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Reducible actions of $D_4 \ltimes T^2$: superlattice patterns and hidden symmetries

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Abstract

We study steady-state pattern-forming instabilities on $\mathbb{R}^2$. A uniform initial state that is invariant under the Euclidean group $E(2)$ of translations, rotations and reflections of the plane loses linear stability to perturbations with a non-zero wavenumber k_c. We identify branches of solutions that are periodic on a square lattice that inherits a *reducible* action of the symmetry group $D_4 \ltimes T^2$. Reducible group actions occur naturally when we consider solutions that are periodic on real-space lattices that are much more widely spaced than the wavelength of the pattern-forming instability. They thus apply directly to computations in large domains where periodic boundary conditions are applied.

The normal form for the bifurcation is calculated, taking the presence of various 'hidden' symmetries into account and making use of previous work by Crawford [8]. We compute the stability (relative to other branches of solutions that exist on this lattice) of the solution branches that we can guarantee by applying the equivariant branching lemma. These computations involve terms higher than third order in the normal form, and are affected by the hidden symmetries. The effects of hidden symmetries that we elucidate are relevant also to bifurcations from fully nonlinear patterns.

In addition, other primary branches of solutions with submaximal symmetry are found always to exist; their existence cannot be deduced by applying the equivariant branching lemma. These branches are stable in open regions of the space of normal form coefficients.

The relevance of these results is illustrated by numerical simulations of a simple pattern-forming PDE.

Mathematics Subject Classification: 37G40

1. Introduction

Steady-state pattern-forming instabilities occur in many branches of physical science; for example, Rayleigh–Bénard convection, directional solidification and reaction–diffusion systems [10]. A common approach to their analysis is to restrict attention to patterns that are spatially periodic, motivated by the observation of spatially periodic patterns in experiments. This is often done implicitly when comparisons are made with numerical simulations employing periodic horizontal boundary conditions. This restriction enables the centre manifold theorem to be applied, and hence a finite-dimensional bifurcation problem can be rigorously derived.

In modelling these physical situations the original uniform state of the system is taken to be of infinite extent and so is invariant under the group $E(2)$ of Euclidean symmetries (reflections, rotations and translations) of the plane $\mathbb{R}^2$. The restriction imposed in considering only patterns with a prescribed spatial periodicity reduces the symmetries of the problem and may also lead to subtle effects in the restricted problem. These effects stem from the original Euclidean symmetry and have been termed 'hidden' symmetries [8, 17].

How these subtle effects affect the resulting bifurcation problem depends on the way in which the spatial periodicity is imposed. Generally, the spatial periodicity is described by a lattice $\mathcal{L} \subset \mathbb{R}^2$ such that all the variables u in the underlying pattern-forming PDEs satisfy $u(\boldsymbol{x}) = u(\boldsymbol{x} + \boldsymbol{\ell})$ where $\boldsymbol{\ell} \in \mathcal{L}$ is a spatial translation. If $\mathcal{L}$ is a square lattice then the restricted problem inherits the symmetry group $\Gamma = D_4 \ltimes T^2$ from the original $E(2)$ symmetric one. The group D_4 is the eight-element symmetry group of a square, and the two-torus of spatial translations modulo the lattice $\mathcal{L}$ is denoted T^2. The group Γ is the semi-direct product of these two groups, and is equal to $N(\mathcal{L})/\mathcal{L}$. Here, $N(\mathcal{L})$ is the normalizer of the lattice group $\mathcal{L}$: $N(\mathcal{L}) = \{\sigma \in E(2) : \sigma^{-1}\mathcal{L}\sigma = \mathcal{L}\} = D_4 \ltimes \mathbb{R}^2$, where $\mathbb{R}^2$ is the group of translations in the plane. Hidden symmetries are elements of $E(2) - N(\mathcal{L})$ and can affect the bifurcation problem in various ways.

Many previous studies of these pattern-forming instabilities (e.g. [13, 14]) choose lattices such that the restricted problem (for $\mathcal{L}$-periodic functions) is equivariant under an irreducible action of Γ. In these cases the hidden symmetries enlarge the symmetry groups of branches of solutions that are guaranteed by the equivariant branching lemma [19]; they do not, however, affect the existence of these branches.

In this paper we study bifurcations on square lattices $\mathcal{L}$ where the associated representation of Γ is *reducible* and we greatly extend the work of Crawford [8]. Crawford [8] showed that hidden rotation and reflection symmetries, which are in $E(2)$ but not in $D_4 \ltimes \mathbb{R}^2$, lead to additional constraints on the normal form. The hidden symmetries are much more significant than in the irreducible case and several features of the analysis are unexpected. For example, we demonstrate the existence (for all combinations of normal form coefficients) of a branch of solutions whose existence cannot be deduced from the equivariant branching lemma. We rely heavily on the results of Crawford; his results show exactly how hidden symmetries affect, for example, the computation of the normal form. A general treatment of the effect of hidden symmetries is beyond the scope of this paper. However, we hope that the analysis of this example will clarify some of the issues that a general theory would have to include.

Much of our notation follows that of Crawford for ease of reference. The structure of the paper is as follows. Section 2 introduces the bifurcation problem and the group action. In section 3 we apply the equivariant branching lemma and deduce the existence of various (axial) branches of solutions. Section 4 contains a derivation of the normal form for the bifurcation in the simplest case. This is then used to compute the stability of the axial branches in section 5. The existence and stability of various non-axial branches is considered in section 6. Section 7 illustrates the theory with a discussion of numerical solutions of a particular pattern-forming PDE. We conclude with a discussion in section 8.

2. Geometry of the lattice and action of $D_4 \ltimes T^2$

Consider a planar Euclidean-equivariant nonlinear PDE

$$\frac{\partial u}{\partial t} = F(u, \mu), \tag{1}$$

where $u(\boldsymbol{x}, t) \in \mathbb{R}^n$, $\boldsymbol{x} = (x, y) \in \mathbb{R}^2$, F is a smooth nonlinear operator acting on a suitable function space and μ is a real (bifurcation) parameter. We assume that for all μ there is a trivial solution $u = 0$ to (1), and that this solution loses stability in a steady-state bifurcation to perturbations with a non-zero wavenumber k_c when $\mu = 0$. Melbourne [24] has proved that it is possible to reduce a planar Euclidean-equivariant PDE (1) to a single PDE (i.e. the case $n = 1$). The resulting single PDE transforms under either the *scalar* or *pseudoscalar* action of $E(2)$. In this paper we consider only the scalar action; this action is appropriate (for example) to reaction–diffusion equations and (after the reduction has been performed) to the Boussinesq equations for thermal convection. In the scalar case the action of $E(2)$ on the single range variable u is trivial. $E(2)$-equivariant bifurcations in the pseudoscalar case are discussed in [2]. In the light of this discussion we set $n = 1$ to simplify the notation in what follows.

Euclidean equivariance implies that the trivial solution loses stability at $\mu = 0$ to plane waves $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ as long as $|\boldsymbol{k}| = k_c$: all horizontal directions are equivalent. This uncountable multiplicity presents a serious difficulty, which we resolve in the familiar way: to obtain rigorous results on the existence of solutions we consider only the subset of solutions that are periodic with respect to a square lattice. It is now possible to apply the centre manifold theorem [4] and hence obtain a finite-dimensional bifurcation problem whose branches of solutions correspond to solutions of the original PDEs. In addition, the symmetry group of the problem is now $D_4 \ltimes T^2$. We define the square lattice $\mathcal{L}$ and the corresponding dual lattice $\mathcal{L}^*$ to be

$$\mathcal{L} = \left\{ n\boldsymbol{\ell}_1 + m\boldsymbol{\ell}_2 : (n, m) \in \mathbb{Z}^2,\ \boldsymbol{\ell}_1 = \left(\frac{2\pi s}{k_c}, 0\right),\ \boldsymbol{\ell}_2 = \left(0, \frac{2\pi s}{k_c}\right) \right\}, \tag{2}$$

$$\mathcal{L}^* = \left\{ n\boldsymbol{q}_1 + m\boldsymbol{q}_2 : (n, m) \in \mathbb{Z}^2,\ \boldsymbol{q}_1 = \left(\frac{k_c}{s}, 0\right),\ \boldsymbol{q}_2 = \left(0, \frac{k_c}{s}\right) \right\}. \tag{3}$$

which ensures that $\boldsymbol{\ell}_i \cdot \boldsymbol{q}_j = 2\pi\delta_{ij}$. The real parameter s selects the spacing of the lattice relative to the critical wavelength of the instability. Having defined $\mathcal{L}$, we let $\Omega \equiv \mathbb{R}^2/\mathcal{L} = [0, (2\pi s/k_c)]^2$ be the fundamental domain (with periodic boundary conditions) on which these $\mathcal{L}$-periodic functions are defined. We can now distinguish between the infinite-dimensional space $\mathbb{E}^c(\mathbb{R}^2)$, which is the linear centre eigenspace of $DF(0, 0)$ (see (1)) and the finite-dimensional space $\mathbb{E}^c(\Omega)$, which is the centre eigenspace of the space of $\mathcal{L}$-periodic functions. The centre eigenspace $\mathbb{E}^c(\Omega)$ contains only those functions that are linear combinations of the finite number of plane waves $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ satisfying $|\boldsymbol{k}| = k_c$ and $\boldsymbol{k} \in \mathcal{L}^*$.

Different choices of s lead to different numbers of intersections between the critical circle $|\boldsymbol{k}| = k_c$ and the dual lattice $\mathcal{L}^*$, and hence to different representations of the symmetry group $D_4 \ltimes T^2$ acting on the set of mode amplitudes. A representation of a compact group Γ on $\mathbb{R}^n$ is a homomorphism $\Gamma \mapsto GL(n)$, which respects the group structure, i.e. $R_{\gamma_1\circ\gamma_2} = R_{\gamma_1}R_{\gamma_2}$, and which associates an $n \times n$ matrix with each element $\gamma \in \Gamma$: $\gamma \mapsto R_\gamma$. The representation is defined to be *irreducible* if there are no nontrivial proper Γ-invariant subspaces of $\mathbb{R}^n$; it is said to be *absolutely irreducible* if any matrix commuting with all matrices R_γ is forced to be a real multiple of the identity matrix. It is easy to check that any absolutely irreducible representation is irreducible [19].

Setting $s = 1$ leads to only four intersection points between the critical circle and the dual lattice $\mathcal{L}^*$ in Fourier space. This leads to the 'fundamental' representation of $D_4 \ltimes T^2$ on the two mode amplitudes $\{z_1, z_2\}$; perturbations to the original solution $u = 0$ for (1) are in the form $u = \mathrm{Re}[z_1 \mathrm{e}^{\mathrm{i}k_c x} + z_2 \mathrm{e}^{\mathrm{i}k_c y}]$. The resulting representation of $D_4 \ltimes T^2$ on $\mathbb{C}^2 \cong \{(z_1, z_2)\}$ is irreducible. In this case there are two axial branches of solutions, of the form $(z_1, z_2) = (z, 0)$ (usually called rolls) and $(z_1, z_2) = (z, z)$ (usually called squares). If both of these bifurcate supercritically, exactly one is stable for a generic set of normal form coefficients. The dynamics of the bifurcation are equivalent to those of the Hopf bifurcation with $O(2)$ symmetry, and have been extensively studied [9, 18].

By fixing a dual lattice $\mathcal{L}^*$ which intersects $|\boldsymbol{k}| = k_c$ in exactly eight points (for example $s = \sqrt{5}$) we obtain the higher-dimensional irreducible representations of $D_4 \ltimes T^2$ studied by Dionne and others [13, 14]. With these choices of lattice Dionne *et al* have proved the existence of a countable infinity of superlattice patterns, named super-squares and anti-squares, and have tested the stability of the simplest roll and square patterns to a wider class of perturbations. All these patterns bifurcate simultaneously from the trivial state $u = 0$ as μ passes through zero.

It is clear, though, that there are further choices for s, which may lead to the existence of more patterns. In this paper we analyse reducible representations of $D_4 \ltimes T^2$ on $\mathbb{E}^c(\Omega)$ by considering dual lattices that intersect the critical circle $|\boldsymbol{k}| = k_c$ at twelve points. The easiest way to organize this is to introduce Pythagorean triples of integers P, Q, R such that $0 < P, Q < R$, $P^2 + Q^2 = R^2$ and P, Q and R have no common factor. It is well known (see, e.g., [1]) that every such triple is generated by a pair of coprime integers $a > b > 0$ that are not both odd, by setting

$$P = a^2 - b^2, \qquad Q = 2ab, \qquad R = a^2 + b^2. \tag{4}$$

In the remainder of the paper we concentrate on the case $s = R$, and in addition, for convenience, we rescale lengths in the PDE (1) so that $k_c = R$. The relation $P^2 + Q^2 = R^2$ implies the existence of 12 intersection points: a set of four at $\pm\boldsymbol{k}_{1,2}$ where $\boldsymbol{k}_1 = (k_c, 0)$ and $\boldsymbol{k}_2 = (0, k_c)$ and a set of eight at $\pm\boldsymbol{k}_{3,4,5,6}$ where $\boldsymbol{k}_3 = (Q, P)$, $\boldsymbol{k}_4 = (Q, -P)$, $\boldsymbol{k}_5 = (P, Q)$ and $\boldsymbol{k}_6 = (P, -Q)$. The planform for perturbations to the uniform state $u = 0$ now takes the form

$$u = \mathrm{Re}\big[z_1 \mathrm{e}^{\mathrm{i}Rx} + z_2 \mathrm{e}^{\mathrm{i}Ry} + w_1 \mathrm{e}^{\mathrm{i}(Qx+Py)} + w_2 \mathrm{e}^{\mathrm{i}(Qx-Py)} + w_3 \mathrm{e}^{\mathrm{i}(Px+Qy)} + w_4 \mathrm{e}^{\mathrm{i}(Px-Qy)}\big] \tag{5}$$

and this is illustrated in figure 1 for the natural first case $a = 2$, $b = 1$ (and hence $P = 3$, $Q = 4$ and $R = 5$). The space of such $\mathcal{L}$-periodic perturbations is (isomorphic to) the centre manifold $\mathbb{E}^c(\Omega) = \{(z_1, z_2, w_1, w_2, w_3, w_4)\} \cong \mathbb{C}^6$ on which the action of $D_4 \ltimes T^2$ is generated by

$$\begin{aligned}
m_x\colon \quad & (x, y) \to (x, -y), \\
& (\boldsymbol{z}, \boldsymbol{w}) \to (z_1, \bar{z}_2, w_2, w_1, w_4, w_3), && (6)\\
m_d\colon \quad & (x, y) \to (y, x), \\
& (\boldsymbol{z}, \boldsymbol{w}) \to (z_2, z_1, w_3, \bar{w}_4, w_1, \bar{w}_2), && (7)\\
[\xi, \eta]\colon \quad & (x, y) \to (x + \xi, y + \eta), \\
& (\boldsymbol{z}, \boldsymbol{w}) \to \big(z_1 \mathrm{e}^{\mathrm{i}R\xi}, z_2 \mathrm{e}^{\mathrm{i}R\eta}, w_1 \mathrm{e}^{\mathrm{i}(Q\xi+P\eta)}, w_2 \mathrm{e}^{\mathrm{i}(Q\xi-P\eta)}, w_3 \mathrm{e}^{\mathrm{i}(P\xi+Q\eta)}, w_4 \mathrm{e}^{\mathrm{i}(P\xi-Q\eta)}\big), && (8)
\end{aligned}$$

where the reflections m_x and m_d generate the group of symmetries of a square D_4, the translations $[\xi, \eta]$ generate the group T^2 and $(\boldsymbol{z}, \boldsymbol{w}) \equiv (z_1, z_2, w_1, w_2, w_3, w_4)$. Since we are working with this specific family of representations, identified by P, Q, R, we will drop the distinction between the group elements γ and the matrices R_γ that form the group

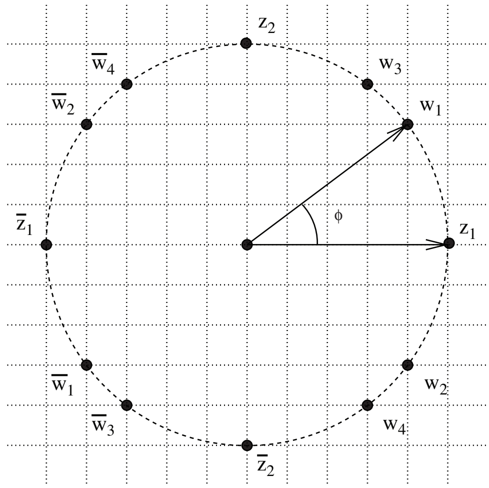

Figure 1. Sketch of the dual lattice for the case $a = 2$, $b = 1$ (hence $P = 3$, $Q = 4$ and $s = R = 5$). The resulting representation of $D_4 \ltimes T^2$ on $\mathbb{E}^c(\Omega)$ is reducible. The angle between the wave-vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_3$ for the z_1 and w_1 modes is $\phi = \tan^{-1}(P/Q) = \tan^{-1}(3/4)$.

representation. We denote the other reflection elements of D_4 by $m_y = m_d \circ m_x \circ m_d : (x, y) \to (-x, y)$ and $m_{d'} = m_x \circ m_d \circ m_x : (x, y) \to (-y, -x)$. A rotation of 90° anticlockwise is denoted by $\rho = m_d \circ m_x$ (as is conventional, we use left-composition of group elements).

