S^1$ invariants

Let $I = z_1^m z_2^n z_3^p z_4^q w_1^r w_2^s w_3^t w_4^u$ be a general invariant term, where $m, \ldots, u$ are integers. We introduce the usual convention that $z_1^m \equiv \bar{z}_1^{|m|}$ if $m < 0$. The order $\mathcal{O}(I)$ of an invariant I is defined to be $\mathcal{O}(I) = |m| + |n| + |p| + |q| + |r| + |s| + |t| + |u|$. S^1 invariance immediately implies that all invariants must be of even order. Furthermore, the only invariants of order 2 are the usual ones $|z_1|^2, \ldots, |w_4|^2$. We assume that the expression I has had all possible order 2 invariants removed from it. Then T^2 invariance implies

$$(m - p + r - t)\alpha + (q - n + s - u)\beta = 0 \tag{15}$$

$$(n - q + s - u)\alpha + (m - p - r + t)\beta = 0 \tag{16}$$

since ξ and η are independent. Since α and β are relatively prime we deduce

$$m - p + r - t = j\beta, \qquad q - n + s - u = -j\alpha \tag{17}$$

$$n - q + s - u = k\beta, \qquad m - p - r + t = -k\alpha \tag{18}$$

where j and k are integers. Define $A = m - p$, $B = r - t$, $C = n - q$ and $D = s - u$ so that these conditions become

$$A + B = j\beta, \qquad A - B = -k\alpha \tag{19}$$

$$C + D = k\beta, \qquad C - D = j\alpha \tag{20}$$

which have a solution

$$A = (j\beta - k\alpha)/2, \qquad B = (j\beta + k\alpha)/2 \tag{21}$$

$$C = (k\beta + j\alpha)/2, \qquad D = (k\beta - j\alpha)/2. \tag{22}$$

However, $A, \ldots, D$ are integers, hence $j\beta - k\alpha$, etc are all even. As exactly one of α and β is even, this further implies that both j and k must be even. So we may define $j' = j/2$ and $k' = k/2$. We now look for low-order invariants in the three cases: (a) $j' = k' = 0$, (b) $j' = 0$, $k' \neq 0$ and (c) $j' \neq 0$, $k' = 0$.

Case (a) produces all possible invariants of order 4. For example, to find all order 4 invariants containing z_1, we fix $m = 1$. Then, as $j' = k' = 0$ implies $A = B = C = D = 0$ we must set $p = 1$ and one pair of (n, q), (r, t) and (s, u) equal to $(-1, -1)$, hence $\mathcal{O}(I) = 4$. This results in the three invariants

$$z_1 z_3 \bar{z}_2 \bar{z}_4, \qquad z_1 z_3 \bar{w}_1 \bar{w}_3, \qquad z_1 z_3 \bar{w}_2 \bar{w}_4. \tag{23}$$

All other order 4 invariants are produced by considering the remaining possibilities.

In case (b) it suffices to consider $j' = 0$, $k' = 1$ since all other integers for k' merely introduce multiples of the order 4 invariants found in case (a). This results in four distinct invariants of order $2(\alpha + \beta)$:

$$\bar{z}_1^{\alpha} z_2^{\beta} w_1^{\alpha} \bar{w}_4^{\beta}, \qquad \bar{z}_1^{\alpha} \bar{z}_4^{\beta} w_1^{\alpha} w_2^{\beta} \tag{24}$$

$$z_2^{\beta} z_3^{\alpha} \bar{w}_3^{\alpha} \bar{w}_4^{\beta}, \qquad z_3^{\alpha} \bar{z}_4^{\beta} w_2^{\beta} \bar{w}_3^{\alpha}. \tag{25}$$

A further four invariants of order $2(\alpha + \beta)$ are derived in case (c):

$$z_1^{\beta} z_2^{\alpha} \bar{w}_2^{\alpha} \bar{w}_3^{\beta}, \qquad z_1^{\beta} \bar{z}_4^{\alpha} \bar{w}_3^{\beta} w_4^{\alpha} \tag{26}$$

$$z_2^{\alpha} \bar{z}_3^{\beta} w_1^{\beta} \bar{w}_2^{\alpha}, \qquad \bar{z}_3^{\beta} \bar{z}_4^{\alpha} w_1^{\beta} w_4^{\alpha}. \tag{27}$$

To prove that these three cases produce all invariants of order less than $2(\alpha+\beta+1)$ we consider all the remaining values of the pair (j', k'). These divide into two cases; either the product $j'k' > 0$ or $j'k' < 0$. In the case $j'k' > 0$ we have (since $\alpha > \beta > 0$):

$$\begin{aligned}
\mathcal{O}(I) &\geqslant |A| + |B| + |C| + |D| \geqslant |A + D| + |B + C| \\
&\geqslant |(j' + k')(\beta - \alpha)| + |(j' + k')(\alpha + \beta)| \\
&\geqslant 2|\alpha - \beta| + 2|\alpha + \beta| \geqslant 2(\alpha + \beta + 1).
\end{aligned} \tag{28}$$

Similarly, when $j'k' < 0$ we have

$$\begin{aligned}
\mathcal{O}(I) &\geqslant |A| + |B| + |C| + |D| \geqslant |A - D| + |B - C| \\
&\geqslant |(j' - k')(\alpha + \beta)| + |(j' - k')(\beta - \alpha)| \\
&\geqslant 2|\alpha - \beta| + 2|\alpha + \beta| \geqslant 2(\alpha + \beta + 1).
\end{aligned} \tag{29}$$

In summary there are exactly eight order 2 invariants $|z_1|^2, \ldots, |w_4|^2$, three invariants of order 4 (23) and eight invariants of order $2(\alpha + \beta)$, listed in (24)–(27).

4.2. Amplitude equations in normal form

Assuming that the origin is stable below the bifurcation point, and using the information on the invariant functions $h_j(\boldsymbol{w})$, the third-order truncation of the amplitude equation for $\dot{z}_1$ is

$$\begin{aligned}
\dot{z}_1 = z_1\big[&\mu + \mathrm{i}\omega(\mu) + \lambda_1|z_1|^2 + \lambda_2|z_3|^2 + \lambda_3|z_2|^2 + \tilde{\lambda}_3|z_4|^2 + \lambda_4|w_1|^2 \\
&+\lambda_5|w_2|^2 + \lambda_6|w_3|^2 + \lambda_7|w_4|^2\big] + \alpha_1\bar{z}_3 z_2 z_4 + \alpha_2\bar{z}_3 w_1 w_3 + \alpha_3\bar{z}_3 w_2 w_4
\end{aligned} \tag{30}$$

