

# Nilpotent symplectic alternating algebras III

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In this paper we finish our classification of nilpotent symplectic alternating algebras of dimension 10 over any field  $\mathbb{F}$ .

## 1 Introduction

A symplectic alternating algebra (SAA) is a symplectic vector space  $L$ , whose associated alternating form is non-degenerate, that is furthermore equipped with a binary alternating product  $\cdot : L \times L \mapsto L$  with the extra requirement that

$$(x \cdot y, z) = (y \cdot z, x)$$

for all  $x, y, z \in L$ . This condition can be expressed equivalently by saying that  $(u \cdot x, v) = (u, v \cdot x)$  for all  $u, v, x \in L$  or in other words that multiplication from the right is self-adjoint with respect to the alternating form.

Symplectic alternating algebras originate from a study of powerful 2-Engel groups [1], [5] and there is a 1-1 correspondence between a certain rich class of powerful 2-Engel 3-groups of exponent 27 and SAAs over the field  $\text{GF}(3)$ . We refer to [2] and [6] for a more detailed discussion of this as well as some general background to SAAs.

In this paper we finish our classification of nilpotent symplectic alternating algebras of dimension 10 that we started in [3]. Our approach like there

relies on a general theory that we developed in [2]. Before giving some account of this we introduce some notation. Firstly we can always pick a basis  $x_1, y_1, \dots, x_n, y_n$  with the property that  $(x_i, x_j) = (y_i, y_j) = 0$  and  $(x_i, y_j) = \delta_{ij}$  for  $1 \leq i \leq j \leq n$ . We refer to a basis of this type as a *standard basis*. It turns out that for any nilpotent symplectic alternating algebra one can always choose a suitable standard basis such that the chain of subspaces

$$0 = I_0 < I_1 < \dots < I_n < I_{n-1}^\perp < \dots < I_0^\perp = L,$$

with  $I_k = \mathbb{F}x_n + \dots + \mathbb{F}x_{n-k+1}$ , is a central chain of ideals. One can furthermore see from this that  $x_i y_j = 0$  if  $j \leq i$  and that  $I_{n-1}^\perp$  is abelian. It follows that a number of the triple values  $(uv, w)$  are trivial. Listing only the values that are possibly non-zero it suffices to consider

$$\mathcal{P} : \quad (x_i y_j, y_k) = \alpha_{ijk}, \quad (y_i y_j, y_k) = \beta_{ijk}, \quad 1 \leq i < j < k \leq n$$

for some  $\alpha_{ijk}, \beta_{ijk} \in \mathbb{F}$ . Such a presentation is called a *nilpotent presentation*. Conversely any such presentation describes a nilpotent SAA. The algebras that are of maximal class turn out to have a rigid ideal structure. In particular when  $2n \geq 10$  we can choose our chain of ideals above such that they are all characteristic and it turns out that  $I_0, I_2, I_3, \dots, I_{n-1}, I_{n-1}^\perp, I_{n-2}^\perp, \dots, I_0^\perp$  are unique and equal to both the terms of the lower and upper central series ( see [2] Theorems 3.1 and 3.2 ). The algebras of maximal class can be identified easily from their nilpotent presentations. In fact, if  $\mathcal{P}$  is any nilpotent presentation of  $L$  with respect to a standard basis  $\{x_1, y_1, \dots, x_n, y_n\}$ , and  $2n \geq 8$ , we have that  $L$  is of maximal class if and only if  $x_i y_{i+1} \neq 0$  for all  $i = 2, \dots, n-2$ , and  $x_1 y_2, y_1 y_2$  are linearly independent ( see [2] Theorem 3.4). In [3] we found all the nilpotent algebras with a non-isotropic centre and we gave a full classification of the nilpotent SAAs with an isotropic centre of dimension 5 and 3. Here we deal with the two remaining cases, namely when the centre has dimension 2 or 4. In the latter our approach leads to an interesting geometrical setup.

## 2 Algebras with an isotropic centre of dimension 4

The situation here is far more complicated and we will need to consider several subcases. Let  $L$  be a nilpotent SAA of dimension 10 with an isotropic

centre of dimension 4. We can pick our standard basis such that

$$Z(L) = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2.$$

Then

$$L^2 = Z(L)^\perp = Z(L) + \mathbb{F}x_1 + \mathbb{F}y_1$$

and by [2, Lemma 2.6] we know that  $L^3 = L^2L \leq Z(L)$ . As  $L^2 \not\leq Z(L)$  we have  $L^3 \neq \{0\}$  and by [2, Propositions 2.9 and 2.10] we then have that  $3 \leq \dim L^3 \leq 4$ . We will consider the cases  $L^3 < Z(L)$  and  $L^3 = Z(L)$  separately.

## 2.1 The algebras where $L^3 < Z(L)$

In this case we can choose our basis such that

$$\begin{array}{c} L^3 \\ Z(L) \end{array} \begin{array}{|c|c|} \hline x_5 & y_5 \\ \hline x_4 & y_4 \\ \hline x_3 & y_3 \\ \hline x_2 & y_2 \\ \hline x_1 & y_1 \\ \hline \end{array} \begin{array}{c} Z_2(L) \\ L^2 \end{array} \begin{array}{l} L^3 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 \\ Z(L) = L^3 + \mathbb{F}x_2 \\ L^2 = Z(L)^\perp = Z(L) + \mathbb{F}x_1 + \mathbb{F}y_1 \\ Z_2(L) = (L^3)^\perp = L^2 + \mathbb{F}y_2 \end{array}$$

Notice that, as  $Z_2(L) \cdot L^2 = \{0\}$ , we have that  $Z_2(L)$  is abelian and thus in particular  $x_1y_2 = y_1y_2 = 0$ . As  $Z_2(L)$  is an abelian ideal we also have that  $Z_2(L)L$  is orthogonal to  $Z_2(L)$  and thus  $Z_2(L) \cdot L \leq Z_2(L)^\perp = L^3$ . It follows that

$$\mathbb{F}x_2 + \mathbb{F}x_1 + \mathbb{F}y_1 + L^3 = L^2 = \mathbb{F}y_3y_4 + \mathbb{F}y_3y_5 + \mathbb{F}y_4y_5 + L^3.$$

Suppose

$$x_2 + L^3 = \alpha y_3y_4 + \beta y_3y_5 + \gamma y_4y_5 + L^3.$$

Now at least one of  $\alpha, \beta, \gamma$  is nonzero and by the symmetry in  $y_3, y_4, y_5$  we can assume that  $\alpha \neq 0$ . Thus

$$x_2 + L^3 = (y_3 - \frac{\gamma}{\alpha}y_5)(\alpha y_4 + \beta y_5) + L^3.$$

By replacing  $x_4, x_5, y_3, y_4$  by  $\tilde{x}_4 = \frac{1}{\alpha}x_4, \tilde{x}_5 = x_5 - \frac{\beta}{\alpha}x_4 + \frac{\gamma}{\alpha}x_3, \tilde{y}_3 = y_3 - \frac{\gamma}{\alpha}y_5, \tilde{y}_4 = \alpha y_4 + \beta y_5$ , we can then assume that  $x_2 + L^3 = y_3y_4 + L^3$ . In particular  $(y_2y_3, y_4) = 1$ . Suppose that  $(y_3y_4, y_5) = \tau$ . Replacing  $x_2, y_5$  by  $\tilde{x}_2 = x_2 + \tau x_5$

and  $\tilde{y}_5 = y_5 - \tau y_2$  we can furthermore assume that  $(y_3 y_4, y_5) = 0$ . If we let  $L = \mathbb{F}y_3 + \mathbb{F}y_4 + \mathbb{F}y_5$ , it follows that we now have

$$L^2 = \mathbb{F}x_2 + \mathbb{F}x_1 + \mathbb{F}y_1 \text{ and } y_3 y_4 = x_2. \quad (1)$$

By (1) we know that

$$x_1 = \alpha y_3 y_4 + \beta y_3 y_5 + \gamma y_4 y_5 = \alpha x_2 + \beta y_3 y_5 + \gamma y_4 y_5,$$

where without loss of generality we can assume that  $\gamma \neq 0$ . Then

$$x_1 = (y_4 + \frac{\beta}{\gamma} y_3)(-\alpha y_3 + \gamma y_5).$$

Replacing  $x_3, x_5, y_4, y_5$  by  $\tilde{x}_3 = x_3 - \frac{\beta}{\gamma} x_4 + \frac{\alpha}{\gamma} x_5$ ,  $\tilde{x}_5 = \frac{1}{\gamma} x_5$ ,  $\tilde{y}_4 = y_4 + \frac{\beta}{\gamma} y_3$  and  $\tilde{y}_5 = -\alpha y_3 + \gamma y_5$ , we obtain

$$y_4 y_5 = x_1. \quad (2)$$

Notice that (1) is not affected by these changes. Finally we know from (1) and (2) that

$$y_1 = -\alpha x_2 - \beta x_1 + \gamma y_5 y_3$$

for some  $0 \neq \gamma \in \mathbb{F}$ . Then

$$y_1 = (y_5 + \frac{\alpha}{\gamma} y_4)(\gamma y_3 + \beta y_4).$$

Now replace  $x_3, x_4, y_3, y_5$  by  $\tilde{x}_3 = \frac{1}{\gamma} x_3$ ,  $\tilde{x}_4 = x_4 - \frac{\alpha}{\gamma} x_5 - \frac{\beta}{\gamma} x_3$ ,  $\tilde{y}_3 = \gamma y_3 + \beta y_4$  and  $\tilde{y}_5 = y_5 + \frac{\alpha}{\gamma} y_4$ . This gives us

$$y_5 y_3 = y_1. \quad (3)$$

This does not affect (2) but instead of (1) we get  $y_3 y_4 = \gamma x_2$ . Now we make the final change by replacing  $x_2$  and  $y_2$  by  $\gamma x_2$  and  $\frac{1}{\gamma} y_2$  and we can assume that (1), (2) and (3) hold. We had seen earlier that  $Z_2(L)$  is abelian and thus all triple values involving two elements from  $\{x_5, x_4, x_3, x_2, x_1, y_1, y_2\}$  are trivial. Thus all the nontrivial triple values involve two of  $y_3, y_4, y_5$  but from (1),(2) and (3) we know what these are. We have thus proved

**Proposition 2.1** *There is a unique nilpotent SAA of dimension 10 that has an isotropic centre of dimension 4 and where  $L^3 < Z(L)$ . This algebra can be given by the nilpotent presentation*

$$\mathcal{P}_{10}^{(4,1)} : (y_2 y_3, y_4) = 1, (y_1 y_4, y_5) = 1, (x_1 y_3, y_5) = 1.$$

**Remark.** Inspection shows that the algebra with that presentation has a centre of dimension 4 and the property that  $L^3 < Z(L)$ .

