Bifurcations and Instabilities in Dissipative Systems

J.H.P. Dawes, March 21, 2006

The Kuramoto–Sivashinsky Equation

1. Introduction

The Kuramoto-Sivashinsky (KS) equation is a parabolic PDE for a single scalar variable $u(x,t)$:

$$u_t + u_{xx} + u_{xxxx} + \frac{1}{2}(u_x)^2 \quad = \quad 0,$$

In addition to describing the nonlinear evolution of the Benjamin–Feir instability, the KS equation has been studied as a model for interfacial instabilities, for example in flame fronts and solidification problems. Coefficients in front of any of the terms may be removed by appropriate rescalings of u, t and x, leaving the canonical form above. The one remaining parameter is the domain size L: the equation is solved in $0 \leq x \leq L$ with typically either periodic boundary conditions (PBC, used here) or specific fixed values for u and u_x.

The first page of the pictures handout shows the temporal evolution of solutions to both the KS equation and the complex Ginzburg–Landau (CGL) equation, quite far from the Benjamin–Feir instability boundary. The solutions look moderately similar - a proper statistical analysis shows that actually the dynamics are somewhat different. But from our derivation in lectures, we can conclude that close to the Benjamin–Feir instability boundary the KS equation should provide a good description of the CGL behaviour.

The KS equation has obvious similarities with the Navier–Stokes (NS) equations: let $v = u_x$, then

$$v_t + v_{xx} + v_{xxxx} + v v_x \quad = \quad 0.$$

The last three terms correspond, respectively, to energy input at large scales, dissipation at small scales and nonlinear advection. Loose though this correspondence is, it is good enough to ensure striking similarities in some situations. In particular, both KS and NS are symmetric under translations by constant amounts in x, and under a more complicated Galilean symmetry taking $u \to u + U$ and transforming the space and time variables appropriately.

As an alternative to solving the equation in $0 \leq x \leq L$, we could scale the domain size to be constant, say 2π. This would introduce a parameter in front of the u_{xxxx} term which becomes small as L becomes large. Hence the 'large domain limit' is equivalent to the small viscosity, or large 'Reynolds number' limit.

2. Low dimensional dynamics

We now consider varying the domain size L. For small L we hope to have 'low-dimensional' dynamics, so that we could perform something like a centre manifold reduction to derive a reduced set of ODEs that describe the KS dynamics. Being a PDE this properly involves a functional-analytic setting, and a notion of a function space for solutions $u(x,t)$. Using such techniques it has been demonstrated that solutions of the KS equation are exponentially attracted to a finite-dimensional *inertial manifold* in phase space. Deriving bounds on the dimension of such an inertial manifold is now a standard functional-analytic calculation for many equations, including KS; this provides a global estimate of the complexity of the dynamics.

One aspect of such a discussion is the definition of the *dimension* of a set. The usual notion of dimension is that of *topological dimension*, defined inductively as follows (Robinson, 1995, p293). A set X has topological dimension zero if, for each point $x \in X$ there is an arbitrarily small neighbourhood U of x such that $\partial U \cap X = \emptyset$. Then, inductively, a set X is defined to have topological dimension $n > 0$ if, for each point $x \in X$, there is an arbitrarily small neighbourhood U of x such that $\partial U \cap X$ has dimension $n-1$. A more general definition of dimension allows non-integer values - one such definition is the *fractal* *dimension*. For the KS equation it has been proved that a universal attractor X exists and has fractal dimension $\dim_F(X) \le cL^{3/2}$. The fractal dimension $\dim_F(X)$ of a set X is defined to be

$$\dim_F(X) \quad := \quad \limsup_{\varepsilon\to 0} \frac{\log N(X,\varepsilon)}{\log 1/\varepsilon},$$

where $N(X,\varepsilon)$ is the minimum number of ε-balls that cover the set X. As a check, we can confirm that for 'standard' sets of integer topological dimension this definition gives the integer answer we expect. If X has topological dimension d, then $N(X,\varepsilon) = V\varepsilon^{-d}$, for some constant V, because we need of the order of $1/\varepsilon$ balls of radius ε along each 'side' of the set. So then

$$\frac{\log N(X,\varepsilon)}{-\log\varepsilon} = \frac{\log V - d\log\varepsilon}{-\log\varepsilon} \to d \quad \text{as} \quad \varepsilon \to 0.$$

For the middle-third Cantor set we can carry out the calculation and derive a non-integer value for the fractal dimension. Recall the construction: start from the interval $[0,1]$ and remove the middle third to leave two intervals $[0,1/3]$, $[2/3,1]$. Iterate. All the points in $[0,1]$ that do not contain a digit '2' in their base-3 expansion are left. When we take $\varepsilon = 1/3^n$ we will need 2^n ε-balls to cover the set. So

$$\frac{\log N(X,\varepsilon)}{-\log\varepsilon} = \frac{n\log 2}{n\log 3} = \frac{\log 2}{\log 3} \approx 0.63.$$

Howver, this kind of result tells us very little about the details of the dynamics, which is really what we would like to examine.

