Isaac Newton Institute, Cambridge

Training Course: Pattern Formation

Lectures on Symmetric Bifurcation Theory

J.H.P. Dawes

These notes are intended to supplement the lectures by providing more detail in a various places and references to the literature. We start from the basic dynamical systems ideas given in the Introductory Lectures; the main aim is to see how these ideas are constrained by the introduction of symmetry, and how the symmetry can be used to simplify and understand the resulting bifurcation problems.

There are various textbooks that cover some or all of the ground sketched in these notes. In particular it is worth seeking out

- M. Golubitsky, I.N. Stewart & D.G. Schaeffer, *Singularities and Groups in Bifurcation Theory. Volume II.* Springer, Applied Mathematical Sciences Series **69** (1988)
- M. Golubitsky & I.N. Stewart *The Symmetry Perspective.* Progress in Mathematics, volume 200. Birkhäuser, Boston (2002)
- R.B. Hoyle, *Pattern Formation.* CUP, due out in 2005.

In style these notes are closer to Hoyle's book than the other two but the influence of all three can be seen at different points! Notation and definitions in these notes tries not to deviate far from the conventions in these references. There are many other general references, including the books by Field [22], Chossat & Lauterbach [10] and the volume *Pattern Formation in Continuous and Coupled Systems* edited by Golubitsky, Luss & Strogatz [23].

Although the book by Golubitsky, Stewart & Schaeffer is possibly the most often cited reference for symmetric bifurcation theory, the subject has roots that go a good deal further back. Results for steady-state bifurcations were obtained by Sattinger [49], Cicogna [11] and Vanderbauwhede [54]. The earlier papers of Ruelle [48] (1973) and Field [19] (1970) explore the general influence of symmetry on dynamical behaviour and bifurcations.

The simplest example of the constraining effects of symmetry is widely recognised: the pitchfork bifurcation. Given that the generic (i.e. codimension-one) steady-state bifurcation from an equilibrium point is a saddle-node bifurcation it should be surprising that we ever observe pitchfork bifurcations in experiments or numerical simulations. In nearly every case where a pitchfork bifurcation is observed it is due to symmetry. In the 1D case $\dot{x} = f(x)$ pitchfork bifurcations arise when $f(-x) = -f(x)$ is an odd function. This constraint is an example of an *equivariance condition* for $f(x)$ and is an expression of the intuitive and fundamental idea of what it means for a set of ODEs to be symmetric:

'If, whenever $x(t)$ is a solution trajectory, so is $g[x(t)]$,
then the operation g is a symmetry of the system.'

Naturally we would like to be able to compose group elements, so that $g_2 \circ g_1$ is a symmetry operation whenever g_1 and g_2 are. Thus we arrive at group theory, and the action of groups on vector spaces (here meaning nothing more than $\mathbb{R}^n$), leading to the idea of a *group representation.*

1 Lecture 1: Groups, representations and linear theory

In this lecture we set the scene by examining how symmetries affect a set of ODEs. This leads to restrictions on the Jacobian matrix at the bifurcation point, and hence to new 'typical' bifurcations. A crucial feature is the existence of invariant subspaces for the trajectories of the ODEs. We may then study the dynamics within each subspace separately. In the case of one-dimensional subspaces this leads to an important result for the existence of new branches of solutions near the bifurcation point.

1.1 Groups

A group is a set $\mathcal{G}$, plus a binary operation of composition of elements, that satisfies the following axioms:

- there exists an identity element I: $g \circ I = I \circ g = g \quad \forall g \in \mathcal{G}$
- there exist inverses: $g^{-1} \circ g = I = g \circ g^{-1} \quad \forall g \in \mathcal{G}$
- closure: $g_2 \circ g_1 \in \mathcal{G} \quad \forall g_1, g_2 \in \mathcal{G}$
- associativity: $g_3 \circ (g_2 \circ g_1) = (g_3 \circ g_2) \circ g_1 \quad \forall g_1, g_2, g_3 \in \mathcal{G}$.

We will usually omit the $\circ$ from now on; the composition rule will be clear from the context. In these notes then convention will be to do the rightmost element first in a product of elements. A group is said to be abelian if $g_1 g_2 = g_2 g_1 \; \forall g_{1,2} \in \mathcal{G}$. A subset $H \subset \mathcal{G}$ is a subgroup if it is closed under the group operation. A subgroup N of a group $\mathcal{G}$ is said to be a normal subgroup if $g^{-1} N g = N$ for all $g \in \mathcal{G}$ (where $g^{-1} N g = \{g^{-1} n g : n \in N\}$). The order of a group $\mathcal{G}$ is the number of elements of $\mathcal{G}$ and is denoted $|\mathcal{G}|$.

Examples of groups

We will concentrate on a small number of groups; the main ideas will be presented for finite groups, and the theory extends naturally to a much larger class of 'nice' groups: compact Lie (continuous) groups.

