# The Dynamics of Mode Interactions: Patterns, Bifurcations and Symmetry

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# Outline

- Bifurcation theory and mode interactions without symmetry

- Incomplete and sketchy introduction to symmetric bifurcation theory

- Planar pattern-forming instabilities

- Three examples:

  - Steady-state patterns on a square superlattice

  - Bursts in rotating Rayleigh-Bénard convection

  - Instabilities on very different spatial scales in low-viscosity magnetoconvection

## Bifurcation theory without symmetry

**In the absence of symmetry or other non-generic features, nonlinear systems undergo steady-state and oscillatory bifurcations.**

**We'll consider the nonlinear ODEs $\dot{x} = f(x, \mu)$ for $x \in \mathbb{R}^n$ and with a real parameter $\mu$.**

**Saddle-node bifurcation:** $\dot{x} = \mu - x^2 \quad x \in \mathbb{R}$

**Hopf bifurcation:** $\dot{w} = (\mu + \mathrm{i}\omega) - w|w|^2 \quad w \in \mathbb{C}$

**If $x = 0$ is required (eg by the physics) to be an equilibrium point for all $\mu$ then a steady-state instability generically will be a transcritical bifurcation $\dot{x} = \mu x - x^2$.**

**If there is a symmetry $x \to -x$ then we expect a steady-state instability generically to be a pitchfork bifurcation, $\dot{x} = \mu x \pm x^3$.**

## Degenerate bifurcations without symmetry

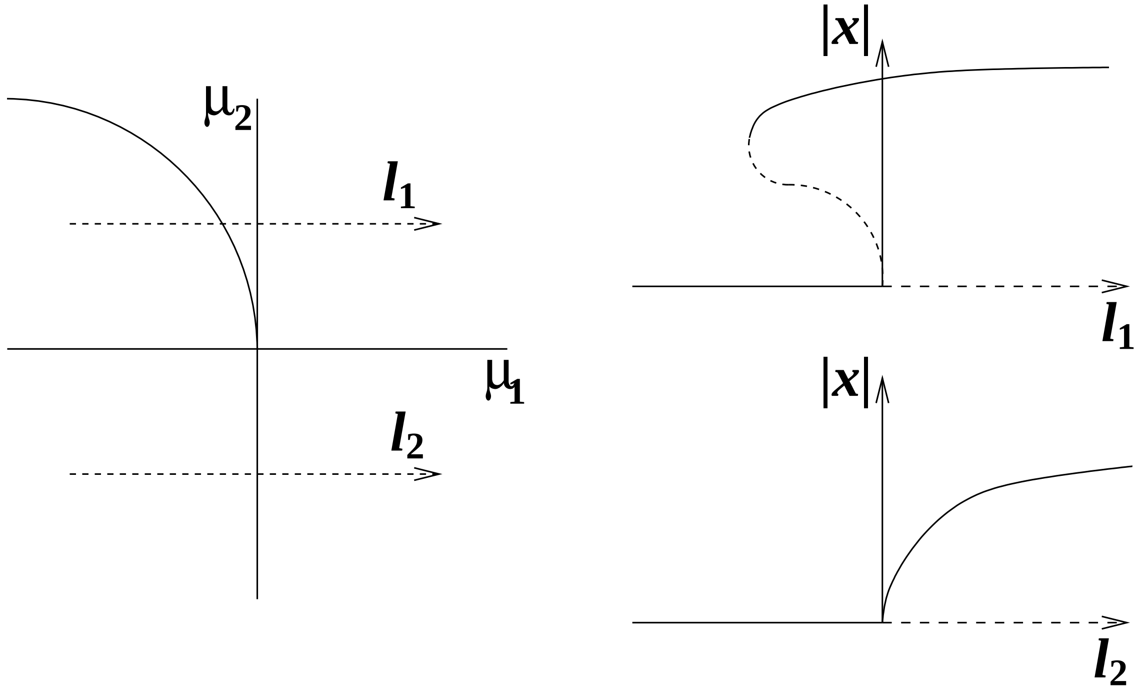
**How can we move beyond these simplest examples?**

**Two ways:**

- **linear degeneracies (mode interactions)**
- **nonlinear degeneracies**

**Example of a nonlinear degeneracy (pitchfork):**

$$\dot{x} \;=\; \mu_1 x + \mu_2 x^3 - x^5$$




**Transition from subcritical to supercritical bifurcation. Codimension 2 (2 unfolding parameters required).**

## Linear degeneracies - mode interactions

**Three codimension-2 possibilities:**

- **steady-state/steady-state (Takens–Bogdanov)**

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- **Hopf / steady-state**

$$\left(\begin{array}{cc|c} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

- **Hopf / Hopf**

$$\left(\begin{array}{cc|cc} 0 & -\omega_1 & 0 & 0 \\ \omega_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\omega_2 \\ 0 & 0 & \omega_2 & 0 \end{array}\right)$$

**Guckenheimer & Holmes (second edition, 1986), chapter 7.**

## Hopf / steady-state mode interaction

**Consider an eqm point with linearisation**

$$\left(\begin{array}{cc|c} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

**"simultaneously" undergoes Hopf and steady-state (assume saddle-node) bifurcations in orthogonal directions. Normal form:**

$$\begin{aligned} \dot{z} &= \mathrm{i}\omega z + (a + \mathrm{i}b)xz + x^2 z + O(4) \\ \dot{x} &= x^2 \pm |z|^2 + O(4) \end{aligned}$$

$z \in \mathbb{C}$, $x \in \mathbb{R}$. **Normal form symmetry** $(z, x) \to (z\mathrm{e}^{\mathrm{i}\theta}, x)$, **broken at higher orders.**

**As before, need to introduce unfolding (bifurcation) parameters to examine all possible qualitatively distinct types of dynamics near** $x = z = 0$. **Two parameters again suffice here:**

$$\begin{aligned} \dot{z} &= (\mu_1 + \mathrm{i}\omega)z + (a + \mathrm{i}b)xz + x^2 z + O(4) \\ \dot{x} &= \mu_2 + x^2 \pm |z|^2 + O(4) \end{aligned}$$

**Now compute location and stability of equilibria, periodic orbits etc.**

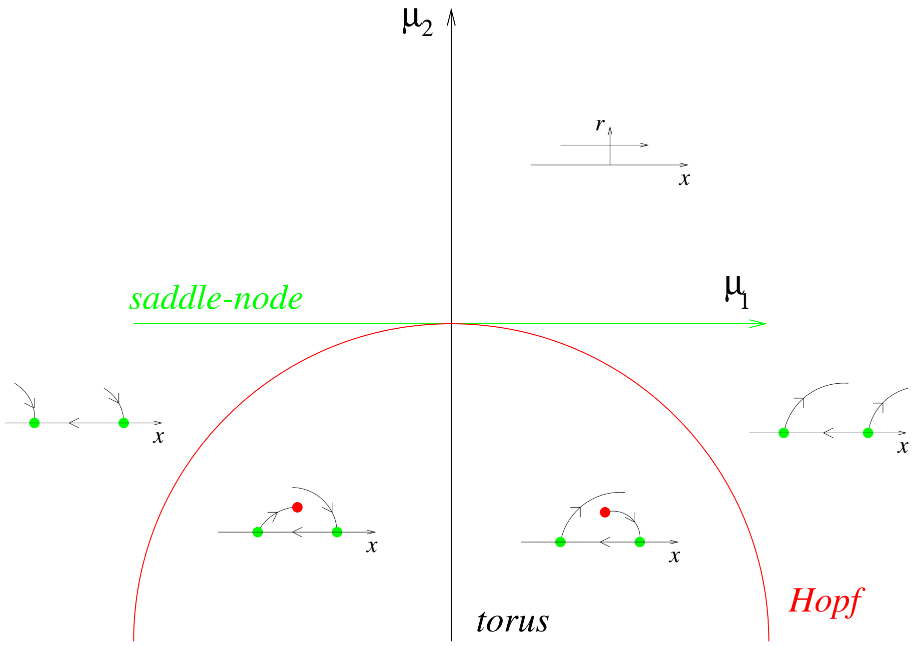
## Hopf / steady-state mode interaction

**Drop the cubic $x^2 z$ term (higher order)**

**Change co-ordinates using $z = re^{i\theta}$:**

$$
\begin{aligned}
\dot{r} &= r(\mu_1 + ax) \\
\dot{x} &= \mu_2 + x^2 \pm r^2 \\
\dot{\theta} &= \omega + bx
\end{aligned}
$$

**$\theta$ decouples! $(r, x)$ dynamics in the '+' case, $a < 0$ are:**



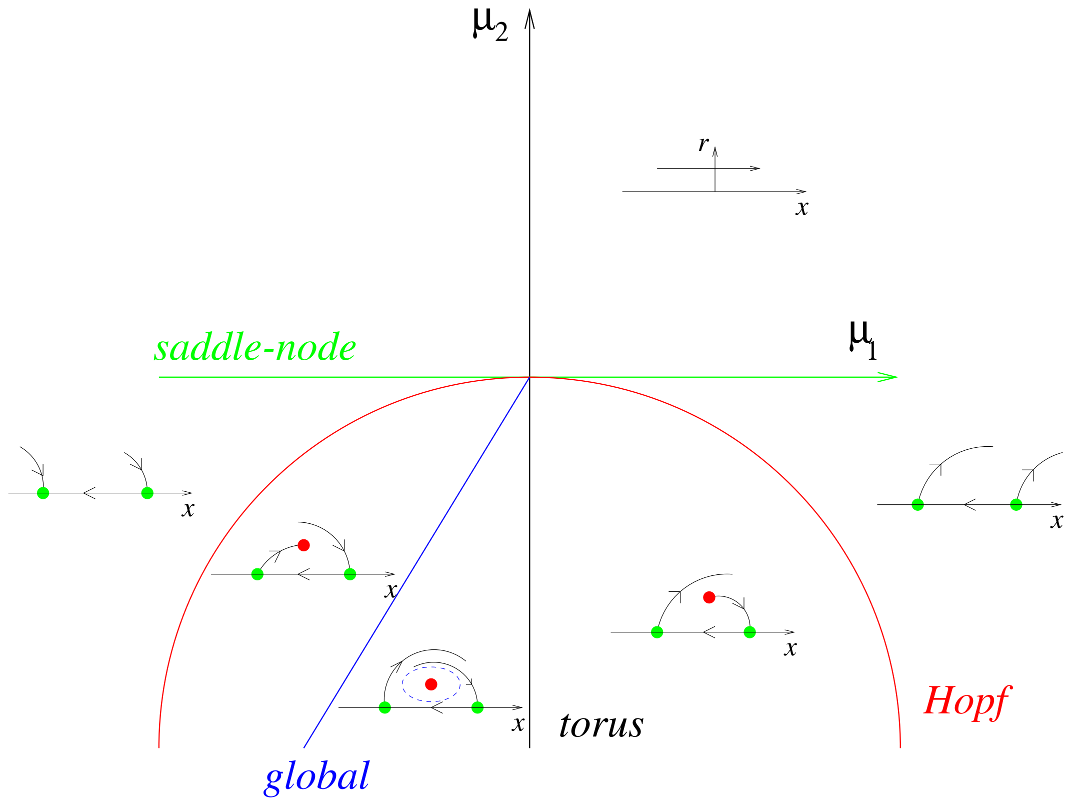

**Questions:**

- **Torus bif'n supercritical or subcritical?**

- **Global bif'ns?**

## Hopf / steady-state mode interaction

**Find that the torus bif'n is degenerate - need cubic term $x^2z$ to break degeneracy. Then a global bif'n also turns up, to destroy torus:**




