Comment on 'The effect of rotation on the Rayleigh–Bénard stability threshold' by A. Prosperetti *Phys. Fluids* **24**,114101 (2012)

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In this Comment we discuss the results presented in Refs. [5] and [4] for the classic problem of Rayleigh–Bénard convection, i.e. the thermal instability of a horizontal layer of viscous fluid heated uniformly from below. In the case of fixed temperature and stress-free horizontal boundaries the critical dimensionless temperature difference (Rayleigh number R) for the onset of motion can be determined exactly ([6]). For fixed temperature, but rigid, boundaries the computation becomes more involved. This case was treated by Chandrasekhar, see Chapter II, section 17 of [1]. He formulated the stability problem as a variational principle and obtained an approximate solution through the use of an appropriate solution *ansatz*. This leads to the following approximate expression for (an upper bound to) the critical Rayleigh number, his equation (311):

$$R(k) = \frac{\beta^6}{k^2}\left[1 - \frac{16\pi^2 k\cosh^2(k/2)}{\beta^4(k+\sinh k)}\right]^{-1}, \tag{1}$$

where $\beta^2 \equiv \pi^2 + k^2$ and k is the horizontal wavenumber of the perturbation to the conduction solution. This expression was re-derived in Ref. [4] through a similar approximation procedure, but without being formulated as a variational problem, and without reference to its earlier existence in Chandrasekhar's book.

We now turn to the effect of rotation on the Rayleigh–Bénard instability for rigid, fixed temperature boundaries. Chandrasekhar considers the effect of rotation about a vertical axis in Chapter III of [1]. This leads naturally to a very similar instability problem, with the overall effect of rotation being to stabilise the flow. Although well-studied, interest in obtaining simplified descriptions of the linear instability continues to attract attention, see for example [3]. We denote the dimensionless rotation rate (Taylor number) by T. The linear instability of thermal convection with rotation can also be formulated as a variational problem, leading again to an approximation for the marginal stability curve $R(k,T)$. As in the non-rotating case, the variational formulation guarantees that any estimate of the critical Rayleigh number will be an upper bound on the true value. Chandrasekhar does not present as simple an explicit expression as (1) but implicitly it is his equation (159), and it is evaluated numerically in his Table VIII. The one-term approximation he presents can be written in the reasonably compact form

$$R(k,T) = \frac{\beta^6 + \pi^2 T}{k^2}\left[1 + \frac{4\pi^2 F}{\beta^2(\beta^6+\pi^2 T)}\right]^{-1}, \tag{2}$$

where

$$F = \frac{\sum_{\text{cyclic}} q_1\beta_1^4(q_3^2-q_2^2)(q_2^2-k^2)(q_3^2-k^2)\tanh(q_1/2)}{\sum_{\text{cyclic}}(q_1^2-k^2)q_2q_3(q_3^2-q_2^2)\tanh(q_2/2)\tanh(q_3/2)}, \tag{3}$$

is the part of the expression that indicates the effect of the rigid boundaries (note that the exact result for the stress-free case is recovered if $F=0$). In (3) the sums are taken over the three terms obtained by cyclically permuting the subscripts (1 2 3), and we define $\beta_j = \pi^2 + q_j^2$. The coefficients q_1, q_2, q_3 are defined as follows: q_1^2, q_2^2 and q_3^2 are the distinct roots of the cubic polynomial $(q^2-k^2)^3 + Tq^2 = 0$.

The need to compute the roots of the cubic is an additional complication, Using standard methods, and defining $Y = (12\sqrt{12T^3 + 81T^2k^4} - 108Tk^2)^{1/3}$, we obtain, exactly,

$$q_1^2 = k^2 + Y/6 - 2T/Y, \qquad q_2^2 = k^2 - \mathrm{e}^{\mathrm{i}\pi/3}(Y/6 + 2T/Y), \qquad q_3^2 = k^2 - \mathrm{e}^{-\mathrm{i}\pi/3}(Y/6+2T/Y). \tag{4}$$

Alternatively, one can compute very good explicit approximations to the roots in a number of ways, for example through applying a Shanks transformation to the first three terms of the Taylor series solution, using T^{-1} as the small parameter [2]. The Shanks transformation yields the simpler, but approximate, expressions

$$q_1^2 = \frac{k^6}{T+3k^4}, \qquad q_2^2 = k^2 + \frac{5k^2T^{1/2}+4\mathrm{i}T}{4T^{1/2}-3\mathrm{i}k^2}, \qquad q_3^2 = k^2 + \frac{5k^2T^{1/2}-4\mathrm{i}T}{4T^{1/2}+3\mathrm{i}k^2}. \tag{5}$$