This action of $D_4 \ltimes T^2$ on the centre manifold is reducible: the group action decomposes $\mathbb{E}^c(\Omega)$ into a direct sum of vector spaces $V_1 \oplus V_2$ where $V_1 = \{(z_1, z_2, 0, 0, 0, 0)\} \cong \mathbb{C}^2$ and $V_2 = \{(0, 0, w_1, w_2, w_3, w_4)\} \cong \mathbb{C}^4$ and $D_4 \ltimes T^2$ acts separately on V_1 and V_2. However, the original action of the Euclidean group $E(2)$ on the centre manifold $\mathbb{E}^c(\mathbb{R}^2)$ is irreducible, and this leads to the existence of 'hidden' rotation symmetries that link the dynamics on V_1 and V_2. Hidden symmetries were first discussed in [17], in a slightly different context. In the specific setting relevant to this paper they were discussed in detail by Crawford [8] and we rely on Crawford's results in what follows. Hidden symmetries arise here by requiring the normal form f for the dynamics on $\mathbb{E}^c(\Omega)$ to be the restriction to $\mathbb{E}^c(\Omega)$ of an $E(2)$-equivariant vector field $\mathcal{F}$ defined on $\mathbb{E}^c(\mathbb{R}^2)$. The rotation R_ϕ anticlockwise through an angle $\phi = \tan^{-1} P/Q$ is an example of a hidden symmetry. The hidden symmetries are crucial to the derivation of sensible amplitude equations, even at the linear level.

We remark that in the terminology of Crawford [8] this is an example of a 'binary mode interaction', and was named the '[4, 8] mode interaction' by Crawford [8]. In that paper Crawford analyses the effect of hidden symmetries on the normal form in the three possible cases where the action of $E(2)$ on $\mathbb{E}^c(\mathbb{R}^2)$ is irreducible but $\mathbb{E}^c(\Omega)$ is a direct sum of exactly two subspaces V_1 and V_2 upon each of which there is an action of $D_4 \ltimes T^2$. In the '[4, 8]' case the subspaces V_j are isomorphic to $\mathbb{C}^2$ and $\mathbb{C}^4$, respectively, and the action of $D_4 \ltimes T^2$ on the larger subspace is translation-free. An action of $D_4 \ltimes T^2$ is said to be translation-free if the only element of T^2 that acts as the identity is the identity element itself.

3. Axial branches

As in bifurcation problems with irreducible representations of $D_4 \ltimes T^2$, we expect to be able to apply the equivariant branching lemma to deduce the existence of solution branches in the form of rolls, squares, rhombs, and possibly various superlattice patterns. We distinguish between branches that bifurcate from the origin at $\mu = 0$, called *primary branches*, and the subset of these, which we call *axial branches*, whose existence can be deduced by a direct application of the equivariant branching lemma. In fact we will distinguish in what follows between two kinds of primary but non-axial branches: those that are forced to exist for all combinations of normal form coefficients, and those that exist only in certain open regions of coefficient space. Similar distinctions have been made by many authors, for example, see definition 3.5.3 in the paper by Field and Richardson [16].

In this section we discuss axial branches, and defer further comments on non-axial primary branches to section 6. We use the notation $z_j = x_j + \mathrm{i}y_j$ and $w_j = u_j + \mathrm{i}v_j$ to write the complex mode amplitudes in real and imaginary parts and recall the following two definitions. The isotropy subgroup $\Sigma_z \subset \Gamma$ of a point $z \in \mathbb{C}^n$ is defined to be

$$\Sigma_z = \{\sigma \in \Gamma : \sigma z = z\}. \tag{9}$$

We use the notation $\langle g_1, \ldots, g_n \rangle$ to denote the group generated by the elements $g_1, \ldots, g_n$. The fixed point subspace of a subgroup $\Sigma \subset \Gamma$ is denoted $\mathrm{Fix}(\Sigma)$:

$$\mathrm{Fix}(\Sigma) = \{z \in \mathbb{C}^n : \sigma z = z \; \forall \, \sigma \in \Sigma\}. \tag{10}$$

This is always a linear subspace of $\mathbb{C}^n$.

As noted in section 2, hidden symmetries play an important role in the analysis that follows. Of particular importance are the largest subspaces on which the hidden rotations R_ϕ (defined to be an anticlockwise rotation through an angle $\tan^{-1} P/Q$) and R_ϕ^{-1} act; we denote these by $\mathcal{U}_1$ and $\mathcal{U}_2$:

$$\begin{aligned}
\mathcal{U}_1 &= \mathbb{E}^c(\Omega) \cap [R_\phi \mathbb{E}^c(\Omega)] \\
&= \{(z_1, z_2, w_1, 0, 0, w_4)\} = \mathrm{Fix}\left(\left[\frac{2\pi Q}{R}, \frac{2\pi P}{R}\right]\right),
\end{aligned} \tag{11}$$

$$\begin{aligned}
\mathcal{U}_2 &= \mathbb{E}^c(\Omega) \cap [R_\phi^{-1} \mathbb{E}^c(\Omega)] \\
&= \{(z_1, z_2, 0, w_2, w_3, 0)\} = \mathrm{Fix}\left(\left[\frac{2\pi P}{R}, \frac{2\pi Q}{R}\right]\right).
\end{aligned} \tag{12}$$

It is worth noting that the notation $R_\phi \mathbb{E}^c(\Omega)$ is a slight abuse since the action of R_ϕ is not defined on every mode amplitude z_j or w_j but rather refers to rotating the corresponding wave-vector $\boldsymbol{k}_j$ through an angle ϕ. Likewise, the expression $\mathbb{E}^c(\Omega) \cap [R_\phi \mathbb{E}^c(\Omega)]$ refers to coincidence of points in the dual lattice $\mathcal{L}^*$ and points in the rotated version of $\mathcal{L}^*$, restricted to the centre subspace $\mathbb{E}^c(\mathbb{R}^2)$ in both cases.

Given this interpretation, clearly $R_\phi : \mathcal{U}_2 \to \mathcal{U}_1$. From the action of the translations (8) we see that these are fixed-point subspaces of these translation elements for all choices of the Pythagorean triple (P, Q, R) as long as

$$\mathrm{hcf}(Q^2 - P^2, R) = \mathrm{hcf}(2PQ, R) = 1. \tag{13}$$

We use the notation $\mathrm{hcf}(p, q)$ to denote the highest common factor (greatest common divisor) of the integers p and q. Conditions (13) ensure that the translation $[(2\pi Q/R), (2\pi P/R)]$ does not fix either w_2 or w_3 and that the translation $[(2\pi P/R), (2\pi Q/R)]$ does not fix either w_1 or w_4. That the conditions (13) hold for all Pythagorean triples (P, Q, R) is not immediately obvious, so for completeness we include the proof below.

Proof. We have that $P = a^2 - b^2$, $Q = 2ab$ and $R = a^2 + b^2$. In fact, as remarked above, any pair (a, b) that are coprime (i.e. $\mathrm{hcf}(a, b) = 1$) and not both odd will generate a Pythagorean triple where $\mathrm{hcf}(P, Q) = 1$, P is odd and Q is even. So we can use the pair (Q, P) themselves to generate a triple, thinking of them as a new pair (a, b). The pair (Q, P) generates the triple $Q^2 - P^2$, $2QP$, $Q^2 + P^2$: these must therefore be (pairwise) coprime since (Q, P) are coprime and not both odd. Hence $\mathrm{hcf}(Q^2 - P^2, Q^2 + P^2) = 1 = \mathrm{hcf}(Q^2 - P^2, R^2)$, and so $\mathrm{hcf}(Q^2 - P^2, R) = 1$ since R and R^2 have identical factors. $\square$

Since $\mathcal{U}_1$ and $\mathcal{U}_2$ are fixed-point subspaces they are flow-invariant. The hidden symmetries $m_x \circ R_\phi^{-1}$ and $m_d \circ R_\phi$ act on $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively:

$$\begin{aligned} m_x \circ R_\phi^{-1}&: (z_1, z_2, w_1, 0, 0, w_4) \to (w_1, w_4, z_1, 0, 0, z_2), \\ m_d \circ R_\phi&: (z_1, z_2, 0, w_2, w_3, 0) \;\;\to (w_3, w_2, 0, z_2, z_1, 0). \end{aligned}$$

Before we list the axial branches in turn, we state the key result in determining the existence of bifurcating branches in steady-state bifurcations: the equivariant branching lemma of [31, 6].

Theorem (The equivariant branching lemma). *Let Γ be a compact Lie group acting on $\mathbb{R}^n$ with* $\mathrm{Fix}(\Gamma) = \{0\}$ *and let $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu)$ be a Γ-equivariant smooth bifurcation problem with $\boldsymbol{f}(\boldsymbol{0}, 0) = 0$ and $D\boldsymbol{f}|_{(\boldsymbol{0},0)} = 0$. Then, for every isotropy subgroup Σ satisfying* $\dim \mathrm{Fix}(\Sigma) = 1$ *there is a unique solution branch $(\boldsymbol{x}(\mu), \mu)$ as long as $D\boldsymbol{f}_\mu|_{(\boldsymbol{0},0)}(\boldsymbol{v_0}) \neq 0$ for non-zero $\boldsymbol{v_0} \in$* $\mathrm{Fix}(\Sigma)$.

We have denoted $\partial \boldsymbol{f}/\partial\mu$ by $\boldsymbol{f}_\mu$, so the condition $D\boldsymbol{f}_\mu|_{(\boldsymbol{0},0)}(\boldsymbol{v_0}) \neq 0$ is the usual non-degeneracy condition, requiring that the eigenvalue corresponding to the eigenvector $\boldsymbol{v_0}$ passes through the origin with non-zero speed as μ is varied. As stated, the theorem does not require the representation of Γ to be absolutely irreducible (as is often done), or even irreducible. Since we are dealing in this paper with reducible representations, it is in the above form that we need the theorem. The 'usual' form of the theorem follows easily from this one. The relationship between the two is discussed in [19, pp 80–4], and a proof of the above, stronger, version is also given there. The usefulness of the theorem comes from the way it turns the analytic problem of the existence of solutions into a (usually easier) group-theoretic one (the calculation of isotropy subgroups of Γ and their fixed point subspaces).

3.1. Construction of finer lattices

The computations of axial branches that we present later in this section show that the isotropy subgroups of many of these branches contain nontrivial translations. This implies that these solutions are periodic on finer real-space lattices than $\mathcal{L}$. In this section we define the finer lattice $\mathcal{L}_1$ that is of particular interest as it corresponds to the subspace $\mathcal{U}_1$. The construction of a finer lattice corresponding to $\mathcal{U}_2$ is very similar and is omitted.

We recall that the dual lattice $\mathcal{L}^*$ is generated by the wave-vectors $\boldsymbol{q}_1 = (k_c/s, 0)$ and $\boldsymbol{q}_2 = (0, k_c/s)$, and that $s = R = a^2 + b^2$. Consider the wave-vectors

$$\boldsymbol{e}_1 = \left(\frac{ak_c}{R}, -\frac{bk_c}{R}\right) \qquad \text{and} \qquad \boldsymbol{e}_2 = \left(\frac{bk_c}{R}, \frac{ak_c}{R}\right),$$

which generate the (dual) lattice we denote by $\mathcal{L}_1^* = \{n\boldsymbol{e}_1 + m\boldsymbol{e}_2 : (n, m) \in \mathbb{Z}^2\}$. The corresponding new real-space lattice is

$$\mathcal{L}_1 = \left\{ n\boldsymbol{v}_1 + m\boldsymbol{v}_2 : (n, m) \in \mathbb{Z}^2,\ \boldsymbol{v}_1 = \left(\frac{2\pi a}{k_c}, -\frac{2\pi b}{k_c}\right),\ \boldsymbol{v}_2 = \left(\frac{2\pi b}{k_c}, \frac{2\pi a}{k_c}\right) \right\}. \tag{14}$$

It is easy to check that $a\boldsymbol{v}_1 + b\boldsymbol{v}_2 = \boldsymbol{\ell}_1$ and $a\boldsymbol{v}_2 - b\boldsymbol{v}_1 = \boldsymbol{\ell}_2$ and so any $\mathcal{L}_1$-periodic function is also $\mathcal{L}$-periodic: $\mathcal{L}_1$-periodicity requires $u(\boldsymbol{x} + n\boldsymbol{v}_1 + m\boldsymbol{v}_2) = u(\boldsymbol{x})$ for all $(n, m) \in \mathbb{Z}^2$, and the

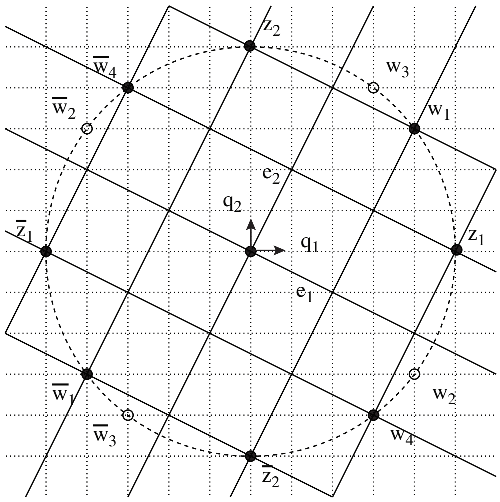

Figure 2. The original dual lattice $\mathcal{L}^*$ generated by the wave-vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ (- - - -) and the new lattice $\mathcal{L}_1^*$ (——) generated by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, which generates patterns that are periodic on a finer real-space lattice.

choices $(n, m) = (a, b)$ and $(n, m) = (-b, a)$ imply $u(\boldsymbol{x} + \boldsymbol{\ell}_1) = u(\boldsymbol{x})$ and $u(\boldsymbol{x} + \boldsymbol{\ell}_2) = u(\boldsymbol{x})$ and hence $u(\boldsymbol{x})$ is $\mathcal{L}$-periodic.