with the evolution equations for $z_2, \ldots, w_4$ related by the interchange symmetries (4) and (6). The frequency ω will in general depend on the bifurcation parameter μ; at the bifurcation point $\omega(0) = 0$ since the frequency ω_0 of linear oscillations has already been factored out in equation (8). The normal form (30) does not take into account the action of the hidden symmetry (12): this forces the coefficients λ_3 and $\tilde{\lambda}_3$ to be equal. In this way the restriction of the third-order truncation of the amplitude equations to the subspace $\mathcal{S}_1$ is equal to that derived for Hopf bifurcations on the simpler square lattice by Knobloch and Silber [5]. In addition, terms at order $2(\alpha+\beta)-1$ on the right-hand side are needed to determine the stability of the four translation-free $\mathbb{C}$-axial branches. For the $\dot{z}_1$ equation these are

$$\dot{z}_1 = \cdots + b_1 \bar{z}_1^{\alpha-1} z_2^{\beta} w_1^{\alpha} \bar{w}_4^{\beta} + b_2 \bar{z}_1^{\alpha-1} \bar{z}_4^{\beta} w_1^{\alpha} w_2^{\beta} + b_3 \bar{z}_1^{\beta-1} \bar{z}_2^{\alpha} w_2^{\alpha} w_3^{\beta} + b_4 \bar{z}_1^{\beta-1} z_4^{\alpha} w_3^{\beta} \bar{w}_4^{\alpha} \tag{31}$$

with similar, symmetrically related, expressions for each of the $\dot{z}_2, \ldots, \dot{w}_4$ equations. Using the result of [9], branches of small-amplitude periodic solutions correspond to solutions of

$$\phi(\boldsymbol{w}, \mu, \tau) \equiv \boldsymbol{f}(\boldsymbol{w}, \mu) - \mathrm{i}(\omega(\mu) + \tau)\boldsymbol{w} = 0 \tag{32}$$

where τ is the correction to the frequency of the periodic solution away from the bifurcation point. The oscillation frequency of a branch of solutions is then given by $\omega_0 + \omega(\mu) + \tau$. The linear (orbital) stability of the non-translation-free $\mathbb{C}$-axial branches is generically determined by the third-order truncation $\Phi(\boldsymbol{w}, \mu, \tau)$ of $\phi(\boldsymbol{w}, \mu, \tau)$.

5. Stability of the $\mathbb{C}$-axial branches

The computation of conditions for the (orbital) stability of the non-translation-free $\mathbb{C}$-axial branches proceeds along familiar lines: all stability conditions depend only on the coefficients of the third-order truncation. The details of the calculations are not presented here: the reader is referred to [5] for further details. The computation of the stability of the four translation-free branches is more involved and hence is presented much more fully.

5.1. Non-translation-free $\mathbb{C}$-axial branches

Table 8 summarizes the stability criteria for the non-translation-free solution branches. These results may be derived by considering the isotypic decomposition of $W \cong \mathbb{C}^8$ for corresponding isotropy subgroups or equivalently by considering the restriction of the third-order truncation (30) to each four-dimensional subspace that contains the periodic solution, as listed in table 7. The dynamics of the third-order truncation within each of these four-dimensional subspaces is either $O(2) \times S^1$-equivariant or $D_4 \times S^1$-equivariant and the relevant stability criteria for these problems are well known. Note that the full amplitude equations restricted to one of subspaces 37, 39, 41 or 42 are not $D_4 \times S^1$-equivariant, but only $\mathbb{Z}_2^2 \times S^1$-equivariant; the third-order truncation of the normal form contains an extra, spurious, symmetry.

5.2. Standing super-squares and standing anti-squares

Standing super-squares and standing anti-squares are distinguished only by the terms at $\mathcal{O}(2(\alpha+\beta)-1)$ in the normal form. Furthermore, they are related by the parameter symmetry which takes $(b_1, b_2, b_3, b_4) \to -(b_1, b_2, b_3, b_4)$ and leaves all other coefficients unchanged. Hence we can easily derive the stability criteria for standing anti-squares by changing the signs of the $b_1, \ldots, b_4$ coefficients in the stability criteria for standing super-squares. Moreover, some of the eigenvalues of the Jacobian matrix must involve the coefficients $b_1, \ldots, b_4$ to yield the relative stability of the two branches. Due to these complications not present in the calculations for the non-translation-free branches we present the stability calculation in full. The approach is of a standard nature; similar calculations to these are given, for different bifurcation problems, in [6] and [7].

Table 8. Stability criteria for the 13 $\mathbb{C}$-axial non-translation-free branches. A branch of solutions to the third-order truncation $\Phi(w, \mu, \tau)$ is stable when all stability criteria are negative. Superscript r denotes 'the real part of'. $B_1 = \lambda_1 + \lambda_2 - 2\lambda_3$, $B_2 = \lambda_1 + \lambda_2 - \lambda_4 - \lambda_6$, $B_3 = \lambda_1 + \lambda_2 - \lambda_5 - \lambda_7$.

