2 Lecture 2: Amplitude equations and Hopf bifurcations

This lecture completes the brief discussion of steady-state bifurcations by discussing vector fields that describe the dynamics near a bifurcation. From such a set of 'amplitude equations' we can calculate conditions for stability of the bifurcating branches in terms of the coefficients of the nonlinear terms. In very many cases it is sufficient to calculate the form of the nonlinear terms only up to cubic order in the variables. Although a very general, systematic theory exists for determining the structure of these amplitude equations we will deal instead in specific examples to fix the ideas clearly. For these simple examples the axial isotropy subgroups can be found by inspection. For more complicated examples there are better methods for finding axial isotropy subgroups, using the *trace formula*, see [25], pp76–78.

The second half of the lecture contains a few remarks on oscillatory (Hopf) bifurcations with symmetry. The central message is very similar to that in the steady-state case, but there are a few subtleties worth pointing out. One example of Hopf bifurcation is given; with $O(2)$ symmetry.

2.1 Worked examples

The trivial representation

The trivial representation is reducible for all $n > 1$. In the case $n = 1$ we have a one dimensional bifurcation problem with only a trivial symmetry group. So the generic bifurcation is a saddle-node bifurcation: two solution branches bifurcate, one stable and one unstable, and both have the full symmetry group of the problem.

D_2 symmetry: one dimensional representations

From the character table in figure 1 we see that D_2 has four one dimensional real (and hence absolutely irreducible) representations. If the symmetry group elements act as in the trivial representation χ_1 on the one-dimensional centre manifold, the Taylor series expansion of $f(x)$ at the bifurcation point (without loss of generality) $\mu = 0$, $x = 0$ is unconstrained by equivariance, hence, in the absence of any other information (for example that the solution $x = 0$ persists on both sides of the bifurcation point) we expect to have a saddle-node bifurcation: $\dot{x} = \mu + cx^2 + \text{h.o.t.}$, for c a real constant. If however we knew also from physical considerations that the $x = 0$ solution remained a solution for all μ close to the bifurcation point then we would expect to see a transcritical bifurcation instead: $\dot{x} = \mu x - cx^2 + \text{h.o.t.}$

In the other cases $\chi_2, \ldots, \chi_4$ the presence of symmetries acting as -1 removes even powers of x from the Taylor series for f near the bifurcation point: equivariance implies $f(-x) = -f(x)$ and so $\dot{x} = \mu x \pm cx^3 + \text{h.o.t.}$ and hence a pitchfork bifurcation takes place on the centre manifold. In each case the new solution branches retain the symmetries that act as $+1$ in the representation. Stability is computed just as for 'standard' pitchfork bifurcations: there are subcritical and supercritical cases.

$D_3 \cong S_3$ symmetry: 1D and 2D irreps

As discussed in section 1.2 (example 2) the 'natural permutation action' of S_3 on $\mathbb{R}^3$ is reducible. On the subspace $V = \{(x, x, x)\}$ all elements of S_3 leave points unchanged (they 'act as multiplication by $+1$') because $(1, 1, 1)^T$ is an eigenvector for all the permutation matrices. Hence if we had a bifurcation where the centre manifold was V it would generically be a saddle-node bifurcation.

On the orthogonal complement $V^\perp$ we see geometrically that the permutation matrices have order two or three and correspond to reflections and rotations of that plane, respectively. The action on $V^\perp$ is irreducible and is isomorphic to the symmetry group $D_3 = \langle \rho, m_x \rangle$ of an equilateral triangle, see figure 2. This is representation χ_3 in figure 2. Using, for convenience, complex notation

$z = x + \mathrm{i}y$ for points in $\mathbb{R}^2$, the 2D absolutely irreducible representation of

$$D_3 = \{I, \rho, \rho^2, m_x, m_d, m_{d'}\}$$

is generated by

$$\rho(z) = \mathrm{e}^{2\pi\mathrm{i}/3} z, \qquad m_x(z) = \bar{z}.$$

Up to conjugacy there is one axial isotropy subgroup $\Sigma = \{I, m_x\} \cong \mathbb{Z}_2$; $\mathrm{Fix}(\Sigma) = (x, 0)$ which is one dimensional. Applying the EBL we can assert the existence of a branch of solutions with $\mathbb{Z}_2$ symmetry in a generic bifurcation with this irrep of D_3. By conjugacy there are also axial isotropy subgroups $\{I, m_d\}$ and $\{I, m_{d'}\}$ which have fixed point spaces $x(\cos 2\pi/3, \sin 2\pi/3)$ and $x(\cos 4\pi/3, \sin 4\pi/3)$. To investigate this bifurcation further we calculate the first few terms of the equivariant ODEs $\dot{z} = f(x)$. Requiring equivariance with respect to this action of D_3 leads to the normal form