In Ref. [5] the alternative expression

$$R = \left(\frac{\beta^6}{k^2} + (1-b)\,T\right)\left[1 - \frac{8\pi^2 k(1+\cosh k)}{\beta^4(k+\sinh k)}\right]^{-1}, \tag{6}$$

is proposed, see equation (25) in that paper, and where

$$b = k\left(\frac{8\pi^2}{\beta^4}\frac{1+\cosh k}{k+\sinh k} + \frac{k-\sinh k}{(k+\tanh k)k^2\tanh(k/2)} - \frac{1}{k+\sinh k} + \frac{4}{\beta^2\tanh(k/2)}\right), \tag{7}$$

i.e. in the notation of [5], $b = \beta^6\tilde{B}_{11}/k^2$ where $\tilde{B}_{11}$ is defined in [5], Appendix A, equation (A23). The expressions (6) - (7) are not as complicated as (2) - (3), but they are also not as accurate. This is demonstrated clearly in the tables below where we compare the critical Rayleigh numbers R_c and wavenumbers k_c over a range of values of T.

Table 1: Comparison of estimates for the marginal stability threshold $R_c(T)$ and the corresponding critical wavenumber $k_c(T)$ for the onset of convection in the presence of rotation. Figures in **bold** differ from the (exact) numerical values quoted in Table I of [5].

	Numerical		(2)-(3)-(5)		(6)-(7)	
T	$R_c(T)$	$k_c(T)$	$R_c(T)$	$k_c(T)$	$R_c(T)$	$k_c(T)$
10^2	1 756.3	**3.16**	1 763.8	3.16	1 753.3	3.15
10^3	2 151.3	3.49	2 159.3	3.48	2 074.7	3.41
10^4	4 712.0	4.79	4 717.1	4.78	4 420.6	4.69
10^5	16 719	7.17	16 743	7.18	17 100	7.56
10^6	71 085	10.82	71 587	10.88	78 569	11.92
10^7	324 510	16.34	328 180	16.47	371 470	18.16
10^8	1 525 100	24.64	1 543 900	24.83	1 760 700	27.20
10^9	7 244 600	37.01	7 326 200	37.27	8 316 800	40.40
10^{10}	**34 498 000**	**55.40**	34 817 000	55.71	39 118 000	59.74

Table 2: Comparison of percentage errors between numerical values and those given by different approximation expressions for $R_c(T)$ and $k_c(T)$. Figures in **bold** indicate substantial differences from those quoted in Table I of [5].

	Numerical		(2)-(3)-(5)		(6)-(7)	
T	$R_c(T)$	$k_c(T)$	% R_c	% k_c	% R_c	% k_c
10^2	1 756.3	**3.16**	0.42	0.93	-0.18	-0.41
10^3	2 151.3	3.49	0.37	0.06	-3.56	-2.12
10^4	4 712.0	4.79	0.11	0.02	-6.19	-1.90
10^5	16 719	7.17	0.14	0.20	2.28	5.47
10^6	71 085	10.82	0.70	0.54	10.53	10.21
10^7	324 510	16.34	1.13	0.76	14.47	11.14
10^8	1 525 100	24.64	1.24	0.79	15.45	10.42
10^9	7 244 600	37.01	1.13	0.69	14.80	9.17
10^{10}	**34 498 000**	**55.40**	0.93	0.55	13.39	7.83

Table 1 shows the values for the critical Rayleigh number R_c at which R attains a minimum over wavenumbers k at fixed T, computed numerically, and estimated by the two different methods. Table 2 shows the percentage errors in both R_c and the corresponding critical wavenumber k_c for both approximate expressions. While those for (2) using (3) and (5) for the roots are around 1% accurate (or better) at all values of T shown, expression (6) is substantially worse, particularly at moderately high T. Qualitatively, we remark also that (i) the approximate values computed using the formula (6) of Ref [5] are neither consistently above nor consistently below the true values, and (ii) that the approximation appears to improve only relatively slowly at large T; as shown in [5] the approximation (6) yields the well-known asymptotic limits for R_c and k_c as $T \to \infty$, therefore both methods must become increasingly accurate in the limit $T \to \infty$. On the other hand, comparison of the marginal stability curves produced by the two methods confirms that (6) is more accurate at low Taylor numbers, $T \leq 10^2$, where the exact roots (4) must be used instead of (5) to preserve the accuracy of Chandrasekhar's approximation.

References

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