**High-order terms will also break the normal form symmetry, i.e. picture is not really axisymmetric and chaotic dynamics results.**

## Summary of Hopf / steady-state mode interaction

- **No 'physical' symmetries**
- **Normal form symmetry (Hopf bifurcation part)**
- **Secondary local and global bifurcations**
- **Breaking normal form symmetry leads to complicated dynamics**

## Bifurcation theory in the presence of symmetry

**ODEs $\dot{x} = f(x, \mu)$ are equivariant (symmetric) w.r.t the group $\mathcal{G}$ if**

$$R_g f(x, \mu) \;=\; f(R_g x, \mu)$$

**where $R_g$ is an $n \times n$ matrix giving the action of the group element $g \in \mathcal{G}$ on $\mathbb{R}^n$.**

**If the map $g \mapsto R_g$ is a homomorphism, i.e. $R_g R_h = R_{gh}$, then the collection of matrices $\{R_g\}$ is a representation of $\mathcal{G}$.**

**In the absence of symmetry, simplest possible bif'ns have one zero eigenvalue or one c.c. pair of eigenvalues; in the symmetric setting we look for the analogous simplest possible structure which is still generic.**

**Simplest representations are irreducible ones:**

**A representation of $\mathcal{G}$ is irreducible if there are no $\mathcal{G}$-invariant subspaces of $\mathbb{R}^n$.**

**A representation of $\mathcal{G}$ is absolutely irreducible if the only linear maps that commute with all the $R_g$ are real multiples of the identity matrix.**

## Steady-state symmetric bifurcations

**In generic steady-state bifurcations the action of $\mathcal{G}$ on the centre manifold is absolutely irreducible:**

$$Df(0,\mu) \;=\; c(\mu)Id_n$$

$c(\mu) \in \mathbb{R}$, $c(0) = 0$, $c'(0) \neq 0$

**Multiple zero eigenvalues are expected since if $Df(v) = 0$ then**

$$DfR_g(v) \;=\; R_gDf(v) = R_g(0) = 0$$

**i.e. $R_g(v)$ is also an eigenvector with eigenvalue zero.**

**Not all absolutely irreducible representations are of the same dimension!**

**First problem:** **find the absolutely irreducible representations of $\mathcal{G}$.**

**Second problem:** **analyse dynamics of a steady-state bifurcation in each case.**

# The Equivariant Branching Lemma

**Turns the analytic problem of locating bifurcating branches into the algebraic problem of finding isotropy subgroups of $\mathcal{G}$.**

**Definitions:**

**The isotropy subgroup $\Sigma_x \subset \mathcal{G}$ of a point $x \in \mathbb{R}^n$ is**

$$\Sigma_x \quad = \quad \{\sigma \in \mathcal{G} : \sigma x = x\}$$

**The fixed point subspace of a subgroup $\Sigma \subset \mathcal{G}$ is denoted $\mathrm{Fix}(\Sigma)$:**

$$\mathrm{Fix}(\Sigma) \quad = \quad \{x \in \mathbb{R}^n : \sigma x = x \forall \sigma \in \Sigma\}$$

**Theorem (the Equivariant Branching Lemma): Let $\mathcal{G}$ be a compact Lie group acting absolutely irreducibly on $\mathbb{R}^n$. Let $\dot{x} = f(x, \mu)$ be a smooth $\mathcal{G}$-equivariant bifurcation problem, i.e. $Df(0,0) = 0$ and $Df(0,\mu) = c(\mu)\mathrm{I}$ where $c(0) = 0$ and $c'(0) \neq 0$. Then there exists a unique solution branch $x(\mu)$ to $f(x(\mu), \mu) = 0$ with isotropy subgroup $\Sigma_x$ for every isotropy subgroup satisfying dim $\mathrm{Fix}(\Sigma_x) = 1$.**

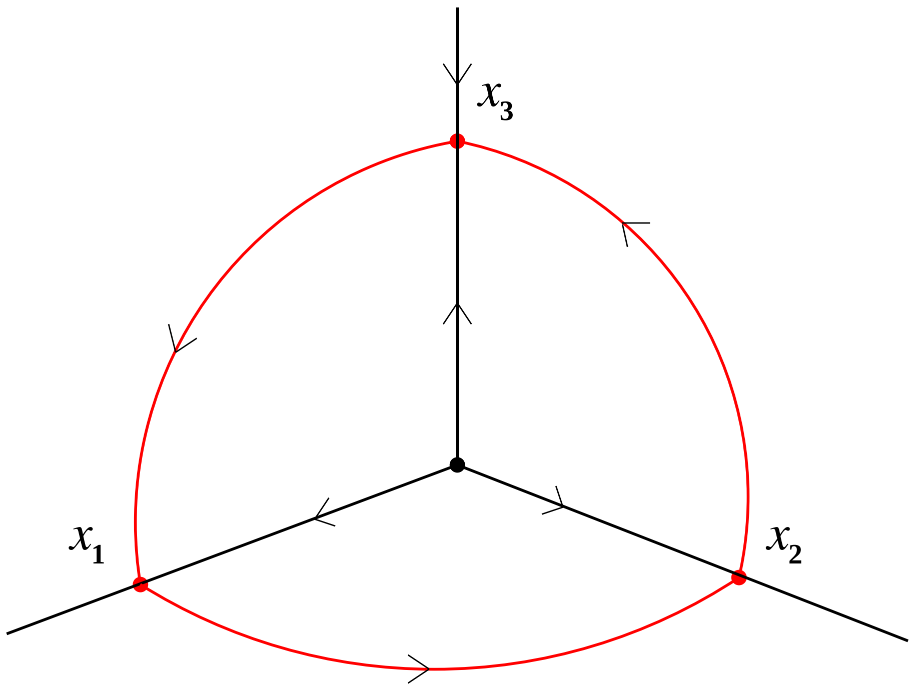
# Axial branches

**Branches of solutions $x(\mu)$ with dim $\mathrm{Fix}(\Sigma_x) = 1$ are termed axial.**

**Morally, the EBL affirms that within $\mathrm{Fix}(\Sigma_x)$ (which is a flow-invariant subspace), we have a 1D problem with an eqm at $x = 0$ and we generically see either a standard transcritical or pitchfork bif'n.**

**Many different subspaces $\Rightarrow$ many axial branches (with different isotropy subgroups)**

**Not the whole story: steady-state symmetric bif'n problems can generate much more exciting dynamics than just branches of equilibra!**



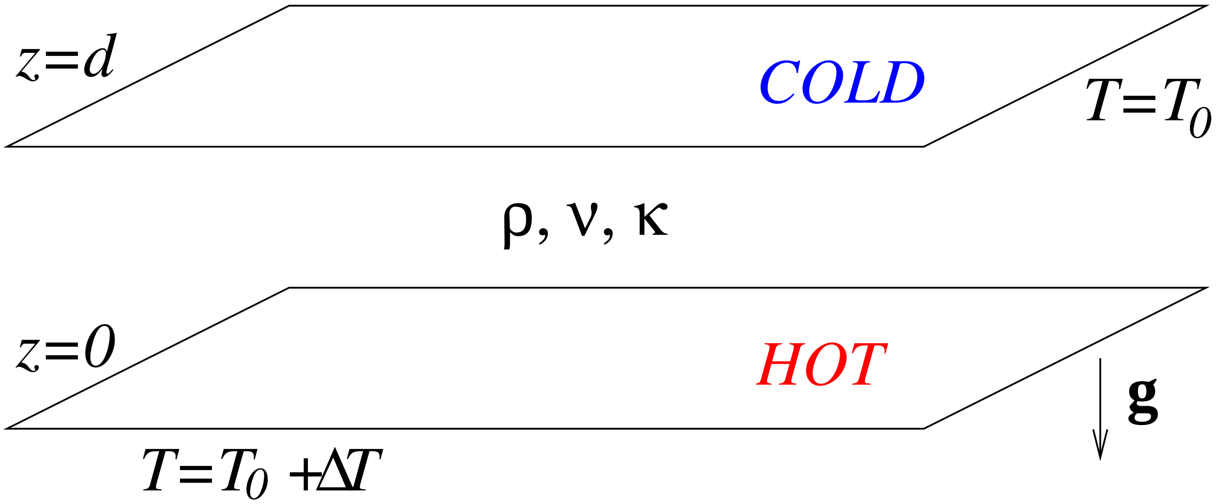

$$
\begin{aligned}
\dot{x}_1 &= x_1[\mu + a x_1^2 + b x_2^2 + c x_3^2] \\
\dot{x}_2 &= x_2[\mu + a x_2^2 + b x_3^2 + c x_1^2] \\
\dot{x}_3 &= x_3[\mu + a x_3^2 + b x_1^2 + c x_2^2]
\end{aligned}
$$