## 2.2 The algebras where $L^3 = Z(L)$

We will see that this case is quite intricate and we will need to consider some subcases.

$$\begin{array}{c}
 \begin{array}{|c|c|}
 \hline
 x_5 & y_5 \\
 \hline
 x_4 & y_4 \\
 \hline
 x_3 & y_3 \\
 \hline
 x_2 & y_2 \\
 \hline
 x_1 & y_1 \\
 \hline
 \end{array} \\
 Z(L) = L^3
 \end{array}
 \quad
 \begin{array}{l}
 Z(L) = L^3 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 \\
 Z_2(L) = L^2 = Z(L)^\perp = Z(L) + \mathbb{F}x_1 + \mathbb{F}y_1 \\
 L^2 = Z_2(L)
 \end{array}$$

In order to clarify the structure further, we will associate to any such algebra a family of new alternating forms that are defined as follows. For each  $\bar{z} = z + Z(L) \in L^2/Z(L)$ , we obtain the alternating form

$$\phi_{\bar{z}} : L/L^2 \times L/L^2 \rightarrow \mathbb{F}$$

given by

$$\phi_{\bar{z}}(\bar{u}, \bar{v}) = (zu, v)$$

where  $\bar{u} = u + L^2$  and  $\bar{v} = v + L^2$ . Notice that this is a well defined function as  $L^2$  is abelian.

**Remarks.** (1) If  $0 \neq \bar{z} = z + Z(L) \in L^2/Z(L)$ , then  $\phi_{\bar{z}} \neq 0$ . Otherwise we would have  $(zu, v) = 0$  for all  $u, v \in L$  that would give the contradiction that  $z \in Z(L)$  and thus  $\bar{z} = 0$ .

(2) There is no non-zero element in  $V = L/L^2$  that is common to the isotropic part of  $V = L/L^2$  with respect to all the alternating forms  $\phi_{\bar{z}}$  with  $\bar{z} \in L^2/Z(L)$ . Otherwise there would be some  $0 \neq t \in \mathbb{F}y_5 + \mathbb{F}y_4 + \mathbb{F}y_3 + \mathbb{F}y_2$  such that  $(zt, u) = 0$  for all  $z \in L^2$  and all  $u \in L$ . But then  $ut \in (L^2)^\perp = Z(L)$  for all  $u \in L$  that gives the contradiction that  $t \in Z_2(L) = L^2$ .

We divide the algebras into three categories.

**A.** The algebras where there exists a basis  $\bar{z}, \bar{t}$  for  $L^2/Z(L)$  such that the alternating forms  $\phi_{\bar{z}}, \phi_{\bar{t}}$  are both degenerate.

**B.** The algebras where there exists  $0 \neq \bar{z} \in L^2/Z(L)$  such that  $\phi_{\bar{z}}$  is degenerate but  $\phi_{\bar{t}}$  is non-degenerate for all  $\bar{t} \in L^2/Z(L)$  that are not in  $\mathbb{F}\bar{z}$ .

C. The algebras where  $\phi_{\bar{z}}$  is non-degenerate for all  $0 \neq \bar{z} \in L^2/Z(L)$ .

### 2.2.1 Algebras of type A

Pick  $x_1, y_1 \in L^2 \setminus Z(L)$  such that  $\phi_{\bar{x}_1}$  and  $\phi_{\bar{y}_1}$  are degenerate and such that  $(x_1, y_1) = 1$ . By the remarks above we thus know that the isotropic part of  $L/L^2$  with respect to both the alternating forms  $\phi_{\bar{x}_1}$  and  $\phi_{\bar{y}_1}$  is of dimension 2 and the intersection of the two is trivial. Thus we can pick a basis  $\bar{y}_5 = y_5 + L^2, \bar{y}_4 = y_4 + L^2, \bar{y}_3 = y_3 + L^2, \bar{y}_2 = y_2 + L^2$  for  $L/L^2$  such that

$$\begin{aligned} \mathbb{F}\bar{y}_4 + \mathbb{F}\bar{y}_5 & \text{ is the isotropic part of } L/L^2 \text{ with respect to } \phi_{\bar{x}_1}. \\ \mathbb{F}\bar{y}_3 + \mathbb{F}\bar{y}_2 & \text{ is the isotropic part of } L/L^2 \text{ with respect to } \phi_{\bar{y}_1}. \end{aligned}$$

This shows us that we can pick our standard basis such that

$$\begin{aligned} (x_1y_2, y_3) &= 1 & (y_1y_2, y_3) &= 0 \\ (x_1y_2, y_4) &= 0 & (y_1y_2, y_4) &= 0 \\ (x_1y_2, y_5) &= 0 & (y_1y_2, y_5) &= 0 \\ (x_1y_3, y_4) &= 0 & (y_1y_3, y_4) &= 0 \\ (x_1y_3, y_5) &= 0 & (y_1y_3, y_5) &= 0 \\ (x_1y_4, y_5) &= 0 & (y_1y_4, y_5) &= 1. \end{aligned}$$

To determine the structure fully we are only left with the triples  $(y_iy_j, y_k) = r_{ijk}$  for  $2 \leq i < j < k \leq 5$ . Let

$$\begin{aligned} \tilde{y}_i &= y_i + \alpha_i x_1 + \alpha_i y_1, \\ \tilde{x}_1 &= x_1 - (\alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5), \\ \tilde{y}_1 &= y_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5. \end{aligned}$$

Inspection shows that we can choose  $\alpha_2, \dots, \alpha_5$  such that  $(\tilde{y}_i \tilde{y}_j, \tilde{y}_k) = 0$  for all  $2 \leq i < j < k \leq 5$ . In fact this works for  $\alpha_2 = -r_{245}, \alpha_3 = -r_{345}, \alpha_4 = -r_{234}$  and  $\alpha_5 = -r_{235}$ . We have thus proved the following result.

**Proposition 2.2** *There is a unique nilpotent SAA of dimension 10 with an isotropic centre of dimension 4 where  $L^3 = Z(L)$  and  $L$  is of type A. This algebra can be given by the presentation*

$$\mathcal{P}_{10}^{(4,2)} : (x_1y_2, y_3) = 1, (y_1y_4, y_5) = 1.$$

Notice that inspection shows that the algebra with this presentation indeed has the properties stated in the proposition.

### 2.2.2 Algebras of type B

Suppose that  $\phi_{\bar{x}_1}$  is degenerate and that the isotropic part of  $L/L^2$  with respect to this alternating form is  $\mathbb{F}\bar{y}_4 + \mathbb{F}\bar{y}_5$ . We are now assuming that  $\phi_{\bar{z}}$  is non-degenerate for all  $\bar{z} \notin \mathbb{F}\bar{x}_1$ . Pick  $y_1 \in L^2$  such that  $(x_1, y_1) = 1$ .

**Remark.** We must have  $\phi_{\bar{y}_1}(\bar{y}_4, \bar{y}_5) = 0$ . Otherwise we would get a basis  $\bar{y}_2, \bar{y}_3, \bar{y}_4, \bar{y}_5$  for  $L/L^2$  such that

$$\phi_{\bar{y}_1}(\bar{y}_4, \bar{y}_5) = 1, \quad \phi_{\bar{y}_1}(\bar{y}_2, \bar{y}_3) = 1, \quad \phi_{\bar{x}_1}(\bar{y}_2, \bar{y}_3) = \alpha \neq 0$$

and where  $\phi_{\bar{y}_1}(\bar{y}_i, \bar{y}_j) = 0$  and likewise  $\phi_{\bar{x}_1}(\bar{y}_i, \bar{y}_j) = 0$  for any pair  $\bar{y}_i, \bar{y}_j$  such that  $2 \leq i < j \leq 5$  that is not included above. But then inspection shows that  $\phi_{\alpha\bar{y}_1 - \bar{x}_1}$  is degenerate where the corresponding isotropic part of  $L/L^2$  is  $\mathbb{F}\bar{y}_2 + \mathbb{F}\bar{y}_3$ . But this contradicts our assumptions.

We thus know that  $\phi_{\bar{y}_1}(\bar{y}_4, \bar{y}_5) = 0$ . As  $\phi_{\bar{y}_1}$  is non-degenerate we know that there exists some  $\bar{y}_2 \in L/L^2$  such that

$$\phi_{\bar{y}_1}(\bar{y}_2, \bar{y}_4) = 1. \quad (4)$$

Replacing  $y_5$  and  $y_3$  by some suitable  $y_5 + \alpha y_4$  and  $y_3 + \beta y_4 + \gamma y_2$  we can furthermore assume that

$$\phi_{\bar{y}_1}(\bar{y}_2, \bar{y}_5) = \phi_{\bar{y}_1}(\bar{y}_3, \bar{y}_4) = \phi_{\bar{y}_1}(\bar{y}_2, \bar{y}_3) = 0. \quad (5)$$

As  $\phi_{\bar{x}_1}$  is non-zero we must have  $\phi_{\bar{x}_1}(\bar{y}_2, \bar{y}_3) \neq 0$  and by replacing  $\bar{y}_3$  by a multiple of itself we can assume that

$$\phi_{\bar{x}_1}(\bar{y}_2, \bar{y}_3) = 1. \quad (6)$$

Notice that this does not affect (4) and (5). As  $\phi_{\bar{y}_1}$  is non-degenerate we cannot have that  $\bar{y}_3$  is isotropic to all vectors in  $L/L^2$  with respect to this alternating form. Thus by (5) we must have  $\phi_{\bar{y}_1}(\bar{y}_3, \bar{y}_5) \neq 0$  and by replacing  $\bar{y}_5$  by a multiple of itself we can assume that

$$\phi_{\bar{y}_1}(\bar{y}_3, \bar{y}_5) = 1. \quad (7)$$

Again equations (4),(5) and (6) are not affected. We thus see that we can choose a standard basis such that

$$\begin{aligned} (x_1y_2, y_3) &= 1 & (y_1y_2, y_3) &= 0 \\ (x_1y_2, y_4) &= 0 & (y_1y_2, y_4) &= 1 \\ (x_1y_2, y_5) &= 0 & (y_1y_2, y_5) &= 0 \\ (x_1y_3, y_4) &= 0 & (y_1y_3, y_4) &= 0 \\ (x_1y_3, y_5) &= 0 & (y_1y_3, y_5) &= 1 \\ (x_1y_4, y_5) &= 0 & (y_1y_4, y_5) &= 0. \end{aligned}$$

As in case A we are now only left with the triples  $(y_iy_j, y_k) = r_{ijk}$  for all  $2 \leq i < j < k \leq 5$ . As in that case we let

$$\begin{aligned} \tilde{y}_i &= y_i + \alpha_i x_1 + \alpha_i y_1, \\ \tilde{x}_1 &= x_1 - (\alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5), \\ \tilde{y}_1 &= y_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5. \end{aligned}$$

Inspection shows that we can choose  $\alpha_2, \alpha_3, \alpha_4, \alpha_5$  such that  $(\tilde{y}_i \tilde{y}_j, \tilde{y}_k) = 0$  for  $2 \leq i < j < k \leq 5$ . We thus get the following result.