Figure 2 shows the dual lattice $\mathcal{L}_1^*$ and its relation with the original dual lattice $\mathcal{L}^*$. The bifurcation problem on the lattice $\mathcal{L}_1$ is well defined, and inherits an irreducible action of $N(\mathcal{L}_1)/\mathcal{L}_1 = \hat{D}_4 \ltimes \hat{T}^2$ from the original $E(2)$ symmetries of the problem; this action differs from the action, defined by (6)–(8), of $D_4 \ltimes T^2$ on $\mathrm{Fix}(\mathcal{L}) = \mathbb{E}^c(\Omega)$. However, the action of $\hat{D}_4 \ltimes \hat{T}^2$ on $\mathrm{Fix}(\mathcal{L}_1) = \mathcal{U}_1$ is isomorphic to that generated by the rotation ρ, the reflection $m_x \circ R_\phi^{-1}$ (which is a 'proper' symmetry acting on $\mathcal{U}_1$) and the translations $[\xi, \eta]$ defined by (8) when restricted to $\mathcal{U}_1$ and taken modulo $[2\pi Q/R, 2\pi P/R]$, as this latter translation acts trivially on $\mathcal{U}_1$.

Within $\mathcal{U}_1$ the original bifurcation problem reduces to that studied by Dionne *et al* [14]. In particular, the axial branches within $\mathcal{U}_1$ must be exactly those determined in [14]. Dionne *et al* determined the existence of (group-orbits of) six axial branches: rolls, simple squares (here referred to simply as squares), super-squares, anti-squares and two kinds of rhombs. From table 1 these correspond to the branches labelled R1, S1, SS, AS2, Rh3 and Rh4. Within $\mathcal{U}_1$ the two roll branches R1 and R2 and the two square branches S1 and S2 lie on the same group-orbit and hence have conjugate isotropy subgroups. We distinguish them in the complete bifurcation problem on $\mathbb{E}^c(\Omega)$ because the conditions for their stability differ.

In summary, each axial branch has an associated isotropy subgroup $\Sigma \subset D_4 \ltimes T^2$ computed using the action (6)–(8) on $\mathbb{E}^c(\Omega)$. When, in addition, a branch has a hidden symmetry we define a second group, denoted by $\mathcal{S}$. $\mathcal{S}$ is the isotropy subgroup of the branch with respect to the action of $\hat{D}_4 \ltimes \hat{T}^2$ on $\mathcal{U}_1$. We call $\mathcal{S}$ the 'hidden symmetry group' of the branch since it contains symmetries that do not act on $\mathbb{E}^c(\Omega)$. Although the actions of $D_4 \ltimes T^2$ and $\hat{D}_4 \ltimes \hat{T}^2$ are distinct, $\mathcal{S}$ and Σ may have elements that act in an identical fashion on $\mathcal{U}_1$.

Dionne *et al* also came across hidden symmetries in their analysis; these affect the isotropy subgroups of rolls and squares within $\mathcal{U}_1$ (see [14, section 3.2]). They found that hidden symmetries for these branches did not affect their stability calculations, and so were not forced to examine those hidden symmetries in more detail. We remark briefly that the construction given above can be extended to derive the ultra-fine lattice on which these roll and square branches exist; we denote the subspace of $\mathcal{U}_1$ corresponding to this lattice by $\mathcal{U}_{R2}$:

$$\mathcal{U}_{R2} = \mathcal{U}_1 \cap R_\phi \mathcal{U}_1 = \mathbb{E}^c(\Omega) \cap R_\phi \mathbb{E}^c(\Omega) \cap R_\phi^2 \mathbb{E}^c(\Omega) = \{(0, 0, w_1, 0, 0, w_4)\}$$

In this way we are led to consider hidden symmetry subgroups $\mathcal{S}$ on $\mathcal{U}_1$ and $\tilde{\mathcal{S}}$ on $\mathcal{U}_{R2}$ for the R2 roll branch and the S2 square branch. Further details are given in the relevant subsections below.

In section 5 we will discuss the relevance of the pair of groups $(\Sigma, \mathcal{S})$ to stability calculations. The remainder of this section is devoted to a detailed discussion of each axial branch in turn; these calculations are summarized in table 1 and figure 3.

3.2. Rolls

The simplest solutions that exist are roll solutions of the form $(x, 0, 0, 0, 0, 0)$ and $(0, 0, u, 0, 0, 0)$. These have the following isotropy subgroups and fixed-point subspaces

$$\begin{aligned} \Sigma_{R1} &= \left\langle [0, \eta], \left[\frac{2\pi}{R}, 0\right], m_x, m_y \right\rangle \cong O(2) \times D_R, \\ \mathrm{Fix}(\Sigma_{R1}) &= (x, 0, 0, 0, 0, 0) \end{aligned} \tag{15}$$

and

$$\begin{aligned} \Sigma_{R2} &= \left\langle \left[\xi, -\frac{Q}{P}\xi\right], \left[\frac{2\pi Q}{R^2}, \frac{2\pi P}{R^2}\right], \rho^2 \right\rangle \cong \mathbb{Z}_2[\rho^2] \ltimes \big(SO(2) \times \mathbb{Z}_{R^2}\big), \\ \mathrm{Fix}(\Sigma_{R2}) &= (0, 0, u, 0, 0, 0). \end{aligned} \tag{16}$$

Each of these solutions, in common with all the branches we will consider later, is associated with a complete group-orbit of solutions related by symmetry to the particular representative given. We will consider two solutions to be equivalent if they lie on the same $D_4 \ltimes T^2$ orbit, that is, if they can be transformed into each other by elements of $D_4 \ltimes T^2$. However, this means we must consider R1 and R2 as distinct solutions since they cannot be so related and are part of different $D_4 \ltimes T^2$ group-orbits. Indeed, solutions on the same group-orbit must have conjugate isotropy subgroups and clearly R1 and R2 have non-isomorphic isotropy subgroups. At first sight this looks very odd; for example, Σ_{R2} contains no reflections. We would expect that these discrepancies could be resolved by computing the hidden symmetry subgroup within $\mathcal{U}_1$. Following the analysis of Dionne *et al* [14, table 3 and section 3.2] we are led also to consider the subspace $\mathcal{U}_{R2} = \{(0, 0, w_1, 0, 0, w_4)\}$, as defined above. In fact, the hidden reflection symmetry $R_\phi^2 \circ m_x$ of R2 does not act on $\mathcal{U}_1$, but only on $\mathcal{U}_{R2}$, and $\mathcal{U}_{R2}$ is the largest subspace on which $R_\phi^2 \circ m_x$ acts. For this solution branch we can form distinct hidden symmetry subgroups $\mathcal{S}_{R2}$ acting on $\mathcal{U}_1$ and $\tilde{\mathcal{S}}_{R2}$ acting on $\mathcal{U}_{R2}$:

$$\begin{aligned} \mathcal{S}_{R2} &= \left\langle \rho^2, \left[\xi, -\frac{Q}{P}\xi\right], \left[\frac{2\pi Q}{R^2}, \frac{2\pi P}{R^2}\right] \right\rangle \cong \mathbb{Z}_2[\rho^2] \ltimes \big(SO(2) \times \mathbb{Z}_R\big), \\ \tilde{\mathcal{S}}_{R2} &= \left\langle \rho^2, \left[\xi, -\frac{Q}{P}\xi\right], R_\phi^2 \circ m_x \right\rangle \cong O(2) \times \mathbb{Z}_2[\rho^2 \circ R_\phi^2 \circ m_x]. \end{aligned}$$

These subgroups agree with those computed by Dionne *et al*. These authors found that consideration of $\tilde{\mathcal{S}}_{R2}$ in addition to $\mathcal{S}_{R2}$ did not affect their normal form or stability calculations. The translation $[2\pi Q/R^2, 2\pi P/R^2]$ has order R when restricted to acting on the subspace $\mathcal{U}_1$. On $\mathcal{U}_{R2}$ it acts trivially and the hidden symmetry $R_\phi^2 \circ m_x$ combines with the translation $[\xi, -(Q/P)\xi]$ to form the group $O(2)$. The hidden symmetry group $\tilde{\mathcal{S}}_{R2}$ is the group we expect, and would have derived, for rolls if we had imposed a lattice with the spatial period equal to the wavelength of the instability, corresponding to the choice $s = 1$.

3.3. Squares

From the analysis of simpler problems we also expect branches of simple square solutions of the form $(x, x, 0, 0, 0, 0)$ and $(0, 0, u, 0, 0, u)$ to exist. The corresponding isotropy subgroups are given in table 1. Exactly as for the R2 branch, we find that for complete accuracy we need to define a pair of hidden symmetry subgroups $\mathcal{S}$ and $\tilde{\mathcal{S}}$ for S2 as well. As for R2, the hidden symmetry subgroup $\tilde{\mathcal{S}}_{S2}$ does not affect our later computations, or those of Dionne *et al*. The S2 branch has a hidden symmetry $R_\phi^2 \circ m_x$ that acts on the subspace $\mathcal{U}_{R2}$. S2 has the hidden symmetry group $\mathcal{S}_{S2}$ on $\mathcal{U}_1$ and $\tilde{\mathcal{S}}_{S2}$ on $\mathcal{U}_{R2}$:

$$\mathcal{S}_{S2} = \left\langle \rho, \left[\frac{2\pi Q}{R^2}, \frac{2\pi P}{R^2}\right]\right\rangle \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_R,$$
$$\tilde{\mathcal{S}}_{S2} = \langle \rho, R_\phi^2 \circ m_x \rangle \cong D_4,$$

since the translation $[(2\pi Q/R^2), (2\pi P/R^2)]$ has order R when restricted to $\mathcal{U}_1$ and acts trivially on $\mathcal{U}_{R2}$.

3.4. Rhombs

There are four distinct branches of rhombic patterns in all. The first two of these have isotropy subgroups $\Sigma \subset D_4 \ltimes T^2$ that have one-dimensional fixed-point subspaces and hence the equivariant branching lemma may be directly applied in order to deduce the existence of the axial branches. The other two branches exist within subspaces on which hidden symmetries act and the presence of the hidden symmetries is crucial to the existence of the solution branch. The first two solution branches Rh1 and Rh2 have no hidden symmetries and are of the form $(0, 0, u, u, 0, 0)$ and $(0, 0, u, 0, u, 0)$. We would naturally expect two further branches of rhombs to exist, of the form $(x, 0, x, 0, 0, 0)$ and $(x, 0, 0, 0, x, 0)$. But it is not possible to construct isotropy subgroups of $D_4 \ltimes T^2$ that have these fixed point subspaces. However, these axial branches do exist due to the action of hidden symmetries. For example, consider the subspace

$$\mathrm{Fix}\left(\left\langle\left[\frac{2\pi}{R}, -\frac{2\pi Q}{PR}\right], \rho^2\right\rangle\right) = (x, 0, u, 0, 0, 0) \equiv \mathcal{U}_3 \subset \mathcal{U}_1,$$

on which the hidden symmetry $m_x \circ R_\phi^{-1}$ acts. In fact, $m_x \circ R_\phi^{-1}$ fixes the one-dimensional subspace $(x, 0, x, 0, 0, 0)$ within $\mathcal{U}_3$ and hence an axial branch of solutions of this form exists, denoted rhombs 3 (Rh3). For Rh3,

$$\begin{aligned} \Sigma_{Rh3} &= \left\langle\left[\frac{2\pi}{R}, -\frac{2\pi Q}{PR}\right], \rho^2\right\rangle \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_{PR}, \\ \mathcal{S}_{Rh3} &\cong D_2 \ltimes \mathbb{Z}_P, \end{aligned} \tag{17}$$

where D_2 is generated by ρ^2 and the hidden symmetry $m_x \circ R_\phi^{-1}$, and the group of translations acting on $\mathcal{U}_1$ is reduced to $\mathbb{Z}_P$ from $\mathbb{Z}_{PR}$. The group $\mathcal{S}_{Rh3}$ gives the symmetries of Rh3 considered within the subspace $\mathcal{U}_1$, and Σ_{Rh3} gives the symmetries of Rh3 strictly outside $\mathcal{U}_1$.

Similarly, an axial branch of solutions of the form $(x, 0, 0, 0, x, 0)$ exists within $\mathcal{U}_2$; this branch is denoted Rh4. It has an isotropy subgroup

$$\Sigma_{Rh4} = \left\langle\left[\frac{2\pi}{R}, -\frac{2\pi P}{QR}\right], \rho^2\right\rangle \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_{QR}, \tag{18}$$

which has a two-dimensional fixed point subspace $\mathrm{Fix}(\Sigma_{Rh4}) = (x, 0, 0, 0, u, 0)$. But since this is a subspace of $\mathcal{U}_2$, the hidden symmetry $m_d \circ R_\phi$ must be taken into account, and so we define $\mathcal{S}_{Rh4} \cong D_2 \ltimes \mathbb{Z}_Q$ where the group D_2 is generated by the half-turn rotation ρ^2 and the hidden symmetry $m_d \circ R_\phi$. Within the subspace $\mathcal{U}_2$ the translation group $\mathbb{Z}_{QR}$ is restricted to a cyclic group $\mathbb{Z}_Q$.

Rh3 and Rh4 correspond to the branches $Rh_{s1,\alpha,\beta}$ and $Rh_{s2,\alpha,\beta}$ in [14]; their hidden symmetry subgroups $\mathcal{S}_{Rh3}$ and $\mathcal{S}_{Rh4}$ are isomorphic to the groups $D_2^d \ltimes S_{1,3}$ and $D_2^x \ltimes S_{1,4}$ defined in table 3 of [14]. We note that the orders of the translation elements agree since the pair of integers α, β in the notation of Dionne *et al* [14] is our pair a, b and $P = a^2 - b^2$ and $Q = 2ab$. This implies that each of the translation subgroups $S_{1,3}$ and $S_{1,4}$ in [14] is in fact generated by either of the two elements listed in the footnotes to [14, table 3]; in both cases, the second element listed is a power of the first.

We remark that these four branches of rhombs can be distinguished by the four different angles between wave-vectors that they contain: $2\tan^{-1} P/Q, \pi/2 - 2\tan^{-1} P/Q, \tan^{-1} P/Q$ and $\tan^{-1} Q/P$, respectively.

3.5. Superlattice patterns

Previous related work [13, 14] suggests that axial branches of superlattice patterns might exist. The most likely candidates for axial branches of superlattice patterns have four non-zero mode amplitudes, and the possibilities fall into two cases: branches involving only the four w_j modes, and those involving the two z_j modes and two of the w_j. We discuss these particular cases first, and later in this subsection discuss why other possibilities cannot lead to axial branches.

For solution branches involving the w_j modes only, we might expect branches in the form $(0, 0, u, u, u, u)$ (super-squares) or $(0, 0, u, -u, -u, u)$ (anti-squares). This is because the argument of [13, section 3] can be applied to the irreducible action of $D_4 \ltimes T^2$ on the subspace $(0, 0, w_1, w_2, w_3, w_4)$; to locate branches involving all four modes we are only interested in translation-free branches, and the result of [13, section 3(c)] applies here. This result is that the only possibilities are the super-square and anti-square branches given above. For this problem there is one further consideration; we must check that the isotropy subgroups of these solutions do not fix any part of the $(z_1, z_2, 0, 0, 0, 0)$ subspace.

A proposed super-square solution of the form $(0, 0, u, u, u, u)$ would have isotropy subgroup D_4, but $\mathrm{Fix}(D_4) = (x, x, u, u, u, u)$, which is two-dimensional, and hence no axial solution branch can be guaranteed by the equivariant branching lemma. The dynamics within $\mathrm{Fix}(D_4)$ is explored in more detail in section 6.1, where we discover that solutions closely related to this branch are in fact guaranteed to exist. In contrast, the supposed anti-squares solution branch, labelled AS1 in table 1, turns out to be an axial branch as Σ_{AS1} fixes no non-zero part of $(z_1, z_2, 0, 0, 0, 0)$.