Finite groups:

- S_n, the symmetric group of permutations of n objects.
- $\mathbb{Z}_n$, the cyclic group of order n (for example rotations of a planar n-gon)
- D_n, the dihedral group (of order $2n$), for example rotations and reflections of a planar n-gon

Continuous groups:

- S^1, the group of rotations of a circle
- $O(n)$, the orthogonal group of $n \times n$ matrices A such that $A^T = A^{-1}$
- $SO(n)$, the special orthogonal group: the subgroup of $O(n)$ consisting of matrices with $\det A = 1$
- $E(2)$, the Euclidean group of planar rotations, reflections and translations

Of these, the first three are compact groups and are therefore straightforward to work with, however, $E(2)$ is not and is therefore much harder to work with. We will find ways to cope with this later on. Note that $SO(1)$ is isomorphic to the circle group S^1; such a group isomorphism is written as $SO(1) \cong S^1$.

Direct and semi-direct products, normal subgroups

Formally, given two group G and H we define the direct product group $G \times H = \{(g, h) : g \in G \text{ and } h \in H\}$ with a composition operation given by $(g_2, h_2) \circ (g_1, h_1) = (g_2 \circ g_1, h_2 \circ h_1)$. This notation, while precise, is more cumbersome that we need: we will just write $h \circ g$ (or just hg) instead of $(I_G, h) \circ (g, I_H)$ for composition. Importantly, a direct product contains subgroups (isomorphic to) the original groups G and H and each of these is a normal subgroup of $G \times H$.

A group $\mathcal{G}$ is said to be a semi-direct product group if it contains two subgroups H and N such that

- N is a normal subgroup
- H and N have only the identity element in common
- every element $g \in \mathcal{G}$ can be written as a product $g = nh$ with $n \in N$ and $h \in H$

we then write $\mathcal{G} = H \ltimes N$. Note that the second requirement implies that the decomposition $g = nh$ is unique. By way of illustration:

- $D_n = \mathbb{Z}_2 \ltimes \mathbb{Z}_n$, where the $\mathbb{Z}_2$ is generated by a reflection and the $\mathbb{Z}_n$ corresponds to rotations
- $E(2) = O(2) \ltimes \mathbb{R}^2$, a decomposition into rotations and reflections, and translations

Cosets and quotient groups

For any $g \in \mathcal{G}$ and subgroup $H \subset \mathcal{G}$, the subset $gH = \{gh : \ h \in H\}$ is called the (left) coset of H by g. Distinct cosets partition the set of elements of $\mathcal{G}$, meaning that every element of $\mathcal{G}$ lies in exactly one coset: $H, g_1H, \ldots, g_kH$.

If H is a normal subgroup then the set of cosets itself forms a group, the quotient group which we write as $\mathcal{G}/H$. This quotient group has a formally different composition rule: $(g_1H) \circ (g_2H) = (g_1g_2H)$. Elements of $\mathcal{G}/H$ are therefore cosets but we will sloppily identify the coset g_1H, sometimes written $[g_1]$, with g_1 itself.

1.2 Representations

A real representation of a group is a homomorphism $\tilde{\rho} : G \to GL(n, \mathbb{R})$, i.e. a map that associates to each group element $g \in G$ an invertible $n \times n$ matrix $\rho(g)$ with real entries. The matrix describes geometrically how the group element acts on $\mathbb{R}^n$. Being a homomorphism means that $\tilde{\rho}(g_2)\tilde{\rho}(g_1) = \tilde{\rho}(g_2g_1)$. The product $\tilde{\rho}(g_2)\tilde{\rho}(g_1)$ is standard matrix multiplication. From this definition various natural properties follow, including

- $\tilde{\rho}(I) = I_n$ the $n \times n$ identity matrix,
- $\tilde{\rho}(g)^{-1} = \tilde{\rho}(g^{-1})$.

A great simplification is due to the following theorem: we need only consider representations composed of orthogonal matrices.

Theorem 1 *Every compact Lie group G acting on $\mathbb{R}^n$ may be identified with a subgroup of $O(n)$.*

Proof: see Golubitsky et al. [25] pp30–31.

Examples of representations

To explore this definition futher we give four simple examples of representations.

Example 1: trivial representation. For any group $\mathcal{G}$ the map $\tilde{\rho}(g) = I_n$ mapping every element g to the $n \times n$ identity matrix is a representation.