**Proposition 2.3** *There is a unique nilpotent SAA of dimension 10 with isotropic centre of dimension 4 where  $L^3 = Z(L)$  and  $L$  is of type B. This algebra can be given by the presentation*

$$\mathcal{P}_{10}^{(4,3)} : (x_1y_2, y_3) = 1, (y_1y_2, y_4) = 1, (y_1y_3, y_5) = 1.$$

**Proof** We have already shown that this algebra is the only candidate. Inspection shows that conversely this algebra has isotropic centre of dimension 4 and  $L^3 = Z(L)$ . It remains to see that the algebra is of type B. Thus let  $r \in \mathbb{F}$ . We want to show that  $\phi_{r\bar{x}_1 + \bar{y}_1}$  is non-degenerate. Let  $t = \alpha y_2 + \beta y_3 + \gamma y_4 + \delta y_5$  such that  $\phi_{r\bar{x}_1 + \bar{y}_1}(\bar{t}, \bar{u}) = 0$  for all  $\bar{u} \in L/L^2$  where  $\bar{t} = t + L^2$ . Then

$$\begin{aligned} 0 &= \phi_{r\bar{x}_1 + \bar{y}_1}(\bar{t}, \bar{y}_5) = \beta \\ 0 &= \phi_{r\bar{x}_1 + \bar{y}_1}(\bar{t}, \bar{y}_4) = \alpha \\ 0 &= \phi_{r\bar{x}_1 + \bar{y}_1}(\bar{t}, \bar{y}_3) = r\alpha - \delta = -\delta \\ 0 &= \phi_{r\bar{x}_1 + \bar{y}_1}(\bar{t}, \bar{y}_2) = -r\beta - \gamma = -\gamma. \end{aligned}$$

Thus  $\bar{t} = 0$ .  $\square$

### 2.2.3 Algebras of type C

Here we are assuming that  $\phi_z$  is non-degenerate for all  $0 \neq z \in L^2/Z(L)$ . Let  $L$  be any nilpotent SAA of type C. Notice that  $L^2/Z(L) = L^2/(L^2)^\perp$  naturally becomes a 2-dimensional symplectic vector space with inherited alternating form from  $L$ . Thus  $(u + Z(L), v + Z(L)) = (u, v)$  for  $u, v \in L^2$ . We pick a basis  $x, y$  for  $L^2/Z(L)$  such that  $(x, y) = 1$  and then choose some fixed elements  $x_1, y_1 \in L^2$  such  $x = \bar{x}_1 = x_1 + Z(L)$  and  $y = \bar{y}_1 = y_1 + Z(L)$ . For any vector  $u \in L/L^2$  we will denote by  $\langle u \rangle_1^\perp$  the subspace of  $L/L^2$  consisting of all the vectors that are isotropic to  $u$  with respect to  $\phi_{\bar{x}_1}$ . Likewise we will denote by  $\langle u \rangle_2^\perp$  the subspace of  $L/L^2$  consisting of all the vectors that are isotropic to  $u$  with respect to  $\phi_{\bar{y}_1}$ .

**Definition.** We say that a subspace of  $L/L^2$  is *totally isotropic* if it is isotropic with respect to  $\phi_z$  for all  $z \in L^2/Z(L)$ .

**Lemma 2.4** *For each  $0 \neq u \in L/L^2$  there exists a unique totally isotropic plane through  $u$ .*

**Proof** We know that  $\langle u \rangle_1^\perp$  and  $\langle u \rangle_2^\perp$  are 3-dimensional. Thus if they are not equal then

$$4 = \dim(\langle u \rangle_1^\perp + \langle u \rangle_2^\perp) = \dim \langle u \rangle_1^\perp + \dim \langle u \rangle_2^\perp - \dim(\langle u \rangle_1^\perp \cap \langle u \rangle_2^\perp).$$

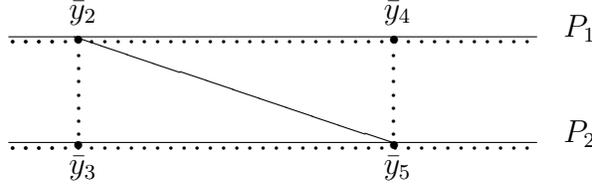
Therefore  $\dim(\langle u \rangle_1^\perp \cap \langle u \rangle_2^\perp) = 6 - 4 = 2$ . Thus the collection of all the elements in  $L/L^2$  that are isotropic to  $u$  with respect to  $\phi_z$  for all  $z \in L^2/Z(L)$ , namely  $\langle u \rangle_1^\perp \cap \langle u \rangle_2^\perp$ , is a plane.

It remains to see that  $\langle u \rangle_1^\perp \neq \langle u \rangle_2^\perp$ . We argue by contradiction and pick a basis  $u, v, w$  for this common subspace and add a vector  $t$  to get a basis for  $L/L^2$ . By replacing  $x_1$  by some suitable  $x_1 + \alpha y_1$ , we can assume that  $\phi_{\bar{x}_1}(u, t) = 0$ . But then  $u$  is isotropic to all elements of  $L/L^2$  with respect to  $\phi_{\bar{x}_1}$  that contradicts the assumption that  $\phi_{\bar{x}_1}$  is non-degenerate.  $\square$

The alternating forms  $\phi_z$  with  $z \in L^2/Z(L)$  will help us understanding the structure of algebras of type C. We will next come up with a special type of presentations for algebras of type C based on the geometry arising from the family of the auxiliary alternating forms.

Let  $L$  be an algebra of type C. As a starting point we pick two distinct totally

isotropic planes  $P_1, P_2 \leq L/L^2$  and we pick some non-zero vector  $\bar{y}_2$  on  $P_1$ . By Lemma 2.4, we have that  $P_1 \cap P_2 = \{0\}$  and thus  $L/L^2 = P_1 \oplus P_2$ . Now  $\langle \bar{y}_2 \rangle_1^\perp$  is 3-dimensional and contains  $P_1$ . Thus  $\langle \bar{y}_2 \rangle_1^\perp \cap P_2$  is 1-dimensional and not contained in  $\langle \bar{y}_2 \rangle_2^\perp$  by Lemma 2.4. Thus there is unique vector  $\bar{y}_5 \in P_2$  where  $\phi_{\bar{x}_1}(\bar{y}_2, \bar{y}_5) = 0$  and  $\phi_{\bar{y}_1}(\bar{y}_2, \bar{y}_5) = 1$ .



Similarly we have a unique element  $\bar{y}_3 \in P_2$  such that  $\phi_{\bar{y}_1}(\bar{y}_2, \bar{y}_3) = 0$  and  $\phi_{\bar{x}_1}(\bar{y}_2, \bar{y}_3) = 1$ . By Lemma 2.4 we have  $\langle \bar{y}_5 \rangle_2^\perp \neq \langle \bar{y}_3 \rangle_2^\perp$ . Thus there exists a unique  $\bar{y}_4 \in P_1$  such that  $\phi_{\bar{y}_1}(\bar{y}_4, \bar{y}_5) = 0$  and  $\phi_{\bar{y}_1}(\bar{y}_3, \bar{y}_4) = 1$ . Notice also that  $\phi_{\bar{x}_1}(\bar{y}_4, \bar{y}_5) \neq 0$  and that  $\bar{y}_2, \bar{y}_3, \bar{y}_4, \bar{y}_5$  form a basis for  $L/L^2$ . It follows from the discussion that, for some  $\alpha, \beta \in \mathbb{F}$  with  $\beta \neq 0$ , we have

$$\begin{aligned} \phi_{\bar{x}_1}(\bar{y}_2, \bar{y}_3) &= 1, & \phi_{\bar{y}_1}(\bar{y}_2, \bar{y}_3) &= 0, & \phi_{\bar{x}_1}(\bar{y}_2, \bar{y}_4) &= 0, & \phi_{\bar{y}_1}(\bar{y}_2, \bar{y}_4) &= 0, \\ \phi_{\bar{x}_1}(\bar{y}_2, \bar{y}_5) &= 0, & \phi_{\bar{y}_1}(\bar{y}_2, \bar{y}_5) &= 1, & \phi_{\bar{x}_1}(\bar{y}_3, \bar{y}_4) &= \alpha, & \phi_{\bar{y}_1}(\bar{y}_3, \bar{y}_4) &= 1, \\ \phi_{\bar{x}_1}(\bar{y}_3, \bar{y}_5) &= 0, & \phi_{\bar{y}_1}(\bar{y}_3, \bar{y}_5) &= 0, & \phi_{\bar{x}_1}(\bar{y}_4, \bar{y}_5) &= \beta, & \phi_{\bar{y}_1}(\bar{y}_4, \bar{y}_5) &= 0. \end{aligned} \quad (8)$$