Translational symmetry implies that Fourier modes provide a good set of 'basis eigenfunctions' for a projection of the PDE on to a set of spatial modes. This procedure corresponds very closely to a spectral numerical scheme, also called a Galerkin truncation.

Write

$$u(x,t) \quad = \quad \sum_{k=-\infty}^{\infty} a_k(t)\exp\left(\frac{2\pi\mathrm{i}kx}{L}\right) \equiv \sum_{-\infty}^{\infty} a_k(t)\phi_k(x),$$

where the coefficients $a_k(t) \in \mathbb{C}$ are the *mode amplitudes*. Because $u(x,t)$ is real, $a_{-k}(t) = \bar{a_k}(t)$. We have the usual orthogonality relations between Fourier modes, so, on substituting this *ansatz* into the KS equation, multiplying by $\phi_l(x)$ and integrating we obtain an ODE for the amplitude a_l:

$$\dot{a}_l \quad = \quad \left(\frac{2\pi l}{L}\right)^2 a_l - \left(\frac{2\pi l}{L}\right)^4 a_l + \frac{1}{2}\left(\frac{2\pi}{L}\right)^2 \sum_j j(l-j)a_j a_{l-j}.$$

Rescaling time by a factor of $(2\pi/L)^2$ we obtain:

$$\dot{a}_l \quad = \quad l^2\left[1 - \left(\frac{2\pi l}{L}\right)^2\right] a_l + \frac{1}{2}\sum_j j(l-j)a_j a_{l-j}.$$

Note that the mode a_0 decouples and decays to zero:

$$\dot{a}_0 \quad = \quad -\frac{1}{2}\sum_j j^2|a_j|^2.$$

Taking all modes with $|j| \le K$ we have a system of K complex ODEs. For example, for $K = 4$ we have the ODEs

$$\begin{aligned}
\dot{a}_1 &= \left[1 - \left(\frac{2\pi}{L}\right)^2\right] a_1 - 2\bar{a}_1 a_2 - 6\bar{a}_2 a_3 - 12\bar{a}_3 a_4, \\
\dot{a}_2 &= 4\left[1 - \left(\frac{4\pi}{L}\right)^2\right] a_2 + \frac{1}{2}a_1^2 - 3\bar{a}_1 a_3 - 8\bar{a}_2 a_4, \\
\dot{a}_3 &= 9\left[1 - \left(\frac{6\pi}{L}\right)^2\right] a_3 + 2a_1 a_2 - \bar{a}_1 a_4, \\
\dot{a}_4 &= 16\left[1 - \left(\frac{8\pi}{L}\right)^2\right] a_4 + 2a_2^2 + 3a_1 a_3.
\end{aligned}$$

Now we will investigate local bifurcations from the origin. The trivial equilibrium $a_j = 0$ for all j is stable for $L < 2\pi$. There is a local bifurcation from the origin at $L = 2m\pi$ for all integers $m \geq 1$. A centre manifold reduction onto a 2-real-dimensional centre manifold yields

$$\dot{a}_m = m^2 \left(\mu - \frac{1}{12}|a_m|^2\right) a_m + O(|a_m|^4),$$

i.e. a 'pitchfork bifurcation of revolution' - the phase of a_m can be ignored, and this corresponds to the x-translational symmetry of the KS PDE. The bifurcation parameter μ is defined via $L = 2\pi m(1+\mu)$. This reduction is most useful for the first bifurcation, $m = 1$. In this case as it is a supercritical bifurcation there must exist a stable, predominantly mode-1, solution for $L > 2\pi$.

Near $L = 4\pi$ ($m = 2$) the origin is already unstable. But if all other modes with $m > 2$ are decaying, it makes sense to eliminate them and leave a_1 and a_2 describing the dynamics on a 4D 'centre-unstable' manifold. Define a new bifurcation parameter by $4[1-(4\pi/L)^2] = 16\mu$ (hence $1-(2\pi/L)^2 = 3/4+\mu$). Then a centre manifold reduction eliminating a_3 and a_4 gives

$$\begin{aligned} a_3 = h_3(a_1, \bar{a}_1, a_2, \bar{a}_2) &= \frac{1}{6} a_1 a_2 + O(3) \\ a_4 = h_4(a_1, \bar{a}_1, a_2, \bar{a}_2) &= \frac{1}{24} a_2^2 + O(3) \end{aligned}$$

