What do we mean by saying that two flows (or maps) have essentially the same structure? The most useful notion of 'essentially the same' turns out to be a topological one; this preserves information about the number, stability, and relative arrangement of invariant sets. It may, however, lose (unimportant) information about transient dynamics - this is OK.

Definition: Two flows $\phi_t^f : X \to X$ and $\phi_\tau^g : Y \to Y$ are **topologically equivalent** if there is a homeomorphism $\mathbf{h} : X \to Y$ (i.e. a continuous bijection with continuous inverse) and a continuous time-rescaling function $\tau(\mathbf{y}, t)$, satisfying $\partial\tau/\partial t > 0 \;\; \forall\, \mathbf{y}$, such that

$$\phi_t^f(\mathbf{x}) = \mathbf{h}^{-1} \circ \phi_\tau^g \circ \mathbf{h}(\mathbf{x})$$

i.e. it is possible to find a map $\mathbf{h}$ between the phase spaces and a map τ that rescales time, in such a way that the trajectories of the two systems can be one-to-one identified.
Clearly topological equivalence maps fixed points to fixed points, and periodic orbits to periodic orbits – up to a change in the period.

Example: The dynamical systems

$$\begin{array}{l} \dot{r} = -r \\ \dot{\theta} = 1 \end{array} \quad \text{and} \quad \begin{array}{l} \dot{\rho} = -2\rho \\ \dot{\psi} = 0 \end{array} \tag{1}$$

are topologically equivalent with $\mathbf{h}(\mathbf{0}) = \mathbf{0}$, $\mathbf{h}(r, \theta) = (r^2, \theta + \ln r)$ for $r \neq 0$ in polar coordinates, and $\tau(\mathbf{x}, t) = t$. To show this, integrate the ODEs to get

$$\phi_t^f(r_0, \theta_0) = (r_0 e^{-t}, \theta_0 + t), \qquad \phi_t^g(\rho_0, \psi_0) = (\rho_0 e^{-2t}, \psi_0)$$

and check

$$\mathbf{h} \circ \phi_t^f = (r_0^2 e^{-2t}, \theta_0 + \ln r_0) = \phi_t^g \circ \mathbf{h}$$

Example: The dynamical systems

$$\begin{array}{l} \dot{r} = 0 \\ \dot{\theta} = 1 \end{array} \quad \text{and} \quad \begin{array}{l} \dot{r} = 0 \\ \dot{\theta} = r + \sin^2\theta \end{array} \tag{2}$$

are topologically equivalent. This should obvious because the trajectories are the same and so we can put $\mathbf{h}(\mathbf{x}) = \mathbf{x}$. All we then need do is stretch the timescale using $d/d\tau = (r + \sin^2\theta)(d/dt)$ along the trajectories to obtain $\tau = \int (r + \sin^2\theta)^{-1}\, d\theta$.

Definition: The flow $\phi_t^f(\mathbf{x})$, arising from integrating the ODEs $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, is **structurally stable** if, for every closed, bounded set $K \subset \mathbb{R}^n$, $\exists\, \varepsilon > 0$ such that $\mathbf{f} + \boldsymbol{\delta}$ is topologically equivalent to $\mathbf{f} \;\; \forall\, \boldsymbol{\delta}(\mathbf{x})$ satisfying

$$|\boldsymbol{\delta}| + \sum_i |\partial\boldsymbol{\delta}/\partial x_i| < \varepsilon \;\; \text{inside } K \text{ and } \boldsymbol{\delta} = 0 \text{ outside } K.$$

Examples: System (1) is structurally stable. System (2) is not (since the periodic orbits are destroyed by a small perturbation $\dot{r} \neq 0$).