The matrix for the alternating form  $\phi_{r\bar{x}_1+s\bar{y}_1}$  with respect to the ordered basis  $(\bar{y}_2, \bar{y}_4, \bar{y}_3, \bar{y}_5)$  is then

$$r \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -\alpha & \beta \\ -1 & \alpha & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{bmatrix} + s \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

with determinant  $(\beta r^2 + \alpha r s + s^2)^2$ . As we are dealing here with algebras of type C this determinant must be non-zero for all  $(r, s) \neq (0, 0)$ . Equivalently we must have that the polynomial

$$t^2 + \alpha t + \beta$$

is irreducible in  $\mathbb{F}[t]$ . Using this and (8) we will now obtain a full presentation for our algebra. As before we are only left with the triples  $(y_i y_j, y_k) = r_{ijk}$  for  $2 \leq i < j < k \leq 5$ . We will see that we can choose a standard basis such that  $x_1 + Z(L) = x$ ,  $y_1 + Z(L) = y$  and  $y_i + L^2 = \bar{y}_i$  for  $2 \leq i \leq 5$ . It turns

out that we do not have to alter our basis elements  $x_5, \dots, x_2$  of the centre. We do this with a change of basis of the form

$$\begin{aligned}\tilde{x}_1 &= x_1 - (\alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5) \\ \tilde{y}_1 &= y_1 + (\alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5) \\ \tilde{y}_i &= y_i + \alpha_i x_1 + \alpha_i y_1.\end{aligned}$$

Inspection shows that the equations  $(\tilde{y}_i \tilde{y}_j, \tilde{y}_k) = 0$ ,  $2 \leq i < j < k \leq 5$  are equivalent to

$$\begin{aligned}\begin{bmatrix} -1 & 1 \\ \beta & \alpha + 1 \end{bmatrix} \begin{bmatrix} \alpha_3 \\ \alpha_5 \end{bmatrix} &= \begin{bmatrix} -r_{235} \\ -r_{345} \end{bmatrix}, \\ \begin{bmatrix} \alpha + 1 & 1 \\ \beta & -1 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \alpha_4 \end{bmatrix} &= \begin{bmatrix} -r_{234} \\ -r_{245} \end{bmatrix}.\end{aligned}$$

Notice that we cannot have that  $\alpha + \beta + 1 = 0$  since otherwise 1 is a root of  $t^2 + \alpha t + \beta$  that is absurd as the polynomial is irreducible. We thus have solution  $(\alpha_2, \alpha_3, \alpha_4, \alpha_5)$  to the equation system and we arrive at a standard basis that gives us the following presentation

$$\mathcal{P}_{10}^{(4,4)}(\alpha, \beta) : (x_1 y_2, y_3) = 1, (x_1 y_3, y_4) = \alpha, (x_1 y_4, y_5) = \beta, (y_1 y_2, y_5) = 1, (y_1 y_3, y_4) = 1,$$

where the polynomial  $t^2 + \alpha t + \beta$  is irreducible. Conversely, inspection shows that any algebra with such presentation where  $t^2 + \alpha t + \beta$  is irreducible, gives us an algebra of type C.

We next turn to the isomorphism problem, that is we want to understand when two pairs  $(\alpha, \beta)$  and  $(\tilde{\alpha}, \tilde{\beta})$  describe the same algebra. As a first step we first prove the following lemma.

**Lemma 2.5** *Let  $x, y$  be elements in  $L^2/Z(L)$  such that  $(x, y) = 1$ . We have that the values of  $\alpha$  and  $\beta$  remain the same for all presentations of the form  $\mathcal{P}_{10}^{(4,4)}(\alpha, \beta)$  where, for the given standard basis,  $x_1 + Z(L) = x$  and  $y_1 + Z(L) = y$ .*

**Proof** Our method for producing  $\alpha$  and  $\beta$  was based on choosing some distinct totally isotropic planes  $P_1, P_2$  and some non-zero vector  $\bar{y}_2$  on  $P_1$ . From this we came up with a procedure that provided us with unique vectors  $\bar{y}_2, \bar{y}_4 \in P_1$  and  $\bar{y}_3, \bar{y}_5 \in P_2$  from which the values  $\alpha$  and  $\beta$  can be calculated as

$$\alpha = \phi_x(\bar{y}_3, \bar{y}_4), \quad \beta = \phi_x(\bar{y}_4, \bar{y}_5).$$

We want to show that if  $x_1 + Z(L), y_1 + Z(L)$  are kept fixed the procedure will always produce the same value for  $\alpha$  and  $\beta$ . As a starting point we will see that the values do not depend on which vector  $\bar{y}_2$  from  $P_1$  we choose. We do this in two steps. First notice that if we choose  $\tilde{y}_2 = a\bar{y}_2$  for some  $0 \neq a \in \mathbb{F}$ , then the procedure gives us the new vectors  $\tilde{y}_3 = \frac{1}{a}\bar{y}_3$ ,  $\tilde{y}_5 = \frac{1}{a}\bar{y}_5$  and  $\tilde{y}_4 = a\bar{y}_4$  and this gives us the values

$$\begin{aligned}\tilde{\alpha} &= \phi_{\bar{x}_1}(\tilde{y}_3, \tilde{y}_4) = \phi_{\bar{x}_1}\left(\frac{1}{a}\bar{y}_3, a\bar{y}_4\right) = \alpha, \\ \tilde{\beta} &= \phi_{\bar{x}_1}(\tilde{y}_4, \tilde{y}_5) = \phi_{\bar{x}_1}(a\bar{y}_4, \frac{1}{a}\bar{y}_5) = \beta.\end{aligned}$$

It remains to consider the change  $\tilde{y}_2 = \bar{y}_4 + a\bar{y}_2$ . Following the mechanical procedure above produces the elements

$$\begin{aligned}\tilde{y}_5 &= \frac{-\beta}{a^2 - a\alpha + \beta} \bar{y}_3 + \frac{a - \alpha}{a^2 - a\alpha + \beta} \bar{y}_5, \\ \tilde{y}_3 &= \frac{a}{a^2 - a\alpha + \beta} \bar{y}_3 + \frac{1}{a^2 - a\alpha + \beta} \bar{y}_5, \\ \tilde{y}_4 &= -\beta \bar{y}_2 + (a - \alpha) \bar{y}_4.\end{aligned}$$

Inspection shows that again we have  $\phi_{\bar{x}_1}(\tilde{y}_3, \tilde{y}_4) = \alpha$  and  $\phi_{\bar{x}_1}(\tilde{y}_4, \tilde{y}_5) = \beta$ .

We have thus seen that for a given pair  $P_1, P_2$  of distinct totally isotropic planes we get unique values  $\alpha(P_1, P_2)$  and  $\beta(P_1, P_2)$  not depending on which vector  $\bar{y}_2$  from  $P_1$  we choose for the procedure above. The next step is to see that  $\alpha(P_2, P_1) = \alpha(P_1, P_2)$  and  $\beta(P_2, P_1) = \beta(P_1, P_2)$ . So suppose we have some standard basis with respect to the pair  $P_1, P_2$  that gives us the presentation  $\mathcal{P}_{10}^{(4,4)}(\alpha, \beta)$ . Now consider  $\tilde{y}_2 = \bar{y}_5, \tilde{y}_4 = \beta\bar{y}_3 \in P_2$  and  $\tilde{y}_3 = \frac{-1}{\beta}\bar{y}_4, \tilde{y}_5 = -\bar{y}_2 \in P_1$ . Inspection shows that this is a standard basis for the pair  $P_2, P_1$  and

$$\begin{aligned}\alpha(P_2, P_1) &= \phi_{\bar{x}_1}(\tilde{y}_3, \tilde{y}_4) = \phi_{\bar{x}_1}\left(\frac{-1}{\beta}\bar{y}_4, \beta\bar{y}_3\right) = \alpha(P_1, P_2) \\ \beta(P_2, P_1) &= \phi_{\bar{x}_1}(\tilde{y}_4, \tilde{y}_5) = \phi_{\bar{x}_1}(\beta\bar{y}_3, -\bar{y}_2) = \beta(P_1, P_2).\end{aligned}$$

Now pick any totally isotropic plane  $P_3$  that is distinct from  $P_1, P_2$ . The aim is to show that  $\alpha(P_3, P_2) = \alpha(P_1, P_2)$  and  $\beta(P_3, P_2) = \beta(P_1, P_2)$ . Take any basis for  $P_3$ . This must be of the form  $u_1 + v_1, u_2 + v_2$  with  $u_1, u_2 \in P_1$  and  $v_1, v_2 \in P_2$ . Notice first that  $u_1 + P_2, u_2 + P_2$  are linearly independent vectors in  $P_1 + P_2/P_2$ . To see this, take  $a, b \in \mathbb{F}$  such that

$$P_2 = au_1 + bu_2 + P_2 = a(u_1 + v_1) + b(u_2 + v_2) + P_2.$$

Then  $a(u_1+v_1)+b(u_2+v_2) \in P_2 \cap P_3 = \{0\}$ . As the vectors  $u_1+v_1, u_2+v_2$  are linearly independent it follows that  $a = b = 0$ . In particular we can choose our basis for  $P_3$  to be of the form  $\bar{y}_2 + u, \bar{y}_4 + v$  with  $u, v \in P_2$ . Inspection shows that for  $\tilde{y}_2 = \bar{y}_2 + u, \tilde{y}_4 = \bar{y}_4 + v \in P_3$  and  $\bar{y}_3, \bar{y}_5 \in P_2$  we have a standard basis with respect to the pair  $P_3, P_2$ . Furthermore the corresponding parameters are  $\phi_{\bar{x}_1}(\bar{y}_3, \tilde{y}_4) = \alpha$  and  $\phi_{\bar{x}_1}(\tilde{y}_4, \bar{y}_5) = \beta$ .