$$\dot{z} = \mu z + b\bar{z}^2 + cz|z|^2 + O(4)$$

(truncated at cubic order) where b and c are coefficients and are required to be real by equivariance. In terms of x and y the truncated normal form is

$$\dot{x} = \mu x + b(x^2 - y^2) + cx(x^2 + y^2) + O(4), \quad (4)$$

$$\dot{y} = \mu y - 2bxy + cy(x^2 + y^2) + O(4). \quad (5)$$

Restricting (4)-(5) to the subspace $\mathrm{Fix}(\Sigma) = (x, 0)$ we see that generically the $\mathbb{Z}_2$-symmetric solution branch bifurcates transcritically. It can be checked that in this case $N\mathcal{G}(\Sigma)/\Sigma \cong \{I\}$. In contrast to a standard transcritical bifurcation without symmetry it can be checked that the solution branch is unstable on both sides of the bifurcation point. This is a case of a more general result for axial branches when the amplitude equations for a bifurcation problem contain quadratic terms, see [27], p39.

2.2 D_4 symmetry: branching and stability

As for D_3 it is easiest to use a single complex coordinate $z = x + \mathrm{i}y$ to describe the only 2D irrep of D_4:

$$\rho(z) = \mathrm{i}z, \qquad m_x(z) = \bar{z}.$$

In this case there are two group orbits of axial branches, with representative isotropy subgroups $\Sigma_1 = \{I, m_x\} \cong \mathbb{Z}_2$ and $\Sigma_2 = \{I, m_d\} \cong \mathbb{Z}_2$. The corresponding fixed point subspaces are $\mathrm{Fix}(\Sigma_1) = (x, 0)$ and $\mathrm{Fix}(\Sigma_2) = (x, x)$. By the EBL there are therefore generically two distinct group orbits of branches of equilibria produced in this bifurcation; Σ_1 and Σ_2 are not conjugate and so the characteristics (for example stability) of these two solution branches will differ.

Normaliser calculations show that both of these branches bifurcate as pitchforks:

$$N(\Sigma_1) = \{I, m_x, \rho^2, m_x\rho^2\} \quad \text{since } \rho^2 m_x \rho^2 = m_x$$

$$N(\Sigma_2) = \{I, m_d, \rho^2, \rho^2 m_d\} \quad \text{since } \rho^2 m_d \rho^2 = m_d$$

so $N(\Sigma_1)/\Sigma_1 \cong \{I, \rho^2\} \cong \mathbb{Z}_2$ and equally for Σ_2. ρ^2 acts as minus the identity matrix on $\mathbb{R}^2$ and forces the bifurcation branches to be pitchforks.

To compute the equivariant normal form $\dot{z} = f(z)$ we suppose that $f(z)$ contains a term $cz^p\bar{z}^q$ in its Taylor series, where c is a constant and p and q are integers. Then equivariance with respect to ρ and m_x respectively implies