For solutions involving both z_j modes and exactly two w_j modes (one of which is taken to be w_1 without loss of generality) there are three sub-cases: $(z_1, z_2, w_1, w_2, 0, 0)$, $(z_1, z_2, w_1, 0, w_3, 0)$ and $(z_1, z_2, w_1, 0, 0, w_4)$. In all of these cases we have to include the action of hidden symmetries in order to have a transitive group action on the mode amplitudes. However, only in the last case is there a fixed point subspace of the bifurcation problem that is strictly smaller than $\mathbb{C}^6$ (and hence it is possible to have hidden symmetries acting) and that contains the four modes. This subspace is exactly the subspace $\mathcal{U}_1$. The hidden symmetry $m_x \circ R_\phi^{-1}$ acts on $\mathcal{U}_1$ as

$$m_x \circ R_\phi^{-1} : (z_1, z_2, w_1, 0, 0, w_4) \to (w_1, w_4, z_1, 0, 0, z_2).$$

The action on this subspace of the group $\hat{D}_4$ generated by ρ and $m_x \circ R_\phi^{-1}$ is irreducible, and the results of [13] apply, enabling us to deduce the existence of two branches of the form $(x, x, x, 0, 0, x)$ and $(x, x, -x, 0, 0, -x)$, which we denote SS (super-squares) and

AS2 (anti-squares 2). These branches have hidden symmetry subgroups

$$\mathcal{S}_{SS} = \langle \rho, m_x \circ R_\phi^{-1} \rangle \cong D_4, \tag{19}$$

$$\mathcal{S}_{AS2} = \langle \rho, [\pi, \pi] \circ m_x \circ R_\phi^{-1} \rangle \cong D_4, \tag{20}$$

which act on $\mathcal{U}_1$. Their isotropy subgroups are identical and include the translations that fix $\mathcal{U}_1$ but omit the hidden symmetries:

$$\Sigma_{SS} = \Sigma_{AS2} = \left\langle \rho, \left[\frac{2\pi Q}{R}, \frac{2\pi P}{R}\right] \right\rangle \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_R. \tag{21}$$

Axial branches containing five or six amplitudes non-zero and of equal magnitude are not possible due to the reducibility of the group action; the reflections and rotations in D_4 do not permute all the amplitudes transitively, which would force them to be equal in magnitude. Hidden symmetries cannot come to the rescue since they do not act on subspaces where five or more amplitudes are non-zero. For the same reason, fixed-point subspaces of symmetries that are compositions of translations and elements of D_4 would also not be one-dimensional. Possibilities involving exactly one z_j mode and three w_j modes are similarly ruled out; again, no hidden symmetry can help since none act on as many as three of the w_j modes.

By the same argument we note that there can be no axial solution branches with exactly three mode amplitudes non-zero. This completes our investigation of axial branches; the details are summarized in table 1 and figure 3. We remark that it is better not to think of the two branches of rolls R1 and R2 as exactly equivalent for this bifurcation problem, even though within the subspace $\mathcal{U}_1$ they are related by the hidden symmetry $m_x \circ R_\phi^{-1}$. Similarly, the two branches of squares are distinct, but can be related within $\mathcal{U}_1$ by the same hidden symmetry. The differences will become more apparent when we compute stability in section 5.

4. The normal form

To calculate the normal form we follow the approach used by (among many others) [14, section 4.1] and [11]. First we compute T^2-invariant polynomials. From these we deduce terms that transform in the same way as z_1 does under the action of T^2. Then we can construct the first component $\dot{z}_1 = f_1(\mu, \boldsymbol{z}, \boldsymbol{w})$. Equivariance with respect to the D_4-action is guaranteed by then requiring $\dot{z}_2 = \rho\dot{z}_1 = f_1(\mu, \rho(\boldsymbol{z}, \boldsymbol{w})) \equiv f_2(\mu, \boldsymbol{z}, \boldsymbol{w})$. In a similar way we construct the equation $\dot{w}_1 = f_3(\mu, \boldsymbol{z}, \boldsymbol{w})$ and require

$$\dot{w}_2 = f_4(\mu, \boldsymbol{z}, \boldsymbol{w}) = f_3(\mu, m_x(\boldsymbol{z}, \boldsymbol{w})),$$
$$\dot{w}_3 = f_5(\mu, \boldsymbol{z}, \boldsymbol{w}) = f_3(\mu, m_d(\boldsymbol{z}, \boldsymbol{w})),$$
$$\dot{w}_4 = f_6(\mu, \boldsymbol{z}, \boldsymbol{w}) = f_3(\mu, \rho^3(\boldsymbol{z}, \boldsymbol{w})).$$

The resulting amplitude equations $(\dot{\boldsymbol{z}}, \dot{\boldsymbol{w}}) = \boldsymbol{f}(\mu, \boldsymbol{z}, \boldsymbol{w})$ are $D_4 \ltimes T^2$-equivariant. Finally, we consider the action of the hidden symmetry R_ϕ on the subspace $\mathcal{U}_1$ and derive extra constraints relating the form of $f_1(\mu, \boldsymbol{z}, \boldsymbol{w})$ to that of $f_3(\mu, \boldsymbol{z}, \boldsymbol{w})$. We know from Crawford [8] that considering other hidden symmetries leads to no further constraints.

In this first subsection we discuss the computation of T^2-invariants in the general case and subsequently concentrate on analysing the particular case $a = 2$ and $b = 1$. This is the natural first case and provides a concrete illustration of several finer points.

4.1. Computation of T^2-invariants

We define the order $O(I)$ of an invariant polynomial

$$I = z_1^{m_0} \bar{z}_1^{m_1} z_2^{n_0} \bar{z}_2^{n_1} w_1^{p_0} \bar{w}_1^{p_1} w_2^{q_0} \bar{w}_2^{q_1} w_3^{r_0} \bar{w}_3^{r_1} w_4^{s_0} \bar{w}_4^{s_1}$$

Table 1. Axial branches, their isotropy subgroups and (where appropriate) hidden symmetry subgroups, for the reducible representation of $\Gamma = D_4 \ltimes T^2$ defined by (6)–(8). We distinguish the two types of roll and square solutions since the conditions for their stability given by this bifurcation problem are different (see table 2). For convenience we define $\hat{\tau} \equiv [(2\pi Q/R^2), (2\pi P/R^2)]$ and $h \equiv R_\phi^2 \circ m_x$. The translation τ_1' is the restriction of τ_1 to the subspace $\mathcal{U}_1$ and τ_2' is similarly the restriction of τ_2 to $\mathcal{U}_2$. For R2 and S2 we give the hidden symmetry subgroups $\tilde{\mathcal{S}}$ that act on $\mathcal{U}_{R2}$.

Solution branch	Fixed-point subspace	Isotropy subgroup Σ	Hidden symmetry subgroup $\mathcal{S}$
Rolls 1 (R1)	$(x, 0, 0, 0, 0, 0)$	$\left\langle [0, \eta], m_x, m_y, \left[\frac{2\pi}{R}, 0\right] \right\rangle \cong O(2) \times D_R$	
Rolls 2 (R2)	$(0, 0, u, 0, 0, 0)$	$\left\langle \rho^2, \left[\xi, -\frac{Q}{P}\xi\right], \hat{\tau} \right\rangle \cong \mathbb{Z}_2[\rho^2] \ltimes (SO(2) \times \mathbb{Z}_{R^2})$	$\tilde{\mathcal{S}}_{R2} = \left\langle \rho^2, \left[\xi, -\frac{Q}{P}\xi\right], h \right\rangle \cong O(2) \times \mathbb{Z}_2[\rho^2 \circ h]$
Squares 1 (S1)	$(x, x, 0, 0, 0, 0)$	$\left\langle m_x, m_d, \left[\frac{2\pi}{R}, 0\right], \left[0, \frac{2\pi}{R}\right] \right\rangle \cong D_4 \ltimes (\mathbb{Z}_R \times \mathbb{Z}_R)$	
Squares 2 (S2)	$(0, 0, u, 0, 0, u)$	$\langle \rho, \hat{\tau} \rangle \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_{R^2}$	$\tilde{\mathcal{S}}_{S2} = \langle \rho, h \rangle \cong D_4$
Rhombs 1 (Rh1)	$(0, 0, u, u, 0, 0)$	$\left\langle m_x, m_y, \left[\frac{\pi}{Q}, \frac{\pi}{P}\right] \right\rangle \cong D_2 \ltimes \mathbb{Z}_{2QP}$	
Rhombs 2 (Rh2)	$(0, 0, u, 0, u, 0)$	$\left\langle m_d, m_{d'}, \left[\frac{2\pi P}{P^2 - Q^2}, \frac{2\pi Q}{Q^2 - P^2}\right] \right\rangle \cong D_2 \ltimes \mathbb{Z}_{\lvert P^2 - Q^2 \rvert}$	
Rhombs 3 (Rh3)	$(x, 0, x, 0, 0, 0)$	$\left\langle \rho^2, \tau_1 \equiv \left[\frac{2\pi}{R}, -\frac{2\pi Q}{PR}\right] \right\rangle \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_{PR}$	$\left\langle \rho^2, m_x \circ R_\phi^{-1}, \tau_1' \right\rangle \cong D_2 \ltimes \mathbb{Z}_P$
Rhombs 4 (Rh4)	$(x, 0, 0, 0, x, 0)$	$\left\langle \rho^2, \tau_2 \equiv \left[\frac{2\pi}{R}, -\frac{2\pi P}{QR}\right] \right\rangle \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_{QR}$	$\langle \rho^2, m_d \circ R_\phi, \tau_2' \rangle \cong D_2 \ltimes \mathbb{Z}_Q$
Super-squares (SS)	$(x, x, x, 0, 0, x)$	$\left\langle \rho, \left[\frac{2\pi Q}{R}, \frac{2\pi P}{R}\right] \right\rangle \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_R$	$\left\langle \rho, m_x \circ R_\phi^{-1} \right\rangle \cong D_4$
Anti-squares 1 (AS1)	$(0, 0, u, -u, -u, u)$	$\langle [\pi, \pi] \circ m_x, \rho \rangle \cong D_4$	
Anti-squares 2 (AS2)	$(x, x, -x, 0, 0, -x)$	$\left\langle \rho, \left[\frac{2\pi Q}{R}, \frac{2\pi P}{R}\right] \right\rangle \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_R$	$\left\langle \rho, [\pi, \pi] \circ m_x \circ R_\phi^{-1} \right\rangle \cong D_4$

to be the sum of the (non-negative) powers of the various amplitudes in it: $O(I) = m_0 + m_1 + \cdots + s_0 + s_1$. As is always the case in problems of this type, the order 2 polynomials $|z_1|^2, \ldots, |w_4|^2$ are T^2-invariant; we refer to them as *trivial* invariants. To remove them from the search for nontrivial invariants (these will introduce dynamics that depend on the relative

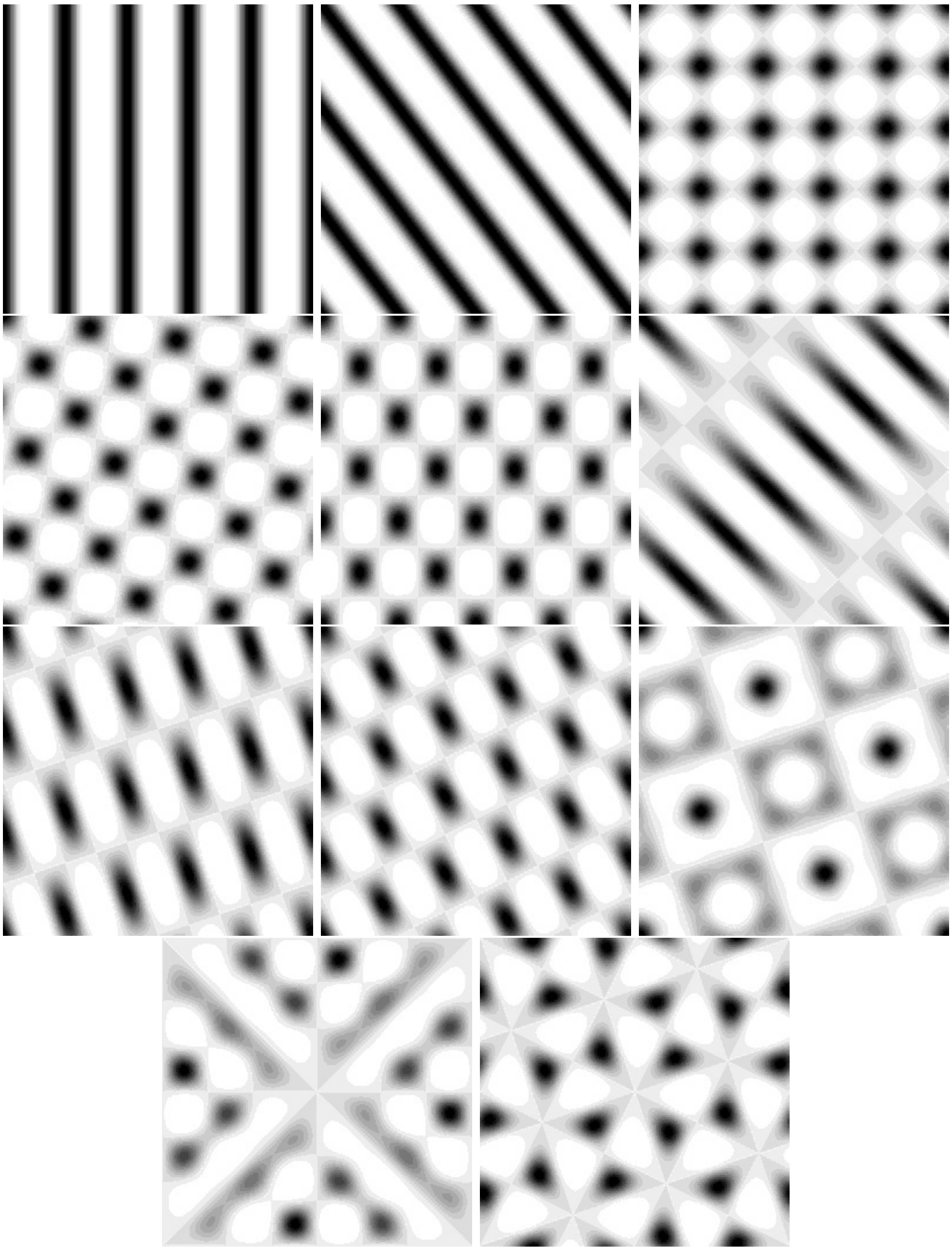

Figure 3. Greyscale images of the eleven axial planforms in the case $a = 2$, $b = 1$. From left to right, top to bottom: rolls 1, rolls 2, squares 1, squares 2, rhombs 1, rhombs 2, rhombs 3, rhombs 4, super-squares, anti-squares 1, anti-squares 2.

phases of the amplitudes) we search for invariants in the form $I = z_1^m z_2^n w_1^p w_2^q w_3^r w_4^s$, where m, n, p, q, r, s are integers but are not constrained to be non-negative. Negative powers are interpreted as powers of the complex conjugate of the amplitude; for example, if $m < 0$ then z_1^m should be interpreted to mean $\bar{z}_1^{-m}$.

Given the action (8) of T^2 on the six mode amplitudes, a polynomial $z_1^m z_2^n w_1^p w_2^q w_3^r w_4^s$ is T^2-invariant when

$$Q(p+q) + P(r+s) + Rm = 0, \tag{22}$$

$$P(p-q) + Q(r-s) + Rn = 0. \tag{23}$$

The task of finding all solutions in integers to (22) and (23) is daunting. A complete description of solutions may be possible using the methods of [29], but the problem analysed in that paper involves far fewer wave-vectors than we have here.