We have now all the input we need to finish the proof of the lemma. Take any four totally isotropic planes  $P_1, P_2, P_3, P_4$  in  $L/L^2$  such that  $P_1 \neq P_2$  and  $P_3 \neq P_4$ . If these planes are not all distinct then we get directly from the analysis above that  $\alpha(P_3, P_4) = \alpha(P_1, P_2)$  and  $\beta(P_3, P_4) = \beta(P_1, P_2)$ . Now suppose the planes are distinct. Then  $\alpha(P_3, P_4) = \alpha(P_1, P_4) = \alpha(P_1, P_2)$  and  $\beta(P_3, P_4) = \beta(P_1, P_4) = \beta(P_1, P_2)$ . This finishes the proof of the lemma.  $\square$

It follows from the lemma that if we want to obtain a new presentation for some given algebra  $L$ , then we must choose different vectors  $x, y$  for  $L^2/Z(L)$ . We thus only need to consider a change of standard basis for  $L$  of the form  $\tilde{x}_1 = ax_1 + by_1, \tilde{y}_1 = cx_1 + dy_1$  where  $1 = (\tilde{x}_1, \tilde{y}_1) = ad - bc$ . Suppose that we have a presentation  $\mathcal{P}_{10}^{(4,4)}(\alpha, \beta)$  with respect to some standard basis  $x_1, y_1, \dots, x_5, y_5$  and let  $\tilde{x}_1, \tilde{y}_1$  be given as above. Going again through the standard procedure with respect to  $P_1 = \mathbb{F}\bar{y}_2 + \mathbb{F}\bar{y}_4, P_2 = \mathbb{F}\bar{y}_3 + \mathbb{F}\bar{y}_5$  and  $\bar{y}_2 \in P_1$  gives us the new basis  $\bar{y}_2, \tilde{y}_3, \tilde{y}_5, \tilde{y}_4$  where

$$\begin{aligned} \tilde{y}_5 &= -b\bar{y}_3 + a\bar{y}_5 \\ \tilde{y}_3 &= d\bar{y}_3 - c\bar{y}_5 \\ \tilde{y}_4 &= \frac{-\alpha bc - \beta ac - bd}{d^2 + cd\alpha + \beta c^2} \bar{y}_2 + \frac{1}{d^2 + \alpha cd + \beta c^2} \bar{y}_4. \end{aligned}$$

From this we can calculate the new parameters  $\tilde{\alpha}$  and  $\tilde{\beta}$  and we obtain the following proposition.

**Proposition 2.6** *Let  $L$  be a nilpotent SAA of dimension 10 with an isotropic centre of dimension 4 where  $L^3 = Z(L)$  and  $L$  is of type C. Then  $L$  has a presentation of the form*

$$\mathcal{P}_{10}^{(4,4)}(\alpha, \beta) : (x_1 y_2, y_3) = 1, (x_1 y_3, y_4) = \alpha, (x_1 y_4, y_5) = \beta, (y_1 y_2, y_5) = 1, (y_1 y_3, y_4) = 1,$$

where the polynomial  $t^2 + \alpha t + \beta$  is irreducible in  $\mathbb{F}[t]$ . Conversely any such presentation gives us an algebra of type C. Furthermore two presentations

$\mathcal{P}_{10}^{(4,4)}(\alpha, \beta)$  and  $\mathcal{P}_{10}^{(4,4)}(\tilde{\alpha}, \tilde{\beta})$  describe the same algebra if and only if

$$\begin{aligned}\tilde{\alpha} &= \frac{(ad + bc)\alpha + 2ac\beta + 2bd}{d^2 + \alpha cd + c^2\beta} \\ \tilde{\beta} &= \frac{b^2 + ab\alpha + a^2\beta}{d^2 + cd\alpha + c^2\beta}\end{aligned}$$

for some  $a, b, c, d \in \mathbb{F}$  where  $ad - bc = 1$ .

### 2.3 Further analysis of algebras of type C and some examples

In order to get a more transparent picture of the algebras of type C, it turns out to be useful to consider the case when the characteristic is 2 and the case when the characteristic is not 2 separately.

**Lemma 2.7** *Let  $L$  be a nilpotent SAA of dimension 10 with an isotropic centre of dimension 4 where  $L^3 = Z(L)$  and  $L$  is of type C over a field  $\mathbb{F}$  of characteristic that is not 2. Then  $L$  has a presentation of the form  $\mathcal{P}_{10}^{(4,4)}(0, \beta)$  with respect to some standard basis, where the polynomial  $t^2 + \beta$  is irreducible in  $\mathbb{F}[t]$ .*

**Proof** By Proposition 2.6 we know that we can choose a standard basis for  $L$  so that  $L$  has a presentation of the form  $\mathcal{P}_{10}^{(4,4)}(\alpha, \beta)$  with respect to this basis. Now let  $a = 0$ ,  $b = 1$ ,  $c = -1$  and  $d = \alpha/2$ . Then  $ad - bc = 1$  and by Proposition 2.6 again we know that there is a presentation for  $L$  of the form  $\mathcal{P}_{10}^{(4,4)}(\tilde{\alpha}, \tilde{\beta})$  where  $\tilde{\alpha} = 0$ .  $\square$

**Proposition 2.8** *Let  $L$  be a nilpotent SAA of dimension 10 with an isotropic centre of dimension 4 where  $L^3 = Z(L)$  and  $L$  is of type C over a field  $\mathbb{F}$  of characteristic that is not 2. Then  $L$  has a presentation of the form*

$$\mathcal{P}(\beta) : (x_1y_2, y_3) = 1, (x_1y_4, y_5) = \beta, (y_1y_2, y_5) = 1, (y_1y_3, y_4) = 1,$$

where  $\beta \notin -\mathbb{F}^2$ . Conversely any such presentation gives us an algebra of type C. Furthermore two such presentations  $\mathcal{P}(\beta)$  and  $\mathcal{P}(\tilde{\beta})$  describe the same algebra if and only if

$$\tilde{\beta}/\beta = (a^2 + b^2\beta)^2$$

for some  $(a, b) \in \mathbb{F} \times \mathbb{F} \setminus \{(0, 0)\}$ .

**Proof** From Lemma 2.7 we know that such a presentation exists and the polynomial  $t^2 + \beta$  is irreducible if and only if  $\beta \notin -\mathbb{F}^2$ . By Proposition 2.6 we then know that any such presentation gives us an algebra of type C and that  $\mathcal{P}(0, \beta)$  and  $\mathcal{P}(0, \tilde{\beta})$  describe the same algebra if and only if there are  $a, b, c, d \in \mathbb{F}$  such that

$$0 = ac\beta + bd, \quad ad - bc = 1$$

and

$$\tilde{\beta} = \frac{b^2 + a^2\beta}{d^2 + c^2\beta}.$$

Solving these together shows that these conditions are equivalent to saying that

$$\tilde{\beta} = \left(\left(\frac{b}{\beta}\right)^2\beta + a^2\right)^2\beta$$

for some  $(a, b) \in \mathbb{F} \times \mathbb{F} \setminus \{(0, 0)\}$ . As  $\frac{b}{\beta}$  is arbitrary, the second part of the proposition follows.  $\square$

**Examples.** (1) If  $\mathbb{F} = \mathbb{C}$  then there are no algebras of type C. This holds more generally for any field  $\mathbb{F}$  whose characteristic is not 2 and where all elements in  $\mathbb{F}$  have a square root in  $\mathbb{F}$ .

(2) Suppose  $\mathbb{F} = \mathbb{R}$ . Here  $\beta \notin -\mathbb{R}^2$  if and only if  $\beta > 0$  in which case there exists  $a \in \mathbb{R} \setminus \{0\}$  such that  $\beta = 1/a^4$ . Hence, by Proposition 2.8,  $\mathcal{P}(\beta)$  describes the same algebra as  $\mathcal{P}(1)$ . This shows that there is only one algebra of type C over  $\mathbb{R}$  that can be given by the presentation

$$\mathcal{P}(1) : (x_1y_2, y_3) = 1, (x_1y_4, y_5) = 1, (y_1y_2, y_5) = 1, (y_1y_3, y_4) = 1.$$

(3) Let  $\mathbb{F}$  be a finite field of some odd characteristic  $p$ . Suppose that  $|\mathbb{F}| = p^n$ . The nonzero elements form a cyclic group  $\mathbb{F}^*$  of order  $p^n - 1$  that is divisible by 2. Thus there are two cosets of  $(\mathbb{F}^*)^2$  in  $\mathbb{F}^*$  and

$$\mathbb{F}^* = -(\mathbb{F}^*)^2 \cup \beta(\mathbb{F}^*)^2$$

for some  $\beta \in \mathbb{F}^*$ . Suppose  $\tilde{\beta} = \beta c^2$  is an arbitrary field element that is not in  $-\mathbb{F}^2$ . As there are  $(|\mathbb{F}| + 1)/2$  squares in  $\mathbb{F}$  we have that the set  $c - \mathbb{F}^2$  and  $\beta\mathbb{F}^2$  intersect. Hence there exist  $a, b \in \mathbb{F}$  such that  $c = a^2 + b^2\beta$  and

thus  $\tilde{\beta} = (a^2 + b^2\beta)^2\beta$ . Hence the situation is like in (2) and we get only one algebra with presentation

$$\mathcal{P}(\beta) : (x_1y_2, y_3) = 1, (x_1y_4, y_5) = \beta, (y_1y_2, y_5) = 1, (y_1y_3, y_4) = 1,$$

where  $\beta$  is any element not in  $-\mathbb{F}^2$ .

**Remark.** Let  $\beta \in \mathbb{F}$  that is not in  $-\mathbb{F}^2$  and consider a splitting field  $\mathbb{F}[\gamma]$  of the polynomial  $t^2 + \beta$  in  $\mathbb{F}[t]$  where  $\gamma^2 = -\beta$ . So  $a^2 + b^2\beta$  is the norm  $N(a + \gamma b)$  of  $a + \gamma b$  that is a multiplicative function and thus

$$G(\beta) = \{(a^2 + b^2\beta)^2 : (a, b) \in \mathbb{F} \times \mathbb{F} \setminus \{(0, 0)\}\}$$

is a subgroup of  $(\mathbb{F}^*)^2$ . Let  $S = \{\beta \in \mathbb{F} : \beta^2 \notin -\mathbb{F}^2\}$ , we now have a relation on  $S$  given by

$$\tilde{\beta} \sim \beta \text{ if and only if } \tilde{\beta}/\beta \in G(\beta).$$

From Proposition 2.8, we know that this is an equivalence relation. We can also see this directly. First notice that

$$(a^2 + b^2\beta)^2 = (a^2 + (b/c)^2\beta c^2)^2$$

for all  $c \in \mathbb{F}^*$ . Hence  $G(\beta) = G(\beta c^2)$  for all  $c \in \mathbb{F}^*$ . In particular we have that  $G(\tilde{\beta}) = G(\beta)$  if  $\tilde{\beta} \sim \beta$ . Let us now see that  $\sim$  is an equivalence relation. Firstly it is reflexive as  $\beta/\beta = 1 \in G(\beta)$ , the latter being a group. To see that  $\sim$  is symmetric, notice that  $G(\tilde{\beta}) = G(\beta)$  is a group and thus  $\tilde{\beta}/\beta \in G(\beta)$  if and only if  $\beta/\tilde{\beta} \in G(\tilde{\beta})$ . Finally to see that  $\sim$  is transitive, let  $\alpha, \beta, \delta \in S$  such that  $\alpha \sim \beta$  and  $\beta \sim \delta$ . Then  $G(\alpha) = G(\beta) = G(\delta)$  and  $\beta/\alpha, \delta/\beta \in G(\alpha)$  implies that their product  $\delta/\alpha \in G(\alpha)$ .