$$\mathrm{i}cz^p\bar{z}^q = c\mathrm{i}^{p-q} z^p \bar{z}^q,$$

$$\overline{cz^p\bar{z}^q} = c\bar{z}^p z^q.$$

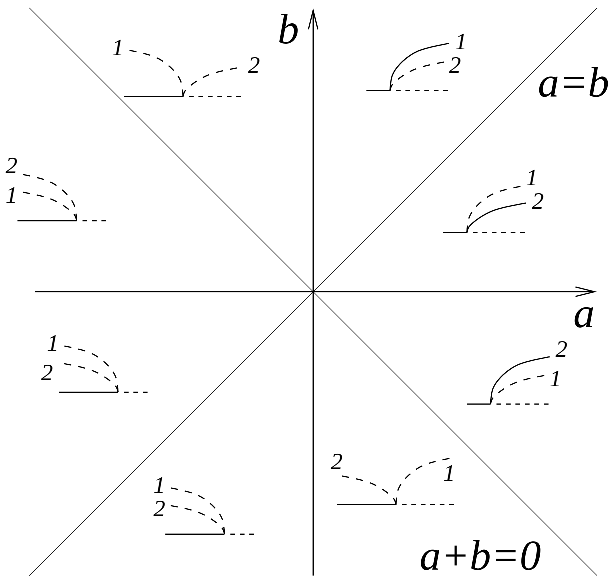

Figure 5: Generic branching behaviour for D_4-symmetric steady state bifurcation in $\mathbb{R}^2$. Solid and dashed lines indicate stable and unstable branches respectively. Type 1 and 2 branches have isotropy subgroups conjugate to Σ_1, Σ_2 respectively.

Hence $c \in \mathbb{R}$ and $p - q \equiv 1 \bmod 4$; any equivariant term must therefore be of odd total order $p + q$. The lowest order polynomial terms that satisfy these constraints are $c_1 z$, $c_2 \bar{z}^3$ and $c_3 z^2 \bar{z}$. Hence the amplitude equation is

$$\dot{z} \;=\; \mu z - \hat{a}\bar{z}^3 - \hat{b}z|z|^2 + O(5), \tag{6}$$

which in real co-ordinates can be written as

$$\begin{aligned} \dot{x} &= \mu x - a x^3 - b x y^2 \\ \dot{y} &= \mu y - a y^3 - b x^2 y \end{aligned}$$

defining $a = \hat{a} + \hat{b}$, $b = \hat{b} - 3\hat{a}$. This calculation can also be carried out in real coordinates throughout, see [32] pp111–118.

Stability of the bifurcating branches follows in the usual way. It turns out that exactly one of the two types of branch $(x, 0)$ (type 1) and (x, x) (type 2) is stable if and only if both branches bifurcate supercritically. The bifurcation possibilities are summarised in figure 5.

2.3 D_5 symmetry: not a transcritical bifurcation

As in the previous two examples the 2D irrep of D_5 is generated by:

$$\rho(z) = \mathrm{e}^{2\pi \mathrm{i}/5} z, \qquad m_x(z) = \bar{z}.$$

which leads to a D_5-equivariant normal form

$$\dot{z} \;=\; \mu z - a z|z|^2 + b\bar{z}^4 + O(5)$$

Just as for the D_3 case, there is one non-zero axial solution branch $(x, 0)$ with isotropy subgroup $\Sigma = \{I, m_x\} \cong \mathbb{Z}_2$. Within $\mathrm{Fix}(\Sigma)$ the dynamics are given by

$$\dot{x} \;=\; \mu x - a x^3 + b x^4 + O(5)$$

and hence the conditions for a pitchfork bifurcation are satisfied, that is:

$$f_\mu = 0, \qquad f_{xx} = 0, \qquad \text{but} \qquad f_{\mu x} \neq 0, \qquad f_{xxx} \neq 0.$$

However, the normaliser calculation gives $N(\Sigma)/\Sigma \cong \{I\}$ because if a power of ρ, $\rho^k \in N(\Sigma)$, $1 \le k \le 4$ then we would require

$$\mathrm{e}^{-2\pi \mathrm{i}k/5} m_x \left(\mathrm{e}^{2\pi \mathrm{i}k/5} z\right) \quad = \quad \bar{z} \qquad \Rightarrow \qquad \mathrm{e}^{-4\pi \mathrm{i}k/5} \bar{z} = \bar{z} \tag{7}$$

which cannot hold. A similar calculation shows $m_x\rho^k \notin N(\Sigma)$. So $N(\Sigma) = \{I, m_x\} = \Sigma$ acts trivially on $\mathrm{Fix}(\Sigma)$. So we would naively expect the dynamics on $\mathrm{Fix}(\Sigma)$, in this case, not to be constrained by symmetry, and so might be expected to be a transcritical, but it is not. This provides an example of the second case mentioned in the remark at the end of section 1.5: when $N(\Sigma)/\Sigma \cong \{I\}$ we cannot infer that the bifurcation is transcritical.