The algebraic analogue of the geometric insight of the finer lattice constructed in section 3.1 is to re-express (22) and (23) using the relations (4). This yields

$$2ab(p+q) + (a^2-b^2)(r+s) + (a^2+b^2)m = 0, \tag{24}$$

$$(a^2-b^2)(p-q) + 2ab(r-s) + (a^2+b^2)n = 0. \tag{25}$$

By multiplying these equations by a and b, respectively, and subtracting, and also after multiplying by b and a, respectively, and adding, we obtain

$$a(m+s) + b(p-n) + \frac{a(a^2-3b^2)r + b(3a^2-b^2)q}{a^2+b^2} = 0, \tag{26}$$

$$b(m-s) + a(p+n) + \frac{b(3a^2-b^2)r + a(3b^2-a^2)q}{a^2+b^2} = 0. \tag{27}$$

Solutions to (26) and (27) correspond exactly to solutions of (24) and (25). If $q = r = 0$ then (26) and (27) are exactly those equations analysed in appendix A.1 of Dionne *et al* [14]. Applying their results we deduce the existence of the T^2-invariants $\bar{z}_1^a z_2^b w_1^b w_4^a$ and $z_1^b z_2^a \bar{w}_1^a w_4^b$, which are clearly of order $2(a+b)$. It is also straightforward to apply the results of this appendix of [14] to the case where $m = n = 0$. This produces T^2-invariants that involve only the w_j modes: $\bar{w}_1^P \bar{w}_2^P w_3^Q w_4^Q$ and $\bar{w}_1^Q w_2^Q w_3^P \bar{w}_4^P$ and are of order $2(P+Q) = 2(a^2-b^2+2ab)$. With more effort we can deduce the following theorem, the proof of which is deferred to the appendix.

Theorem. *All nontrivial invariants are of order at least* $2(a+b)$.

In particular, the lack of nontrivial invariants of order 4 (because $a > b \geqslant 1 \Rightarrow a+b \geqslant 3$, hence $2(a+b) \geqslant 6$) results in amplitude equations that can never contain nontrivial cubic order terms. In fact, the invariants of order $2(a+b)$ are exactly those found by Dionne *et al* [14]. This is also proved in the appendix, as a corollary to the proof of the theorem.

By way of illustration, in the case $a = 2, b = 1$ there are nontrivial T^2-invariants of order $2(a+b) = 6$; for example, $z_1^2 \bar{z}_2 \bar{w}_1 \bar{w}_4^2$. Applying the D_4 symmetries to an invariant yields a 'group-orbit' of T^2-invariants. There are a further four distinct types (unrelated by D_4 symmetries) of order 8 invariant: $z_1 \bar{z}_2^3 w_1 \bar{w}_4^3$, $z_1 \bar{w}_1^3 w_2 w_3^2 \bar{w}_4$, $z_1 \bar{z}_2 w_1^2 \bar{w}_2 \bar{w}_3^2 \bar{w}_4$ and $z_1^2 \bar{w}_1^2 \bar{w}_2^2 w_3 w_4$. In the next subsection we use these results to derive the normal form in the specific case $a = 2, b = 1$.

4.2. The normal form for the case $a = 2$, $b = 1$

Having calculated the invariant polynomials up to a given order, it is straightforward to derive the truncation of the normal form up to and including terms of one order fewer. We have determined the number of nontrivial equivariant terms in the $\dot{z}_1$ equation up to fifteenth order: there are 2 nontrivial equivariant terms at fifth order and a further 19 terms at seventh order that are not products of fifth order equivariants and trivial invariants. Similarly there are 26 new nontrivial terms at ninth order, 57 at eleventh order, 116 at thirteenth order and another

116 at fifteenth order. Most of the stability calculations of the axial branches involve cubic terms. Eigenvalues calculated from the cubic truncation appear with multiplicities that are due both to symmetry and, sometimes, to degeneracies that stem from the omission of terms higher than third order. It is important to determine which of these multiplicities are broken when higher-order terms are included, and it is for this reason that we have computed these higher-order terms in the normal form.

It is clear that none of the cubic terms involve the relative phases of the modes and so the cubic truncation can never be sufficient completely to determine the stability of the axial branches. The inclusion of fifth-order terms is necessary but not sufficient because the fifth-order truncation still has non-generic features. For example, the subspace $(0, 0, w_1, w_2, w_3, w_4)$ is flow-invariant for the fifth-order truncation, but it is not when seventh-order terms are included. We find that there are several more degeneracies, broken as we include successive sets of higher-order terms. For higher values of a and b we expect the picture to be similar: for example, terms of order $2(a + b) - 1$ are always involved in determining the relative stability of SS and AS2.

Using the invariant polynomials and $D_4 \ltimes T^2$-equivariance we are able to compute the amplitude equations for z_1 and w_1. For illustration, we include terms up to seventh order derived from the nontrivial T^2-invariants, and involve products of trivial invariants:

$$\begin{aligned}\dot{z}_1 = {} & z_1\big[\mu + a_1|z_1|^2 + a_2|z_2|^2 + a_3(|w_1|^2 + |w_2|^2) + a_4(|w_3|^2 + |w_4|^2)\big] \\ & + b_1\big[\bar{z}_1 z_2 w_1 w_4^2 + \bar{z}_1\bar{z}_2 w_2 w_3^2\big] + b_2\big[z_2^2 w_2^2\bar{w}_3 + \bar{z}_2^2 w_1^2\bar{w}_4\big] \\ & + c_1\bar{z}_1 w_1^2 w_2^2\bar{w}_3\bar{w}_4 + c_2 w_3 w_4[z_2\bar{w}_1^2 w_2 w_3 + \bar{z}_2 w_1\bar{w}_2^2 w_4] \\ & + c_3\big[\bar{w}_1 w_2^3 w_3\bar{w}_4^2 + w_1^3\bar{w}_2\bar{w}_3^2 w_4\big] + c_8\big[z_2^3\bar{w}_1 w_4^3 + \bar{z}_2^3\bar{w}_2 w_3^3\big] \\ & + c_9 z_1^3\big[\bar{z}_2\bar{w}_1\bar{w}_4^2 + z_2\bar{w}_2\bar{w}_3^2\big] + c_{10}\bar{z}_1^2\big[z_2 w_2^3 w_3 + \bar{z}_2 w_1^3 w_4\big] \\ & + c_{11} z_1^2\big[z_2^2 w_4\bar{w}_1^2 + \bar{z}_2^2\bar{w}_2^2 w_3\big] + O(9),\end{aligned} \tag{28}$$

$$\begin{aligned}\dot{w}_1 = {} & w_1[\tilde{\mu} + \alpha_1|w_1|^2 + \alpha_2|w_2|^2 + \alpha_3|w_3|^2 + \alpha_4|w_4|^2 + \alpha_5|z_1|^2 + \alpha_6|z_2|^2] \\ & + \beta_1\bar{w}_1 z_1 z_2^2 w_4 + \beta_2 z_1^2\bar{z}_2\bar{w}_4^2 + \gamma_1\bar{z}_1 z_2^3 w_4^3 + \gamma_2\bar{w}_1^2\bar{w}_4 z_1^3 z_2 \\ & + \gamma_3\bar{w}_1^2 z_1 w_2 w_3^2\bar{w}_4 + \gamma_4 z_2\bar{w}_2^2 w_3 w_4^3 + \gamma_5\bar{w}_1\bar{w}_2 w_3^3 w_4\bar{z}_2 + \gamma_6\bar{z}_1 w_2^3 w_3\bar{w}_4^2 \\ & + \gamma_7\bar{w}_1\bar{z}_1 z_2 w_2 w_3^2 w_4 + \gamma_8 z_1 z_2 w_2^2\bar{w}_3\bar{w}_4^2 + \gamma_9\bar{w}_1 z_1^2\bar{w}_2^2 w_3 w_4 + \gamma_{10}\bar{z}_2^2 w_2 w_3^2\bar{w}_4^2 \\ & + \gamma_{11}\bar{z}_1\bar{z}_2^2 w_1 w_2^2\bar{w}_3 + \gamma_{12} z_1\bar{z}_2^2 w_1\bar{w}_2^2 w_3 + \gamma_{13} z_1^2 z_2 w_1\bar{w}_2\bar{w}_3^2 + \gamma_{14}\bar{z}_1^2\bar{z}_2 w_1 w_2 w_3^2 \\ & + \gamma_{15}\bar{z}_1\bar{z}_2^2 w_1^3\bar{w}_4 + \gamma_{16}\bar{z}_1^2 z_2 w_1^2 w_4^2 + O(9).\end{aligned} \tag{29}$$

To derive sensible amplitude equations Crawford [8] proved that it is necessary and sufficient to make use of the hidden symmetry $R_\phi : \mathcal{U}_2 \to \mathcal{U}_1$ defined by equations (11) and (12). Hence, following Crawford [8], if

$$\dot{z}_1 = f(z_1, z_2, w_1, w_2, w_3, w_4),$$

$$\dot{w}_1 = h(z_1, z_2, w_1, w_2, w_3, w_4),$$

then, restricting to $\mathcal{U}_2$:

$$R_\phi\frac{\mathrm{d}}{\mathrm{d}t}z_1 = R_\phi f(z_1, z_2, 0, w_2, w_3, 0) = f(w_1, \bar{w}_4, 0, z_1, z_2, 0),$$

but $R_\phi(\mathrm{d}z_1/\mathrm{d}t) = (\mathrm{d}w_1/\mathrm{d}t) = h(z_1, z_2, w_1, 0, 0, w_4)$ also, hence hidden symmetry implies

$$f(w_1, \bar{w}_4, 0, z_1, z_2, 0) = h(z_1, z_2, w_1, 0, 0, w_4).$$

Requiring this relation between (28) and (29) implies the equality of the bifurcation parameters $\mu = \tilde{\mu}$ and in addition

$$\begin{array}{llll} a_1 = \alpha_1, & a_2 = \alpha_4, & a_3 = \alpha_5, & a_4 = \alpha_6, \\ b_1 = \beta_1, & b_2 = \beta_2, & c_8 = \gamma_1, & c_{10} = \gamma_2, \\ c_9 = \gamma_{15}, & c_{11} = \gamma_{16}. & & \end{array}$$

So the $\dot{w}_1$ equation becomes

$$\begin{aligned} \dot{w}_1 = w_1\big[&\mu + a_1|w_1|^2 + a_5|w_2|^2 + a_6|w_3|^2 + a_2|w_4|^2 + a_3|z_1|^2 + a_4|z_2|^2\big] \\ &+ b_1\bar{w}_1 z_1 z_2^2 w_4 + b_2 z_1^2 \bar{z}_2 \bar{w}_4^2 + c_4 \bar{w}_1^2 z_1 w_2 w_3^2 \bar{w}_4 + c_5 z_2 \bar{w}_2^2 w_3 w_4^3 \\ &+ c_6 \bar{w}_1 \bar{w}_2 w_3^3 w_4 \bar{z}_2 + c_7 \bar{z}_1 w_2^3 w_3 \bar{w}_4^2 + c_8 \bar{z}_1 z_2^3 w_4^3 + c_9 \bar{z}_1 \bar{z}_2^2 w_1^3 \bar{w}_4 \\ &+ c_{10} \bar{w}_1^2 \bar{w}_4 z_1^3 z_2 + c_{11} \bar{z}_1^2 z_2 w_1^2 w_4^2 + c_{12} \bar{w}_1 \bar{z}_1 z_2 w_2 w_3^2 w_4 + c_{13} z_1 z_2 w_2^2 \bar{w}_3 \bar{w}_4^2 \\ &+ c_{14} \bar{w}_1 z_1^2 \bar{w}_2^2 w_3 w_4 + c_{15} \bar{z}_2^2 w_2 w_3^2 \bar{w}_4^2 + c_{16} \bar{z}_1 \bar{z}_2^2 w_1 w_2^2 \bar{w}_3 + c_{17} z_1 \bar{z}_2^2 w_1 \bar{w}_2^2 w_3 \\ &+ c_{18} z_1^2 z_2 w_1 \bar{w}_2 \bar{w}_3^2 + c_{19} \bar{z}_1^2 \bar{z}_2 w_1 w_2 w_3^2 + O(9), \end{aligned} \tag{30}$$

where we have re-labelled α_2, α_3 and $\gamma_7, \ldots, \gamma_{14}$ as (respectively) a_5, a_6 and $c_{12}, \ldots, c_{19}$ for consistency: the coefficients of the third-, fifth- and seventh-order terms are now labelled $a_1, \ldots, a_6$, b_1 and b_2 and $c_1, \ldots, c_{19}$ respectively. Equivariance with respect to the half-turn rotation symmetry ρ^2 ensures that all coefficients are real. Consistent with our discussion of axial branches in section 3, the term $c_3[\bar{w}_1 w_2^3 w_3 \bar{w}_4^2 + w_1^3 \bar{w}_2 \bar{w}_3^2 w_4]$ in (28) prohibits the existence of an axial branch of super-squares of the form $(\boldsymbol{z}, \boldsymbol{w}) = (0, 0, u, u, u, u)$, but is identically zero when $(\boldsymbol{z}, \boldsymbol{w}) = (0, 0, u, -u, -u, u)$, in agreement with the existence of the axial branch AS1.

We remark in passing that the cubic truncation of these amplitude equations has a gradient structure that is broken by higher-order terms. The computations below show that there are open regions of the normal form coefficient space where all axial branches bifurcate supercritically yet none is stable; branches of stable non-axial equilibria exist. Moreover, complex dynamics may also be possible, although we do not explore this issue further here.

5. Stability

In this section we discuss the stability of the eleven axial branches listed in table 1, relative to each other, by computing the eigenvalues of the Jacobian matrix found from the normal form truncated to some order.

The usual way of proceeding with the stability calculation of an equilibrium point with isotropy subgroup Σ is to decompose the 12×12 Jacobian matrix into its isotypic components. If V is the 12-dimensional space spanned by the eigenvectors of the Jacobian matrix, then, according to [19, chapter XII, theorem 2.5], V can be decomposed into a finite number of components $W_1, \ldots, W_K$:

$$V = W_1 \oplus \cdots \oplus W_K,$$

where the W_k ($1 \leqslant k \leqslant K$) are called isotypic components of V. Within each isotypic component, the isotropy subgroup acts as one or more isomorphic copies of the same irreducible representation (irrep) of Σ, and each different isotypic component is associated with a different irrep. Correspondingly, the Jacobian matrix can be block diagonalized, with the kth block having the same dimension as W_k. In the case that Σ acts absolutely irreducibly on W_k, the corresponding block of the Jacobian matrix will be diagonal, and the eigenvalue will have multiplicity equal to the dimension of the irrep in question. If there were no hidden symmetries, then this decomposition would be sufficient to compute all the eigenvalues of the Jacobian, and hence to determine the stability of the equilibrium in question.

However, this procedure is made more complicated here by the existence of hidden symmetries, which provide extra constraints on the form of the Jacobian matrix. For example, super-squares (SS), in the case $a = 2$, $b = 1$, have $\Sigma_{SS} \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_5$, which has five irreps, four of dimension 1 and one of dimension 4. However, the Jacobian matrix of SS has eigenvalues (amongst others) of multiplicity 2, which does not correspond to the dimensions of any of the irreps of Σ_{SS}. It is clear then, that consideration of the decomposition with respect to Σ alone is not sufficient. We show below how we have computed the stability of the 11 branches of equilibria, within the context of perturbations that lie on the original lattice (that is with the original imposed spatial periodicity).