Example 2: S_3. Let $\sigma \in S_3$ act on $\mathbb{R}^3$ by permutations of the coordinates $x = (x_1, x_2, x_3)$:

$$\tilde{\rho}(\sigma)(x_1, x_2, x_3) \quad = \quad (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)})$$

where the inverses make the map ρ a homomorphism: let $y = \tilde{\rho}(\sigma_1)x$, i.e. in coordinates $y_j = x_{\sigma_1^{-1}(j)}$, $j = 1, 2, 3$. Then

$$\begin{aligned} \tilde{\rho}(\sigma_2)\tilde{\rho}(\sigma_1)x &= \tilde{\rho}(\sigma_2)y = (y_{\sigma_2^{-1}(1)}, \ldots) \\ &= (x_{\sigma_1^{-1}\sigma_2^{-1}(1)}, \ldots) = (x_{(\sigma_2\sigma_1)^{-1}(1)}, \ldots) = \tilde{\rho}(\sigma_2\sigma_1)x. \end{aligned}$$

Using the cycle notation for permutations, we have explicitly

$$\tilde{\rho}((123)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \tilde{\rho}((12)(3)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 3: D_4. Let D_4 act on $\mathbb{R}^2$ in the natural way, as the symmetries of a square:

$$D_4 \quad = \quad \{I, m_x, m_y, m_d, m_{d'}, \rho, \rho^2, \rho^3\}$$

Then one possible representation is given by $R : D_4 \to GL(n, \mathbb{R})$:

$g \in D_4$:	I	ρ	ρ^2	ρ^3	m_x	m_y	m_d	$m_{d'}$
$R(g)$:	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Example 4: the 1D 'orientational' representation. For a group $\mathcal{G}$ corresponding to symmetries of a geometrical object in $\mathbb{R}^n$, $n > 1$, there is a one-dimensional representation $\tilde{\rho} : \mathcal{G} \to \{+1, -1\}$ given by

$$\begin{aligned} \tilde{\rho}(g) = +1 &\quad \text{if } g \text{ preserves orientation} \\ \tilde{\rho}(g) = -1 &\quad \text{if } g \text{ reverses orientation} \end{aligned}$$

Similarly, for (a subgroup of) the permutation group S_n there is a one-dimensional representation given by

$$\begin{aligned} \tilde{\rho}(g) = +1 &\text{ if } g \text{ is composed of an even number of transpositions,} \\ \tilde{\rho}(g) = -1 &\text{ if } g \text{ is composed of an odd number of transpositions.} \end{aligned}$$

A central idea in representation theory is *irreducibility*. A linear subspace $V \subseteq \mathbb{R}^n$ is $\mathcal{G}$-invariant (or simply invariant) if $\tilde{\rho}(g)v \in V \; \forall g \in \mathcal{G}$ and $v \in V$. A representation $\tilde{\rho}(\mathcal{G})$ is defined to be irreducible if the only invariant subspaces of $\mathbb{R}^n$ are the origin $\{0\}$ and the whole space $\mathbb{R}^n$; there are no proper non-trivial invariant subspaces. Otherwise the representation is said to be reducible. 'Irreducible representation' is often abbreviated to 'irrep'.

Returning to the examples above, the action of S_3 in example 2 is reducible: the subspace $V_1 \equiv \{(x, x, x) : x \in \mathbb{R}\}$ is invariant, and since the representation is orthogonal so is its complement

$V_1^\perp$. So we can simplify the problem and consider the group action on the two subspaces separately. In example 3 there are no invariant one-dimensional lines for the action of D_4 and so this action is irreducible. All one-dimensional representations are automatically irreducible (example 4).

Remark: Usually representation theory for finite (and compact) groups is developed over $\mathbb{C}$, i.e. representations are maps $\tilde{\rho} : G \to GL(n, \mathbb{C})$ into invertible matrices with complex entries. This is mathematically a more natural setting and allows a great deal to be learn about the representations of a particular group. We will stick with real representations, which introduces a slight complication in that there is a further refinement of the notion of irreducibility that we need to introduce.

Let $\mathcal{G}$ act irreducibly on $\mathbb{R}^n$. Then the set D of linear maps that commute with the matrices $\tilde{\rho}(\mathcal{G})$ forms an associative division ring (or 'skew field') [25]. This is one statement of the result known as Schur's Lemma. In the case of $\mathcal{G}$ acting on $\mathbb{R}^n$ there are three possibilities for the set D: $\mathbb{R}$, $\mathbb{C}$ or the quaternions $\mathbb{H}$. The set of commuting maps D being 'isomorphic to $\mathbb{R}$' just means that the only commuting maps are real multiples of the identity matrix. If this is the case then the representation of $\mathcal{G}$ is said to be absolutely irreducible.

Similarly in the $\mathbb{C}$ case the only commuting maps are isomorphic to complex multiples of the identity matrix. This case occurs in the study of oscillatory (Hopf) bifurcations with symmetry but we will not discuss it further here. The $\mathbb{H}$ case does not naturally occur in bifurcation problems.

For further notational convenience we will, once we've fixed on a particular representation, use g rather than $\tilde{\rho}(g)$ to denote the matrix representing the group element g. This will lead to less confusion overall.