Solution branch	Number of zero eigenvalues	Stability criteria			
TR	1	λ_1^r	$\lambda_2^r - \lambda_1^r$	$\lambda_3^r - \lambda_1^r$	$\lambda_4^r - \lambda_1^r$
		$\lambda_5^r - \lambda_1^r$		$\lambda_6^r - \lambda_1^r$	$\lambda_7^r - \lambda_1^r$
SR	2	$\lambda_1^r + \lambda_2^r$	$\lambda_1^r - \lambda_2^r$	$-B_1^r$	$\lvert B_1\rvert^2 - \lvert\alpha_1\rvert^2$
		$-B_2^r$	$\lvert B_2\rvert^2 - \lvert\alpha_2\rvert^2$	$-B_3^r$	$\lvert B_3\rvert^2 - \lvert\alpha_3\rvert^2$
TS	2	$\lambda_1^r + \lambda_3^r$		$\lambda_1^r - \lambda_3^r$	
		$\lambda_2^r + \alpha_1^r - \lambda_1^r - \lambda_3^r$		$\lambda_2^r - \alpha_1^r - \lambda_1^r - \lambda_3^r$	
		$\lambda_4^r + \lambda_5^r - \lambda_1^r - \lambda_3^r$		$\lambda_5^r + \lambda_6^r - \lambda_1^r - \lambda_3^r$	
		$\lambda_6^r + \lambda_7^r - \lambda_1^r - \lambda_3^r$		$\lambda_7^r + \lambda_4^r - \lambda_1^r - \lambda_3^r$	
SS	3	$\lambda_1^r + \lambda_2^r + 2\lambda_3^r + \alpha_1^r$		$\lambda_1^r - \lambda_2^r - \alpha_1^r$	
		$B_1^r - 3\alpha_1^r$		$\mathrm{Re}[B_1\bar{\alpha}_1] - \lvert\alpha_1\rvert^2$	
		$B_1^r - B_2^r - B_3^r - \alpha_1^r$		$\lvert B_1 - B_2 - B_3 - \alpha_1\rvert^2 - \lvert\alpha_2 + \alpha_3\rvert^2$	
AR	3	$\lambda_1^r + \lambda_2^r + 2\lambda_3^r - \alpha_1^r$		$\lambda_1^r - \lambda_2^r + \alpha_1^r$	
		$B_1^r + 3\alpha_1^r$		$-\mathrm{Re}[B_1\bar{\alpha}_1] - \lvert\alpha_1\rvert^2$	
		$B_1^r - B_2^r - B_3^r + \alpha_1^r$		$\lvert B_1 - B_2 - B_3 + \alpha_1\rvert^2 - \lvert\alpha_2 - \alpha_3\rvert^2$	
TRh$_1$	2	$\lambda_1^r + \lambda_4^r$		$\lambda_1^r - \lambda_4^r$	
		$\lambda_3^r + \lambda_5^r - \lambda_1^r - \lambda_4^r$		$\lambda_3^r + \lambda_7^r - \lambda_1^r - \lambda_4^r$	
		$\lambda_2^r + \lambda_6^r - \alpha_2^r - \lambda_1^r - \lambda_4^r$		$\lambda_2^r + \lambda_6^r + \alpha_2^r - \lambda_1^r - \lambda_4^r$	
TRh$_2$	2	$\lambda_1^r + \lambda_6^r$		$\lambda_1^r - \lambda_6^r$	
		$\lambda_3^r + \lambda_5^r - \lambda_1^r - \lambda_6^r$		$\lambda_3^r + \lambda_7^r - \lambda_1^r - \lambda_6^r$	
		$\lambda_2^r + \lambda_4^r - \alpha_2^r - \lambda_1^r - \lambda_6^r$		$\lambda_2^r + \lambda_4^r + \alpha_2^r - \lambda_1^r - \lambda_6^r$	
TRh$_3$	2	$\lambda_1^r + \lambda_5^r$		$\lambda_1^r - \lambda_5^r$	
		$\lambda_3^r + \lambda_4^r - \lambda_1^r - \lambda_5^r$		$\lambda_3^r + \lambda_6^r - \lambda_1^r - \lambda_5^r$	
		$\lambda_2^r + \lambda_7^r - \alpha_3^r - \lambda_1^r - \lambda_5^r$		$\lambda_2^r + \lambda_7^r + \alpha_3^r - \lambda_1^r - \lambda_5^r$	
TRh$_4$	2	$\lambda_1^r + \lambda_7^r$		$\lambda_1^r - \lambda_7^r$	
		$\lambda_3^r + \lambda_6^r - \lambda_1^r - \lambda_7^r$		$\lambda_3^r + \lambda_4^r - \lambda_1^r - \lambda_7^r$	
		$\lambda_2^r + \lambda_5^r - \alpha_3^r - \lambda_1^r - \lambda_7^r$		$\lambda_2^r + \lambda_5^r + \alpha_3^r - \lambda_1^r - \lambda_7^r$	
SRec$_1$	3	$\lambda_1^r + \lambda_2^r + \lambda_4^r + \lambda_6^r + \alpha_2^r$			
		$\lambda_1^r + \lambda_4^r - \lambda_2^r - \lambda_6^r - \alpha_2^r$		$\lambda_1^r + \lambda_6^r - \lambda_2^r - \lambda_4^r - \alpha_2^r$	
		$B_2^r - 3\alpha_2^r$		$\mathrm{Re}[B_2\bar{\alpha}_2] - \lvert\alpha_2\rvert^2$	
		$B_2^r - B_1^r - B_3^r - \alpha_2^r$		$\lvert\alpha_1 + \alpha_3\rvert^2 - \lvert B_1 - B_2 + B_3 + \alpha_2\rvert^2$	
SRec$_2$	3	$\lambda_1^r + \lambda_2^r + \lambda_5^r + \lambda_7^r + \alpha_3^r$			
		$\lambda_1^r + \lambda_5^r - \lambda_2^r - \lambda_7^r - \alpha_3^r$		$\lambda_1^r + \lambda_7^r - \lambda_2^r - \lambda_5^r - \alpha_3^r$	
		$B_3^r - 3\alpha_3^r$		$\mathrm{Re}[B_3\bar{\alpha}_3] - \lvert\alpha_3\rvert^2$	
		$B_3^r - B_1^r - B_2^r - \alpha_3^r$		$\lvert\alpha_1 + \alpha_2\rvert^2 - \lvert B_1 + B_2 - B_3 + \alpha_3\rvert^2$	
ARec$_1$	3	$\lambda_1^r + \lambda_2^r + \lambda_4^r + \lambda_6^r - \alpha_2^r$			
		$\lambda_1^r - \lambda_2^r + \lambda_4^r - \lambda_6^r + \alpha_2^r$		$\lambda_1^r - \lambda_2^r - \lambda_4^r + \lambda_6^r + \alpha_2^r$	
		$B_2^r + 3\alpha_2^r$		$-\mathrm{Re}[B_2\bar{\alpha}_2] - \lvert\alpha_2\rvert^2$	
ARec$_2$	3	$\lambda_1^r + \lambda_2^r + \lambda_5^r + \lambda_7^r - \alpha_3^r$			
		$\lambda_1^r - \lambda_2^r + \lambda_5^r - \lambda_7^r + \alpha_3^r$		$\lambda_1^r - \lambda_2^r - \lambda_5^r + \lambda_7^r + \alpha_3^r$	
		$B_3^r + 3\alpha_3^r$		$-\mathrm{Re}[B_3\bar{\alpha}_3] - \lvert\alpha_3\rvert^2$	

The Γ-equivariance of ϕ implies that the Jacobian matrix $D\phi$ evaluated on the standing super-squares solution branch commutes with the matrices generating the action of Γ:

$$R_\rho D\phi R_\rho^{-1} = D\phi, \qquad R_{m_x} D\phi R_{m_x}^{-1} = D\phi \tag{33}$$

which implies that $D\phi$ is a matrix of the form

$$D\phi_{SSS} = \left(\begin{array}{cccc|cccc} A & B & C & D & E & F & G & H \\ D & A & B & C & H & E & F & G \\ C & D & A & B & G & H & E & F \\ B & C & D & A & F & G & H & E \\ \hline E & H & G & F & A & D & C & B \\ F & E & H & G & B & A & D & C \\ G & F & E & H & C & B & A & D \\ H & G & F & E & D & C & B & A \end{array}\right) \tag{34}$$

(the full lines are to indicate the structure of the matrix), where each of $A, \ldots, H$ is itself a 2×2 matrix:

$$A = \begin{pmatrix} \partial\phi_1/\partial z_1 & \partial\phi_1/\partial\bar{z}_1 \\ \partial\bar{\phi}_1/\partial z_1 & \partial\bar{\phi}_1/\partial\bar{z}_1 \end{pmatrix} \qquad \ldots \qquad H = \begin{pmatrix} \partial\phi_1/\partial w_4 & \partial\phi_1/\partial\bar{w}_4 \\ \partial\bar{\phi}_1/\partial w_4 & \partial\bar{\phi}_1/\partial\bar{w}_4 \end{pmatrix}. \tag{35}$$