**Küppers & Lortz (1969), Busse & Heikes (1980), Guckenheimer & Holmes (1988)**

**Also: "Instant Chaos", Guckenheimer & Worfolk (1992)**

## Steady-state planar pattern forming instabilities

**Rayleigh-Bénard convection is the archetypal example.**

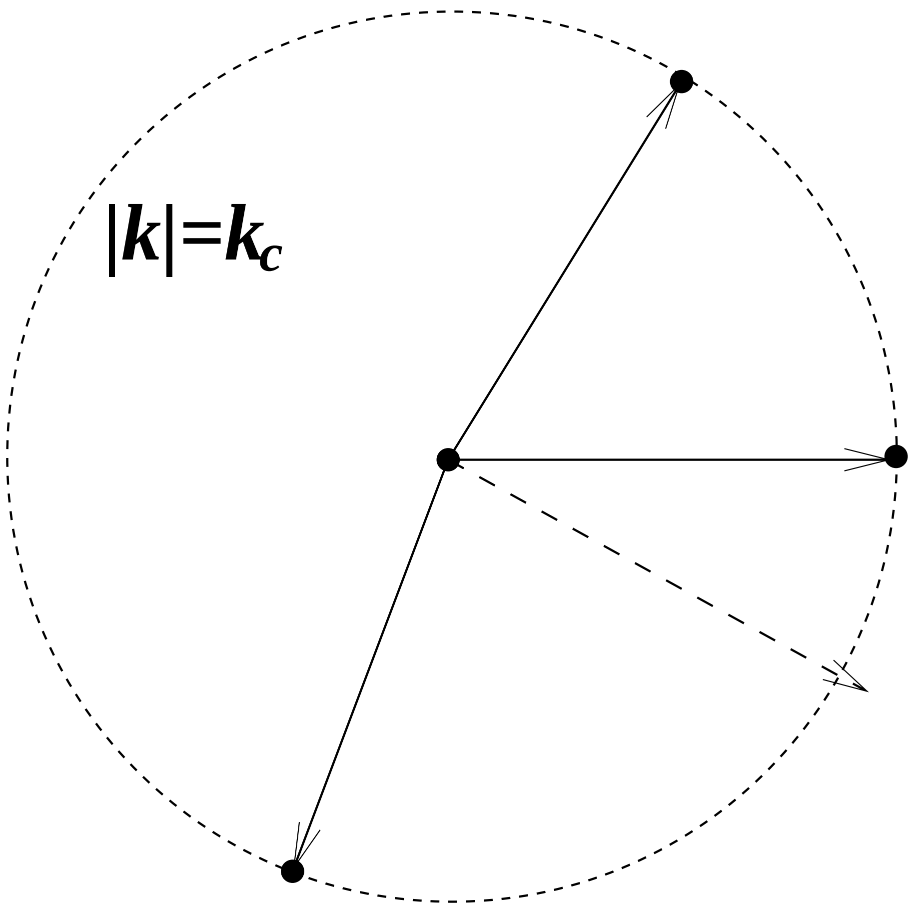



**Define the Rayleigh number** $\mathrm{Ra} = \frac{\hat{\alpha} g \Delta T d^3}{\kappa\nu}$.

**For** $\mathrm{Ra} < \mathrm{Ra}_c$ **the conduction state (no fluid flow) is stable, and symmetric under all rotations, reflections and horizontal translations -** $E(2)$.

**At** $\mathrm{Ra} = \mathrm{Ra}_c$ **there is a linear instability of this state to perturbations with non-zero wavenumber** $|\mathbf{k}| = k_c$.

## Steady-state planar pattern forming instabilities




**Two major problems:**

- **Rotational degeneracy: all horizontal directions are 'equivalent' $\Rightarrow$ 'centre manifold' is infinite-dimensional**

- **No 'spectral gap' - can't apply CMT directly**

## Restriction to a spatially-periodic lattice

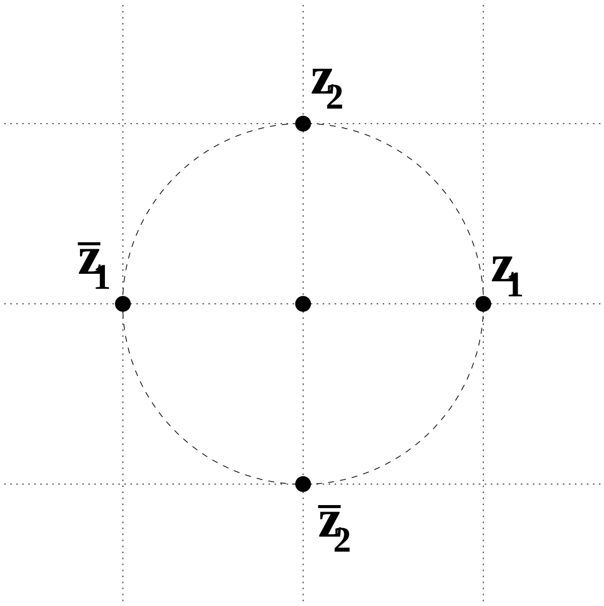
**Solves both of these problems. Equivalent to doing numerics in a square box with periodic b.c.s. All patterns are now periodic w.r.t. the lattice $\mathcal{L}$:**

$$\mathcal{L} = \left\{ n\ell_1 + m\ell_2 : (n,m) \in \mathbb{Z}^2,\ \ell_1 = \left(\frac{2\pi s}{k_c}, 0\right),\ \ell_2 = \left(0, \frac{2\pi s}{k_c}\right) \right\}$$
$$\mathcal{L}^* = \left\{ n\mathbf{q}_1 + m\mathbf{q}_2 : (n,m) \in \mathbb{Z}^2,\ \mathbf{q}_1 = \left(\frac{k_c}{s}, 0\right),\ \mathbf{q}_2 = \left(0, \frac{k_c}{s}\right) \right\}$$

**Symmetry group is now $\mathcal{G} = D_4 \ltimes T^2$, not $E(2)$.**

**Different choices of $s$ lead to different representations of $\mathcal{G}$.**

**Simplest case is $s = 1$:**




**Planform:** $u = \mathrm{Re}[z_1 e^{\mathrm{i}k_c x} + z_2 e^{\mathrm{i}k_c y}]$

## Representation of $D_4 \ltimes T^2$ in the case $s = 1$

$m_x : (x, y) \to (x, -y) : \ (z_1, z_2) \to (z_1, \bar{z}_2)$

$m_d : (x, y) \to (y, x) : \ (z_1, z_2) \to (z_2, z_1)$

$[\xi, \eta] : (x, y) \to (x+\xi/k_c, y+\eta/k_c) : \ (z_1, z_2) \to (z_1 e^{\mathrm{i}\xi}, z_2 e^{\mathrm{i}\eta})$

**This representation is absolutely irreducible. Apply the EBL - find two axial branches:**

- **Rolls:** $\Sigma_R = \langle [0, \eta], m_x, m_y \rangle \cong O(2) \times \mathbb{Z}_2$

  $\mathrm{Fix}(\Sigma_R) = (x, 0)$**, dim** $\mathrm{Fix}(\Sigma_R) = 1$

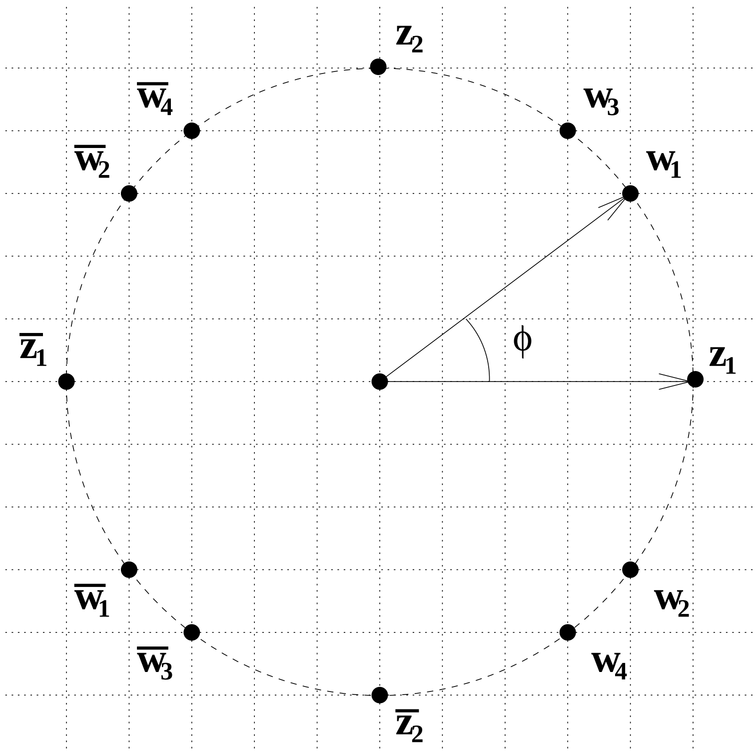
- **Squares:** $\Sigma_S = \langle m_x, m_d \rangle \cong D_4$

  $\mathrm{Fix}(\Sigma_S) = (x, x)$**, dim** $\mathrm{Fix}(\Sigma_S) = 1$

**Use equivariance of $f(z_1, z_2, \mu)$ to construct the normal form 'amplitude equations'. Coefficients determine relative stability of these two branches.**

## Example 1: A reducible representation of $D_4 \ltimes T^2$

**Rescale lengths so that $k_c = R$ where $P^2 + Q^2 = R^2$, and take $s = R$ (e.g. in the case $P = 3$, $Q = 4$, $R = 5$):**




$$
\begin{aligned}
u &= \mathrm{Re}[z_1 e^{\mathrm{i}Rx} + z_2 e^{\mathrm{i}Ry} + w_1 e^{\mathrm{i}(Qx+Py)} + w_2 e^{\mathrm{i}(Qx-Py)} \\
&\quad + w_3 e^{\mathrm{i}(Px+Qy)} + w_4 e^{\mathrm{i}(Px-Qy)}]
\end{aligned}
$$

**Centre manifold is now**

$$
\mathbb{E}^c(\Omega) = \{(\mathbf{z}, \mathbf{w})\} \equiv \{(z_1, z_2, w_1, w_2, w_3, w_4)\} \cong \mathbb{C}^6
$$

## A reducible representation of $D_4 \ltimes T^2$

**Representation of $D_4 \ltimes T^2$ is generated by**

$$\begin{aligned} m_x &: (x,y) \to (x,-y) \\ &: (\mathbf{z},\mathbf{w}) \to (z_1, \bar{z}_2, w_2, w_1, w_4, w_3) \end{aligned}$$

$$\begin{aligned} m_d &: (x,y) \to (y,x) \\ &: (\mathbf{z},\mathbf{w}) \to (z_2, z_1, w_3, \bar{w}_4, w_1, \bar{w}_2) \end{aligned}$$

$$\begin{aligned} [\xi,\eta] &: (x,y) \to (x+\xi, y+\eta) \\ &: (\mathbf{z},\mathbf{w}) \to (z_1 e^{\mathrm{i}R\xi}, z_2 e^{\mathrm{i}R\eta}, w_1 e^{\mathrm{i}(Q\xi+P\eta)}, \\ &\quad w_2 e^{\mathrm{i}(Q\xi-P\eta)}, w_3 e^{\mathrm{i}(P\xi+Q\eta)}, w_4 e^{\mathrm{i}(P\xi-Q\eta)}) \end{aligned}$$

$\mathbb{E}^c(\Omega) = V_1 \oplus V_2$**:**

$V_1 = \{(z_1, z_2)\} \cong \mathbb{C}^2$**,** $\qquad V_2 = \{(w_1, w_2, w_3, w_4)\} \cong \mathbb{C}^4$

$D_4 \ltimes T^2$ **acts separately on** $V_{1,2}$**, so the representation is reducible.**