Let us now move to the case when the characteristic of  $\mathbb{F}$  is 2. We first see that the algebras here split naturally into two classes.

**Lemma 2.9** *Let  $L$  be a nilpotent SAA of dimension 10 with an isotropic centre of dimension 4 where  $L^3 = Z(L)$  and  $L$  is of type C over a field  $\mathbb{F}$  of characteristic 2. Then  $L$  cannot have both a presentation of the form  $\mathcal{P}_{10}^{(4,4)}(0, \beta)$  and  $\mathcal{P}_{10}^{(4,4)}(\alpha, \gamma)$  where  $\alpha \neq 0$  and  $t^2 + \beta, t^2 + \alpha t + \gamma$  are irreducible polynomials in  $\mathbb{F}[t]$ .*

**Proof** We argue by contradiction and suppose we have an algebra satisfying both types of presentations. By Proposition 2.6 we then have  $0 =$

$(ad + bc)\alpha = (ad - bc)\alpha = \alpha$  that contradicts our assumption that  $\alpha \neq 0$ .  $\square$

For the algebras where  $\alpha = 0$ , the same analysis works as for algebras with  $\text{char } \mathbb{F} \neq 2$  and we get the same result as in Proposition 2.8. This leaves us with algebras where  $\text{char } \mathbb{F} = 2$  and where  $\alpha \neq 0$ . Notice that Proposition 2.6 tells us here that  $\mathcal{P}_{10}^{(4,4)}(\alpha, \beta)$  and  $\mathcal{P}_{10}^{(4,4)}(\tilde{\alpha}, \tilde{\beta})$  describe the same algebra if and only if

$$\begin{aligned}\tilde{\alpha} &= \frac{\alpha}{d^2 + cd\alpha + c^2\beta} \\ \tilde{\beta} &= \frac{b^2 + ab\alpha + a^2\beta}{d^2 + cd\alpha + c^2\beta}\end{aligned}$$

for some  $a, b, c, d \in \mathbb{F}$  where  $ad + bc = 1$ .

We don't take the analysis further but end by considering an example, the finite fields of characteristic 2. Let  $\mathbb{F}$  be the finite field of order  $2^n$ . As a first step we show that we can always in that case, choose our presentation such that  $\beta = 1$ . To see this take first some arbitrary  $\alpha$  and  $\beta$  such that  $L$  satisfies the presentation  $\mathcal{P}_{10}^{(4,4)}(\alpha, \beta)$ . The groups of units  $\mathbb{F}^*$  is here a cyclic group of odd order  $2^n - 1$  and thus  $(\mathbb{F}^*)^2 = \mathbb{F}^*$ . Now pick  $b \in \mathbb{F}^*$  such that  $b^2 = \beta/\alpha$  and let  $a = 0$ ,  $c = 1/b$  and  $d = b$ . Then  $ad + bc = 1$  and

$$\tilde{\beta} = \frac{b^2}{b^2 + \alpha + \beta/b^2} = 1.$$

Thus we can assume from now on that  $\beta = 1$ . Now let  $b \in \mathbb{F}$  be arbitrary and let  $a = b + 1$ ,  $c = b$  and  $d = b + 1$ . Then  $ad + bc = 1$ ,  $\tilde{\beta} = 1$  and  $\tilde{\alpha} = \frac{\alpha}{b(b+1)\alpha+1}$ . The number of such values for  $\tilde{\alpha}$  is  $2^{n-1}$  that gives us all possible values such that  $t^2 + \tilde{\alpha}t + 1$  is irreducible (easy to count that the number of reducible polynomials of the form  $t^2 + ut + 1$  is  $2^{n-1}$ ). We thus conclude that there is only one algebra of type C in this case.

### 3 Algebras with an isotropic centre of dimension 2

In this section we will be assuming that  $Z(L)$  is isotropic of dimension 2. Notice that if  $L = \mathbb{F}u + \mathbb{F}v + L^2$ , then  $L^2 = \mathbb{F}uv + L^3$ . It follows that

$L^2 = Z(L)^\perp$  is of dimension 8 and that  $L^3$  is of dimension 7. We can then pick our standard basis such that

$$\begin{aligned} Z(L) &= \mathbb{F}x_5 + \mathbb{F}x_4, \\ L^2 &= \mathbb{F}x_5 + \cdots + \mathbb{F}x_1 + \mathbb{F}y_1 + \mathbb{F}y_2 + \mathbb{F}y_3, \\ L^3 &= \mathbb{F}x_5 + \cdots + \mathbb{F}x_1 + \mathbb{F}y_1 + \mathbb{F}y_2. \end{aligned}$$

Furthermore  $L^3 = \mathbb{F}uvu + \mathbb{F}uvv + L^4$  and thus  $\dim L^4 \in \{5, 6\}$ . Let  $k$  be the nilpotence class of  $L$ . We know that the maximal class is 7 and as  $\dim L^k \neq 1$  and  $L^k \leq Z(L)$ , we must have that  $L^k = Z(L)$ . Moreover, we know that  $\dim L^s \neq 2$  for  $1 \leq s \leq 4$  and thus  $5 \leq k \leq 7$ . If  $L^5 = Z(L)$  then  $\dim Z_2(L) - \dim Z(L) = \dim L^2 - \dim L^3 = 1$  and we get the contradiction that  $L^4 = Z_2(L)$  is of dimension 3. Thus  $6 \leq k \leq 7$ . We will deal with the two cases separately.

### 3.1 The algebras of class 6

As the class is 6, it follows that  $(L^4, L^4) = (L^7, L) = 0$  and thus  $L^4$  is isotropic. We have seen that the dimension of  $L^4$  is at least 5 and thus  $\dim L^4 = 5$ . We can thus now furthermore choose our standard basis such that

$$\begin{array}{l} L^6 = Z(L) \\ L^5 = Z_2(L) \\ L^4 = Z_3(L) \end{array} \begin{array}{|c|c|} \hline x_5 & y_5 \\ \hline x_4 & y_4 \\ \hline x_3 & y_3 \\ \hline x_2 & y_2 \\ \hline x_1 & y_1 \\ \hline \end{array} \begin{array}{l} L^2 = Z_5(L) \\ L^3 = Z_4(L) \end{array} \begin{array}{l} Z(L) = L^6 = \mathbb{F}x_5 + \mathbb{F}x_4 \\ Z_2(L) = L^5 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 \\ Z_3(L) = L^4 = \mathbb{F}x_5 + \cdots + \mathbb{F}x_1 \\ Z_4(L) = L^3 = L^4 + \mathbb{F}y_1 + \mathbb{F}y_2 \\ Z_5(L) = L^2 = L^4 + \mathbb{F}y_1 + \mathbb{F}y_2 + \mathbb{F}y_3. \end{array}$$

As  $L^4 Z_4(L) = 0$  we must have

$$x_1 y_2 = 0.$$

It then follows that  $L^4 L^3 = 0$  and then

$$L^3 L^3 = \mathbb{F}y_1 y_2.$$

Notice that  $L^3 L^3 \neq 0$  since this would imply that  $(L^6, L) = (L^3, L^4) = (L^3 L^3, L) = 0$  and we would get the contradiction that the class of  $L$  is at most 5. Next let us see that  $x_1 y_3$  and  $x_2 y_3$  are linearly independent. To see this we argue by contradiction and suppose that  $(ax_1 + bx_2)y_3 = 0$  for some  $a, b \in \mathbb{F}$  where not both  $a, b$  are zero. But this would imply that  $(ax_1 + bx_2)L \in$

$Z(L)$  and we would thus get the contradiction that  $ax_1 + bx_2 \in Z_2(L) = L^5$ . Thus

$$L^4L^2 = \mathbb{F}x_1y_3 + \mathbb{F}x_2y_3 = Z(L).$$

Notice that  $L^3L^3 = \mathbb{F}y_1y_2$  is a one-dimensional characteristic subspace of  $Z_2(L)$ . We consider two possibilities:  $L^3L^3 \leq Z(L)$  and  $L^3L^3 \not\leq Z(L)$ .