By conjugating $D\phi_{SSS}$ with a change-of-basis matrix it can be put into a block-upper-triangular form, with 2×2 or 4×4 blocks on the diagonal. The eigenvalues of $D\phi_{SSS}$ are then the eigenvalues of these submatrices: the four 2×2 matrices are

$$A + B + C + D + E + F + G + H$$

$$A + B + C + D - E - F - G - H$$

$$A - B + C - D + E - F + G - H$$

$$A - B + C - D - E + F - G + H$$

of which the first contains a zero eigenvalue corresponding to translations in the direction tangential to the periodic orbit. For stability we require the traces of these 2×2 matrices to be negative and their determinants to be positive. These conditions yield the following stability criteria when evaluated on the third-order truncation of the normal form:

$$\lambda_1^r + \lambda_2^r + 2\lambda_3^r + \lambda_4^r + \lambda_5^r + \lambda_6^r + \lambda_7^r + \alpha_1^r + \alpha_2^r + \alpha_3^r < 0 \tag{36}$$

$$B_2^r + B_3^r - B_1^r + \alpha_1^r - 3(\alpha_2^r + \alpha_3^r) < 0 \tag{37}$$

$$|\alpha_2 + \alpha_3|^2 - \mathrm{Re}[(B_2 + B_3 - B_1 + \alpha_1)(\bar{\alpha}_2 + \bar{\alpha}_3)] > 0 \tag{38}$$

$$B_1^r + B_3^r - B_2^r + \alpha_2^r - 3(\alpha_1^r + \alpha_3^r) < 0 \tag{39}$$

$$|\alpha_1 + \alpha_3|^2 - \mathrm{Re}[(B_1 + B_3 - B_2 + \alpha_2)(\bar{\alpha}_1 + \bar{\alpha}_3)] > 0 \tag{40}$$

$$B_1^r + B_2^r - B_3^r + \alpha_3^r - 3(\alpha_1^r + \alpha_2^r) < 0 \tag{41}$$

$$|\alpha_1 + \alpha_2|^2 - \mathrm{Re}[(B_1 + B_2 - B_3 + \alpha_3)(\bar{\alpha}_1 + \bar{\alpha}_2)] > 0 \tag{42}$$

where a superscript r denotes the real part of a coefficient, and

$$B_1 = \lambda_1 + \lambda_2 - 2\lambda_3 \tag{43}$$

$$B_2 = \lambda_1 + \lambda_2 - \lambda_4 - \lambda_6 \tag{44}$$

$$B_3 = \lambda_1 + \lambda_2 - \lambda_5 - \lambda_7. \tag{45}$$

The stability criteria (36)–(42) can also be deduced from the dynamics within subspaces 37, 38 and 39.

The other half of the Jacobian matrix in the new basis contains two 4×4 blocks:

$$\begin{pmatrix} A-C+H-F & E-G-B+D & 0_2 & 0_2 \\ E-G+B-D & A-C+F-H & 0_2 & 0_2 \\ 0_2 & 0_2 & A-C+E-G & B-D+H-F \\ 0_2 & 0_2 & D-B+H-F & A-C-E+G \end{pmatrix}$$

$$\equiv \begin{pmatrix} M_1 & 0_4 \\ 0_4 & M_2 \end{pmatrix} \tag{46}$$

where 0_n represents an $n \times n$ block of zeros. Each of the 4×4 blocks M_1, M_2 must contain one zero eigenvalue since the standing super-squares solution has three in total: they can be thought of as the neutral stability of the solution to shifts in the horizontal directions and in time. Evaluating the eigenvalues of M_1 and M_2 using the third-order truncation $\Phi(\boldsymbol{w}, \mu, \tau)$ yields the same four eigenvalues for each matrix:

$$0,\ 0,\ \lambda_1^r - \lambda_2^r - \alpha_1^r - \alpha_2^r - \alpha_3^r \pm \left[(\lambda_4^r - \lambda_6^r)^2 + (\lambda_5^r - \lambda_7^r)^2\right]^{1/2}. \tag{47}$$

One of these zero eigenvalues was expected, due to the directions of neutral stability. The other is a degeneracy caused by ignoring terms higher than third order: when terms of order $2(\alpha+\beta)-1$ are included this second zero eigenvalue will no longer be zero, and will select exactly one of standing super-squares and standing anti-squares to be stable. Thinking of the $\mathcal{O}(2(\alpha+\beta)-1)$ terms as a small perturbation to the third-order truncation (as is the case near the bifurcation point $\mu = 0$) we use the characteristic polynomial $\mathcal{P}(m)$ of M_1 to derive a condition guaranteeing that the second zero eigenvalue creates a negative root of $\mathcal{P}(m)$ when the higher-order terms are introduced.

The matrices M_1 and M_2 are of the form

$$\begin{pmatrix} P & Q & R & S \\ \bar{Q} & \bar{P} & \bar{S} & \bar{R} \\ T & U & V & W \\ \bar{U} & \bar{T} & \bar{W} & \bar{V} \end{pmatrix} \tag{48}$$

where $P, \ldots, W$ are complex entries. Nevertheless, the characteristic polynomial $\mathcal{P}(m)$ is entirely real, hence it is guaranteed to have four real roots. From figure 2 a necessary and sufficient condition for the perturbed eigenvalue to be negative is

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}\mathcal{P}(m)}{\mathrm{d}m}\bigg|_{m=0} &\equiv -P_r(|V|^2 - |W|^2) - V_r(|P|^2 - |Q|^2) + \mathrm{Re}[S(\bar{U}V - T\bar{W} + \bar{P}\bar{U} - \bar{Q}\bar{T}) \\ &\quad + R(\bar{P}T - \bar{Q}U + T\bar{V} - \bar{U}W)] > 0. \end{aligned} \tag{49}$$

This results in the following criterion for the stability of standing super-squares:

$$\begin{aligned} &4(Y_1^r Y_2^r - \delta_r^2)[\alpha(b_1^r + b_2^r) + \beta(b_3^r + b_4^r)] + (b_1^i + b_2^i)[K_1(Y_2^r - Y_1^r) + K_2(Y_1^r + Y_2^r + 2\delta_r)] \\ &\quad + (b_3^i + b_4^i)[K_2(Y_2^r - Y_1^r) + K_1(Y_1^r + Y_2^r - 2\delta_r)] > 0 \end{aligned} \tag{50}$$

where $\delta = \lambda_4 - \lambda_6$ and

$$Y_1 = \lambda_1 - \lambda_2 - \lambda_5 + \lambda_7 - \alpha_1 - \alpha_2 - \alpha_3 \tag{51}$$

$$Y_2 = \lambda_1 - \lambda_2 + \lambda_5 - \lambda_7 - \alpha_1 - \alpha_2 - \alpha_3 \tag{52}$$