**But each representation on** $V_{1,2}$ **separately is irreducible - a 'binary' mode interaction.**

## Hidden symmetries

**2 absolutely irreducible representations for** $(\mathbf{z}, \mathbf{w}) \in V_1 \oplus V_2$

$\Rightarrow$ **linear terms look like**

$$\begin{pmatrix} \dot{\mathbf{z}} \\ \dot{\mathbf{w}} \end{pmatrix} = \left(\begin{array}{c|c} \mu \mathrm{I} & 0 \\ \hline 0 & \hat{\mu} \mathrm{I} \end{array}\right) \begin{pmatrix} \mathbf{z} \\ \mathbf{w} \end{pmatrix}$$

**But we expect** $\mu = \hat{\mu}$ **since all wave-vectors go unstable simultaneously!**

**Turns out there are 'hidden' symmetries that force** $\mu = \hat{\mu}$**. So there is only one bifurcation parameter (even though this is a codimension-2 bifurcation).**

# Hidden symmetries

**'Obvious' symmetries are in the normaliser**

$$N(\mathcal{L}) = \{\sigma \in E(2) : \sigma^{-1}\mathcal{L}\sigma = \mathcal{L}\} = D_4 \ltimes T^2 \times \mathcal{L}.$$

**Hidden symmetries are elements of $E(2) - N(\mathcal{L})$. May affect bif'n problem, though!**

**Hidden symmetries occur because the ODEs for the lattice problem (which give the dynamics on the centre manifold $\mathbb{E}^c(\Omega)$) are the restriction of an $E(2)$-equivariant vector field $\mathcal{F}$ defined on the 'full' centre manifold $\mathbb{E}^c(\mathbb{R}^2)$.**

**E.g. the rotation $R_\phi$ anticlockwise through an angle $\phi = \tan^{-1} P/Q$: define the subspace**

$$\mathcal{U} = \{(z_1, z_2, w_1, 0, 0, w_4)\} = \mathrm{Fix}\left(\left[\frac{2\pi Q}{R}, \frac{2\pi P}{R}\right]\right)$$

**then the hidden symmetry**

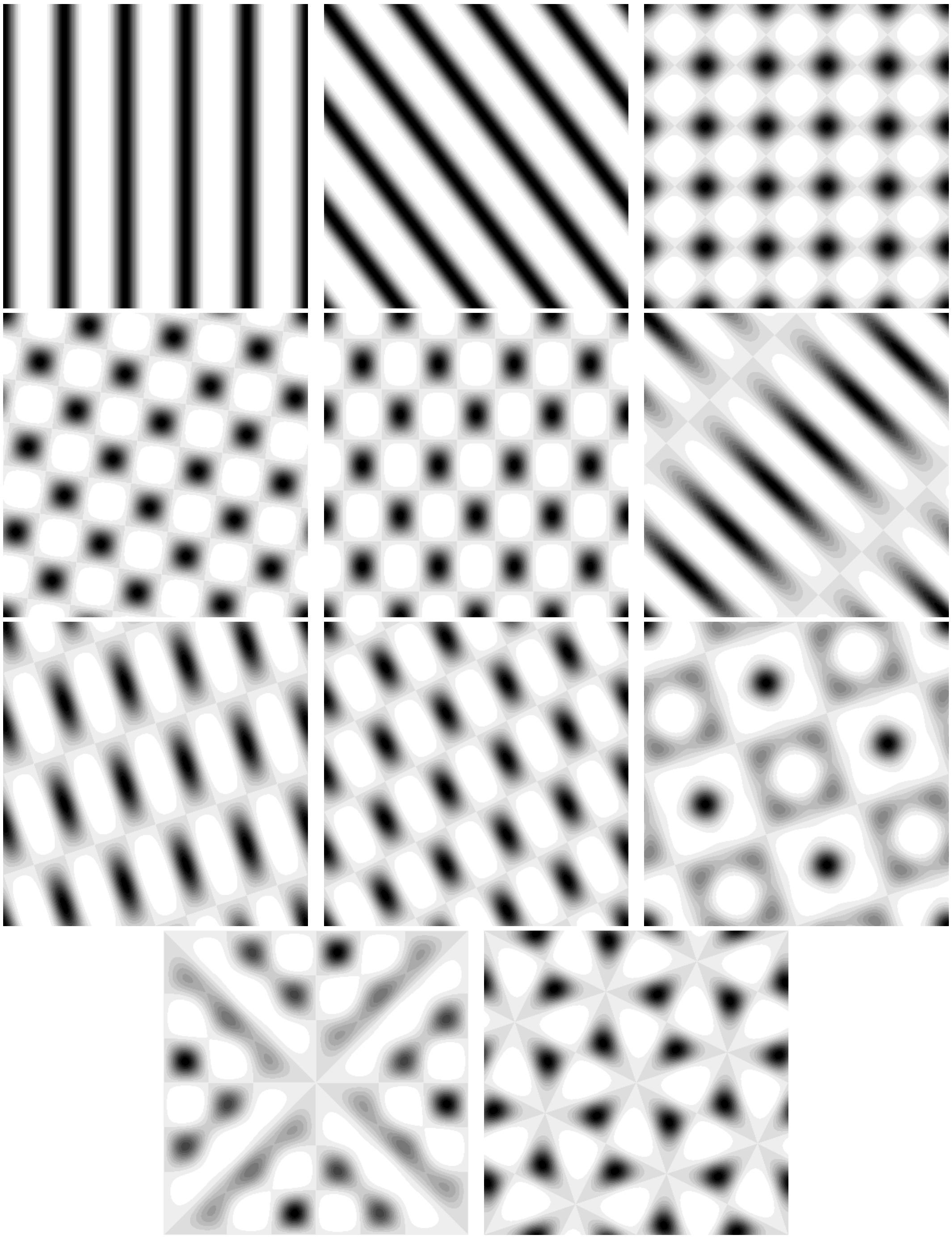
$$m_x \circ R_\phi^{-1} : \; (z_1, z_2, w_1, 0, 0, w_4) \; \to (w_1, w_4, z_1, 0, 0, z_2),$$

**acts on this subspace (but not on all of $\mathbb{E}^c$)**

**$m_x \circ R_\phi^{-1}$ links the amplitude equations (restricted to $\mathrm{u}_1$) for $z_1$ and $w_1$, hence forces $\mu = \hat{\mu}$**

**Further details: Crawford, Nonlinearity (1994)**

# Axial branches



**Greyscale images of the eleven axial planforms, in the case $P = 3$, $Q = 4$, $R = 5$. From left to right, top to bottom: rolls 1, rolls 2, squares 1, squares 2, rhombs 1, rhombs 2, rhombs 3, rhombs 4, super-squares, anti-squares 1, anti-squares 2.**

## Non-axial branches

**Are there branches with all six modes non-zero?**

**Look within the 2D subspace** $\mathrm{Fix}(D_4) = (x, x, u, u, u, u)$**:**

**Normal form equations are**

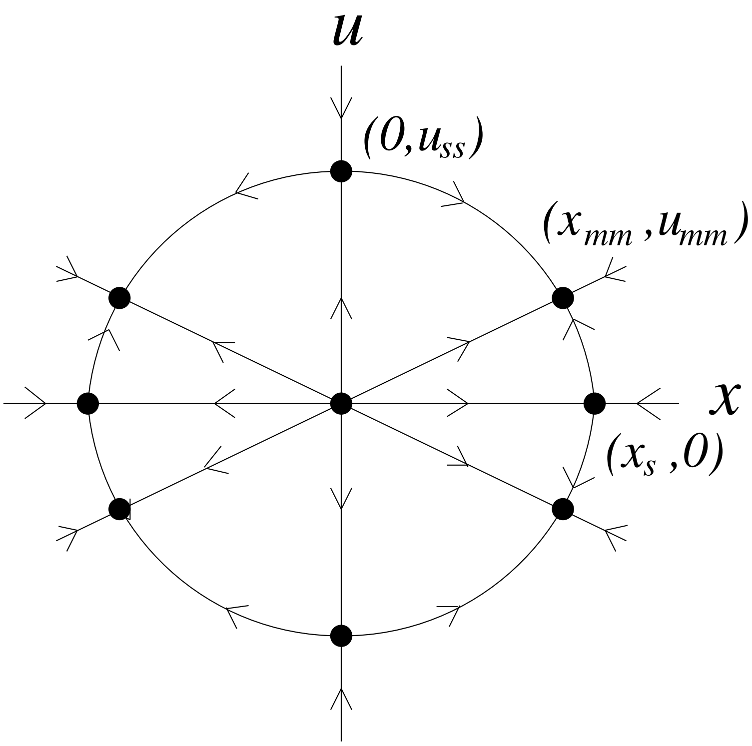
$$\begin{aligned} \dot{x} &= x[\mu + Ax^2 + 2Bu^2] + xh_1(x, u) + Cu^7 \\ \dot{u} &= u[\mu + Du^2 + Bx^2] + uh_2(x, u) \end{aligned}$$

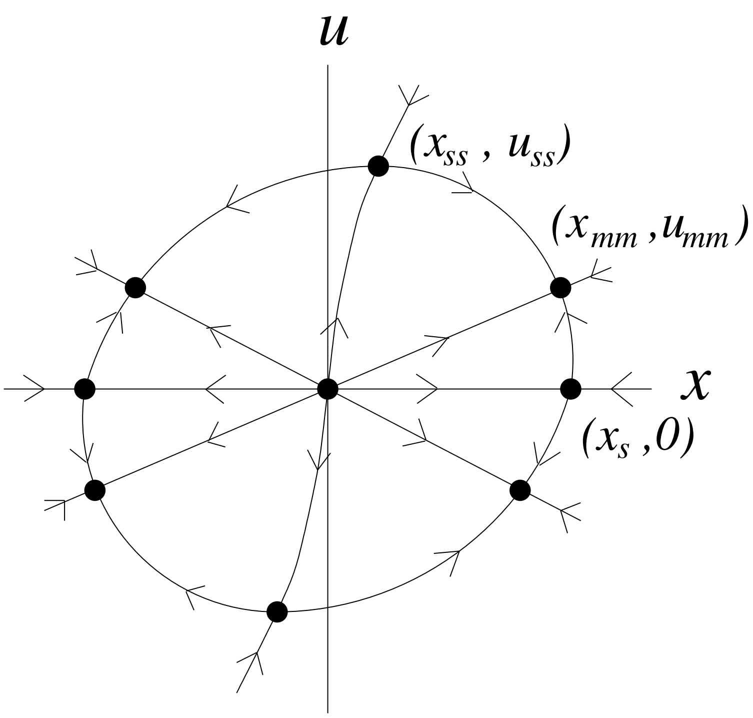
**Cubic truncation has three distinct types of equilibria:**