Consider an equilibrium point with isotropy subgroup Σ and hidden symmetry group $\mathcal{S}$, which acts on a subspace $\mathcal{U}$ (with $\mathcal{U}$ the largest such subspace). Since $\mathcal{U}$ is a flow-invariant fixed-point subspace, eigenvectors of the Jacobian matrix must lie either within $\mathcal{U}$ or outside $\mathcal{U}$, and the number of eigenvectors within $\mathcal{U}$ will be equal to the dimension of $\mathcal{U}$. Since $\mathcal{S}$ acts on $\mathcal{U}$, the isotypic decomposition of the subspace spanned by eigenvectors within $\mathcal{U}$ will correspond to irreducible actions of $\mathcal{S}$. The remaining eigenvectors, which lie strictly outside $\mathcal{U}$, span a space that is isotypically decomposed according to irreducible actions of Σ. However, only those irreps of Σ in which group elements of Σ that fix $\mathcal{U}$ act nontrivially will appear.

As an example, consider super-squares $(x, x, x, 0, 0, x)$, in the case $a = 2$, $b = 1$, which have $\Sigma_{SS} \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_5$ and $\mathcal{S}_{SS} \cong D_4$, acting on $\mathcal{U} = \mathcal{U}_1 = \mathrm{Fix}(\langle[(2\pi Q/R), (2\pi P/R)]\rangle) = \{(z_1, z_2, w_1, 0, 0, w_4)\}$. There is a decomposition of the 12-dimensional space $V = \mathbb{E}^c(\Omega)$:

$$
\begin{aligned}
V &= W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5 \oplus W_6,\\
W_1 &= \mathbb{R}\{(1, 1, 1, 0, 0, 1)\},\\
W_2 &= \mathbb{R}\{(1, 1, -1, 0, 0, -1)\},\\
W_3 &= \mathbb{R}\{(1, -1, 1, 0, 0, -1)\},\\
W_4 &= \mathbb{R}\{(1, -1, -1, 0, 0, 1)\},\\
W_5 &= \mathbb{R}\{(\mathrm{i}R, 0, \mathrm{i}Q, 0, 0, \mathrm{i}P), (0, \mathrm{i}R, \mathrm{i}P, 0, 0, -\mathrm{i}Q),\\
&\qquad (-\mathrm{i}Q, -\mathrm{i}P, \mathrm{i}R, 0, 0, 0), (-\mathrm{i}P, \mathrm{i}Q, 0, 0, 0, \mathrm{i}R)\},\\
W_6 &= \mathbb{C}\{(0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0)\},
\end{aligned}
$$

where W_1, W_2, W_3 and W_4 correspond to the four different one-dimensional irreps of $\mathcal{S} \cong D_4$, W_5 is four-dimensional and corresponds to two copies of the two-dimensional irrep of $\mathcal{S}$, and W_6 is four-dimensional and corresponds to the four-dimensional irrep of $\Sigma \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_5$, in which the element $[(2\pi Q/R), (2\pi P/R)]$ acts nontrivially. Note that each subspace apart from W_5 corresponds to absolutely irreducible representations of $\mathcal{S}$ or Σ, and so the corresponding blocks of the Jacobian matrix are diagonal. The eigenvalues corresponding to W_5 have multiplicity two, and one pair of these is the pair of zero eigenvalues arising from the neutral stability of the pattern to translations. The other pair of eigenvalues is (at leading order) proportional to $-(b_2 + 2b_1)$.

We have computed the eigenvalues of the 11 axial branches by constructing decompositions as described above. The results are presented in tables 2 and 3, which show, for each axial branch, the equation that determines the amplitude of the solution branch and the combinations of coefficients that determine the eigenvalues, together with the multiplicity of each eigenvalue. Note that the roll solutions have one zero eigenvalue and all other planforms have two zero eigenvalues, due to the translational symmetry. All eigenvalues are real. The stability of the roll and square solutions is fully determined at cubic order; for these branches the results in table 2 are valid for all values of $a > b > 0$.

Table 2. Stability criteria for axial branches: part 1. Numbers in square brackets give the multiplicities of the eigenvalues. Roll solutions have one zero eigenvalue and all other solutions have two. Multiplicity [2 + 2] indicates that this eigenvalue is split by higher-order terms, as discussed in the text.

Axial branch and branching equation	Stability criteria (stable near $\mu = 0$ if all quantities given are negative)		
Rolls 1 $0 = \mu + a_1x^2 + \cdots$	a_1 [1], $a_3 - a_1$ [4],	$a_2 - a_1$ [2], $a_4 - a_1$ [4]	
Rolls 2 $0 = \mu + a_1x^2 + \cdots$	a_1 [1], $a_3 - a_1$ [2],	$a_2 - a_1$ [2], $a_4 - a_1$ [2],	$a_5 - a_1$ [2], $a_6 - a_1$ [2]
Squares 1 $0 = \mu + (a_1 + a_2)x^2 + \cdots$	$a_1 + a_2$ [1], $a_1 - a_2$ [1],	 $-a_1 - a_2 + a_3 + a_4$ [8]	
Squares 2 $0 = \mu + (a_1 + a_2)x^2 + \cdots$	$a_1 + a_2$ [1], $a_1 - a_2$ [1],	$-a_1 - a_2 + a_3 + a_4$ [4], $-a_1 - a_2 + a_5 + a_6$ [4]	
Rhombs 1 $0 = \mu + (a_1 + a_5)x^2 + \cdots$	$a_1 + a_5$ [1], $a_1 - a_5$ [1], $-a_1 - a_5 + a_2 + a_6$ [4]	$-a_1 - a_5 + 2a_3$ [2], $-a_1 - a_5 + 2a_4$ [2],	
Rhombs 2 $0 = \mu + (a_1 + a_6)x^2 + \cdots$	$a_1 + a_6$ [1], $a_1 - a_6$ [1],	$-a_1 - a_6 + a_2 + a_5$ [2+2], $-a_1 - a_6 + a_3 + a_4$ [2+2]	
Rhombs 3 $0 = \mu + (a_1 + a_3)x^2 + \cdots$	$a_1 + a_3$ [1], $a_5 - a_1$ [2],	$a_1 - a_3$ [1], $-a_1 - a_3 + a_2 + a_4$ [2+2]	$-a_1 - a_3 + a_4 + a_6$ [2],
Rhombs 4 $0 = \mu + (a_1 + a_4)x^2 + \cdots$	$a_1 + a_4$ [1], $a_5 - a_1$ [2],	$a_1 - a_4$ [1], $-a_1 - a_4 + a_2 + a_3$ [2+2]	$-a_1 - a_4 + a_3 + a_6$ [2],

Table 3. Stability criteria for axial branches: part 2. Numbers in square brackets give the multiplicities of the eigenvalues. These three branches all have two zero eigenvalues each. Multiplicity [2 + 1 + 1] indicates that this eigenvalue is split by higher-order terms, as discussed in the text. Underlined stability criteria depend on coefficients of terms of fifth and higher orders. $\hat{C} = a_1 + a_2 - 2a_3 - 2a_4 + a_5 + a_6$.

Axial branch and branching equation	Stability criteria (stable near $\mu = 0$ if all quantities given are negative)	
SS $0 = \mu + (a_1 + a_2 + a_3 + a_4)x^2$ $+(b_1 + b_2)x^4 + \cdots$	$a_1 + a_2 + a_3 + a_4$ [1], $a_1 + a_2 - a_3 - a_4$ [1], $-a_1 - a_2 + a_5 + a_6$ [4], $\underline{-(b_2 + 2b_1)}$ [2]	$a_1 - a_2 + a_3 - a_4$ [1], $a_1 - a_2 - a_3 + a_4$ [1],
AS1 $0 = \mu + (a_1 + a_2 + a_5 + a_6)x^2$ $+ \cdots$	$a_1 + a_2 + a_5 + a_6$ [1], $a_1 + a_2 - a_5 - a_6$ [1], $-\hat{C}$ [2+1+1], $\underline{c_3(4c_4 + 3c_5 + 3c_6 + 4c_7)/\hat{C}}$ [2]	$a_1 - a_2 + a_5 - a_6$ [1], $a_1 - a_2 - a_5 + a_6$ [1],
AS2 $0 = \mu + (a_1 + a_2 + a_3 + a_4)x^2$ $-(b_1 + b_2)x^4 + \cdots$	$a_1 + a_2 + a_3 + a_4$ [1], $a_1 + a_2 - a_3 - a_4$ [1], $-a_1 - a_2 + a_5 + a_6$ [4], $\underline{(b_2 + 2b_1)}$ [2]	$a_1 - a_2 + a_3 - a_4$ [1], $a_1 - a_2 - a_3 + a_4$ [1],

A recurring difficulty with these calculations involves deciding whether or not the level of truncation of the normal form has led to erroneous stability conclusions, and if so, determining a sufficiently high level of truncation to resolve the situation. In most cases, each isotypic component corresponds to an irreducible action either of $\mathcal{S}$ or of Σ (as appropriate), and the resulting eigenvalue is distinct from all other eigenvalues associated with other isotypic components. In these cases, the cubic truncation is sufficient. However, two different problems arise. First, eigenvalues corresponding to different isotypic components could be equal in value at the cubic level of truncation. Since the eigenvalues are associated with different components, this degeneracy is spurious and is removed by the addition of higher-order terms. We refer to this as eigenvalues being 'split' by higher-order terms. Second, the action of $\mathcal{S}$ or of Σ in a component might correspond to two (or more) copies of an irreducible action, leading to several different eigenvalues that will depend on diagonal and off-diagonal entries in that Jacobian block. If any of these entries is zero, there is a possibility that going to higher order in the normal form might lead to a non-zero entry, casting doubt on the original estimate of the eigenvalues.

The first problem arises in rhombs 2, rhombs 3, rhombs 4 and anti-squares 1. In each case, there are eigenvalues that are four-fold degenerate at the cubic level of truncation, but that correspond to two different two-dimensional isotypic components (rhombs) or three different isotypic components of dimensions 1, 1 and 2 (anti-squares 1). Since there are no off-diagonal entries linking blocks in different isotypic components, increasing the degree of the truncation sufficiently resolves this spurious degeneracy without any additional complications—though for rhombs 2 it proves necessary to go to fifteenth order to break the degeneracy (for the case $a = 2$, $b = 1$). The affected eigenvalues are indicated by [2 + 2] and [1 + 1 + 2] in tables 2 and 3.

The second problem arises in the cases of super-squares and anti-squares 1 and 2. In the cases of super-squares and anti-squares 2, the affected eigenvalues are associated with four-dimensional isotypic components, in which two copies of the two-dimensional irrep of D_4 are acting (as in the subspace W_5 described above). In these two cases (as noted by Dionne *et al* [14]), one pair of eigenvalues is zero, corresponding to translations along the group-orbit, and the other pair is determined unambiguously at fifth order. Indeed, a considerable number of the entries in table 3 can be obtained from table 8 of Dionne *et al* [14]. For example, for the super-square and anti-squares 2 solutions, the stability problem within the invariant subspace $\mathcal{U}_1$ reduces to that studied in [14]. The case of anti-squares 1 is more complicated: three isotypic components are reducible. Two of these are two-dimensional, with two copies each of one-dimensional irreducible actions of $\Sigma = D_4$; the other one is six-dimensional, with three copies of the two-dimensional irreducible action of $\Sigma = D_4$, and the neutral eigenvalues are associated with this component. In the two two-dimensional cases, the diagonal entries of the blocks are proportional to μ (the bifurcation parameter), while the off-diagonal entries are proportional to μ^3, so the signs of the eigenvalues will be unaffected by retaining higher-order terms in the normal form. The six-dimensional component factorizes into two copies of a three-dimensional matrix, and one eigenvalue of each of these is zero. Since the eigenvectors of the zero eigenvalues are known, a two-dimensional matrix for the other eigenvalues can be extracted. After coordinate transformations, this matrix has diagonal entries proportional to μ and μ^6, and off-diagonal entries proportional to μ^3. We therefore expect eigenvalues of order μ and μ^5; expressions giving the signs of these are in table 3.

Note that, while we have given explicit stability criteria depending on coefficients of quintic and higher terms in the normal form (underlined in table 3), these were computed without including terms that are products of lower-order invariant and equivariant terms. The implication is that the stability criteria that are underlined are correct for the normal form as given in equations (28) and (30), but are not correct in general because of the missing fifth and higher-order terms.

The stability results lead to a number of conclusions. If rolls bifurcate subcritically then all solutions are unstable. However, it is possible for any other solution branch to be subcritical while other branches are stable; for example, Rh1 can be stable when SS is subcritical. There are several combinations of solutions that can be simultaneously stable. For example, S1 can be simultaneously stable with any of S2, Rh1, Rh2, Rh3, Rh4 or AS1. All three of R1, Rh1 and Rh2 can be stable. The maximum number of simultaneously stable solutions seems to be four: S1, S2, Rh2 and either Rh3 or Rh4.

An interesting result is that it is possible for all axial branches to be supercritical while none of them is stable. This is not possible for the square superlattice problem considered by Dionne *et al* [14, section 4], but it is the case in other problems: notably it occurs for the corresponding bifurcation on a hexagonal superlattice in the degenerate situation when the quadratic terms vanish. That bifurcation problem was analysed in section 5.1 of [14] and their note (3) remarks that it is possible for all axial branches to bifurcate supercritically yet none to be stable if the normal form coefficients satisfy a certain inequality. It turns out that for those coefficient values a primary but non-axial branch (identified by Silber and Proctor [27]) exists and is stable. This branch has submaximal symmetry and exists for all combinations of normal form coefficients.

In the case

$$a_1 < a_2 = a_3 = a_4 = a_5 = a_6 < 0, \tag{31}$$

all axial branches bifurcate supercritically, but they are unstable (at least near $\mu = 0$) because each solution branch has at least one stability criterion that is equal to $a_2 - a_1 + O(|\boldsymbol{z}, \boldsymbol{w}|^2)$. In section 7 we show that stable non-axial branches of solutions occur for a PDE for which the condition (31) holds.

6. Non-axial branches

In this section we comment briefly on two subspaces that contain non-axial branches. The subspace $\mathrm{Fix}(D_4)$ is of interest since within this subspace we can guarantee the existence of a primary (but non-axial) branch, an unusual feature of this bifurcation problem. The second subspace we discuss is of interest since, in addition to several axial branches, it contains branches that resemble hexagonal solutions, despite this analysis being carried out on a square lattice.

6.1. The dynamics within the subspace $\mathrm{Fix}(D_4)$

$\mathrm{Fix}(D_4)$ is the two-dimensional subspace (x, x, u, u, u, u); within this subspace the amplitude equations (28) and (30) reduce to

$$\dot{x} = x[\mu + Ax^2 + 2Bu^2] + xh_1(x, u) + Cu^7 + O(9), \tag{32}$$

$$\dot{u} = u[\mu + Du^2 + Bx^2] + uh_2(x, u), \tag{33}$$

where $A = a_1 + a_2$, $B = a_3 + a_4$, $C = 2c_3$, $D = A + a_5 + a_6$ and the smooth functions h_1 and h_2 are $O(|x, u|^4)$. These equations have a symmetry $(x, u) \to (-x, -u)$ and in addition the line $u = 0$ is invariant.

The cubic truncation of (32) and (33) has three distinct types of equilibria:

- a squares solution $x_s^2 = -\mu/A$, $u = 0$ (which corresponds to the axial branch S1);
- a 'super-squares' solution $x_{ss} = 0$, $u_{ss}^2 = -\mu/D$;
- mixed-mode solutions $x_{mm}^2 = \mu(D - 2B)/(2B^2 - AD)$, $u_{mm}^2 = \mu(A - B)/(2B^2 - AD)$.