$$K_1 = (\alpha+\beta)Y_1^i - (\alpha-\beta)Y_2^i + 2\beta\delta_i \tag{53}$$

$$K_2 = (\alpha+\beta)Y_1^i + (\alpha-\beta)Y_2^i - 2\alpha\delta_i \tag{54}$$

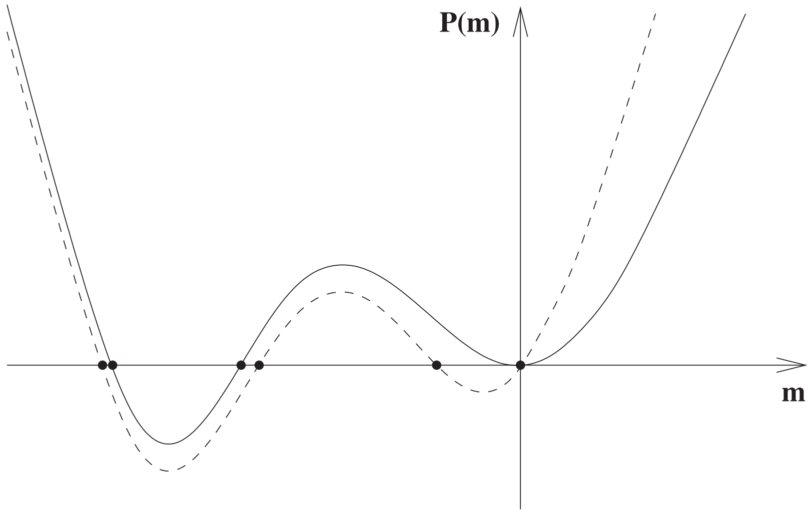

Figure 2. Sketch of the characteristic polynomial $\mathcal{P}(m)$ of the matrix M_1. The full curve corresponds to evaluation of M_1 using the third-order truncation only. The broken curve includes terms of order $2(\alpha+\beta)-1$, breaking the degeneracy between standing super-squares and standing anti-squares.

and superscripts r and i denote real and imaginary parts, respectively. In the special case $b_1^i = b_2^i = b_3^i = b_4^i = 0$, using the degenerate condition at third order given by (47), the condition (50) reduces to requiring

$$\alpha(b_1^r + b_2^r) + \beta(b_3^r + b_4^r) > 0 \tag{55}$$

which is the analogous condition for stability to that derived for the steady-state bifurcation problem on a square superlattice, studied in [1, section 4.2, table 8]. The other 4×4 block M_2 yields the same stability criteria as M_1, so this completes the stability analysis of standing super-squares. The criteria for the stability of standing anti-squares given by the cubic truncation are exactly the same, namely (36)–(42) and (47), but the opposite criterion to (50) applies due to the parameter symmetry $(b_1, b_2, b_3, b_4) \to -(b_1, b_2, b_3, b_4)$ between standing super-squares and standing anti-squares.

5.3. Alternating super-squares and alternating anti-squares

A very similar analysis goes through for alternating super-squares and alternating anti-squares. These two solution branches are also related by the parameter symmetry which changes the sign of $b_1, \ldots, b_4$; as in the case of standing super-squares and standing anti-squares, they are not distinguished by the third-order truncation of the normal form, and hence terms at order $2(\alpha+\beta)-1$ must be brought in to distinguish between them.

The Jacobian matrix for the alternating super-squares solution takes the form

$$D\phi_{ASS} = \left(\begin{array}{cccc|cccc} A & B & C & D & E & F & G & H \\ D^- & -A^- & B^- & -C^- & H^- & -E^- & F^- & -G^- \\ C & D & A & B & G & H & E & F \\ B^- & -C^- & D^- & -A^- & F^- & -G^- & H^- & -E^- \\ \hline E & H & G & F & A & D & C & B \\ F^- & -E^- & H^- & -G^- & B^- & -A^- & D^- & -C^- \\ G & F & E & H & C & B & A & D \\ H^- & -G^- & F^- & -E^- & D^- & -C^- & B^- & -A^- \end{array}\right) \tag{56}$$

due to requiring equivariance with respect to the action of the group D_4^{ASS}, where again, each entry is a 2×2 matrix and the matrices A and A^- are related as follows; if

$$A = \begin{pmatrix} a & a' \\ \bar{a}' & \bar{a} \end{pmatrix} \qquad \text{then} \qquad A^- = \begin{pmatrix} -a & a' \\ \bar{a}' & -\bar{a} \end{pmatrix}. \tag{57}$$

By applying a change of basis transformation to $D\phi_{ASS}$ it can be rendered block-upper-triangular, but this time with four 4×4 blocks on the diagonal. These matrices are

$$N_1 = \begin{pmatrix} a+c+g+e & b'+d'+f'+h' & -b-d-f-h & a'+c'+e'+g' \\ \bar{b}'+\bar{d}'+\bar{f}'+\bar{h}' & \bar{a}+\bar{c}+\bar{e}+\bar{g} & \bar{a}'+\bar{c}'+\bar{e}'+\bar{g}' & -\bar{b}-\bar{d}-\bar{f}-\bar{h} \\ b+f+d+h & a'+c'+e'+g' & a+c+e+g & -b'-d'-f'-h' \\ \bar{a}'+\bar{c}'+\bar{e}'+\bar{g}' & \bar{b}+\bar{d}+\bar{f}+\bar{h} & -\bar{b}'-\bar{d}'-\bar{f}'-\bar{h}' & \bar{a}+\bar{c}+\bar{e}+\bar{g} \end{pmatrix} \tag{58}$$

$$N_2 = \begin{pmatrix} a+c-e-g & b'+d'-f'-h' & -b-d+f+h & a'+c'-e'-g' \\ \bar{b}'+\bar{d}'-\bar{f}'-\bar{h}' & \bar{a}+\bar{c}-\bar{e}-\bar{g} & \bar{a}'+\bar{c}'-\bar{e}'-\bar{g}' & -\bar{b}-\bar{d}+\bar{f}+\bar{h} \\ b-f+d-h & a'+c'-e'-g' & a+c-e-g & -b'-d'+f'+h' \\ \bar{a}'+\bar{c}'-\bar{e}'-\bar{g}' & \bar{b}+\bar{d}-\bar{f}-\bar{h} & -\bar{b}'-\bar{d}'+\bar{f}'+\bar{h}' & \bar{a}+\bar{c}-\bar{e}-\bar{g} \end{pmatrix} \tag{59}$$

$$N_3 = \begin{pmatrix} a-c+e-g & a'-c'+e'-g' & d-b+f-h & d'-b'+f'-h' \\ \bar{a}'-\bar{c}'+\bar{e}'-\bar{g}' & \bar{a}-\bar{c}+\bar{e}-\bar{g} & \bar{d}'-\bar{b}'+\bar{f}'-\bar{h}' & -\bar{b}+\bar{d}+\bar{f}-\bar{h} \\ d-b+h-f & -d'+b'+f'-h' & a-c-e+g & c'-a'-g'+e' \\ \bar{b}'-\bar{d}'+\bar{f}'-\bar{h}' & \bar{d}-\bar{b}-\bar{f}+\bar{h} & \bar{c}'-\bar{a}'-\bar{g}'+\bar{e}' & \bar{a}-\bar{c}-\bar{e}+\bar{g} \end{pmatrix} \tag{60}$$