2.4 D_n symmetry: the general case, $n \ge 5$

The results above for D_3 and D_4 are not really typical of the general situation for bifurcation symmetric under the 2D irrep of D_n. A detailed discussion of the cases $n \ge 5$ can be found in [25] pp97–103. For the action

$$\rho(z) = \mathrm{e}^{2\pi \mathrm{i}/n} z, \qquad\qquad m_x(z) = \bar{z}.$$

we find that the axial isotropy subgroups $\Sigma_1 = \{I, m_x\}$ and $\Sigma_2 = \{I, \rho m_x\}$ are conjugate if n is odd. They are not conjugate if n is even. However because the normal form takes the general form

$$\dot{z} \quad = \quad p(u, v, \mu)z + q(u, v, \mu)\bar{z}^{n-1}$$

(where p and q are polynomials in the invariants $u = |z|^2$ and $v = z^n + \bar{z}^n$ and in μ the bifuration parameter), both branches, in the case n even, bifurcate in the same direction. So the bifurcation diagrams look very similar in the cases of odd and even n, but the group-theoretic details differ.

2.5 Steady-state bifurcation with $O(2)$ symmetry

In the 'limit $n \to \infty$' we have a continuous rotation symmetry, and the group D_n is replaced by $O(2)$ - the rotations and reflections of a circle. The group $O(2)$ arises naturally when describing instabilities of axisymmetric flows, for example when confined in a cylindrical domain.

Compact (i.e. nice) continuous symmetry groups have countably many irreps. In the case of $O(2)$ the most interesting irreps are those given by

$$\tau_\theta(z) = \mathrm{e}^{\mathrm{i}m\theta} z, \qquad\qquad m_x(z) = \bar{z},$$

where m_x and $\theta \in [0, 2\pi)$ generate the group $O(2) \cong \mathbb{Z}_2 \ltimes S^1$ and $m \in \mathbb{Z}$ indicates the possible m-fold actions of τ_θ on $\mathbb{R}^2$. For this irrep the calculations are very similar to those above: there is an axial isotropy subgroup $\Sigma = \{m_x, \tau_{2\pi/m}\}$ with a one-dimensional fixed point subspace $\mathrm{Fix}(\Sigma) = (x, 0)$. For the m-fold action of the rotation, this new solution branch has D_m symmetry and the amplitude equation is

$$\dot{z} \quad = \quad p(|z|^2, \mu)z = \mu z - az|z|^2 + O(5)$$

where, as above, p is a polynomial in μ and the invariant $|z|^2$. This bifurcation is sometimes known as a 'pitchfork of revolution' because all points z with the same phase are equivalent. Equilibrium points $|z|^2 = \mu/a$ (at leading order) all have a zero eigenvalue in their Jacobians. This corresponds to perturbations around the circle $|z|^2 = \mu/a$ and shows that the continuum of equilibria has a direction of 'neutral stability'. The underlying reason for this is the continuous rotation symmetry, which leads to a continuous group orbit. This is important for physical systems as these zero eigenvalues are often involved in secondary bifurcations leading to solutions that drift along this group orbit. Physically the steady solution breaks a reflection symmetry and rotates, typically with a constant rotation rate, in the azimuthal direction.

2.6 Hopf bifurcations with symmetry

First we digress to discuss the phase-shift symmetry that naturally arises in Hopf bifurcation problems in 'normal form'.

Normal form symmetry

The idea of a normal form is to take a system of ODEs in $\mathbb{R}^n$ at a bifurcation point and apply successive near-identity coordinate transformations to try to simplify the terms in the Taylor series expansion as much as possible. More specifically, write

$$\dot{x} \quad = \quad f(x) \equiv Lx + f_k(x) + h_k(x) + h.o.t.$$

where L is a matrix describing the linear terms, $f_k(x)$ contains terms of order 2 up to $k-1$ that we assume we have already dealt with, and $h_k(x)$ contains all the polynomial terms of order k. A near-identity coordinate transformation $x \to y \approx x$, $|x|, |y| \ll 1$ looks like

$$x \quad = \quad y + P_k(y)$$

where $P_k(y)$ contains polynomial terms of order k (the same order as h_k). Then, can we deduce a simpler ODE for y than the one we start with for x? We have

$$\begin{aligned} y &= x - P_k(y) \\ &= x - P_k(x - P_k(y)) \\ &= x - P_k(x) + O(k+1) \end{aligned}$$

so

$$\begin{aligned} \dot{y} &= \dot{x} - DP_k\dot{x} + O(k+1) \\ &= Lx + f_k(x) + h_k(x) - DP_kLx + O(k+1) \end{aligned}$$

where the derivative matrix DP_k is the usual one:

$$DP_k \quad = \quad \begin{pmatrix} \frac{\partial P_{k_1}}{\partial x_1} & \frac{\partial P_{k_1}}{\partial x_2} & \cdots & \frac{\partial P_{k_1}}{\partial x_n} \\ \frac{\partial P_{k_2}}{\partial x_1} & \cdots & & \vdots \\ \vdots & \cdots & \cdots & \frac{\partial P_{k_n}}{\partial x_n} \end{pmatrix} = O(k-1).$$