- **a squares solution** $x_s^2 = -\mu/A$**,** $u = 0$ **(axial)**
- **a 'super-squares' solution** $x_{ss} = 0$**,** $u_{ss}^2 = -\mu/D$ **(not axial)**
- **mixed-mode solutions** $x_{mm}^2 = \mu(D-2B)/(2B^2-AD)$**,** $u_{mm}^2 = \mu(A-B)/(2B^2-AD)$ **(not axial)**

## Non-axial branches may persist

**But if these two non-axial branches are hyperbolic equilibria for the cubic truncation they will persist for the full normal form.**



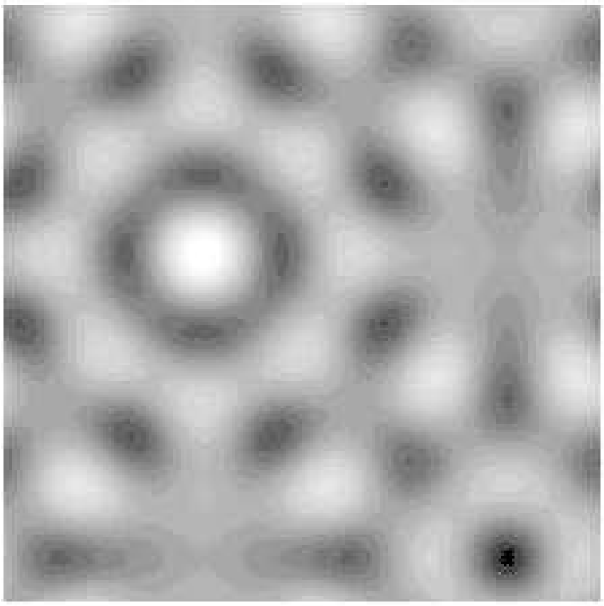



## Non-axial branches appear in numerics

**Numerical solutions of the PDE**

$$\frac{\partial u}{\partial t} = -\nabla^2 \left( ru - (1+\nabla^2)^2 u - pu\nabla^2 u - q|\nabla u|^2 - su^2 - u^3 \right)$$

**form 'mixed-mode' patterns $(x_{mm}, u_{mm})$:**



**Greyscale plot of a numerically obtained stable stationary solution. Parameters are $s = 2.2$, $p = 3.0$, $q = 1.6$, $r = 0.005$.**

## Summary of Example 1

- **Steady-state / steady-state mode interaction**

- **'Hidden' symmetries**

- **Natural sequence of more complicated mode interaction problems on finer and finer dual lattices $\mathcal{L}^*$ (bigger and bigger periodic domains).**

**For more details:**

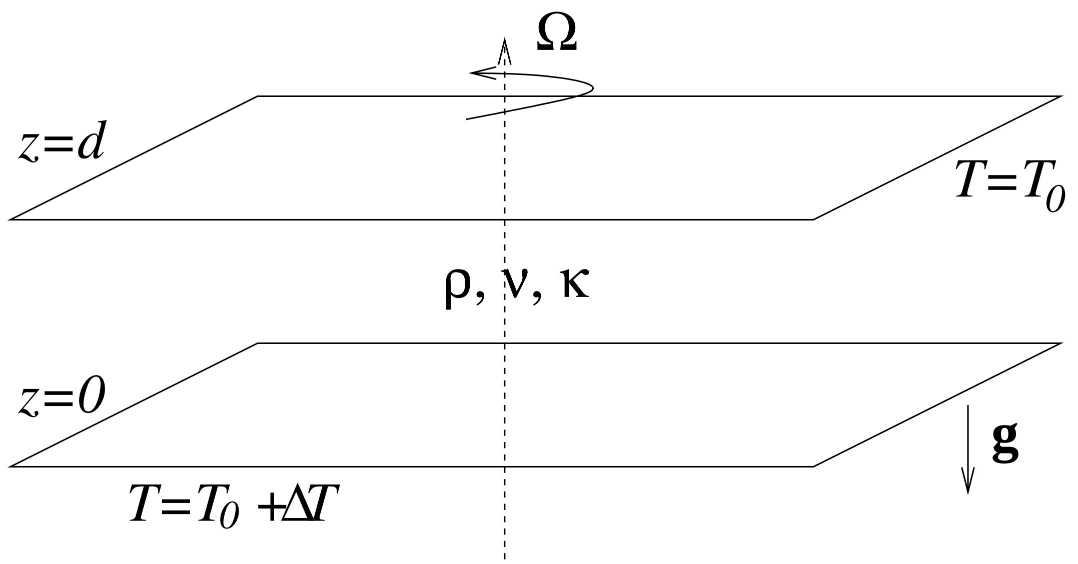
**J.H.P. Dawes, P.C. Matthews and A.M. Rucklidge, Reducible actions of $D_4 \ltimes T^2$: superlattice patterns and hidden symmetries. Preprint.**

## Example 2: $1 : \sqrt{2}$ Hopf/steady–state mode interaction

**Hopf bifurcations with symmetry generalise in a similar way to steady-state bif'ns.**

**Won't present details here.**

**Rotating Rayleigh-Bénard convection:**




**Dimensionless parameters are:**

- **Prandtl number** $\sigma = \frac{\nu}{\kappa}$
- **Taylor number (rotation rate)** $\tau^2 = \frac{4\Omega^2 d^4}{\nu^2}$

**...and Rayleigh number** $\mathrm{Ra} \sim \Delta T$ **as before.**

## Linear Theory

**Linear stability criteria produced by consideration of perturbations**

$\sim \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}+(\lambda+\mathrm{i}\omega)t} f(z)$

**The condition for marginal stability** $\lambda = 0 \Rightarrow$ **a critical** $\mathrm{Ra}_c = \mathrm{Ra}_c(\tau, \sigma)$ **and** $k_c = k_c(\tau, \sigma)$ **for** $\omega = 0$ **(steady instability) and** $\omega \neq 0$ **(oscillatory instability).**

**Steady instability:**

$$R_s = \frac{(\pi^2 + k^2)^3 + \tau^2\pi^2}{k^2}$$

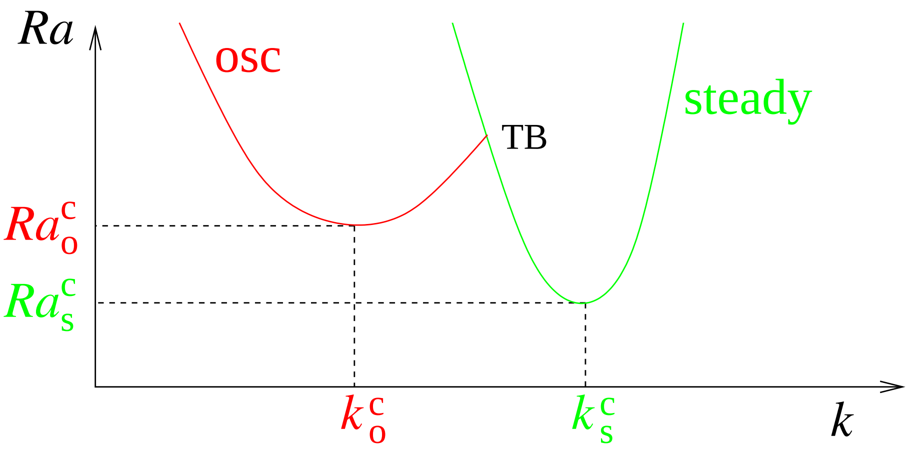
**Oscillatory instability:**

$$R_o = \frac{2(\pi^2 + k^2)^3(1+\sigma)}{k^2} + \frac{2\sigma^2\tau^2\pi^2}{k^2(1+\sigma)}$$

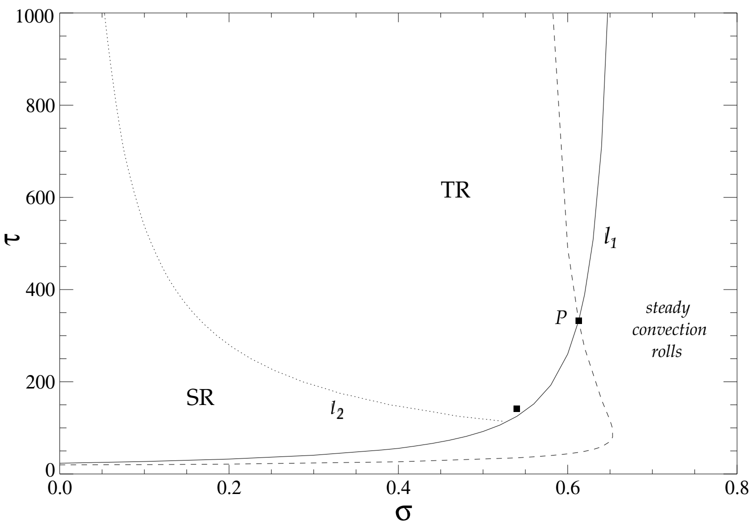
$$\omega^2 = (k^2 + \pi^2)^2\sigma^2 \left[\frac{\tau^2\pi^2(1-\sigma)}{(k^2+\pi^2)^3(1+\sigma)} - 1\right]$$

# Linear Theory

## At fixed $\sigma$ and $\tau$:




## Type of initial bifurcation (steady/oscillatory) using most unstable wavenumber:



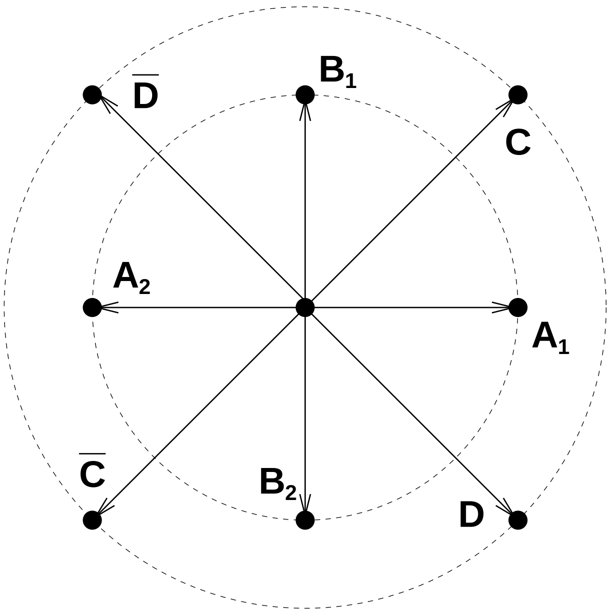

**$\ell_1$ separates regions of initial steady and oscillatory bif'ns: steady-state instability occurs to the right of $\ell_1$. Dashed line indicates a critical wavenumber ratio of exactly $1 : \sqrt{2}$.**