$$N_4 = \begin{pmatrix} a-c-e+g & a'-c'-e'+g' & d-b-f+h & d'-b'-f'+h' \\ \bar{a}'-\bar{c}'-\bar{e}'+\bar{g}' & \bar{a}-\bar{c}-\bar{e}+\bar{g} & \bar{d}'-\bar{b}'-\bar{f}'+\bar{h}' & -\bar{b}+\bar{d}-\bar{f}+\bar{h} \\ d-b-h+f & -d'+b'-f'+h' & a-c+e-g & c'-a'+g'-e' \\ \bar{b}'-\bar{d}'-\bar{f}'+\bar{h}' & \bar{d}-\bar{b}+\bar{f}-\bar{h} & \bar{c}'-\bar{a}'+\bar{g}'-\bar{e}' & \bar{a}-\bar{c}+\bar{e}-\bar{g} \end{pmatrix} \tag{61}$$

and will be examined in turn. When evaluated using the third-order truncation, N_1 contains one zero eigenvalue and three real eigenvalues corresponding to requiring the branch of solutions to bifurcate supercritically, and the stability criteria within subspace 38, hence for stability we require

$$\lambda_1^r + \lambda_2^r + 2\lambda_3^r + \lambda_4^r + \lambda_5^r + \lambda_6^r + \lambda_7^r - \alpha_1^r + \alpha_2^r - \alpha_3^r < 0 \tag{62}$$

$$B_1^r + B_3^r - B_2^r + \alpha_2^r + 3(\alpha_1^r + \alpha_3^r) < 0 \tag{63}$$

$$|\alpha_1 + \alpha_3|^2 + \mathrm{Re}[(B_1 + B_3 - B_2 + \alpha_2)(\bar{\alpha}_1 + \bar{\alpha}_3)] > 0. \tag{64}$$

Matrix N_2 contains no zero eigenvalues. Using the third-order truncation, necessary and sufficient conditions for the four real eigenvalues of N_2 to be negative are

$$X_1^r + 2\hat{Y}_1^r < 0 \tag{65}$$

$$X_2^r + 2\hat{Y}_2^r < 0 \tag{66}$$

$$\mathrm{Re}[X_1\bar{\hat{Y}}_1] > 0 \tag{67}$$

$$\mathrm{Re}[X_2\bar{\hat{Y}}_2] > 0 \tag{68}$$

where

$$X_1 = B_1 + B_2 - B_3 + \alpha_1 - \alpha_2 - \alpha_3 \tag{69}$$

$$X_2 = B_2 + B_3 - B_1 - \alpha_1 - \alpha_2 + \alpha_3 \tag{70}$$

$$\hat{Y}_1 = \alpha_1 - \alpha_2 \tag{71}$$

$$\hat{Y}_2 = \alpha_3 - \alpha_2. \tag{72}$$

The third block and fourth blocks have eigenvalues analogous to those of M_1 and M_2 in the previous section, but with the signs of α_1 and α_3 changed. This is due to the parameter symmetry between standing super-squares and alternating super-squares (and similarly between standing anti-squares and alternating anti-squares). The eigenvalues of N_3 and N_4 are equal, and when they are evaluated using the third-order truncation Φ we obtain

$$0,\ 0,\ \lambda_1^r - \lambda_2^r + \alpha_1^r - \alpha_2^r + \alpha_3^r \pm \left[(\lambda_4^r - \lambda_6^r)^2 + (\lambda_5^r - \lambda_7^r)^2\right]^{1/2}. \tag{73}$$

As for standing super-squares, the second zero eigenvalue is a degeneracy caused by omitting higher-order terms. We calculate the movement of this eigenvalue when these higher-order terms are introduced in exactly the same way. This leads to the condition

$$\begin{aligned}4(Y_1^rY_2^r - \delta_r^2)[\alpha(b_1^r + b_2^r) + \beta(b_3^r + b_4^r)] + (b_1^i + b_2^i)[K_1(Y_2^r - Y_1^r) + K_2(Y_1^r + Y_2^r + 2\delta_r)]\\ +(b_3^i + b_4^i)[K_2(Y_2^r - Y_1^r) + K_1(Y_1^r + Y_2^r - 2\delta_r)] > 0\end{aligned} \tag{74}$$

for the stability of alternating super-squares, where $\delta = \lambda_4 - \lambda_6$ and

$$Y_1 = \lambda_1 - \lambda_2 - \lambda_5 + \lambda_7 + \alpha_1 - \alpha_2 + \alpha_3 \tag{75}$$

$$Y_2 = \lambda_1 - \lambda_2 + \lambda_5 - \lambda_7 + \alpha_1 - \alpha_2 + \alpha_3 \tag{76}$$

$$K_1 = (\alpha + \beta)Y_1^i - (\alpha - \beta)Y_2^i + 2\beta\delta_i \tag{77}$$

$$K_2 = (\alpha + \beta)Y_1^i + (\alpha - \beta)Y_2^i - 2\alpha\delta_i. \tag{78}$$

If the inequality (74) is reversed then alternating anti-squares are stable instead. As we expect, the stability condition (74) is related to (50) by the parameter symmetry $(\alpha_1, \alpha_3) \to (-\alpha_1, -\alpha_3)$.

This completes the stability calculations for the four 'superlattice' patterns.

6. Complex dynamics

The normal form for the D_4-symmetric Hopf bifurcation on $\mathbb{C}^2$ was studied by Swift [8]. In this much simpler problem it is well known that, in addition to the $\mathbb{C}$-axial solutions, branches of quasiperiodic solutions may bifurcate from the origin as μ passes through zero.

For the Hopf bifurcation problem corresponding to the representation of $D_4 \ltimes T^2$ on $\mathbb{C}^4$ studied by Knobloch and Silber [5], Swift's analysis was used to understand the dynamics in a fixed-point subspace (corresponding to subspace 22 here) where the resulting flow is D_4-symmetric. The dynamics in this subspace could be transformed onto the surface of a sphere, denoted the associated spherical system. Since the dynamics is then two-dimensional, they can be completely understood, and chaotic dynamics is not possible as long as the normal form symmetry is imposed. Swift's numerical results indicated that chaotic dynamics was indeed possible when the normal form symmetry was not imposed. More recent work [13] on the $D_4 \ltimes T^2$-symmetric Hopf bifurcation on $\mathbb{C}^4$ has highlighted the existence of other doubly and triply periodic solution branches that may be created in the bifurcation, and which live outside the D_4-symmetric subspace. However, neither asymptotically stable heteroclinic cycles nor chaotic dynamics have been found.