Isotypic decomposition

If the representation of $\mathcal{G}$ on $\mathbb{R}^n$ is reducible then there is a nice way to decompose $\mathbb{R}^n$ into subspaces on which the matrices of the representation are block-diagonal. This fact turns out to be very helpful when computing stability since it enables us to block-diagonalise the Jacobian matrix. For the moment we will just see how this is done; later we will put it to use.

By repeatedly identifying subspaces $U_1, \ldots, U_m$ on which $\mathcal{G}$ acts irreducibly (and hence $\mathcal{G}$ acts irreducibly on the orthogonal complement) we can decompose $\mathbb{R}^n$ into a finite sum of irreducible subspaces $\mathbb{R}^n = U_1 \oplus U_2 \oplus \cdots \oplus U_m$. This decomposition is, however, not unique since the representations of $\mathcal{G}$ might be isomorphic on two or more of the U_j. We group those U_j on which $\mathcal{G}$ acts by the same representation (up to isomorphism) into larger subspaces $W_1, \ldots, W_k$: these are the isotypic components of $\mathbb{R}^n$. We can then prove

Theorem 2 *Let $\mathcal{G}$ act on $\mathbb{R}^n = W_1 \oplus W_2 \oplus \cdots \oplus W_k$, isotypically decomposed. Then for any linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ that commutes with $\mathcal{G}$, $A(W_j) \subseteq W_j$, i.e. the matrix corresponding to A is block-diagonal in these coordinates.*

Proof: First we argue that for any commuting linear map A, the linear subspace $\ker(A) \equiv \{x \in \mathbb{R}^n : Ax = 0\}$ is an invariant subspace. Let $x \in \ker(A)$, then $Ax = 0$, so $gAx = 0$ for all $g \in \mathcal{G}$. Then, by commutivity $A(gx) = 0$ and so $gx \in \ker(A)$. So $\ker(A)$ is $\mathcal{G}$-invariant.

Now consider the restriction $A|_{U_k^{(j)}}$ where $W_j = U_1^{(j)} \oplus \cdots \oplus U_{m_j}^{(j)}$ are the m_j irreducible subspaces of the isotypic component W_j. Then $\ker(A|_{U_k^{(j)}})$ is an invariant subspace by the first part of this argument. Then since the action of $\mathcal{G}$ on $U_k^{(j)}$ is assumed to be irreducible, the subspace $\ker(A|_{U_k^{(j)}})$ must either be $\{0\}$ or the whole subspace $U_k^{(j)}$. Hence $A|_{U_k^{(j)}}$ must either be invertible (and hence the image $A(U_k^{(j)}) \cong U_k^{(j)}$) or it must be identically zero. In both cases the image $A(U_k^{(j)})$ must lie inside the isotypic component W_j that contains all irreps isomorphic to the action of $\mathcal{G}$ on $U_k^{(j)}$. Hence $A(W_j) \subseteq W_j$ for all $j = 1, \ldots, k$. $\square$

A minor point is that we have not shown rigorously that a direct-sum decomposition of $\mathbb{R}^n$ as a sum of irreducibles is possible. See [27], pp37–38 for a proof of this.

1.3 Character tables

This section is a slight digression - there are ways of computing all the irreps of a given finite group and this gives a great deal of information about the structure of a group. But we will not need to go into detail about these methods.

Representations by matrices in $GL(n, \mathbb{C})$ have many nice properties not shared by real representations. Among these are a number of orthogonality properties that, for a specific group, enable a list of all irreducible representations to be calculated. This list of irreducible representations is summarised by a character table. The character of a representation is a map $\chi : \mathcal{G} \to \mathbb{C}$ defined by taking the trace of the matrices for each group elements: $\chi(g) = \mathrm{tr}\rho(g)$. Because traces are invariant under conjugacy operations by invertible matrices, many different collections of matrices acting irreducibly will have the same characters; concentrating on characters eliminates this rather spurious multiplicity. Two group elements h_1 and h_2 are said to be conjugate if there exists $g \in \mathcal{G}$ such that $h_2 = g^{-1}h_1 g$. Conjugacy is an equivalence relation on the group elements, and so divides the elements of $\mathcal{G}$ into equivalence classes, called conjugacy classes. It follows that all elements in a conjugacy class have the same character. We now give three very useful results that we state without proof concerning irreps.