**The intersection point $P$ defines the $1 : \sqrt{2}$ point $(\sigma_0, \tau_0)$.**

**$\ell_2$ bounds the regions of stability (for the 2D problem) of Travelling Rolls (TR) and Standing Rolls (SR).**

## Planform (3D)

**Codimension-two bifurcation:** $\mathrm{Ra}_s = \mathrm{Ra}_o$ **and** $\mathbf{k}_s^c = \mathbf{k}_o^c\sqrt{2}$




**Planform:**

$$
\begin{aligned}
u_z(x,y,z,t) &= Re(A_1 e^{\mathrm{i}(kx-\omega t)} + A_2 e^{-\mathrm{i}(kx+\omega t)} \\
&\quad + B_1 e^{\mathrm{i}(ky-\omega t)} + B_2 e^{-\mathrm{i}(ky+\omega t)} \\
&\quad + C e^{\mathrm{i}k(x+y)} + D e^{\mathrm{i}k(x-y)}) f(z)
\end{aligned}
$$

**Restricted to a square lattice again - symmetry group is now** $\mathbb{Z}_4 \ltimes T^2 \times S^1$ **(**$S^1$ **is normal form symmetry).**

**No hidden symmetries occur.**

## Representation of $\mathbb{Z}_4 \ltimes T^2 \times S^1$

$$\rho: \qquad (x, y) \rightarrow (y, -x)$$

$$(A_1, A_2, B_1, B_2, C, D) \rightarrow (B_1, B_2, A_2, A_1, \bar{D}, C)$$

$$[(\xi, \eta), \phi]: \qquad (x, y, t) \rightarrow (x + \xi/k_c, y + \eta/k_c, t + \phi/\omega)$$

$$\begin{array}{l}(A_1, A_2, B_1, B_2, C, D) \rightarrow \\ (A_1 e^{\mathrm{i}(\xi-\phi)}, A_2 e^{-\mathrm{i}(\xi+\phi)}, B_1 e^{\mathrm{i}(\eta-\phi)}, B_2 e^{-\mathrm{i}(\eta+\phi)}, \\ C e^{\mathrm{i}(\xi+\eta)}, D e^{\mathrm{i}(\xi-\eta)})\end{array}$$

**Impose additional Boussinesq (up/down) symmetry - removes quadratic terms from the normal form.**

**Normal form:**

$$\begin{aligned}\dot{A}_1 &= A_1[\hat{\mu}_1 + a|A_1|^2 + b|A_2|^2 + c|B_1|^2 + d|B_2|^2 \\ &\quad + \lambda_1|C|^2 + \lambda_2|D|^2] + e\bar{A}_2 B_1 B_2 + \lambda_3 A_2 C D\end{aligned}$$

$$\begin{aligned}\dot{C} &= C[\mu_2 + \beta_1|C|^2 + \beta_2|D|^2 + \beta_3|A_1|^2 + \bar{\beta}_3|A_2|^2 \\ &\quad + \beta_4|B_1|^2 + \bar{\beta}_4|B_2|^2] + \nu_1 B_1 \bar{B}_2 D + \nu_2 A_1 \bar{A}_2 \bar{D}\end{aligned}$$

**Depend on relative phases of osc. modes**

**Couple steady and osc. modes**

## Mode interaction

**Representation of $\mathbb{Z}_4 \ltimes T^2 \times S^1$ acts on $V_1 \oplus V_2$:**

$V_1 = \{A_1, A_2, B_1, B_2\}$

$V_2 = \{C, D\}$

**Action on $V_1$ is irreducible but not absolutely irreducible (Hopf bifurcation)**

**Action on $V_2$ is absolutely irreducible**

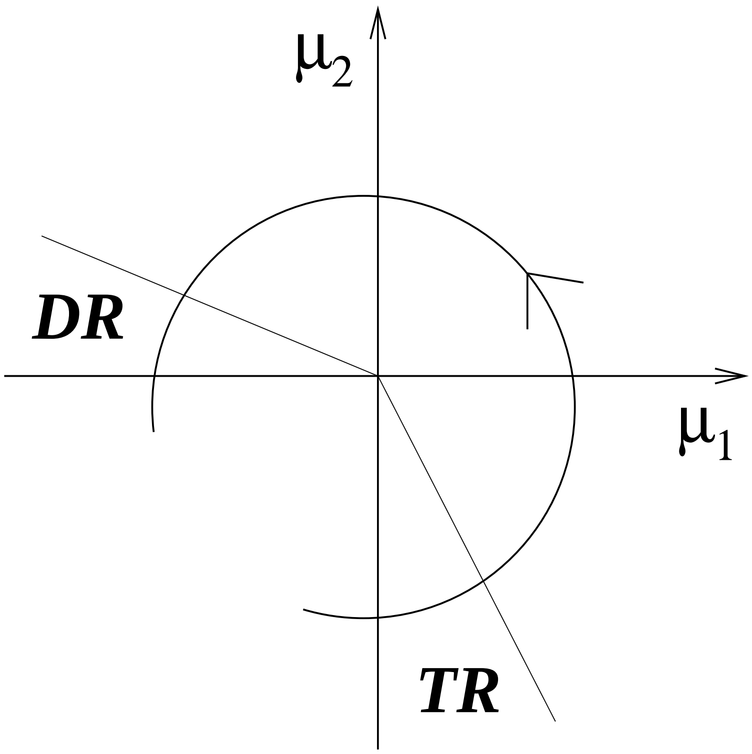
**Linearised system must look like**

$$\begin{pmatrix} \dot{\mathbf{z}} \\ \dot{\mathbf{w}} \end{pmatrix} = \left(\begin{array}{c|c} \mathrm{i}\omega \mathrm{I} & 0 \\ \hline 0 & \hat{\mu}_2 \mathrm{I} \end{array}\right) \begin{pmatrix} \mathbf{z} \\ \mathbf{w} \end{pmatrix}$$

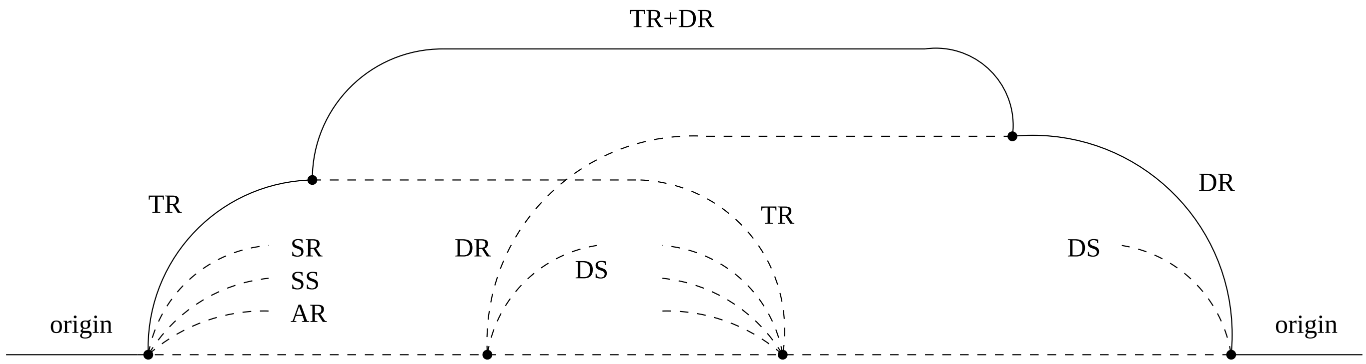
**at the bifurcation point, where $\mathbf{z} = (A_1, A_2, B_1, B_2)$ and $\mathbf{w} = (C, D)$.**