Now note that the terms at orders 2 up to k are unchanged by this change of variable, in that

$$\begin{aligned} f_k(x) &= f_k(y + P_k(y)) \\ &= f_k(y) + O(k+1), \text{ and,} \\ h_k(x) &= h_k(y) + O(k+1) \end{aligned}$$

so

$$\begin{aligned} \dot{y} &= L(y + P_k(y)) + f_k(y) + h_k(y) - DP_kLy + O(k+1) \\ &= \underbrace{Ly + f_k(y)}_{\text{same as before}} + \underbrace{h_k(y) + LP_k(y) - DP_kLy}_{\text{terms of order } k} + O(k+1). \end{aligned}$$

Now we can freely choose the coefficients of the polynomial terms in P_k, and we can use them to eliminate some (and possibly all) of the terms of order k in $h_k(y)$. Because $x \approx y$ for small x there will be an open disc containing the origin on which the transformation is invertible. This guarantees that the new and old systems of ODEs have qualitatively equivalent dynamics. This process is inhibited by resonances between the eigenvalues of L and so in general not all terms are removable at every order. On the other hand, if the eigenvalues are non-resonance at all orders k (say they are 1 and $\sqrt{2}$), then in principle all nonlinear terms could be removed by successive near-identity coordinate transformations. However, there may not be a neighbourhood of the equilibrium where this infinite sequence of coordinate transformations are all invertible, so we usually truncate the procedure at some order N. Once as many nonlinear terms as possible have been removed, the system is said to be in Birkhoff normal form (truncated at some finite order N):

$$\dot{y} = Ly + \text{polynomials up to order N} = Ly + g_N(y). \tag{8}$$

There is freedom to choose how to eliminate terms in many cases, but there is one choice that yields a transformed system with an extra symmetry property - this is particularly useful for Hopf bifurcations.

Theorem 10 (Normal form symmetry) *Let the system of ODEs (8) have an equilibrium at $y = 0$ that has all eigenvalues of the linearisation L lying on the imaginary axis. Then there is a choice of near-identity coordinate changes after which the nonlinear part of the truncated normal form commutes with the matrices* $\exp(sL^T)$ *for all $s \in \mathbb{R}$, i.e.*

$$g_N\left(\exp(sL^T)y\right) = \exp(sL^T)g_N(y)$$

The matrix $\exp(sL^T)$ is called the normal form symmetry. This result is due to Elphick et al. *Physica* D **29**, 95–127 (1987). See also Wiggins, pp290–301.

Note that it is possible (for example in the Takens–Bogdanov bifurcation) that the linearisation L does not commute with $\exp(sL^T)$.

Hopf bifurcation without symmetry

At a Hopf bifurcation point, after we have chosen coordinates to put the linearisation in normal form, $L = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$ for some $\omega \in \mathbb{R}$. Then

$$\exp(sL^T) = \begin{pmatrix} \cos s\omega & \sin s\omega \\ -\sin s\omega & \cos s\omega \end{pmatrix}$$

which geometrically is just a rotation matrix. So we can make coordinate changes so that the only nonlinear terms that survive in the normal form are those that commute with this rotation. These are of the form

$$(y_1^2 + y_2^2)^m \begin{pmatrix} A_m & -B_m \\ B_m & A_m \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

where A_m and B_m are real constants, for each $m \geq 1$, and the terms have order $2m + 1$, i.e. in particular all even order terms are removable. In complex notation, writing $z = y_1 + \mathrm{i}y_2$ this result means that the normal form looks like

$$\dot{z} = \mathrm{i}\omega z + (A_1 + \mathrm{i}B_1)z|z|^2 + (A_2 + \mathrm{i}B_2)z|z|^4 + O(|z|^7)$$

or, equivalently, in polar coordinates $z = r\mathrm{e}^{\mathrm{i}\theta}$:

$$\begin{aligned} \dot{r} &= A_1 r^3 + A_2 r^5 + O(r^7), \\ \dot{\theta} &= \omega + B_1 r^2 + B_2 r^4 + O(r^6). \end{aligned}$$

Note that the $\dot{r}$ equation contains no θ-dependence: in the truncated normal form they are decoupled by the special form of the terms. We must remember that arbitrarily high-order terms coupling θ to r still exist in the original ODEs, but (by the Implicit Function Theorem) these higher-order terms do not affect our conclusions about the existence of periodic orbits near the bifurcation, although they do affect the qualitative structure of the dynamics in more complicated bifurcations.