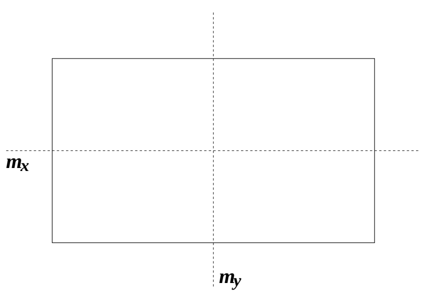
Theorem 3 (The dimension sum) *Let $\mathcal{G}$ have k irreducible representations $\rho_1, \ldots, \rho_k$ in $GL(n, \mathbb{C})$, and let $d_1, \ldots, d_k$ be the dimensions of these irreps. Then*

$$\sum_{j=1}^{k} d_j^2 = |G|.$$

Theorem 4 *$\mathcal{G}$ has the same number of irreps as conjugacy classes.*

Theorem 5 (Orthogonality properties of character tables) *Columns of character tables are orthogonal with respect to the usual 'dot product'. Rows are orthogonal also, when each term in the inner product is weighted by the number of elements in each conjugacy class.*

These theorems can be verified to hold in the three cases in figures 1 – 3, for the groups D_2, D_3 and D_4.

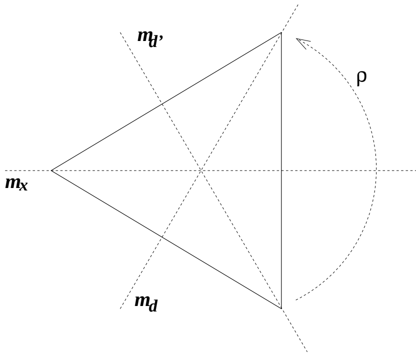


(a)

	Conjugacy classes			
	$\{Id\}$	$\{m_x\}$	$\{m_y\}$	$\{m_x \circ m_y\}$
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	1	-1	-1
χ_4	1	-1	-1	1

(b)

Figure 1: (a) A D_2-symmetric object, illustrating the group generators. (b) The character table for the group D_2.

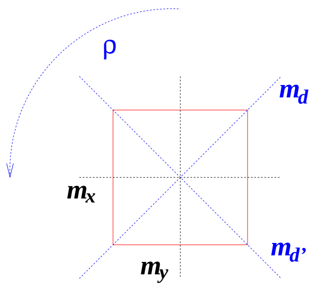


(a)

	Conjugacy classes		
	$\{Id\}$	$\{\rho,\ \rho^2\}$	$\{m_x,\ m_d,\ m_{d'}\}$
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

(b)

Figure 2: (a) A D_3-symmetric object, illustrating the group generators. (b) The character table for the group D_3.



(a)

Irrep	Conjugacy classes				
	$\{I\}$	$\{m_x, m_y\}$	$\{m_d, m_{d'}\}$	$\{\rho, \rho^3\}$	$\{\rho^2\}$
χ_1	1	1	1	1	1
χ_2	1	-1	-1	1	1
χ_3	1	-1	1	-1	1
χ_4	1	1	-1	-1	1
χ_5	2	0	0	0	-2

(b)

Figure 3: (a) A D_4-symmetric object, illustrating the group generators. (b) The character table for the group D_4.

1.4 Symmetric ODEs and bifurcations

Now we can return to the initial observation of 'system symmetry' and make it more precise. Suppose that $gx(t)$ is a solution trajectory of $\dot{x} = f(x)$ whenever $x(t)$ is. Then

$$\frac{d}{dt}gx(t) \quad = \quad f(gx(t))$$

which implies

$$f(gx(t)) = \frac{d}{dt}gx(t) = g\frac{d}{dt}x(t) = gf(x(t))$$

since g commutes with the time derivative. Hence we have the equivariance condition; the vector field $f(x)$ is equivariant under the (given representation of the) group $\mathcal{G}$ if

$$f(gx, \mu) = gf(x, \mu) \tag{1}$$

for all $g \in \mathcal{G}$, introducing a parameter μ.

From differentiating the equivariance condition we find

$$Df|_{(gx,\mu)}g \quad = \quad gDf|_{(x,\mu)} \tag{2}$$

and so $Df|_{(0,\mu)}$ (the Jacobian matrix at $x = 0$) is a linear map that commutes with the representation.

We now discuss the case where $\mathcal{G}$ acts absolutely irreducibly. Then we show that this is the generic context for steady-state bifurcations, in that if $\mathcal{G}$ does not act absolutely irreducibly then there are small perturbations of the bifurcation problem that do not undergo the same bifurcation, or even have a steady-state bifurcation at all! This is a key point in the theory:

generic steady-state bifurcations with symmetry
occur when the group acts absolutely irreducibly

In a preliminary step, we suppose that at $\mu = 0$ there is a steady-state bifurcation and all eigenvalues of $Df|_{(0,0)}$ lie on the imaginary axis; i.e. we have already performed a centre manifold reduction to eliminate other variables and are now working on the centre manifold.

The absolutely irreducible case

If $\mathcal{G}$ acts absolutely irreducibly and trivially, in the sense that the centre manifold is one dimensional but $\rho(g) = 1$ for all $g \in \mathcal{G}$, then symmetry does not restrict the form of the bifurcation at all. So we expect to see a saddle-node bifurcation as usual. From now on we will ignore the 'trivial' case as assume that $\mathcal{G}$ does not act trivially.