## Dynamics of the third order truncation

**Codimension-2 bifurcation: two parameters, $\mu_1$, $\mu_2$.**

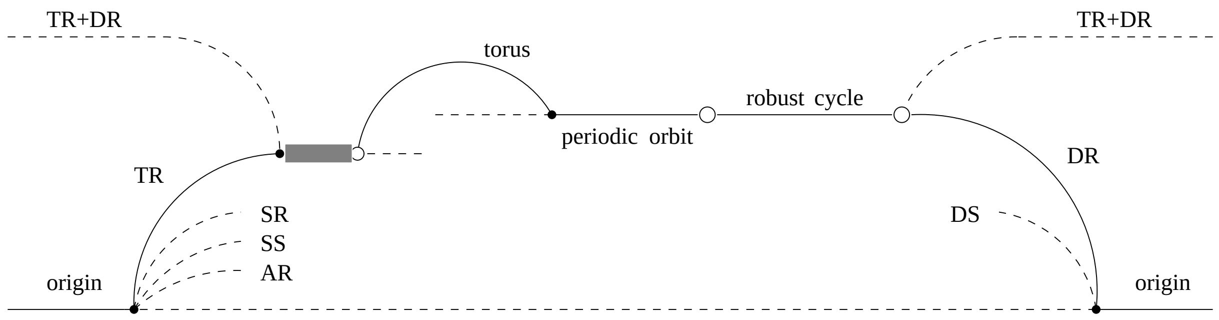



**One possibility is a single intermediate branch of '$TR+DR$' mixed-modes:**

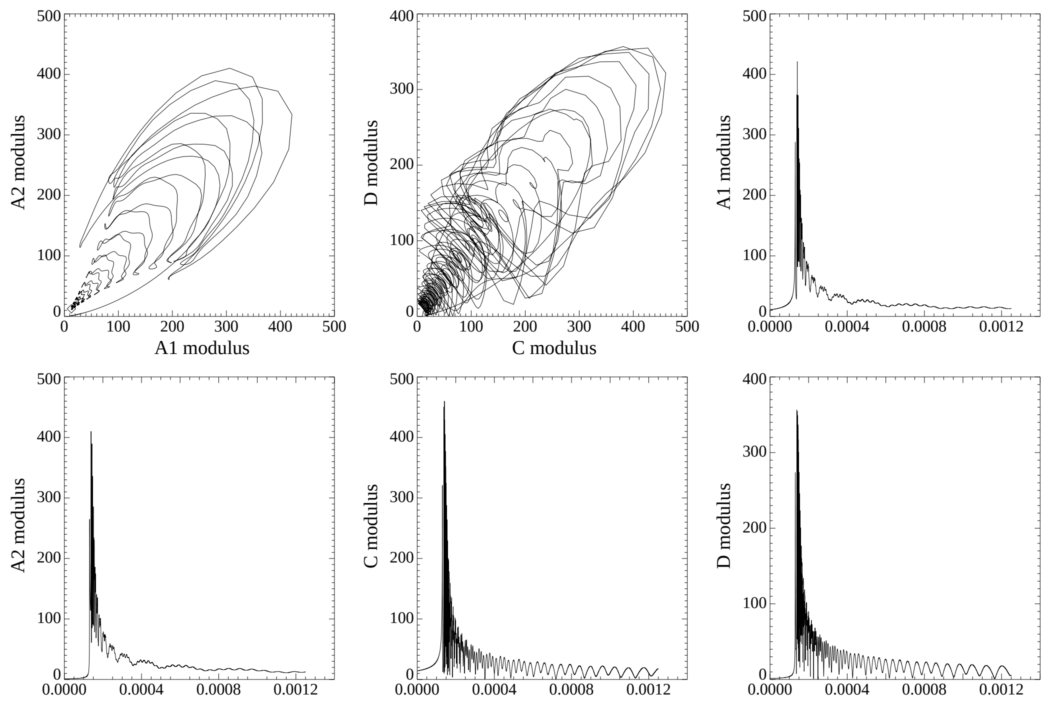


# Dynamics of the third order truncation

**Actually find a subcritical bif'n to '$TR+DR$', and much more complicated dynamics:**

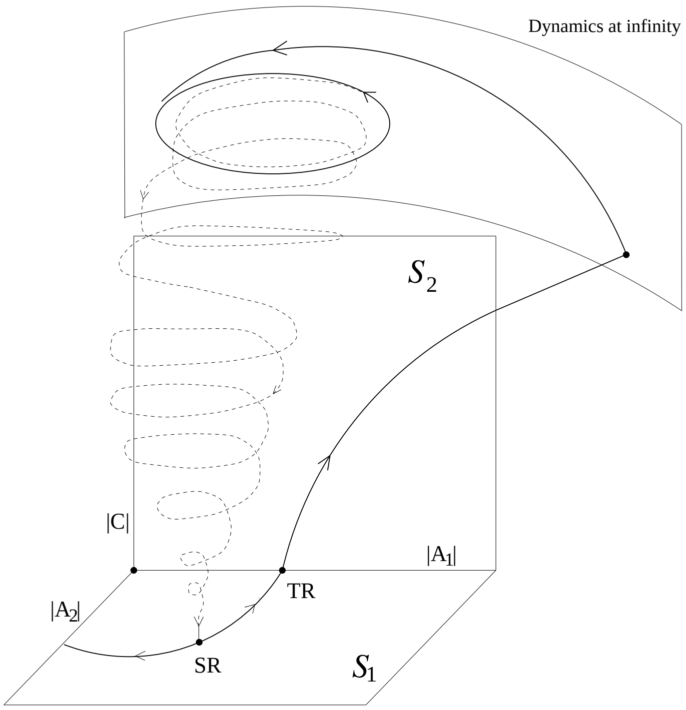



## Bursts:

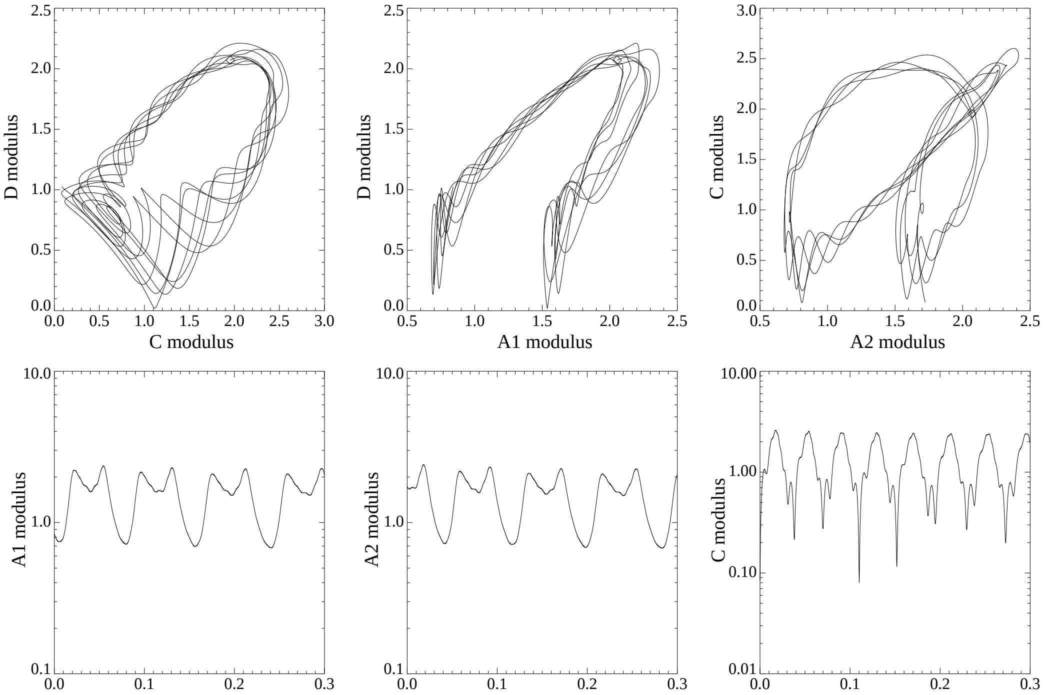



**Bursting dynamics: two phase space projections and four time traces.**
**Numerics: RK4 with $\Delta t = 5 \times 10^{-8}$.**

# Dynamics of the third order truncation

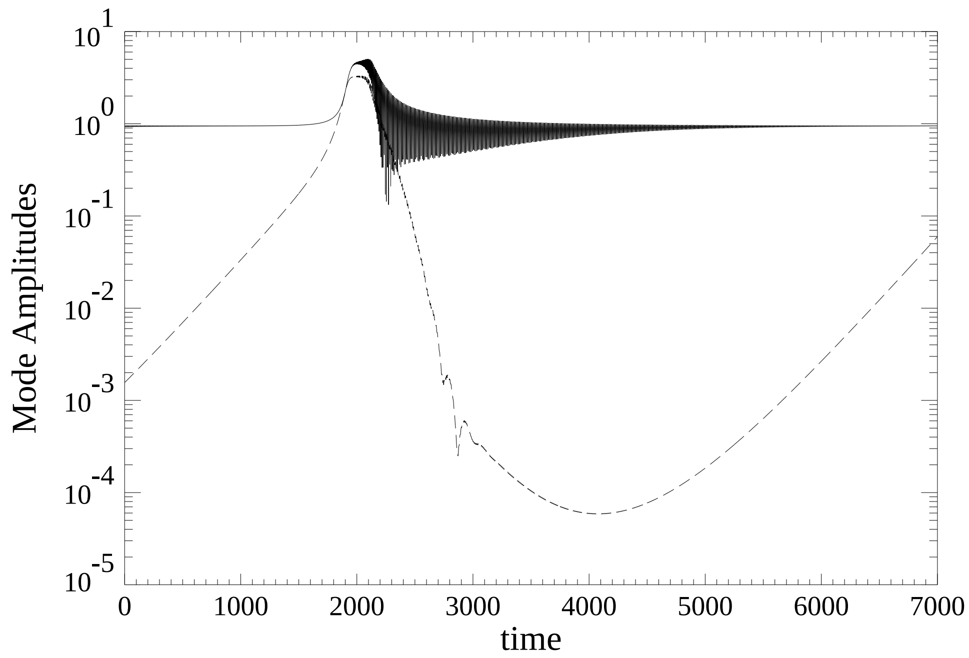



**Schematic representation of the heteroclinic connection to infinity. Thick lines show heteroclinic connections within invariant subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$. Dashed trajectory spirals in towards SR.**




**Quasiperiodic solution trajectory after the bursting ceases.**

# Numerical integration of the PDEs




**Bursting seen in numerical integrations of the PDEs for rotating thermal convection.**

**Solid line** – $|A_1| + |A_2|$**:**

- **constant** – **TR**
- **oscillating** – **SR**

**Dashed line** – $|C|$

**Very close to** $\mathrm{Ra} = \mathrm{Ra}_c$**;** $\tau = 300$**,** $\sigma = 0.6$**. Small domain** ($L_x = L_y = 2\pi/k_c$)**. Resolution:** $12 \times 12 \times 65$**.**

## Summary of Example 2

- **Hopf/steady-state mode interaction with wavenumber ratio $1 : \sqrt{2}$**

- **Complicated dynamics, unexpected from a simple combination of stable TR and steady rolls.**

- **Dynamics of third-order truncation seem to (qualitatively) explain numerical simulations.**

**More details:**

**J.H.P. Dawes, A Hopf/steady-state mode interaction in rotating convection: bursts and heteroclinic cycles in a square periodic domain. Physica D, 149 197–209 (2001)**

## Example 3: Mode interaction between small-scale and large-scale patterns (2D)

**Small-scale steady patterns and large-scale oscillatory patterns (TWs).**

**Physical motivation: Rayleigh-Bénard convection in a low viscosity conducting fluid, with a vertical magnetic field.**

**Strong magnetic field causes steady convection cells to become thinner**

**Oscillating convection cells remain square-ish.**

**Derive coupled PDE–ODEs for the amplitudes of these steady and oscillatory modes.**

**"$1:\infty$" spatial mode interaction!**

## Linear theory for convection with a magnetic field

**Governing PDEs:**

- **Navier-Stokes (incompressible fluid) + Boussinesq approx**
- **Lorenz force acts back on velocity field**
- **temperature and induction equations**

**Dimensionless parameters:**

- **Prandtl number** $\sigma = \frac{\nu}{\kappa}$
- **Chandrasekhar number (field strength)** $Q = \frac{|\mathbf{B_0}|^2 d^2}{\mu_0 \rho_0 \nu \eta}$
- **Rayleigh number** $\mathrm{Ra} \sim \Delta T$ **as before.**

**Consider the inviscid limit** $\nu \to 0$**, i.e.**

$\sigma \to 0$, $Q \to \infty$ **but** $\sigma Q = s \sim 1$

## Linear theory for convection with a magnetic field

**Steady convection:**

$$R_s = \frac{(k_s^2+\pi^2)^3}{k_s^2} + \frac{Q\pi^2(k_s^2+\pi^2)}{k_s^2}$$

**Critical wavenumber $k_s$ solves**

$$2k_s^6 + 3\pi^2 k_s^4 = \pi^6 + \pi^4 Q \qquad \sim \pi^4 Q$$

**Oscillatory convection (need $\zeta < 1$):**

$$R_o = \frac{(k_o^2+\pi^2)^3}{k_o^2}\frac{(\sigma+\zeta)(1+\zeta)}{\sigma} + \frac{Q\pi^2(k_o^2+\pi^2)}{k_o^2}\frac{(\sigma+\zeta)\zeta}{1+\sigma}$$

**Critical wavenumber $k_o$ solves**

$$2k_o^6 + 3\pi^2 k_o^4 = \pi^6 + \frac{\pi^4\sigma\zeta Q}{(1+\sigma)(1+\zeta)} \sim \pi^6 + \frac{\pi^4 s\zeta}{1+\zeta}$$