### 3.1.1 Algebras where $L^3L^3 \leq Z(L)$

We pick our standard basis such that

$$L^3L^3 = \mathbb{F}y_1y_2 = \mathbb{F}x_5. \quad (9)$$

We have seen above that  $Z(L) = \mathbb{F}x_2y_3 + \mathbb{F}x_1y_3 = L^4L^2$ . In order to clarify the structure of  $L$  we introduce the characteristic subspace

$$W = \{x \in L^4 : xL^2 \leq L^3L^3\}.$$

Notice that  $W$  is the kernel of the surjective linear map  $L^4 \rightarrow Z(L)/L^3L^3$ ,  $x \mapsto xy_3 + L^3L^3$  and thus  $W$  is of codimension 1 in  $L^4$ . Also  $L^5 < W$ . We can thus pick our standard basis such that

$$W = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2.$$

From this one sees that we have a chain of characteristic ideals of  $L$

$L^3L^3$	$x_5$	$y_5$	
$L^6 = Z(L)$	$x_4$	$y_4$	$(L^3L^3)^\perp$
$L^5 = Z_2(L)$	$x_3$	$y_3$	$L^2 = Z_5(L)$
$W$	$x_2$	$y_2$	$L^3 = Z_4(L)$
$L^4 = Z_3(L)$	$x_1$	$y_1$	$W^\perp$

Notice that  $\mathbb{F}x_2y_3 = \mathbb{F}x_5$ . We continue considering characteristic subspaces. Let

$$S = \{x \in L^3 : x \cdot L^2 \leq L^3L^3\}.$$

Notice that  $L^3L^2 = Z(L)$  and that  $S$  is the kernel of the surjective linear map  $L^3 \rightarrow Z(L)/L^3L^3$ ,  $x \mapsto x \cdot y_3 + L^3L^3$  and is thus of codimension 1 in  $L^3$ . Notice also that  $x_1 \notin S$  whereas  $W \leq S$ . It follows that we can pick our standard basis such that

$$S = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2 + \mathbb{F}y_1 + \mathbb{F}y_2.$$

In particular we have  $y_1y_3, y_2y_3 \in L^3L^3$ . Notice that

$$S^\perp = L^5 + \mathbb{F}y_1$$

and that  $L^2S^\perp = L^2y_1 = \mathbb{F}y_1y_2 + \mathbb{F}y_1y_3 = L^3L^3$ . Let

$$T = \{x \in L^2 : xS^\perp = 0\}.$$

Then  $T$  is the kernel of the surjective linear map  $L^2 \rightarrow L^3L^3$ ,  $x \mapsto y_1x$ . Notice that  $W^\perp \leq T$  but that  $y_2 \notin T$ . We can then pick our standard basis such that

$$T = W^\perp + \mathbb{F}y_3.$$

In particular  $y_1y_3 = 0$ . We now have a characteristic isotropic subspace  $T^\perp = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_2$  where  $T^\perp \cdot (L^3L^3)^\perp = x_2(L^3L^3)^\perp = L^3L^3$ . We now let

$$R = \{x \in (L^3L^3)^\perp : xT^\perp = 0\}.$$

This is the kernel of the surjective linear map  $(L^3L^3)^\perp \rightarrow L^3L^3$ ,  $x \mapsto x_2x$  that contains  $L^3$ . We now refine our standard basis such that

$$R = L^3 + \mathbb{F}y_4$$

and we have in particular  $x_2y_4 = 0$ . Let us summarize. For every standard basis that respects the list of characteristic subspaces above, we have

$$\begin{aligned} x_2y_4 &= 0 \\ \mathbb{F}x_2y_3 &= \mathbb{F}x_5 \\ x_1y_2 &= 0 \\ \mathbb{F}x_5 + \mathbb{F}x_1y_3 &= \mathbb{F}x_5 + \mathbb{F}x_4 \\ \mathbb{F}x_5 + \mathbb{F}x_1y_4 &= \mathbb{F}x_5 + \mathbb{F}x_3 \\ \mathbb{F}y_1y_2 &= \mathbb{F}x_5 \\ y_1y_3 &= 0 \\ y_2y_3 &\in \mathbb{F}x_5 \end{aligned}$$

It is not difficult to see that we can furthermore refine our basis such that

$$x_2y_3 = x_5, x_2y_4 = 0, x_1y_2 = 0, x_1y_3 = x_4, x_1y_4 = -x_3, y_1y_2 = x_5, y_1y_3 = 0. \quad (10)$$

This deals with all triple values apart from

$$\begin{aligned} (y_1y_4, y_5) &= b, & (y_2y_4, y_5) &= e, & (x_3y_4, y_5) &= r, \\ (y_3y_4, y_5) &= f, & (y_2y_3, y_5) &= c. \end{aligned}$$

Replacing  $x_1, y_1, x_2, y_2, x_4, y_4, x_5, y_5, y_3$  by  $r^2x_1, (1/r)^2y_1, (1/r)x_2+(b/r)x_4, ry_2-crx_2-ex_3-bcrx_4, r^2x_4, (1/r)^2y_4-(b/r^2)y_2, (1/r)x_5, ry_5, y_3-(e/r)x_2-(1/r)(f+bc)x_3-(b/r)ex_4$  and we can assume that  $b = f = e = c = 0$  and  $r = 1$ . We thus arrive at a unique presentation.

**Proposition 3.1** *There is a unique nilpotent SAA  $L$  of dimension 10 with an isotropic centre of dimension 2 that has the further properties that  $L$  is nilpotent of class 6 and  $L^3L^3 \leq Z(L)$ . This algebra can be given by the presentation*

$$\mathcal{P}_{10}^{(2,1)} : \quad (x_3y_4, y_5) = 1, \quad (x_2y_3, y_5) = 1, \quad (x_1y_3, y_4) = 1, \quad (y_1y_2, y_5) = 1.$$

One readily verifies that the algebra with the presentation above has the properties stated.

### 3.1.2 Algebras where $L^3L^3 \not\leq Z(L)$

Recall that  $x_1y_2 = 0$ . As we had observed before,  $\mathbb{F}y_1y_2 = L^3L^3 \leq Z_2(L) = Z(L) + \mathbb{F}x_3$ . We had also seen that  $\mathbb{F}x_1y_3 + \mathbb{F}x_2y_3 = Z(L)$ . We can now pick our standard basis such that

$$x_1y_2 = 0, \quad y_1y_2 = x_3, \quad x_1y_3 = x_4, \quad x_2y_3 = x_5. \quad (11)$$

This leaves us with the following list of triple values to determine.

$$\begin{aligned} (x_1y_4, y_5) &= a, & (y_2y_3, y_4) &= c, & (y_1y_3, y_4) &= f, & (y_1y_4, y_5) &= h, \\ (x_2y_4, y_5) &= b, & (y_2y_3, y_5) &= d, & (y_1y_3, y_5) &= g, & (y_3y_4, y_5) &= k, \\ (x_3y_4, y_5) &= r, & (y_2y_4, y_5) &= e, \end{aligned}$$

where  $r \neq 0$  as  $x_3 \notin Z(L)$ . We show that we can further refine the basis so that  $a = b = c = d = e = f = g = h = k = 0$ . Let  $\alpha = ac - e + bd$ ,  $\beta = c - g$  and replace  $x_1, y_1, x_2, y_2, y_3, y_4, y_5$  by  $x_1 - (a/r)x_3, y_1 - cx_2 - ((h - bc)/r)x_3, x_2 - (b/r)x_3 - fx_4 + \beta x_5, y_2 - cx_1 - dx_2 + (\alpha/r)x_3, y_3 - (h/r)x_1 - (e/r)x_2 - (k/r)x_3 + (a/r)y_1 + (b/r)y_2, y_4 - cfx_1 - dfx_2 + (\alpha f/r)x_3 + fy_2, y_5 + c\beta x_1 + d\beta x_2 - (\beta\alpha/r)x_3 - \beta y_2$ . We thus arrive at a family of algebras given by the presentation  $\mathcal{P}_{10}^{(2,2)}$  given in the next proposition.

**Proposition 3.2** *Let  $L$  be a nilpotent SAA of dimension 10 with an isotropic centre of dimension 2 with the further properties that  $L$  is of nilpotence class 6 and  $L^3L^3 \not\subseteq Z(L)$ . Then  $L$  has a presentation of the form*

$$\mathcal{P}_{10}^{(2,2)}(r) : (x_3y_4, y_5) = r, (x_2y_3, y_5) = 1, (x_1y_3, y_4) = 1, (y_1y_2, y_3) = 1,$$

where  $r \neq 0$ . Furthermore the presentations  $\mathcal{P}_{10}^{(2,2)}(r)$  and  $\mathcal{P}_{10}^{(2,2)}(s)$  describe the same algebra if and only if  $s/r \in (\mathbb{F}^*)^4$ . Conversely any algebra with such a presentation has the properties stated.

**Proof** We have already seen that all such algebras have a presentation of the form  $\mathcal{P}_{10}^{(2,2)}(r)$  for some  $0 \neq r \in \mathbb{F}$ . Straightforward calculations show that conversely any algebra with such a presentation has the properties stated in the proposition. It remains to prove the isomorphism property. To see that it is sufficient, suppose that we have an algebra  $L$  with presentation  $\mathcal{P}_{10}^{(2,2)}(r)$  with respect to some given standard basis. Let  $s$  be any element in  $\mathbb{F}^*$  such that  $s/r = b^4 \in (\mathbb{F}^*)^4$ . Replace the basis for  $L$  with a new standard basis  $\tilde{x}_1, \dots, \tilde{y}_5$  where  $\tilde{x}_1 = x_1$ ,  $\tilde{y}_1 = y_1$ ,  $\tilde{x}_2 = (1/b)x_2$ ,  $\tilde{y}_2 = by_2$ ,  $\tilde{x}_3 = bx_3$ ,  $\tilde{y}_3 = (1/b)y_3$ ,  $\tilde{x}_4 = (1/b)x_4$ ,  $\tilde{y}_4 = by_4$ ,  $\tilde{x}_5 = (1/b^2)x_5$  and  $\tilde{y}_5 = b^2y_5$ . Direct calculations show that  $L$  has the presentation  $\mathcal{P}_{10}^{(2,2)}(s)$  with respect to this new basis. It remains to see that the condition is necessary. Consider again an algebra  $L$  with presentation  $\mathcal{P}_{10}^{(2,2)}(r)$  and suppose that  $L$  has also a presentation  $\mathcal{P}_{10}^{(2,2)}(s)$  with respect to some other standard basis  $\tilde{x}_1, \dots, \tilde{y}_5$ . We want to show that  $s/r \in (\mathbb{F}^*)^4$ . We know that  $L = \mathbb{F}\tilde{y}_5 + \mathbb{F}\tilde{y}_4 + L^2 = \mathbb{F}y_5 + \mathbb{F}y_4 + L^2$ . Thus

$$\begin{aligned} \tilde{y}_4 &= ay_4 + by_5 + u_4 \\ \tilde{y}_5 &= cy_4 + dy_5 + u_5, \end{aligned}$$

for some  $u_4, u_5 \in L^2$  and  $a, b, c, d \in \mathbb{F}$  where  $ad - bc \neq 0$ . We know that  $L^2L^2 \subseteq L^4$  and thus

$$\begin{aligned} \tilde{y}_5\tilde{y}_4\tilde{y}_4 &= (cy_4 + dy_5)(ay_4 + by_5)(ay_4 + by_5) + w \\ \tilde{y}_4\tilde{y}_5\tilde{y}_5 &= (ay_4 + by_5)(cy_4 + dy_5)(cy_4 + dy_5) + z, \end{aligned}$$

where  $w, z \in L^4$ . We use the fact that  $L^7 = 0$  and  $L^3L^3 \subseteq L^5$  in the following calculation. We have

$$s^3 = (\tilde{y}_4\tilde{y}_5\tilde{y}_5(\tilde{y}_4\tilde{y}_5), \tilde{y}_5\tilde{y}_4\tilde{y}_4) = (ad - bc)^4 r^3.$$

Hence  $s/r \in (\mathbb{F}^*)^4$ .  $\square$

**Remarks.**(1) It thus depends on the field  $\mathbb{F}$ , how many algebras there are of this type. When  $(\mathbb{F}^*)^4 = \mathbb{F}^*$  there is just one algebra. This includes the case when  $\mathbb{F}$  is algebraically closed or a finite field of characteristic 2.