If $\mathcal{G}$ acts absolutely irreducibly, then from (1) we see that $f(0, \mu) = gf(0, \mu)$ for all $g \in \mathcal{G}$, and hence $f(0, \mu) = 0$ since otherwise the vector $f(0, \mu)$ would span a one-dimensional invariant subspace of $\mathbb{R}^n$ and so the representation would not be irreducible. So $x = 0$ is an equilibrium point for all values of the parameter μ. Further, by (2)

$$Df|_{(0,\mu)} \quad = \quad c(\mu)I_n$$

(for $c \in \mathbb{R}$) is a real multiple of the $n \times n$ identity matrix. Hence when $c(\mu) = 0$ the linearisation at $x = 0$ has n zero eigenvalues; when $c(\mu) \neq 0$ the origin is hyperbolic. Without loss of generality we may take $c(0) = 0$, i.e. the origin loses hyperbolicity, and we expect a steady-state bifurcation, when $\mu = 0$.

Non absolutely irreducible cases

Suppose that $\mathcal{G}$ acts *reducibly.* Then we can write $\mathbb{R}^n = V \oplus V^{\perp}$ where V is a non-trivial invariant subspace, say $\dim(V) = m \geq 1$. Then $Df|_{(0,\mu)}$ takes the form

$$Df|_{(0,\mu)} \quad = \quad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

which can be perturbed to

$$\tilde{Df}|_{(0,\mu)} \quad = \quad \begin{pmatrix} A + \varepsilon I_m & 0 \\ 0 & B \end{pmatrix}$$

which keeps $n - m$ zero eigenvalues in B when $\mu = 0$, but moves the remaining m of them off the imaginary axis when $\mu = 0$. This perturbed version of Df still commutes with the group action. So near $\mu = 0$ we have a different bifurcation problem: using the centre manifold theorem we could reduce the problem to one posed on $V^{\perp}$. So this case does not occur generically. It is still of interest however, since typically the representations of $\mathcal{G}$ on V and $V^{\perp}$ are different, and the proximity of the two bifurcation problems may generate secondary bifurcations on the branches of solutions that are produced in the bifurcation problems reduced to each subspace separately. Hence it is termed a 'mode interaction' (between two steady-state instabilities in this case).

Finally, suppose that $\mathcal{G}$ acts irreducibly but not absolutely irreducibly. Then the set of matrices that commute with the action of $\mathcal{G}$ is (isomorphic to) either $\mathbb{C}$ or $\mathbb{H}$ (which contains $\mathbb{C}$). So there is another real matrix, say M, that is non-zero, is not a multiple of Df, and also commutes with the group action. Then the perturbed Jacobian matrix

$$\tilde{Df}|_{(0,\mu)} = Df|_{(0,\mu)} + \varepsilon M \tag{3}$$

also commutes with the group action (by linearity) and has no zero eigenvalues. In the case where the set of commuting matrices is isomorphic to $\mathbb{C}$ the perturbation M takes the form

$$M = \begin{pmatrix} 0 & -\omega I_{n/2} \\ \omega I_{n/2} & 0 \end{pmatrix}$$

and the zero eigenvalues are 'pushed out' along the imaginary axis, see figure 4.

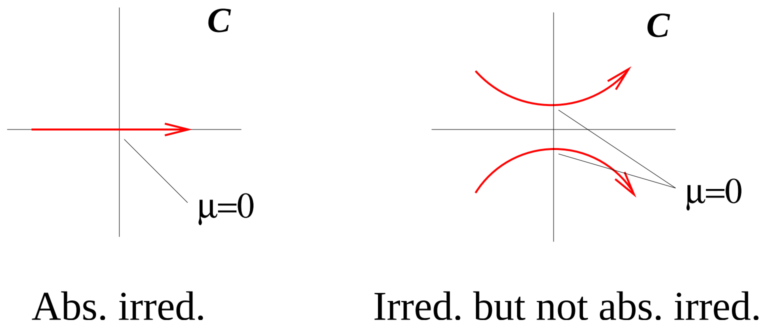


Figure 4: Sketches of the complex plane showing the behaviour of eigenvalues as μ crosses through zero. (a) The absolutely irreducible case: the eigenvalues are forced to be real, and all forced to cross the imaginary axis for one value of μ. (b) The generic situation in the irreducible but not absolutely irreducible case. There are no zero eigenvalues.

1.5 The Equivariant Branching Lemma

So far we have shown that generic steady-state bifurcations involve absolutely irreducible representations of $\mathcal{G}$ but we have said nothing about what happens at such a bifurcation point. The rest of these lectures will essentially be a discussion of various possibilities for the dynamics near these bifurcations. Most importantly, although we have discussed *system symmetry* in some detail we now need to turn to describing *solution symmetry*. This lecture will finish with an important result that gives an algebraic (i.e. in terms of the group action) answer to what appears to be a very analytic question:

What branches of solutions typically appear
in a steady-state bifurcation with symmetry?