The first and second of these exist for all combinations of coefficients. Squares are stable within $\mathrm{Fix}(D_4)$ when $B < A < 0$, and super-squares are likewise stable when $2B < D < 0$. Note that the non-axial $(0, u_{ss})$ solution is entirely distinct from the axial branch of super-squares solutions discussed in section 3. The mixed-mode solutions correspond to two different real-space patterns, given by $x > 0, u > 0$ and $x > 0, u < 0$. They exist when $(D-2B)(A-B) > 0$ and are stable when $Ax_{mm}^2 + Du_{mm}^2 < 0$ and $2B^2 - AD < 0$. In the cubic truncation of the full problem, these 'solution branches' would have six zero eigenvalues in their Jacobians, and we would not be able to guarantee that they persisted to give solution branches for the complete normal form. However, as long as we remain within $\mathrm{Fix}(D_4)$, these branches of equilibria are hyperbolic (for generic choices of the coefficients). So we may apply the implicit function theorem and assert that any branch of equilibria $(\hat{x}(\mu), \hat{u}(\mu))$ that exists and is hyperbolic in the cubic truncation within $\mathrm{Fix}(D_4)$ (for a given set of coefficients $A, \ldots, D$) must persist as a branch of solutions to the full amplitude equations for sufficiently small $|\mu|$.

Figure 4(*a*) illustrates the dynamics of the cubic truncation of (32) and (33) in the case that all three branches of equilibria bifurcate supercritically and the mixed-mode branch (x_{mm}, u_{mm}) is stable. Figure 4(*b*) illustrates the full dynamics of (32) and (33), indicating the persistence of the branches of equilibria. Note that the super-squares branch (x_{ss}, u_{ss}) has been perturbed off the axis, and the four mixed-mode solutions, which were all related by symmetry in the cubic truncation, have split into two distinct pairs under the influence of the higher-order terms.

There is a clear contrast between this analysis and the results of section 3 within this subspace; from table 1 the existence of only the axial squares branch S1 can be deduced. The 'super-squares' branch $(0, u_{ss})$ for the cubic truncation generically persists (when it is hyperbolic) to yield a branch of solutions where all six modes are non-zero; the corresponding branch of solutions to (32) and (33) has $x_{ss} \approx Cu_{ss}^5/(D - 2B)$ to leading order, due to the term Cu^7 in (32). It is an example of a primary but non-axial branch of solutions.

Moreover, for particular choices of the coefficients of the cubic terms in the normal form we find that the mixed-mode solution (x_{mm}, u_{mm}), where both x and u are $O(\mu^{1/2})$, is stable, at least within this subspace. The existence and stability of the mixed-mode solution

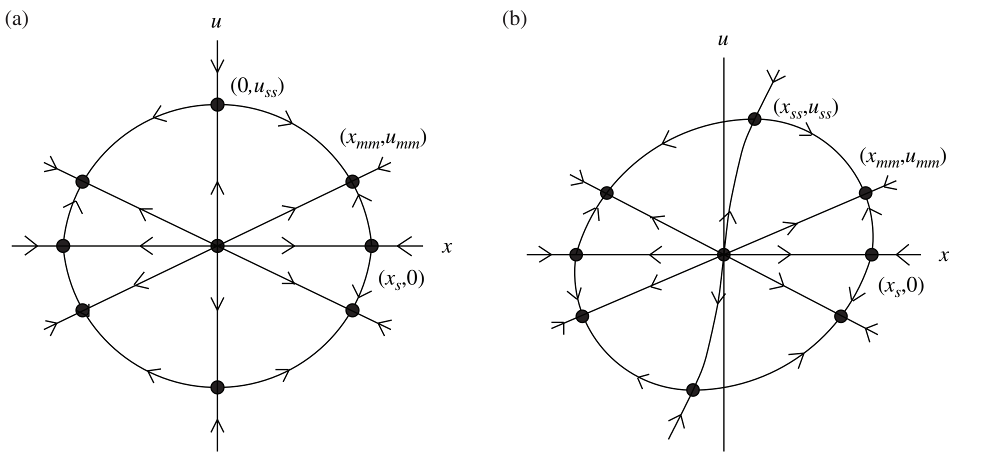

Figure 4. The dynamics within $\mathrm{Fix}(D_4)$ for $\mu > 0$, in the case that all solution branches bifurcate supercritically and the mixed-mode solution exists and is stable. (*a*) A sketch of the phase portrait of the cubic truncation (32) and (33) showing the squares $(x_s, 0)$, the super-squares $(0, u_{ss})$ and the mixed-mode (x_{mm}, u_{mm}). (*b*) The dynamics of the non-truncated ODEs (32) and (33) illustrating the persistence of the equilibria and the existence of the primary but non-axial branch (x_{ss}, u_{ss}).

is independent of the nature of the higher-order terms, and so is independent of the choice of a and b. Furthermore, we would expect the values of the cubic coefficients to vary only relatively little if a and b were to vary in a way that keeps the ratio a/b nearly constant, since the coefficients depend only on the angles between modes, which depend only on the ratio a/b. In section 7 we discuss numerical solutions of a model PDE that appear to be in the form of this stable mixed-mode solution.

6.2. *'Nearly hexagonal' solution branches*

Among the many fixed-point subspaces of the problem, the subspace $\mathrm{Fix}([0, 2\pi/Q]) = (z_1, 0, 0, 0, w_3, w_4)$ is of particular interest since exactly three modes have non-zero amplitudes within this subspace. Furthermore, the angle between the wave-vectors corresponding to the modes z_1, w_3 and w_4 can be made arbitrarily close to 60° by choosing a and b so that a^2/b^2 is sufficiently close to 3. The dynamics within this subspace gives rise to solutions that, despite being periodic on a square lattice, resemble hexagons modulated on a much longer length-scale. A quick calculation of T^2-invariant polynomial terms which involve only z_1, w_3 and w_4 reveals that the lowest-order nontrivial invariant is $I_1 = z_1^{2P}\bar{w}_3^R\bar{w}_4^R$. The dynamics within this subspace are governed at leading order by the cubic truncation of (28) and the equations for $\dot{w}_3$ and $\dot{w}_4$ obtained by applying symmetries to (30). The cubic truncation contains no information on the relative phases of the modes, and so we add the higher-order terms that result from the nontrivial invariant I_1. In fact, the calculation of invariant terms involving only z_1, w_3 and w_4 shows that all higher-order nontrivial terms are composed of products of trivial invariants and powers of I_1, so it is enough to keep only the lowest-order nontrivial terms. After writing $z_1 = A\mathrm{e}^{\mathrm{i}\theta_1}$, $w_3 = B\mathrm{e}^{\mathrm{i}\theta_2}$ and $w_4 = C\mathrm{e}^{\mathrm{i}\theta_3}$ and substituting we obtain

$$\dot{A} = A[\mu + a_1A^2 + a_4(B^2 + C^2)] + b_3A^{2P-1}B^RC^R\cos\phi, \tag{34}$$

$$\dot{B} = B[\mu + a_1B^2 + a_4A^2 + a_5C^2)] + b_4A^{2P}B^{R-1}C^R\cos\phi, \tag{35}$$

$$\dot{C} = C[\mu + a_1C^2 + a_4A^2 + a_5B^2)] + b_4A^{2P}B^RC^{R-1}\cos\phi, \tag{36}$$

$$\dot{\phi} = -\sin\phi[2Pb_3A^2 + Rb_4(B^2 + C^2)]A^{2P-2}(BC)^{R-2}, \tag{37}$$

where $\phi = 2P\theta_1 - R(\theta_2 + \theta_3)$ and b_3 and b_4 are coefficients of the terms at $O(2(P + R) - 1)$. Note that (34)–(36) contain only terms of odd order, and that they contain three subspaces where exactly one of the variables is zero and the other two are non-zero; within each of these subspaces a rhomb solution (conjugate to either Rh1 or Rh4) exists. The structure of these equations is, therefore, very different to the normal form for a steady-state bifurcation on a hexagonal lattice, as discussed in [3], for example.

In the case $a = 2$, $b = 1$ these equations were investigated for various combinations of coefficient values. We will summarize results for one combination in the remainder of this section. The coefficient values chosen are $a_1 = -2$, $a_4 = -1$, $a_5 = -1.1$, $b_3 = 10^5$, $b_4 = 1.1 \times 10^5$. From (37) we see that all stable solutions have $\phi = 0$. The values of a_4 and a_5 are expected to be close together, but not equal since they depend entirely on the angle between the relevant pair of wave-vectors: the angle between the wave-vectors corresponding to z_1 and w_3 is close to, but not equal to, that between w_3 and w_4. Similarly, the coefficients b_3 and b_4 are expected to be close. We have chosen coefficients several orders of magnitude greater for b_3 and b_4 simply to bring any additional bifurcation structure due to the higher-order term closer to the initial bifurcation at $\mu = 0$, which makes numerical investigations easier. By rescaling the amplitudes A, B and C we could have set both b_3 and b_4 to be ± 1. We take $a_1 < a_4$ to ensure that roll solutions are unstable to solutions involving several modes. For simplicity we have omitted the large number of terms due to the trivial invariants that are also present in the complete amplitude equations.

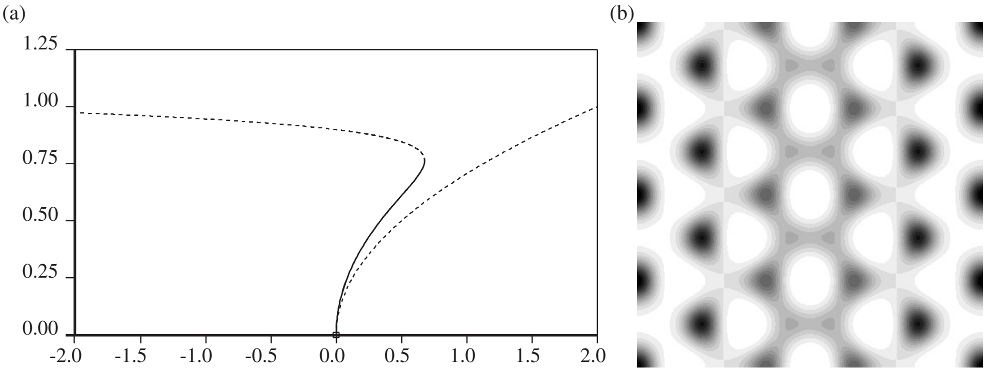

Figure 5. (*a*) Bifurcation structure within the subspace $(z_1, 0, 0, 0, w_3, w_4)$, shown in the $(\mu, \|z\|+\|w\|)$ plane: stable and unstable branches are shown as solid and dashed lines, respectively. (*b*) Greyscale image of a 'nearly hexagonal' solution with $\phi = 0$ and $A \approx B = C$.

Figure 5 shows the bifurcation structure in the $(\mu, \|z\| + \|w\|)$ plane, computed using the bifurcation and continuation package AUTO [15]. The unstable branch existing only in $\mu > 0$ corresponds to rolls. The other branch, which exists stably in $0 < \mu < 0.6$, contains solutions that look 'nearly hexagonal', i.e. $A \approx B = C$. After the saddle-node bifurcation at $\mu \approx 0.6$, the branch continues into $\mu < 0$ and is unstable there. The point we wish to emphasise is the existence of coefficient values at which the stable dynamics within this subspace is reminiscent of hexagonal solutions. Moreover, there are open regions of the coefficient space in which all the axial branches are unstable and the nearly hexagonal branch is attracting both within and transverse to this subspace. This occurs, for example, when $a_1 = a_2 = -2.0$, $a_3 = a_6 = -1.8$ and $a_4 = a_5 = -1.0$; these conditions ensure that for each axial branch at least one of the stability conditions that is determined at cubic order is not satisfied, and so even without computing higher-order terms we have ensured that all the axial branches are unstable.

Clearly this is an area where much more can be done, and we leave a more detailed analysis to be the subject of future work. In particular, it would be interesting to pursue the relationship between these solutions and the analysis of nearly-hexagonal solutions in square boxes carried out by Matthews [22].

7. Application to a model PDE

In this section we compute the coefficients in the cubic truncation of the amplitude equations (28) and (30) and describe some numerical solutions, for the PDE

$$\frac{\partial u}{\partial t} = -\nabla^2(ru - (1+\nabla^2)^2 u - pu\nabla^2 u - q|\nabla u|^2 - su^2 - u^3). \qquad (38)$$

Equations of this type can be used to model pattern-forming systems with a conserved quantity, such as convection with fixed-flux boundaries [23]. Equation (38) has a number of interesting properties and is discussed in detail elsewhere [7]. Note that (38) is similar to the Swift–Hohenberg equation, but with an additional ∇^2 acting on the right-hand side. This modification allows (38) to have more exotic stable solutions than the Swift–Hohenberg equation; we return to this point in the concluding discussion.

7.1. Computation of coefficients

The linear growth rate σ for a Fourier mode of wavenumber k in (38) is $\sigma = k^2(r-(1-k^2)^2)$. The state $u = 0$ is unstable for $r > 0$, to a band of wavenumbers centred at $k = 1$.

To determine the coefficients in the cubic truncation of the amplitude equations (28) and (30) it is sufficient to restrict attention to a rhombic lattice. We set $r = \varepsilon^2 \ll 1$ and consider the expansion

$$u = \varepsilon\big(z_1 e^{ix} + z_2 e^{i(x\cos\theta + y\sin\theta)}\big) + O(\varepsilon^2), \tag{39}$$

consisting of two Fourier modes on a rhombic lattice with lattice angle θ.

At second order, four modes are driven by products of z_1 and z_2, and we can solve for these modes provided that $\cos\theta \neq \pm\frac{1}{2}$. At third order, the amplitude equations for z_1 and z_2 are obtained in the form

$$\dot{z_1} = z_1 + a_1|z_1|^2 z_1 + a_\theta |z_2|^2 z_1, \tag{40}$$

$$\dot{z_2} = z_2 + a_1|z_2|^2 z_2 + a_\theta |z_1|^2 z_2, \tag{41}$$

where the dot denotes differentiation with respect to the slow time $\varepsilon^2 t$. In general, a_θ is a complicated function of the parameters and the angle θ. However, if we choose the coefficients in (38) such that $q = 2(p-s)$, then, the angular dependence vanishes and a_1 and a_θ are given by

$$a_1 = -3 - \nu, \qquad a_\theta = -6 - 4\nu, \qquad \text{where } \nu = (2s-p)(s-p). \tag{42}$$

Since there is no dependence on θ, the coefficients $a_2 \cdots a_6$ in (28) and (30) are all equal to a_θ. Applying the results of tables 2 and 3, it follows that rolls are stable if $\nu > -1$, but the axial solutions are all supercritical and unstable if $-\frac{21}{13} < \nu < -1$. For these values of the coefficients, as discussed in section 6.1, the cubic truncation of (28) and (30) permits a solution (the 'mixed-mode' branch) in which all six mode amplitudes are non-zero and of the same order in μ. Furthermore, this solution is stable if $-\frac{11}{7} < \nu < -1$. We therefore expect that (38) exhibits a stable pattern involving all six modes in this parameter regime; however, the cubic truncation does not determine the relative phases of the modes.

7.2. Numerical simulations

To investigate the solution involving all six modes, (38) was solved numerically using a Fourier spectral method on a 48×48 grid. The size of the periodic domain was fixed at 10π so that the first modes to become unstable as r increases through zero are the $(5, 0)$ and $(4, 3)$ modes. Each computation was started from a small-amplitude random initial condition, and continued until a steady state was reached. This requires a very long integration time because the relative phases of the modes, and hence the character of the pattern, is not determined at cubic order.

Figure 6(*a*) shows the stationary solution obtained when $r = 0.005$, $s = 2.2$, $p = 3.0$, $q = 1.6$. For these parameter values, $\nu = -1.12$, so all axial branches are unstable and the six-mode solution is stable according to the cubic truncation. The solution found exhibits D_4 symmetry (since the boundaries are periodic we can simply shift the pattern until it is centred at the centre of the computational domain), and is therefore on the same group-orbit as the mixed-mode branch located within $\mathrm{Fix}(D_4)$. The amplitudes of the $(5, 0)$ and $(0, 5)$ modes are 0.0484 and the amplitudes of the $(4, 3)$ and $(3, 4)$ modes are 0.0429. The relative difference between the mode amplitudes decreases as r decreases. A pattern very similar to that shown in figure 6(*a*) was sketched by Gomes *et al* [20]; however, this is the first time that such a pattern has been obtained analytically or numerically in a PDE.