2.7 The Equivariant Hopf Theorem

We have seen that generic steady-state bifurcations with symmetry are given by absolutely irreducible representations of the symmetry group. In this section we discuss the corresponding generic case for the group action when we have complex conjugate eigenvalues crossing the imaginary axis. We begin by proving that single copies of absolutely irreducible representations of $\mathcal{G}$ will not give rise to Hopf bifurcations.

Theorem 11 ([27], p90, Lemma 4.2) *Suppose the Jacobian matrix at the bifurcation point $L = Df_{(0,0)}$ has a non-real eigenvalue. Then either*

- *$\mathcal{G}$ acts non-absolutely irreducibly on $\mathbb{R}^n$, or*
- *there exists subspaces V_1, V_2 such that $V_1 \oplus V_2 \subseteq \mathbb{R}^n$ and $\mathcal{G}$ acts isomorphically and absolutely irreducibly on V_1 and V_2.*

Proof: Suppose neither of these conditions is true, then we can write $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k$, an isotypic decomposition in which each $\mathcal{G}$ acts absolutely irreducibly, and distinctly, on each of the V_j. From Theorem 2 we can conclude that $L(V_j) \subseteq V_j$ and, because $\mathcal{G}$ acts absolutely irreducibly on each V_j we must have

$$L|_{V_j} \quad = \quad c_j I_{\dim(V_j)}$$

i.e. the Jacobian is a real multiple of the identity matrix when restricted to each V_j. So $L = Df|_{(0,0)}$ is diagonal and has only real eigenvalues. □

So in order to have a Hopf bifurcation, and eigenvalues $\pm i\omega$, one or other of the conclusions in the theorem must hold. As one might imagine, it transpires [25] that one or other of these cases holds generically for Hopf bifurcations; these are the simplest possible situations to arrange conditional on allowing purely imaginary eigenvalues. In either of these cases the group action is said to be $\mathcal{G}$-simple. The first case ($\mathcal{G}$ acts irreducibly but not absolutely irreducibly) arises when the group $\mathcal{G}$ contains only rotations, for example $\mathcal{G} = SO(2)$. For problems involving orthogonal groups or their subgroups which contain reflections (as will be the case in problems considered later), the second case occurs.

Commuting matrices

Another view of the dichotomy of the generic representations of $\mathcal{G}$ for Hopf bifurcation is to start from the full symmetry group of the bifurcation problem: $\mathcal{G} \times S^1$ where S^1 is the circle group of phase shifts discussed above. The action of $\mathcal{G} \times S^1$ on the center manifold $\mathbb{R}^n$ must generically be irreducible; if it is not, we can split the action into actions on a subspace V and its complement $V^\perp$. Then, as in the steady-state case, small perturbations to the linearisation matrix $Df|_{V^\perp}$ move some but not all of the eigenvalues off the imaginary axis and we have a different bifurcation problem. Given that the action of $\mathcal{G} \times S^1$ is irreducible we can investigate the possible actions of G on $\mathbb{R}^n$ that are compatible with this. We will assume that the set of commuting matrices is isomorphic to $\mathbb{C}$ and ignore the $\mathbb{H}$ case here.

In suitable coordinates, $\theta \in S^1$ acts on $\mathbb{R}^{n/2} \oplus \mathbb{R}^{n/2}$ by the rotation

$$R_\theta = \begin{pmatrix} \cos\theta I & -\sin\theta I \\ \sin\theta I & \cos\theta I \end{pmatrix}.$$

Any matrix L that commutes with S^1 must then take the form

$$L = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \tag{9}$$

(including the matrices $\tilde{\rho}(g)$ representing elements $g \in \mathcal{G}$). The eigenvalues of L are then the (complex) eigenvalues of the matrix $A + \mathrm{i}B$.