First we need to introduce a few more terms: these will be essential to all the later analysis. The isotropy subgroup $\Sigma_x \subseteq \mathcal{G}$ of a point $x \in \mathbb{R}^n$ is defined to be

$$\Sigma_x = \{g \in \mathcal{G} \ : \ gx = x\}$$

By linearity $\Sigma_0 = \mathcal{G}$, and, if $\mathcal{G}$ acts absolutely irreducibly, and not by the trivial one dimensional representation, then no other point has 'full' isotropy $\mathcal{G}$. A closely related definition is that of the fixed point subspace $\mathrm{Fix}(\Sigma)$ of a subgroup (not necessarily an isotropy subgroup) $\Sigma \subseteq \mathcal{G}$:

$$\mathrm{Fix}(\Sigma) = \{x \in \mathbb{R}^n \ : \ \sigma x = x \ \forall \sigma \in \Sigma\}$$

The isotropy subgroup contains all symmetries that leave a point invariant. A fixed point subspace $\mathrm{Fix}(\Sigma)$ is a linear subspace of $\mathbb{R}^n$ made up of points that are invariant under symmetries in Σ. Note that there are many subgroups H of $\mathcal{G}$ which are not isotropy subgroups: if $x \in \mathrm{Fix}(H)$ then we can conclude that certainly $H \subseteq \Sigma_x$.

Moreover, some isotropy subgroups are conjugate to each other, i.e. they are equivalent under the symmetry of the system. For a point $x \in \mathbb{R}^n$ we define $\mathcal{G}x = \{gx : g \in \mathcal{G}\}$ to be the group orbit of the point x under the action of $\mathcal{G}$.

Theorem 6 *Points on the same group orbit have conjugate isotropy subgroups.*

Proof: Let x and gx be points on the same group orbit. Then

$$\begin{aligned}
\Sigma_{gx} &= \{\sigma \in \mathcal{G} : \ \sigma g x = gx\} \\
&= \{\sigma \in \mathcal{G} : \ g^{-1}\sigma g x = x\} \\
&= g\{g^{-1}\sigma g \in \mathcal{G} : \ g^{-1}\sigma g x = x\}g^{-1} \\
&= g\Sigma_x g^{-1}
\end{aligned}$$

where the first and last lines use the definition of an isotropy subgroup and the middle two lines involve multiplying by g and g^{-1} in various places. Hence $\Sigma_{gx} = g\Sigma_x g^{-1}$ and so the isotropy subgroups are conjugate. $\square$

From the equivariance condition (1) it is easy to show

Theorem 7 *Fixed point subspaces are flow invariant.*

Proof: Let f be $\mathcal{G}$-equivariant and let Σ be a subgroup of $\mathcal{G}$. Then if $x_0 \in \mathrm{Fix}(\Sigma)$, using the definition of the evolution operator $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ that defines solution curves, we see that

$$\begin{aligned}
\sigma\phi_t(x_0) &= \sigma\left(x_0 + \int_0^t f(x(s))ds\right) \\
&= \sigma x_0 + \int_0^t \sigma f(x(s))ds = \sigma x_0 + \int_0^t f(\sigma x(s))ds \\
&= \phi_t(\sigma x_0) = \phi_t(x_0)
\end{aligned}$$

for every $\sigma \in \Sigma$, using linearity and equivariance in the first two lines and then that fact that $x_0 \in \mathrm{Fix}(\Sigma)$ in the last line. Hence $\phi_t(x_0) \equiv x(t) \in \mathrm{Fix}(\Sigma)$ whenever $x_0 \in \mathrm{Fix}(\Sigma)$. $\square$

As as corollary, isotropy subgroups are constant along solution curves: the above result shows that $x(t) \in \mathrm{Fix}(\Sigma_{x(0)})$ and hence $\Sigma_{x(0)} \subseteq \Sigma_{x(t)}$. But because the evolution operator ϕ_t is invertible we can run time backwards and also find $\Sigma_{x(t)} \subseteq \Sigma_{x(0)}$. So we have equality: $\Sigma_{x(t)} = \Sigma_{x(0)}$

So one possible line of attack is to restrict the dynamics to each fixed point subspace in turn and analyse each of these simpler (well, certainly lower-dimensional) problems in turn. This approach turns out to work extremely well in the simplest case, and we therefore distinguish those isotropy subgroups Σ for which $\dim \mathrm{Fix}(\Sigma) = 1$. These are called axial isotropy subgroups.

Theorem 8 (The Equivariant Branching Lemma) *Let $\mathcal{G} \subseteq O(n)$ be a compact Lie group acting absolutely irreducibly on $\mathbb{R}^n$, and let $\dot{x} = f(x, \mu)$ be a $\mathcal{G}$-equivariant set of differential equations. Then it follows that*

- $f(0, \mu) = 0 \ \forall \mu$,
- $Df|_{(0,\mu)} = c(\mu) I_n$.