The pattern changes in character as r is increased. This is because in a relatively large domain, there are other modes near the marginal circle. For example, the $(5, 1)$ mode has a

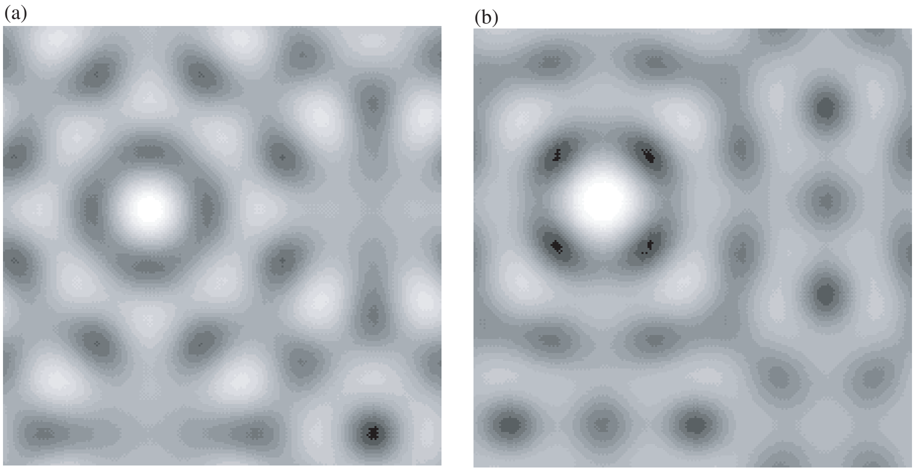

Figure 6. Greyscale plot of numerically obtained stable stationary solutions of (38), with $s = 2.2$, $p = 3.0$, $q = 1.6$. (*a*) $r = 0.005$; (*b*) $r = 0.02$.

positive growth rate for $r > 0.0016$. Figure 6(*b*) shows the stationary solution for $r = 0.02$. In this case the leading Fourier mode is in fact the (5, 1) mode, but the pattern retains D_4 symmetry (as before, shifting the centre of symmetry of the pattern to lie at the centre of the computational domain). As r is increased further, this pattern becomes unstable, with loss of the D_4 symmetry. The D_4-symmetric state is stable for $r \leqslant 0.03$ but by $r = 0.04$ it has undergone a pitchfork bifurcation to a solution with D_2 symmetry (generated by the diagonal reflections m_d and $m_{d'}$).

8. Discussion and directions for future work

We have analysed steady-state pattern-forming instabilities in systems with planar Euclidean symmetry, in large domains on which periodic boundary conditions are applied in both directions. The dynamics of the problem are restricted by the periodic boundary conditions; this is equivalent to restricting our class of solutions to those that are periodic with respect to a square lattice. Previous studies of the dynamics of such instabilities have focused on the cases where the resulting compact symmetry group $\Gamma = D_4 \ltimes T^2$ acts irreducibly. In this paper we concentrate on cases where Γ acts reducibly. This leads to several subtleties in the bifurcation-theoretic machinery we employ. For example, we use a form of the equivariant branching lemma appropriate to the reducible action of Γ, and we are forced to account for the existence and influence of so-called 'hidden' symmetries [8], which turn up in a natural way. The hidden symmetries are in some sense the natural result of attempting to solve a Euclidean-symmetric PDE in a domain that allows only the much smaller symmetry group Γ.

The importance of hidden symmetries has been noted in previous related work (for example [13, 14, 17]). In the steady-state bifurcation on a square superlattice analysed by Dionne *et al* [14] there are axial solution branches that have isotropy subgroups of $D_4 \ltimes T^2$ with one-dimensional fixed-point subspaces. Dionne *et al* then note that these solutions have more symmetries than those in the isotropy subgroups—hidden symmetries. For that problem, the hidden symmetries do not play a role in determining the axial branches initially, they just enlarge the symmetry group of the branch that has already been shown to exist by the equivariant branching lemma.

However, in the analysis in [14] of the corresponding hexagonal superlattice bifurcation, Dionne *et al* note that the existence of a hidden symmetry is crucial to asserting the existence of an axial branch of rhombs, labelled Rh_{h0} (see [14, table 5]). In exactly the same way, having fixed our lattice, hidden symmetries are fundamental to asserting the existence of the axial branches Rh3, Rh4, SS and AS2 here. However, the existence of these solution branches, when instead the finest lattice that supports them is chosen, follows directly from the results of Dionne and Golubitsky [13]. We expect similar effects to be present when even larger domains are considered; this bifurcation problem highlights difficulties that are not encountered when irreducible representations are considered, but that are probably typical of any, more complex, superlattice bifurcation problem.

The axial branches we have found comprise all the roll, square and rhomb branches that one might expect, along with various superlattice patterns. We also note the existence, for all combinations of normal form coefficients, of a primary (but non-axial) branch of slightly perturbed super-squares, of the form (x, x, u, u, u, u) with $|x| \ll |u|$. The corresponding anti-squares (AS1) branch $(0, 0, u, -u, -u, u)$ appears, however, as an axial branch. In addition, there exist axial branches of super-squares of the form $(x, x, x, 0, 0, x)$ and anti-squares (AS2) of the form $(x, x, -x, 0, 0, -x)$. Several other primary but non-axial branches exist for at least an open region of the normal form coefficient space; of particular interest are solutions that resemble twelve-mode quasipatterns and others that resemble hexagons modulated on a longer length-scale. Each of these is stable for choices of the normal form coefficients within a particular open region of the coefficient space.

We have taken great care to explain how our calculation of the stability of the axial branches is affected by hidden symmetries. In particular, we have discussed how the presence of hidden symmetries alters the usual isotypic decomposition of the Jacobian matrix: instead of an isotypic decomposition based on the irreps of the isotropy subgroup of an equilibrium point, we found it necessary to consider a decomposition based on two groups: the hidden symmetry group, for perturbations within a subspace in which the hidden symmetry acts, and the isotropy subgroup, for perturbations outside that subspace. Of course, this does not amount to a general treatment of the effect of hidden symmetries, but it suffices for this particular problem, within the context of perturbations that have the same spatial periodicity as the imposed lattice. The connection between irreducible representations and the various modes of instability of an axial branch does not depend on the weakly nonlinear framework used in this paper, and has been exploited in recent related work [25, 30].

The results of this paper throw up many possible directions for future work, of which we briefly describe three. First, there are two more 'binary mode interaction' problems investigated by Crawford, the '[8, 4]' and '[8, 8]' mode interactions. Hidden symmetries play corresponding roles in the dynamics of these problems. Briefly, in the '[8, 4]' case the subspaces V_j are (as in the '[4, 8]' case) isomorphic to $\mathbb{C}^4$ and $\mathbb{C}^2$, but the action of $D_4 \ltimes T^2$ on $\mathbb{C}^2$ is not translation-free; this subspace is spanned by modes that lie along the diagonals of the square lattice. It turns out that the results for the '[8, 4]' mode interaction are essentially the same as for the '[4, 8]' problem. In the '[8, 8]' case both subspaces V_j are isomorphic to $\mathbb{C}^4$; the square lattice intersects the critical circle $|\boldsymbol{k}| = k_c$ in two sets of eight points, none of the points lying along the axes or diagonals of the lattice. We intend to analyse the details of these bifurcation problems in a future paper. Clearly, by picking ever larger values of s (the real number that defines the real-space lattice $\mathcal{L}$), we can produce lattices that intersect the critical circle $|\boldsymbol{k}| = k_c$ in arbitrarily many sets of eight intersection points, plus four more lying along the axes, in cases where s is an integer.

Second, it is possible to set up codimension-two problems that look superficially very similar to the problems studied in this paper, but in which the hidden symmetries that would arise if all the wave-vectors were of equal magnitude are broken, while the $D_4 \ltimes T^2$ symmetry is kept intact. An example of this is given by the interaction of modes with wave-vectors $\{(\pm 2, \pm 1)k_c/\sqrt{5}\}$ and modes with wave-vectors $\{(\pm k_c, 0), (0, \pm k_c)\}$; this could be referred to as a $2 : \sqrt{5}$ mode interaction.

Finally, we remark that in PDEs such as the Swift–Hohenberg equation or the 'long-wavelength' model discussed by Knobloch [21] and Skeldon and Silber [28], it can be shown that if the coefficient a_θ in (40) does not depend on the lattice angle θ, then $a_\theta = 2a_1$, in which case only rolls can be stable. Even if a_θ does depend on θ, then $a_\theta \to 2a_1$ as $\theta \to 0$. This means that small-angle rhombs are always unstable, as, we speculate, are super-square and anti-square patterns involving a small angle. We speculate also that the PDE (38) does not suffer from this restriction because of the ∇^2 in front of the linear term. This may provide at least a partial explanation of why it is possible that stable solutions to this PDE can involve several modes, with small angles between them. Work on this, and related issues, is in progress.

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Appendix

In this appendix we give the proof of the theorem concerning T^2-invariant polynomials, stated in section 4.1, and also prove a corollary that states that the only invariants of order exactly $2(a + b)$ are those found by Dionne *et al* [14]. For convenience we first re-state the theorem.

Theorem. *All nontrivial invariants are of order at least* $2(a + b)$.

Proof. We distinguish the three cases (i) $m = n = 0$, (ii) $m = 0$, $n \neq 0$ (the case $m \neq 0$, $n = 0$ is exactly analogous) and (iii) $mn \neq 0$. In each case we prove a bound on the order $O(I)$ of an invariant I; the least of these is $O(I) = 2(a + b)$.

(i) $m = n = 0$. Equations (22) and (23) reduce to

$$Q(p + q) + P(r + s) = 0,$$
$$P(p - q) + Q(r - s) = 0,$$

where P and Q are coprime and not both odd. The result of [14] alluded to above implies that the lowest order invariants in this case are of order $2(P + Q) = 2(a^2 - b^2 + 2ab) = 2(a + b)(a - b) + 4ab \geqslant 4(a + b) + 2$, using the relation $2ab \geqslant a + b + 1$, which comes from re-arranging the inequality $a(b - 1) + b(a - 1) \geqslant 1$ (since $a > b \geqslant 1$).

(ii) $m = 0, n \neq 0$. From (22) we deduce $p + q = \alpha P$, $r + s = \alpha Q$, since P and Q are coprime. Now we distinguish the two sub-cases (a) $\alpha = 0$ and (b) $\alpha \neq 0$.

In case (a) we have $q = -p$, $r = -s$, and so equation (23) implies

$$2Pp + 2Qr + Rn = 0 \qquad (43)$$

and hence $n = 2\tilde{n} \neq 0$ (say) is even, as R must be odd. Then, substituting $P = a^2 - b^2$ and $Q = 2ab$, equation (43) becomes

$$bp + ar \equiv 0 \bmod R,$$

but because $R = a^2+b^2$, we must have $|p|+|r| \geqslant a+b$. Therefore, $I = |p|+|q|+|r|+|s|+|n| \geqslant 2(a+b)+2$.

In case (b) we are able to derive the same bound more easily, because $I \geqslant |p+q| + |r+s| + |n| \geqslant P + Q + 1 = a^2 - b^2 + 2ab + 1 \geqslant (a+b)(a-b) + a + b + 2 \geqslant 2(a+b) + 2$, since $a > b \geqslant 1$.

(iii) $m \neq 0$, $n \neq 0$. Taking (24) modulo R we find

$$a(p+q) - b(r+s) \equiv 0 \bmod R,$$
$$b(p+q) + a(r+s) \equiv 0 \bmod R$$

and hence either

$$p + q \equiv \alpha b \bmod R, \qquad r + s \equiv \alpha a \bmod R, \tag{44}$$

or

$$p + q \equiv \beta a \bmod R, \qquad r + s \equiv -\beta b \bmod R, \tag{45}$$

for some integers α and β that are not both zero. Similarly, taking (25) modulo R we obtain

$$b(p-q) - a(r-s) \equiv 0 \bmod R,$$
$$a(p-q) + b(r-s) \equiv 0 \bmod R,$$

which leads to either

$$p - q \equiv \gamma a \bmod R, \qquad r - s \equiv \gamma b \bmod R, \tag{46}$$

or

$$p - q \equiv \delta b \bmod R, \qquad r - s \equiv -\delta a \bmod R, \tag{47}$$

where at least one of the integers γ and δ is also non-zero. We now divide the analysis into four sub-cases given by the conditions (a) $\alpha\gamma \neq 0$, (b) $\alpha\delta \neq 0$, (c) $\beta\gamma \neq 0$ and (d) $\beta\delta \neq 0$. The analysis of the third and fourth of these is very similar to that of the first two.

In case (a) we substitute (44) into (24) and cancel a factor $a^2 + b^2$ to obtain $a\alpha + m = 0$. It is not possible to decrease the order of the invariant by adding multiples of $R = a^2 + b^2$ to $p + q$ since these would increase $|m|$, and hence also the order of the invariant. Similarly we substitute (46) into (25) and again cancel a factor of $a^2 + b^2$ to obtain $a\gamma + y = 0$. Hence,

$$O(I) \geqslant |p+q| + |r+s| + |m| + |n| \geqslant a + b + 2a \geqslant 2(a+b). \tag{48}$$

In case (b) we proceed exactly as before; the substitution of (44) into (24) produces $a\alpha + m = 0$. The substitution of (47) into (25) produces $y = b\delta$, so in this case

$$O(I) \geqslant |p+q| + |r+s| + |m| + |n| \geqslant 2(a+b), \tag{49}$$

and we cannot improve on the bound $2(a+b)$.

Cases (c) and (d) follow exactly analogously, but in both cases the bound $2(a+b)$ cannot be improved on. In case (c) we deduce $n + a\gamma = 0$ and $m + b\beta = 0$, substituting (45) and (46) into (24) and (25). In case (d) we use (45) and (47) and note that we deduce $b\beta + m = 0$ and $n - b\beta = 0$, which implies $|m| + |n| \geqslant 2b$, a weaker result than in cases (a)–(c). However, the bound for the order of the invariant remains the same:

$$O(I) \geqslant |p+q| + |r-s| + |m| + |n| \geqslant 2(a+b). \tag{50}$$

This concludes the analysis of case (iii), and hence we have shown that there are no nontrivial invariants of order less than $2(a+b)$. $\square$

Corollary. *The only invariants of order exactly* $2(a+b)$ *are of the same form as those found by [14].*

Proof. We prove the corollary by considering case (iii) of the proof of the theorem in more detail. The proof of the theorem above shows that cases (i), (ii) and (iii)(a) produce invariants that are always of strictly higher order than $2(a+b)$. Setting $|\alpha| = |\delta| = 1$ in case (iii)(b) results in the solution $p = b$, $q = r = 0$, $s = a$, $m = -a$, $n = b$, which is of the form discussed in the paragraph after equation (27). If either $|\alpha| > 1$ or $|\delta| > 1$ then the resulting invariant must have order

$$O(I) \geqslant \max(|\alpha|, |\delta|)(a+b) + a|\alpha| + b|\delta| > 2(a+b).$$

Case (iii)(c) gives exactly analogous results, and so cases (iii)(b) and (iii)(c) produce all the invariants found by Dionne *et al* [14]. Case (iii)(d) gives slightly different results; because $a \pm b$ is odd, no solutions with $|\beta| = |\delta| = 1$ exist. Hence $|\beta| + |\delta| > 2$ and so

$$O(I) \geqslant (|\beta| + |\delta|)(a+b) > 2(a+b). \qquad \square$$

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