- If the action of $\mathcal{G}$ is irreducible then the eigenvalues of L have geometric multiplicity $n/2$ (suppose they are $a \pm \mathrm{i}b$) and, given L takes the form 9, L must in fact be in the Jordan normal form

$$L = \begin{pmatrix} aI & bI \\ -bI & aI \end{pmatrix}$$

- If the action of $\mathcal{G}$ is not irreducible yet generic, then there are exactly two copies of the same irreducible representation by theorem 11. Suppose then R_g takes the form

$$R_g = \begin{pmatrix} P_g & 0 \\ 0 & P_g \end{pmatrix}$$

  where P_g is an irreducible representation of $\mathcal{G}$. Now L commutes with R_g if and only if

$$P_g A = A P_g, \qquad \text{and} \qquad P_g B = B P_g$$

  but A and B are independent $n/2 \times n/2$ matrices, so for the set of commuting matrices to be isomorphic to $\mathbb{C}$ we require $A = aI$ and $B = bI$ to be real multiples of the identity matrix, that is, P_g must be an absolutely irreducible representation, not just an irreducible one.

In summary, in both cases, after choosing a nice basis, a commuting matrix $L = Df$ can be assumed to take the form

$$Df = \begin{pmatrix} \mu I_{n/2} & -\omega I_{n/2} \\ \omega I_{n/2} & \mu I_{n/2} \end{pmatrix}$$

for real parameters μ and ω.

The Equivariant Hopf Theorem

The analogous result for the existence of periodic branches of solutions to Hopf bifurcation problems is the Equivariant Hopf Theorem [26]:

Theorem 12 (The Equivariant Hopf Theorem) *Let $\mathcal{G}$ be a (compact Lie) group acting irreducibly and $\mathcal{G}$-simply on $\mathbb{R}^n$ and let $\dot{x} = f(x, \mu)$ be a $\mathcal{G}$-equivariant smooth bifurcation problem with*

$$Df|_{(0,0)} = \begin{pmatrix} 0 & -I_{n/2} \\ I_{n/2} & 0 \end{pmatrix},$$

where $I_{n/2}$ denotes the $n/2 \times n/2$ identity matrix. Then there exist real functions $c(\mu)$ and $\omega(\mu)$ such that the eigenvalues of $Df|_{(0,\mu)}$ are $c(\mu) \pm \mathrm{i}\omega(\mu)$ and, after time rescaling, $\omega(0) = 1$. We also assume that there is a non-degenerate bifurcation at $\mu = 0$: $c(0) = 0$ and $c'(0) \neq 0$. Then there exist branches of periodic solutions with period close to 2π having isotropy subgroup $\Sigma \subset \mathcal{G} \times S^1$ whenever $\dim \mathrm{Fix}(\Sigma) = 2$.

Proof: See [26], or [25], p275, Theorem 4.1.

A key feature of the Equivariant Hopf Theorem is that isotropy subgroups Σ are subgroups of $\mathcal{G} \times S^1$, not subgroups of $\mathcal{G}$, but in the statement of the theorem we require $f(x, \mu)$ to be only $\mathcal{G}$-equivariant, not $\mathcal{G} \times S^1$-equivariant. This is an important distinction. If $f(x, \mu)$ is, in fact, $\mathcal{G} \times S^1$-equivariant it is said to be in 'normal form', as discussed above. The effects of (high-order) terms which break the S^1 symmetry are discussed in general by [25], and by [53] for the particular example of a Hopf bifurcation with D_4 symmetry. Most of the time only the cubic order truncation of the normal form is considered in the literature: in this case the subspace $\text{Fix}(\Sigma)$ is invariant for the dynamics of $f(x, \mu)$ and the proof of the Equivariant Hopf Theorem loosely amounts to restricting the dynamics to each two-dimensional fixed point subspace $\text{Fix}(\Sigma)$ within which $\dot{x} = f(x, \mu)$ describes a generic Hopf bifurcation without symmetry, giving rise to a periodic orbit with symmetry group Σ. Indeed, since the phase shifts in the S^1 group act on any fixed point subspace of $\mathcal{G}$ (they are in the normaliser since they commute with every element of $\mathcal{G}$), it is not possible to have odd-dimensional fixed-point subspaces.