As table 7 indicates, there are huge numbers of possibilities for heteroclinic orbits in the problem studied here. A complete investigation would be very time consuming, and has not been performed. A few calculations have indicated that it appears unlikely that the number of conditions required to ensure both existence and asymptotic stability are all satisfied simultaneously for one collection of normal form coefficients.

Of more interest is the existence of stable chaotic dynamics in these amplitude equations without needing to consider breaking the normal form symmetry. These dynamics are located within the subspace $\mathcal{S}_4 = \mathrm{Fix}(\rho^2) = \{(z_1, z_2, z_1, z_2, w_1, w_2, w_1, w_2)\} \cong \mathbb{C}^4$ and we discuss this briefly in the remainder of this section. Numerical integrations indicate two things: firstly the existence of chaotic dynamics in the third-order truncation of the normal form when several normal form coefficients are set to zero, and secondly the persistence of the dynamics when these coefficients are small but non-zero, and when the $\mathcal{O}(2(\alpha + \beta) - 1)$ terms are included. The effect of breaking the normal form symmetry has not been investigated.

6.1. Reduction to a six-dimensional set of ODEs in $\mathcal{S}_4$

The third-order truncation of the amplitude equation (30) and its symmetrically related counterparts, restricted to $\mathcal{S}_4$, take the form

$$\dot{z}_1 = z_1[\mu + \mathrm{i}\omega + (\lambda_1 + \lambda_2)|z_1|^2 + 2\lambda_3|z_2|^2 + \lambda_4|w_1|^2] + \alpha_1\bar{z}_1 z_2^2 \tag{79}$$

$$\dot{z}_2 = z_2[\mu + \mathrm{i}\omega + (\lambda_1 + \lambda_2)|z_2|^2 + 2\lambda_3|z_1|^2 + \lambda_4|w_2|^2] + \alpha_1\bar{z}_2 z_1^2 \tag{80}$$

$$\dot{w}_1 = w_1[\mu + \mathrm{i}\omega + (\lambda_1 + \lambda_2)|w_1|^2 + 2\lambda_3|w_2|^2 + \lambda_4|z_1|^2] + \alpha_1\bar{w}_1 w_2^2 \tag{81}$$

$$\dot{w}_2 = w_2[\mu + \mathrm{i}\omega + (\lambda_1 + \lambda_2)|w_2|^2 + 2\lambda_3|w_1|^2 + \lambda_4|z_2|^2] + \alpha_1\bar{w}_2 w_1^2 \tag{82}$$

where we have in addition set $\lambda_5 = \lambda_6 = \lambda_7 = \alpha_2 = \alpha_3 = 0$. Now we transform into modulus and argument variables

$$z_1 = r_1 \mathrm{e}^{\mathrm{i}\theta_{z_1}}, \qquad w_1 = r_3 \mathrm{e}^{\mathrm{i}\theta_{w_1}} \tag{83}$$

$$z_2 = r_2 \mathrm{e}^{\mathrm{i}\theta_{z_2}}, \qquad w_2 = r_4 \mathrm{e}^{\mathrm{i}\theta_{w_2}}. \tag{84}$$

After substituting these into the amplitude equations we note that the phase variables θ_j only occur in the combinations $\phi_1 = 2(\theta_{z_2} - \theta_{z_1})$ and $\phi_2 = 2(\theta_{w_2} - \theta_{w_1})$. The resulting system of six real ODEs is

$$\dot{r}_1 = r_1[\mu + (\lambda_1^r + \lambda_2^r)r_1^2 + (2\lambda_3^r + \alpha_1^r\cos\phi_1 - \alpha_1^i\sin\phi_1)r_2^2 + \lambda_4^r r_3^2] \tag{85}$$

$$\dot{r}_2 = r_2[\mu + (\lambda_1^r + \lambda_2^r)r_2^2 + (2\lambda_3^r + \alpha_1^r\cos\phi_1 + \alpha_1^i\sin\phi_1)r_1^2 + \lambda_4^r r_4^2] \tag{86}$$

$$\dot{\phi}_1 = 2(2\lambda_3^i - \lambda_1^i - \lambda_2^i + 2\alpha_1^i\cos\phi_1)(r_1^2 - r_2^2) - 2\lambda_4^i(r_3^2 - r_4^2) - 2(r_1^2 + r_2^2)\alpha_1^r\sin\phi_1 \tag{87}$$

$$\dot{r}_3 = r_3[\mu + (\lambda_1^r + \lambda_2^r)r_3^2 + (2\lambda_3^r + \alpha_1^r\cos\phi_2 - \alpha_1^i\sin\phi_2)r_4^2 + \lambda_4^r r_1^2] \tag{88}$$

$$\dot{r}_4 = r_4[\mu + (\lambda_1^r + \lambda_2^r)r_4^2 + (2\lambda_3^r + \alpha_1^r\cos\phi_2 + \alpha_1^i\sin\phi_2)r_3^2 + \lambda_4^r r_2^2] \tag{89}$$

$$\dot{\phi}_2 = 2(2\lambda_3^i - \lambda_1^i - \lambda_2^i + 2\alpha_1^i\cos\phi_2)(r_3^2 - r_4^2) - 2\lambda_4^i(r_1^2 - r_2^2) - 2(r_3^2 + r_4^2)\alpha_1^r\sin\phi_2 \tag{90}$$

where superscripts r and i, respectively, denote the real and imaginary parts of a coefficient.