## Linear theory in the limit $\nu \to 0$

$k_s \sim (\pi^4/2)^{1/6} Q^{1/6}$ **becomes large in this limit**

$k_o$ **remains** $\mathcal{O}(1)$

**Both critical Rayleigh numbers scale like $Q$:**

$$
\begin{aligned}
R_s &\sim \pi^2 Q \\
R_o &\sim \left[ \frac{(k_o^2+\pi^2)^3 \zeta (1+\zeta)}{s k_o^2} + \frac{(k_o^2+\pi^2)\zeta^2\pi^2}{k_o^2} \right] Q
\end{aligned}
$$

**Interaction of:**

- **a short-lengthscale ($\mathcal{O}(Q^{-1/6}$) steady pattern**

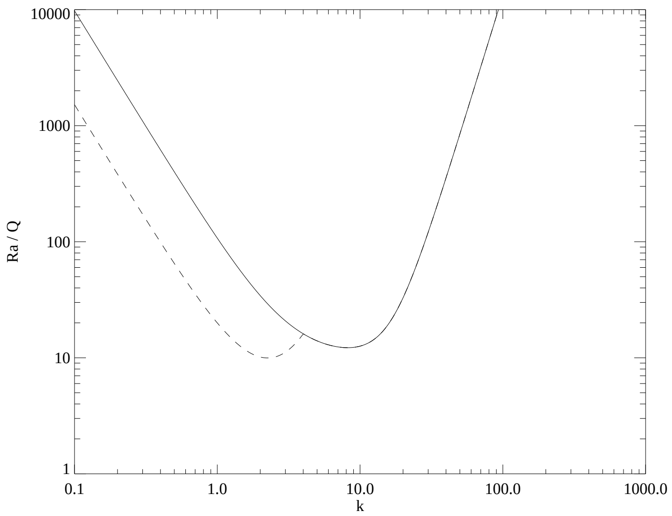
- **an $\mathcal{O}(1)$-lengthscale oscillatory pattern.**

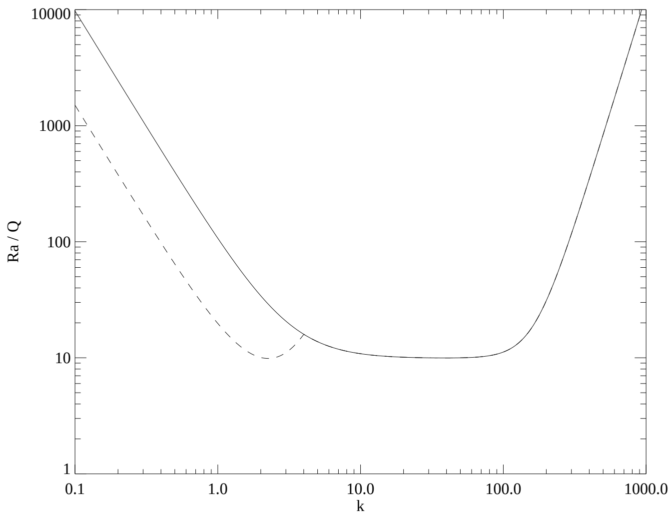
**The oscillation frequency of the $\mathcal{O}(1)$ pattern remains $\mathcal{O}(1)$:**

$$
\begin{aligned}
\omega^2 &= \frac{\pi^2 \sigma \zeta Q (1-\zeta)}{1+\sigma} - \zeta^2 (k_o^2+\pi^2)^2 \\
&\sim \pi^2 s \zeta (1-\zeta) - \zeta^2 (k_o^2+\pi^2)^2
\end{aligned}
$$

## Linear theory in the limit $\nu \to 0$

**Marginal stability curves, fixing $\zeta = 0.1$.**




$\sigma = 10^{-3}, Q = 7550$ $\sigma = 10^{-7}, Q = 7.55 \times 10^{7}$

**Solid line - steady convection**

**Dashed line - oscillatory convection**

## Mode interaction ansatz (2D)

**We assume that the pattern planform can be approximated by**

$$w = \mathrm{Re}[A(x,T)\mathrm{e}^{\mathrm{i}k\xi} + B(T)\mathrm{e}^{\mathrm{i}(\ell x - \omega t))} + C(T)\mathrm{e}^{-\mathrm{i}(\ell x + \omega t))}]$$

**where $\xi = xQ^{-1/6}$ is the characteristic short lengthscale for the steady convection, and $T = \varepsilon^2 t$ is a slow timescale. For simplicity we take $\varepsilon = Q^{-1/6}$, so there is only a single small parameter.**

**Derive amplitude equations for $A(x,T)$ (a PDE) and ODEs for $B(T)$ and $C(T)$ using symmetry arguments (or a perturbation expansion)**

## Amplitude equations for the mode interaction

$$\begin{aligned}
\dot{A} &= A[\lambda + a_1|A|^2 + a_2|B|^2 + \bar{a}_2|C|^2] + c_4 A_{xx} \\
&\quad + A(a_3 B\bar{C}\mathrm{e}^{2\mathrm{i}x} + \bar{a}_3 \bar{B} C \mathrm{e}^{-2\mathrm{i}x}) \\
\dot{B} &= B[\mu + \mathrm{i}\omega + b_1|B|^2 + b_2|C|^2 + b_3\langle|A|^2\rangle] + b_4 C\langle|A|^2\mathrm{e}^{-2\mathrm{i}x}\rangle \\
\dot{C} &= C[\mu + \mathrm{i}\omega + b_1|C|^2 + b_2|B|^2 + b_3\langle|A|^2\rangle] + b_4 B\langle|A|^2\mathrm{e}^{2\mathrm{i}x}\rangle
\end{aligned}$$

**where**

$$\langle f(x)\rangle = \frac{2\pi}{L}\int_0^{L/2\pi} f(x)\mathrm{d}x$$

**is a spatial average.**

**New terms drive modulational instabilities of $A$ and alter amplitudes of the TWs.**

**Compute coefficients using perturbation expansion (!)**

# Simple solutions

**1. Steady convection** $B = C = 0$**,** $A = R\mathrm{e}^{\mathrm{i}Qx}$

- $\lambda - c_4 Q^2 + a_1 R^2 = 0$

- **Usual Eckhaus instability when** $Q^2 = \lambda/3$

- **Simultaneous instability to** $B$ **and** $C$ **modes when** $\tilde{\mu} \equiv \mu + b_3 R^2 = 0$**:**

**Near** $\tilde{\mu} = 0$**, evolution of** $B$ **and** $C$ **modes is given by**

$$\begin{aligned}\dot{B} &= B\left[\tilde{\mu} + \mathrm{i}\omega + d|B|^2 + \left(b_2 - \frac{b_3 a_2^r}{a_1} - \frac{a_3 b_4 R^2}{a_1 R^2 + 2c_4(Q^2-1)}\right)|C|^2\right] \\ \dot{C} &= C\left[\tilde{\mu} + \mathrm{i}\omega + d|C|^2 + \left(b_2 - \frac{b_3 a_2^r}{a_1} - \frac{\bar{a}_3 b_4 R^2}{a_1 R^2 + 2c_4(Q^2-1)}\right)|B|^2\right]\end{aligned}$$

**(cubic truncation).** $d = (b_1 - b_3 a_2^r/a_1)$**. Not quite a Hopf bif'n with** $O(2)$ **symmetry.**

**Solution branches:**

- **steady convection + TW** $A \neq 0$**,** $B \neq 0$**,** $C = 0$ – **STW**

- **steady convection + MW** $A \neq 0$**,** $B \neq C \neq 0$ – **SMW**

## ‘Mixed mode’ branches SMW and STW

**2. SMW (near this bifurcation):**

**SMW has $A$ spatially modulated, and**

$$
\begin{aligned}
|B|^2 &= \frac{(\tilde{\mu})(d_r - v_{1r})}{v_{1r}v_{2r} - d_r^2} \\
|C|^2 &= \frac{(\tilde{\mu})(d_r - v_{2r})}{v_{1r}v_{2r} - d_r^2}
\end{aligned}
$$

**i.e. need $(d_r - v_{1r})(d_r - v_{2r}) > 0$ for existence**

$v_{1,2} =$ **coeffs of $B|C|^2$ and $C|B|^2$ respectively.**

**3. STW - more progress can be made: $C = 0$, $A = R\mathrm{e}^{\mathrm{i}Qx}$**

$$
\begin{aligned}
|B|^2 &= \frac{a_1\mu - b_3^r(\lambda - c_4 Q^2)}{a_2 b_3^r - a_1 b_1^r} \\
R^2 = |A|^2 &= \frac{b_1^r(\lambda - c_4 Q^2) - a_2\mu}{a_2 b_3^r - a_1 b_1^r}
\end{aligned}
$$

## Stability of STW

**Let**

$$\begin{aligned} A &= R\mathrm{e}^{\mathrm{i}Qx}(1 + \alpha_1 \mathrm{e}^{2\mathrm{i}x} + \bar{\alpha}_2 \mathrm{e}^{-2\mathrm{i}x}) \\ B &= B_0(1 + b) \end{aligned}$$

**in the usual way ($C$ is also small). Then, linearising,**

$$\begin{aligned} \dot{\alpha}_1 &= a_1 R^2(\alpha_1 + \alpha_2) - 4c_4(1 + Q)\alpha_1 + a_3 B_0 \bar{C} \\ \dot{\alpha}_2 &= a_1 R^2(\alpha_1 + \alpha_2) - 4c_4(1 - Q)\alpha_2 + a_3 B_0 \bar{C} \\ \dot{C} &= C[\mu + \mathrm{i}\omega + b_2|B_0|^2 + b_3 R^2] + b_4 B_0 R^2(\bar{\alpha}_1 + \bar{\alpha}_2) \end{aligned}$$

**3 complex equations.**

**A real version of this system was analysed by Proctor & Metzener, Eur. J. Mech B / Fluids (1992).**

**Secondary steady-state and Hopf bifurcations possible.**

## Summary of Example 3

- **Magnetoconvection in the inviscid limit $\nu \to 0$, in 2D**

- **Patterns on widely different scales - unusual interaction terms**

- **Secondary Hopf and steady-state bifurcations**

- **Validity of these averaged amplitude equations?**

## Overall Summary

- **Brief introduction to degeneracies in bifurcations**
- **Bifurcation with symmetry**
- **Three examples of pattern-forming systems:**
  - **Steady-state/steady-state interaction on a fine lattice**
  - $1 : \sqrt{2}$ **Hopf/steady-state interaction in rotating convection**
  - **Large/small-scale Hopf/steady-state interaction in magnetoconvection**
- **Enables location of secondary bifurcations**
- **Explanation of complex dynamics**
- **'More than the sum of their parts'**