Hence

$$\begin{aligned} \dot{a}_1 &= \left(\frac{3}{4} + \mu\right) a_1 - 2\bar{a}_1 a_2 - a_1 |a_2|^2 \\ \dot{a}_2 &= 16\mu a_2 + \frac{1}{2} a_1^2 - \frac{1}{3} a_2 |a_2|^2 - \frac{1}{2} a_2 |a_1|^2 \end{aligned}$$

After rescaling $-2a_2 = \hat{a}_2$ and dropping the hat we obtain:

$$\begin{aligned} \dot{a}_1 &= \left(\frac{3}{4} + \mu\right) a_1 + \bar{a}_1 a_2 - \frac{1}{4} a_1 |a_2|^2 \\ \dot{a}_2 &= 16\mu a_2 - a_1^2 - \frac{1}{12} a_2 |a_2|^2 - \frac{1}{2} a_2 |a_1|^2 \end{aligned}$$

which is now in the standard form that we analysed for the $1:2$ mode interaction. Note that here there is only one bifurcation parameter μ, so the dynamics cut a line through some part of our previous two-parameter bifurcation diagrams, see handout, page 2.

It turns out that numerical investigations of this 2-mode system are almost identical to the PDE dynamics at least up to $L \approx 6\pi$.

3. Heteroclinic dynamics - low dimensional temporal intermittency

The 2-mode system derived at the end of the last section naturally gives rise to intermittent dynamics for sets of coefficients for which the robust heteroclinic cycle is stable. More details of this can be found in the discussion of the $1:2$ mode interaction, for example from lecture notes available from my web page, or as discussed earlier in the course. It is interesting to note that the convective form of the nonlinearity vv_x guarantees that the low-order dynamics is of the 'interesting' kind, i.e. the '$-$' case rather than the '$+$' case in the $1:2$ mode interaction. Moreover, the Fourier mode truncation of the KS equation contains many invariant subspaces (for example, take all the modes with mode numbers that are a multiple of any given integer). Within each of these subspaces there is the possibility of intermittent dynamics in an analogous low-order system of ODEs. Time integrations, as on pages 4 and 5 of the pictures handout, show the intermittency very clearly, particularly in figures 14, 15 and 34.

4. Intermediate size domains

From the pictures on pages 3, 4 and 5 of the handout (taken from Hyman et al. 1986) it is clear that for larger L there is a sequence of alternating stable equilibria, or other simple attractors, and intermittent or chaotic dynamics. The last figure on page 5 shows the power spectrum of the solution averaged over time. This indicates that the solution still has a dominant spatial scale even in large domains, and illustrated in the top figure on page 1. In fact, as the wavelet analysis of Wittenberg & Holmes shows, about 80% of the energy is concentrated in spatial scales around the maximum.

5. Large domains

The power spectrum shows three distinct ranges of scales: at large scales the spectrum is almost flat, at intermediate 'active' scales there is a very short 'inertial range' with a slope $\sim k^{-4}$, roughly for $1/\sqrt{2} < k < 1$. For smaller scales the power spectrum decays exponentially.

Page 6 of the handout shows a solution of the KS equation obtained in a very large domain ($L = 5000$). Although close up (top figure) the solution has a preferred spatial scale and looks 'cellular' in the sense of the figure on page 1 of the handout, it is clear that the solution is much 'rougher' when looked at over large scales. Thus the evolution on long spatial scales is of interest, and is a different kind of problem to the pattern forming instability apparent at the active scales. It has been found that the large scales evolve effectively stochastically, forced by the active scales. The statistics for fluctuations on the large scales are Gaussian, and similar to those found for the forced Burgers equation, also known as the KPZ (Kardar - Parisi - Zhang) equation:

$$h_t = \nu h_{xx} + \frac{\lambda}{2}(h_x)^2 + \eta(x,t)$$

where $\eta(x,t)$ is a delta-correlated gaussian white noise term providing random forcing (i.e. at each timestep a gaussian iid increment is added to the RHS). Convergence of statistics to those of KPZ is only observed in large domains, $L > L_c \approx 2500$ typically, and on long timescales, $t > t_c \approx 7000$. The most useful statistical quantity is the RMS interface width $W(t) = \langle \tilde{h}(x,t)^2 \rangle_x$ where $\tilde{h}(x,t) = h(x,t) - \langle h(x,t) \rangle_x$ is the normalised interface position, and the $\langle \cdots \rangle_x$ denotes a spatial average. On intermediate timescales $t < t_c$ numerics indicate $W(t) \sim t^{1/4}$, a scaling that can be derived from the linearised noisy diffusion equation

$$h_t = \nu h_{xx} + \eta(x,t)$$

which is known as the Edwards-Wilkinson equation. On longer timescales $t > t_c$ there is a crossover to the faster, and nonlinear, KPZ scaling $W(t) \sim t^{1/3}$.