Assume also that

- $c(0) = 0$ *(bifurcation occurs at $\mu = 0$),*
- $\frac{dc}{d\mu}|_{\mu=0} \neq 0$ *(the eigenvalues cross the imaginary axis nondegenerately).*

Then for each axial isotropy subgroup $\Sigma \subseteq \mathcal{G}$ there exists a unique branch of solutions $x(\mu)$ satisfying $f(x(\mu), \mu) = 0$ branching from the origin and having symmetry Σ.

Proof: see Golubitsky et al. [25] pp82–83 and Golubitsky & Stewart [27] p18. The strategy of the proof is to use the above result on the flow invariant of the dynamics in $\mathrm{Fix}(\Sigma)$ to apply the standard one-dimensional bifurcation result to the dynamics within this subspace. Formally, we would apply the Implicit Function Theorem to the restricted bifurcation problem in $\mathrm{Fix}(\Sigma)$. $\square$

Informally we can argue that within $\mathrm{Fix}(\Sigma)$ the Taylor expansion of $\dot{x} = f(x, \mu)$ looks like

$$\dot{y} \quad = \quad \mu y + g_1(\mu) y^2 + g_2(\mu) y^3 + O(y^4)$$

where $y \in \mathbb{R}$ and $g_j(\mu)$ are smooth functions of μ. So we would generically expect to have a transcritical bifurcation within this subspace, or a pitchfork bifurcation if it turned out that g_1 were zero. To discern which of these cases occurs we should look to see whether there are any symmetry constraints on the dynamics within $\mathrm{Fix}(\Sigma)$. Unfortunately a straightforward approach to this (which we will present below) turns out to provide only a partial answer. Later we will give an example (D_5 acting on $\mathbb{R}^2$) that illustrates the partial nature of the answer.

Normalisers and fixed point subspaces

The normaliser $N_{\mathcal{G}}(H)$ of a subgroup $H \subset \mathcal{G}$ is defined to be

$$N_{\mathcal{G}}(H) \quad = \quad \{g \in \mathcal{G} : \; g^{-1} h g \in H \; \forall h \in H\}.$$

Notice that $H \subseteq N_{\mathcal{G}}(H)$ and that $N_{\mathcal{G}}(H)$ is a subgroup of $\mathcal{G}$. Moreover we can check that H is a normal subgroup of $N_{\mathcal{G}}(H)$ so the quotient group $N_{\mathcal{G}}(H)/H$ makes sense.

Theorem 9 *Let H be an isotropy subgroup of $\mathcal{G}$. Then $K \equiv N_{\mathcal{G}}(H)/H$ contains exactly those elements of $\mathcal{G}$ that act as symmetries on the subspace* $\mathrm{Fix}(H)$.

Remark: The dynamics within $\mathrm{Fix}(H)$ is K-equivariant.

Proof:

(1). Let $k \in K$. Then for all $h_1 \in H$ there exists $h_2 \in H$ such that $h_1 k = k h_2$. Then for any $x \in \mathrm{Fix}(H)$ we have

$$h_1 k x = k h_2 x = k x$$

so $kx \in \mathrm{Fix}(H)$ whenever $x \in \mathrm{Fix}(H)$ and so K maps $\mathrm{Fix}(H)$ bijectively to itself, since k is invertible.

(2). Let $k \in \mathcal{G} \setminus H$ be a symmetry that maps $\mathrm{Fix}(H)$ to itself. Consider the action of $p \equiv k^{-1} h k$. By assumption, $px = x$ whenever $x \in \mathrm{Fix}(H)$, i.e. p fixes $\mathrm{Fix}(H)$. Hence $p \in H$ because H is an isotropy subgroup and so contains all group elements that fix $\mathrm{Fix}(H)$. Let $p = \hat{h} \in H$. Then $hk = k\hat{h}$ for h, $\hat{h} \in H$ which implies $k \in N_{\mathcal{G}}(H)$. Since initially we required $k \notin H$ we must have $k \in K \equiv N_{\mathcal{G}}(H)/H$. $\square$

Now for axial branches we have $\dim \mathrm{Fix}(\Sigma) = 1$ and so $K = N_{\mathcal{G}}(\Sigma)/\Sigma \cong \{I\}$ or $K \cong \mathbb{Z}_2 = \{I, m\}$; either there is a single element that maps $x \to -x$ or there is only one element and it is the identity.

- If $K \cong \mathbb{Z}_2$ then the vector field within $\mathrm{Fix}(\Sigma)$ has only odd order terms and the bifurcation is a pitchfork.
- If $K \cong \{I\}$ then we cannot say: it may be a transcritical bifurcation or it may not, when there are other 'hidden' constraints.