The symmetries of branches of periodic orbits are very likely to contain combinations of spatial symmetries and temporal shifts - these are called spatio-temporal symmetries. Solution branches guaranteed by the Equivariant Hopf Theorem are called $\mathbb{C}$-axial branches, by analogy with the steady-state case.

2.8 Hopf bifurcation with $O(2)$ symmetry

We have seen already that the interesting irreducible representations of $O(2)$ are on $\mathbb{R}^2 \cong \mathbb{C}$ and that they are absolutely irreducible. We consider the Hopf bifurcation problem where $O(2)$ acts as two copies of this absolutely irreducible representation: the centre manifold is then $\mathbb{C} \oplus \mathbb{C}$ and we take coordinates (w_1, w_2). Then the linearised system takes the form

$$\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = L\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and so the phase shift $\exp(sL^T)$ acts on $\mathbb{C} \oplus \mathbb{C}$ by

$$\exp(sL^T)(w_1, w_2) = (w_1 \cos s\omega + w_2 \sin s\omega, -w_1 \sin s\omega + w_2 \cos s\omega).$$

Recall that the two copies of the standard action of $O(2)$ on $\mathbb{C} \oplus \mathbb{C}$ are generated by

$$\rho_\theta(w_1, w_2) = (\mathrm{e}^{\mathrm{i}\theta} w_1, \mathrm{e}^{\mathrm{i}\theta} w_2), \qquad m_x(w_1, w_2) = (\bar{w}_1, \bar{w}_2).$$

The linear change of coordinates

$$z_1 = (\bar{w}_1 - \mathrm{i}\bar{w}_2)/2, \qquad z_2 = (w_1 - \mathrm{i}w_2)/2,$$

produces a neater action of $\mathcal{G} \times S^1$, relabelling the phase shift symmetry τ_ϕ:

$$\begin{aligned} \rho_\theta(z_1, z_2) &= (\mathrm{e}^{-\mathrm{i}\theta} z_1, \mathrm{e}^{\mathrm{i}\theta} z_2), \\ m_x(z_1, z_2) &= (z_2, z_1), \\ \tau_\phi(z_1, z_2) &= (\mathrm{e}^{\mathrm{i}\phi} z_1, \mathrm{e}^{\mathrm{i}\phi} z_2). \end{aligned}$$

We denote an element $g \circ \tau_\phi$ of $\mathcal{G} \times S^1$ by square brackets: $[g, \phi]$.

$\mathbb{C}$-axial branches

To apply the Equivariant Hopf Theorem we identify isotropy subgroups with two-dimensional fixed-point subspaces. There are exactly two group orbits of these, with representatives

$$\begin{aligned} \Sigma_{SW} &= \{[I, 0], [m_x, 0], [\rho_\pi, \pi], [m_x\rho_\pi, \pi]\}, \qquad \text{Fix}(\Sigma_{SW}) = (z, z), \\ \Sigma_{RW} &= \{[\rho_\theta, \theta] : \forall 0 \leq \theta \leq 2\pi\}, \qquad \text{Fix}(\Sigma_{RW}) = (z, 0). \end{aligned}$$

The first of these solutions is often referred to as a *standing wave* since at every point in time it has a reflection symmetry, and the second half-period of the oscillation is a symmetric image of the first half-period of the motion. The second solution is often referred to as a *rotating wave*. It has no spatial symmetry, but phase shifts in time act in the same way as shifts in space. The isotropy subgroup $\Sigma_{SW} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is a direct product, and $\Sigma_{RW} \cong S^1$ is isomorphic to the circle group. Equivariance can be used, as before, to deduce the normal form, truncated at cubic order, for the Hopf bifurcation:

$$\begin{aligned} \dot{z}_1 &= z_1[\mu + \mathrm{i}\omega - a_1|z_1|^2 - a_2|z_2|^2], \\ \dot{z}_2 &= z_2[\mu + \mathrm{i}\omega - a_1|z_2|^2 - a_2|z_1|^2], \end{aligned}$$

where a_1 and a_2 are complex coefficients. Writing $z_j = r_j \mathrm{e}^{\mathrm{i}\phi_j}$ we see that the phase variables ϕ_j decouple and we are in fact left with the same equations as for the steady-state bifurcation with D_4 symmetry. So the possible generic bifurcation behaviours can be read off from figure 5!

2.9 Hopf bifurcation with D_4 symmetry

To be added at some later point...