The active scales are, in turn, forced by the large scales - their dynamics are not qualitatively changed if the large-scale dynamics is replaced by stochastic forcing. In the absence of any large-scale dynamics, the active scales revert to spatially ordered solutions, as illustrated by the figure 8 from the paper by Wittenberg & Holmes (1999). Bohr et al. (1998) contains more discussion of this and related points.

Spatiotemporal dynamics in large systems is necessarily high-dimensional because distant parts of the system are effectively decoupled from each other. This is due to the existence of a finite correlation length for the dynamics; one of many ways of quantifying spatiotemporally complicated dynamics. The *correlation length* ξ is usually given as the inverse decay rate of the correlation function: $< u(x)u(x+r) > - < u >^2 \sim \exp(-r/\xi)$. If disturbances to the solution propagate with a finite speed bounded above by c then the correlation length is bounded by Kc/λ where K is a constant and λ is the largest (positive) Lyapounov exponent for the dynamics (the largest exponential rate at which nearby trajectories in phase space separate). Points in the domain that are separated by a distance larger than ξ are effectively decoupled from each other. Hence, heuristically, the dimension of the underlying attractor in phase space should grow linearly with the system volume, here the domain length L.

Bursts in boundary layers

One fluid mechanical example of wide interest that displays intermittent dynamics is the formation of turbulent streaks (streamwise vortices) in the boundary layer flows near a flat wall, as illustrated on page 7 of the handout. Experimental measurements show that the vortices periodically go through a 'burst–sweep' cycle which ejects fluid particles from near the wall out into the main stream of the flow.

The vortices have a well-defined horizontal scale and spacing, and can be analysed through a Galerkin truncation very similar to (though more involved than) that for the KS equation above.

It turns out (and for the details, see the book by Holmes et al.) that the burst–sweep cycle is generated by the interaction of low wavenumber modes essentially in the same way as the 1 : 2 mode interaction comes about in the KS equation. Indeed it contains subspaces where modes interact exactly as in the 1 : 2 resonance:

$$\begin{aligned} \dot{a}_2 &= a_2 + \bar{a}_2 a_4 + a_2|a_2|^2 + a_2|a_4|^2 + d_1(t) \\ \dot{a}_4 &= a_4 + a_2^2 + a_4|a_4|^2 + a_4|a_2|^2 + d_2(t) \end{aligned}$$

where I have omitted coefficients in front of every term on the RHS. The behaviour of these equations is shown in the bifurcation diagram on page 7. Of course, there is stability in other directions to consider; here we find that another kind of heteroclinic cycle between the mixed modes M_+ is possible; this behaviour was not possible in the original plain vanilla 1 : 2 resonance problem.

A summary of the dynamics for varying $\alpha \sim 1/Re$ is as follows:

Range of α	Behaviour
$2.41 < \alpha$	trivial eqm at origin is stable
$2.3 < \alpha < 2.41$	‘almost pure’ mode 2 bifurcates from origin and is stable
$2.0 < \alpha < 2.3$	het cycle as in 1 : 2 resonance problem
$1.61 < \alpha < 2.0$	M_+ mixed mode stable
$1.35 < \alpha < 1.61$	new kind of cycling between M_+ modes
$\alpha < 1.32$	new cycle loses stability: more irregular dynamics take over

There is one further complication in all this: the terms $d_j(t)$ on the RHS represent random forcings of the boundary layer by the flow in the outside region, well above the wall, through the pressure term in the NS equation. Analytically, the problem is to determine the change in the behaviour of trajectories close to a robust heteroclinic cycle when random noise is added to the system. The result is generally that the dynamics within a burst are unaffected, but the durations of the quiescent periods between bursts are random variables, with a probability distribution that can be calculated analytically in simple enough cases. The mean quiescent period scales as

$$T = c_1 - \frac{1}{\lambda_+} \log \varepsilon$$

where c_1 is a constant, λ_+ is the unstable eigenvalue at the saddle point on the cycle, and ε is a measure of the noise amplitude. This formula closely resembles the kind of result we derived using approximations within a small box near a saddle point composed with a global map near the unstable manifold of the saddle, so it is not really that surprising. Page 9 shows the regularising effect of noise or pressure fluctuations on the dynamics; the heteroclinic behaviour in the absence of noise is replaced by (slightly noisy) time-periodic oscillations. The bottom graph shows a close agreement between the analytically derived probability density function for the lengths of quiescent periods and the results for real simulations of the standard 1 : 2 ODEs with gaussian noise added.

To summarise, the bursting dynamics of this boundary layer problem are intrinsic to the fluid flow near the wall; the triggers for each of the burst–sweep cycles are determined by external fluctuations acting as noisy perturbations added to the dynamics.

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