3.2 Example: Basins of Attraction

Consider the two-dimensional system (written in the complex variable $z = x + \mathrm{i}y$):

$$\dot{z} = \mathrm{e}^{\mathrm{i}\alpha}(z - z^3)$$

1. The fixed points are at $z = 0$ and $z = \pm 1$.
2. The Jacobian (again in complex form) is $\mathrm{e}^{\mathrm{i}\alpha}(1 - 3z^2)$. At $z = 0$ the eigenvalues are $\cos\alpha \pm \mathrm{i}\sin\alpha$ and at $z = \pm 1$ they are $-2(\cos\alpha \pm \mathrm{i}\sin\alpha)$. (Check this by taking real and imaginary parts!) Hence if $|\alpha| < \pi/2$ the origin is an unstable focus, whereas ± 1 are stable foci and the only attracting sets.
3. The direction of trajectories is best considered in polars $z = r\mathrm{e}^{\mathrm{i}\theta}$:

   $$\Rightarrow \dot{r} + \mathrm{i}r\dot{\theta} \quad = \quad r\mathrm{e}^{\mathrm{i}\alpha} - r^3\mathrm{e}^{\mathrm{i}(\alpha+2\theta)}$$

   which leads to

   $$\begin{aligned} \dot{r} &= r\cos\alpha - r^3\cos(\alpha + 2\theta), \\ r\dot{\theta} &= r\sin\alpha - r^3\sin(\alpha + 2\theta). \end{aligned}$$

4. There is no point in calculating eigenvectors for foci. We can deduce the sense of rotation from the $\dot{\theta}$ equation; for example when r is small and $\alpha > 0$ we see that $\dot{\theta} > 0$.
5. The behaviour of $\dot{r}$ as $r \to 0$ confirms the stability results for $z = 0$.

All trajectories emerging from $z = 0$ are attracted to $z = \pm 1$ except two which escape to infinity along the directions where $\dot{\theta} = 0$ and $\dot{r} > 0$ i.e. $\theta = (\pi - \alpha)/2$ and $\theta = (3\pi - \alpha)/2$. These trajectories form the boundaries of the basins of attraction of $z = \pm 1$.

Below is the phase portrait for $\alpha = 0.96\pi/2$. Refer to example sheet 2 for additional details.