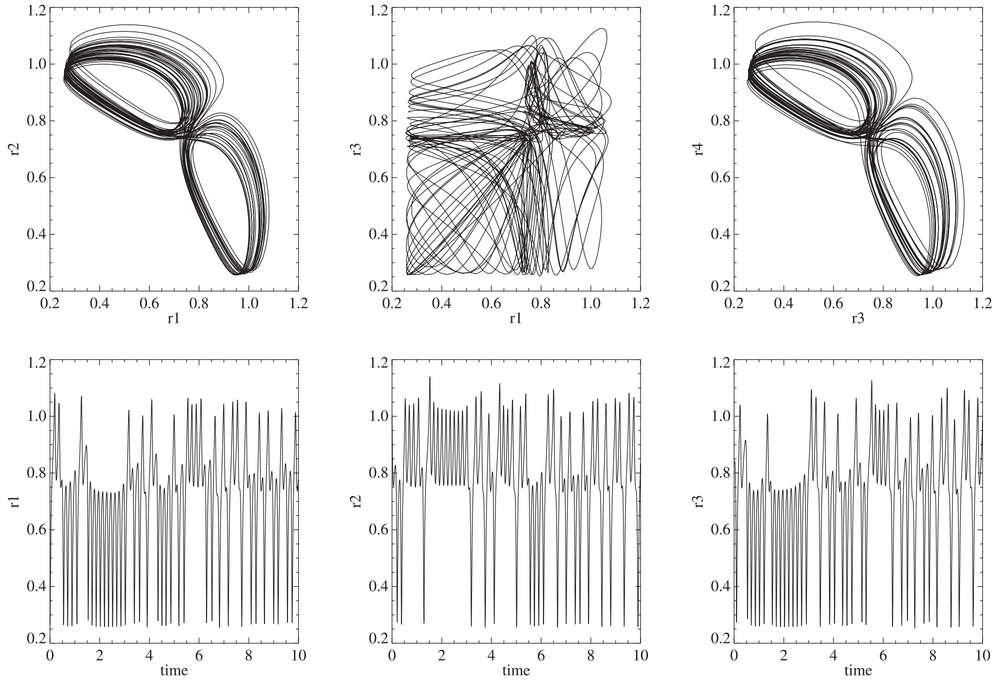

Figure 3. Chaotic dynamics near the diagonal subspace within $\mathcal{S}_4$. The normal form coefficients are $\lambda_1 = -11.70 - 1.54\mathrm{i}$, $\lambda_2 = -0.16 + 1.82\mathrm{i}$, $\lambda_3 = -1.23 - 31.60\mathrm{i}$, $\alpha_1 = -8.99 - 38.29\mathrm{i}$, $\lambda_4 = 1.0$.

6.2. Chaotic dynamics near a global bifurcation

When $\lambda_4 = 0$, equations (85)–(90) decouple into two three-dimensional sets of ODEs. Each of these could be further reduced to a two-dimensional system using the associated spherical system transformation introduced by Swift [8]. Hence the dynamics cannot be chaotic, but, for suitable choices of the coefficients, a periodic orbit can become homoclinic to an equilibrium. As λ_4^r is increased from zero, the (r_1, r_2, ϕ_1) and (r_3, r_4, ϕ_2) systems are coupled together. For many sets of parameter values the diagonal subspace $r_1 = r_3$, $r_2 = r_4$, $\phi_1 = \phi_2$ is transversely attracting, and the dynamics near this global bifurcation again only involves the creation or destruction of periodic orbits (quasiperiodic solutions to (79)–(82)) since it is again equivalent to that of a two-dimensional system. For other parameter values we see dynamics, outside the diagonal subspace, which appears chaotic, as illustrated in figure 3.

Unfortunately, Swift's associated spherical system transformation confers no advantage when applied here to (79)–(82) since the 'radial' directions cannot be eliminated by a time rescaling when λ_4 is non-zero, so the equations within $\mathcal{S}_4$ cannot be reduced from six dimensional to four dimensional.

6.3. A period-doubling cascade

A second unexpected feature of the dynamics of (85)–(90) is the existence of a period-doubling cascade of periodic orbits. This is illustrated in figure 4. When we now fix $\lambda_4^r = -4.0$ and vary λ_4^i in the range $0.1 \leqslant \lambda_4^i \leqslant 1.0$ we find that the diagonal subspace is no longer attracting,

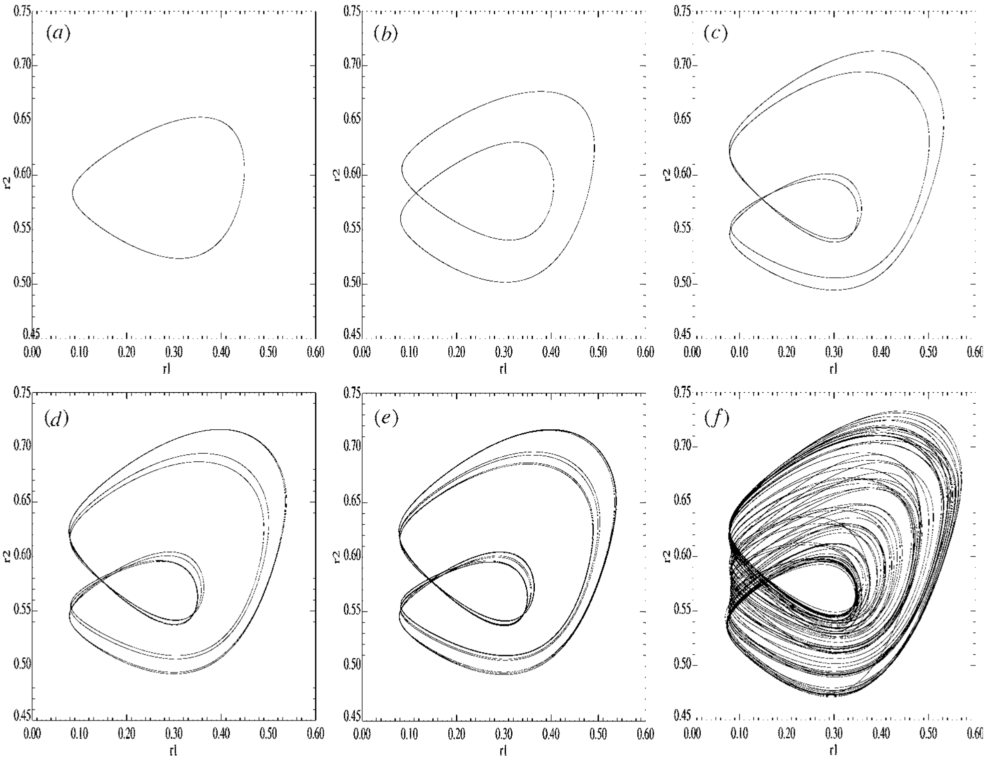

Figure 4. Periodic orbits viewed in the (r_1, r_2)-plane within $\mathcal{S}_4$, in a period-doubling cascade as λ_4^i decreases. (*a*) Period 1 orbit at $\lambda_4^i = 0.7$, (*b*) period 2 orbit at $\lambda_4^i = 0.5$, (*c*) period 4 orbit at $\lambda_4^i = 0.17$, (*d*) period 8 orbit at $\lambda_4^i = 0.157$, (*e*) period 16 orbit at $\lambda_4^i = 0.155$, (*f*) chaotic attractor at $\lambda_4^i = 0.14$. The other normal form coefficients are fixed at $\lambda_1 = -8.63 - 5.45\mathrm{i}$, $\lambda_2 = -5.13 - 11.00\mathrm{i}$, $\lambda_3 = -4.37 - 34.07\mathrm{i}$, $\alpha_1 = -13.04 - 42.08\mathrm{i}$, $\lambda_4^r = -4.0$.

and solutions are attracted to a succession of periodic orbits which undergo period-doubling bifurcations accumulating in the formation of a chaotic attractor.

Further numerical simulations have indicated that both examples of chaotic dynamics are found in the full third-order truncation (i.e. the subspace $\mathcal{S}_4$ is transversely attracting for these combinations of normal form coefficients), and, moreover, that the chaotic dynamics persist when the coefficients λ_5, λ_6, λ_7, α_2 and α_3 are small but non-zero. In addition the chaotic dynamics persist when the higher-order terms (31) for the case $\alpha = 2$, $\beta = 1$ are added in. We conclude that chaotic dynamics are a robust feature of this Hopf bifurcation problem, whereas they have not been observed in the simpler problem studied in [5].

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