(2) Let  $\mathbb{F}$  be a finite field of order  $p^n$  where  $p$  is an odd prime. If  $p \equiv 1 \pmod{4}$  then there are 4 algebras and if  $p \equiv -1 \pmod{4}$  then there are 4 algebras when  $n$  is even and 2 algebras when  $n$  is odd.

(3) For  $\mathbb{F} = \mathbb{R}$  there are two algebras, one for  $r < 0$  and one for  $r > 0$ . For  $\mathbb{F} = \mathbb{Q}$  there are infinitely many algebras.

### 3.2 The algebras of class 7

Here we are dealing with algebras of maximal class and thus we can make use of the general theory concerning these. In particular we know that we can choose our standard basis such that

$$\begin{aligned} L^7 = Z(L) &= \mathbb{F}x_5 + \mathbb{F}x_4, \\ L^6 = Z_2(L) &= \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3, \\ L^5 = Z_3(L) &= \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_2. \end{aligned}$$

We also know that  $L^4 = Z_4(L) = (L^5)^\perp$ ,  $L^3 = Z_5(L) = (L^6)^\perp$  and  $L^2 = Z_6(L) = (L^7)^\perp$ . Furthermore we know that we can also get characteristic ideals of dimensions 1, 5 and 9 in the following way.

Firstly, we know from the general theory that  $x_3y_4, x_2y_3 \neq 0$ . As a result  $L^5L^2 = \mathbb{F}x_2y_3 \neq 0$ . This gives us a characteristic ideal of dimension 1 and then  $(L^5L^2)^\perp$  is a characteristic ideal of dimension 9.

We now turn to the description of a characteristic ideal of dimension 5. From the general theory we also know that  $x_1y_2, y_1y_2$  are linearly independent. Thus  $L^4L^3 = \mathbb{F}x_1y_2 + \mathbb{F}y_1y_2$  is a 2-dimensional characteristic subspace of  $L^6$ . Let  $I_1 = L^5L^2, I_2 = L^7$  and  $I_3 = L^6$ . Let  $k$  be the smallest such that  $I_k \cap L^4L^3 \neq \{0\}$ . Then

$$U = \{x \in L^4 : xL^3 \leq I_k\}$$

is a characteristic ideal of dimension 5. We can thus further refine our basis such that we have the following situation.

$$\begin{array}{rcc}
L^5L^2 & \begin{array}{|c|c|} \hline x_5 & y_5 \\ \hline \end{array} & \\
L^7 = Z(L) & \begin{array}{|c|c|} \hline x_4 & y_4 \\ \hline \end{array} & (L^5L^2)^\perp \\
L^6 = Z_2(L) & \begin{array}{|c|c|} \hline x_3 & y_3 \\ \hline \end{array} & L^2 = Z_6(L) \\
L^5 = Z_3(L) & \begin{array}{|c|c|} \hline x_2 & y_2 \\ \hline \end{array} & L^3 = Z_5(L) \\
U & \begin{array}{|c|c|} \hline x_1 & y_1 \\ \hline \end{array} & L^4 = Z_4(L)
\end{array}$$

There are now few separate cases to consider according to whether  $L^4L^3 = L^7$  or  $L^4L^3 \neq L^7$  and whether or not  $L^4L^3 \cap L^5L^2 \neq \{0\}$ .

### 3.2.1 Algebras where $L^4L^3 = L^7$

In this case we have

$$\mathbb{F}x_2y_3 = \mathbb{F}x_1y_2 = \mathbb{F}x_5, \quad \mathbb{F}x_5 + \mathbb{F}y_1y_2 = \mathbb{F}x_5 + \mathbb{F}x_4.$$

Now consider the characteristic subspace  $UL^2 = \mathbb{F}x_5 + \mathbb{F}x_1y_3$ . There are again two subcases to consider as either  $UL^2$  has dimension 1 or 2.

#### I. Algebras where $UL^2$ is 1-dimensional

In this case we have that  $x_1y_3 \in \mathbb{F}x_5$  and it follows that  $L^4L^2 = UL^2 + y_1L^2 = \mathbb{F}x_5 + y_1L^2 = Z(L)$ . Consider the characteristic subspace

$$V = \{x \in L^2 : L^4x \leq L^5L^2\}.$$

Then  $V$  is of codimension 1 in  $L^2$  and  $L^4 \leq V$ . Also  $y_2 \notin V$ . One sees readily that we can refine our choice of basis further such that

$$V = L^4 + \mathbb{F}y_3.$$

In particular  $y_1y_3 \leq \mathbb{F}x_5$ . Next consider the characteristic subspace

$$W = \{x \in L^4 : xV = 0\}.$$

We have that  $L^6 \leq W$  and that  $x_2 \notin W$ . Also  $W$  is the kernel of the surjective linear map  $L^4 \rightarrow L^5L^2$ ,  $x \mapsto xy_3$  and thus of codimension 1 in  $L^4$ . We can now pick our basis further such that

$$W = L^6 + \mathbb{F}x_1 + \mathbb{F}y_1.$$

It is not difficult to see that such a choice is compatible to what we have done so far. Notice that it follows that  $y_1y_3 = x_1y_3 = 0$ . Next one notices that  $L^3V = Z(L)$  and that  $L^3U \leq L^5L^2$ . Let

$$Z = \{x \in V : L^3x \leq L^5L^2\}.$$

Then  $Z$  is of codimension 1 in  $V$  and  $y_1 \notin Z$ . We can now further refine the basis such that

$$Z = U + \mathbb{F}y_3.$$

The reader can convince himself that this is compatible to our choice so far. In particular  $y_2y_3 \in \mathbb{F}x_5$ . Replacing  $y_2$  by a suitable  $y_2 - ux_2$ , we can furthermore assume that  $y_2y_3 = 0$ . With this choice of basis we thus have  $x_1y_3 = y_1y_3 = y_2y_3 = 0$  as well as  $\mathbb{F}x_1y_2 = \mathbb{F}x_2y_3 = \mathbb{F}x_5$  and  $\mathbb{F}x_5 + \mathbb{F}x_4 = \mathbb{F}x_5 + \mathbb{F}y_1y_2$ . It is not difficult to see that we can further refine our basis such that

$$x_1y_3 = y_1y_3 = y_2y_3 = 0, \quad x_1y_2 = x_2y_3 = x_5, \quad y_1y_2 = x_4. \quad (12)$$

We are then only left with the following triple values

$$\begin{aligned} (x_1y_4, y_5) &= a, & (y_1y_4, y_5) &= c, & (y_3y_4, y_5) &= e, \\ (x_2y_4, y_5) &= b, & (y_2y_4, y_5) &= d, & (x_3y_4, y_5) &= r, \end{aligned}$$

where  $r \neq 0$ . Replacing  $x_5, x_4, x_3, x_2, x_1, y_1, y_2, y_3, y_4, y_5$  by  $(1/r)^2x_5, r^4x_4, rx_3 + brx_4, (1/r)x_2 + (a/r)x_4 - (c/r)x_5, (1/r)^3x_1, r^3y_1 + dr^3x_4, ry_2, (1/r)y_3 - (e/r^2)x_3 - (e/r^2)bx_4, (1/r)^4y_4 - (b/r^4)y_3 - (a/r^4)y_2 + (d/r^4)x_1, r^2y_5 + cr^2y_2$ , gives us a new standard basis where we can assume that  $a = b = c = d = e = 0$  and  $r = 1$ . The reader can check that (12) is not affected by these changes. We thus arrive at unique presentation for  $L$ .

**Proposition 3.3** *There is a unique nilpotent SAA  $L$  of dimension 10 that has isotropic centre of dimension 2 with the further properties that the class is 7,  $L^4L^3 = L^7$  and  $\dim UL^2 = 1$ . This algebra can be given by the presentation*

$$\mathcal{P}_{10}^{(2,3)} : (x_3y_4, y_5) = 1, (x_2y_3, y_5) = 1, (x_1y_2, y_5) = 1, (y_1y_2, y_4) = 1.$$

Direct calculations show that the algebra with this presentation has the properties stated.

## II. Algebras where $UL^2$ is 2-dimensional

In this case  $\mathbb{F}x_2y_3 = \mathbb{F}x_1y_2 = \mathbb{F}x_5$  and  $\mathbb{F}y_1y_2 + \mathbb{F}x_5 = \mathbb{F}x_1y_3 + \mathbb{F}x_5 = \mathbb{F}x_4 + \mathbb{F}x_5$ . It is not difficult to see that we can choose our standard basis such that

$$x_1y_2 = x_2y_3 = x_5, \quad y_1y_2 = x_4. \quad (13)$$

Now  $y_1y_3 = ax_5 + bx_4$  for some  $a, b \in \mathbb{F}$ . Replacing  $x_2, y_1, y_2, y_3$  by  $x_2 + bx_3, y_1 - ax_2, y_2 - ax_1, y_3 - by_2 + abx_1$  gives

$$y_1y_3 = 0, \quad (14)$$

and the changes do not affect (13). Next consider  $y_2y_3 = ax_4 + bx_5$  and replace  $x_1, y_2, y_3$  by  $x_1 - ax_3, y_2 - bx_2, y_3 + ay_1$ . These changes imply that we can assume furthermore that

$$y_2y_3 = 0. \quad (15)$$

Now consider  $x_1y_3 = ax_5 + bx_4$  (where  $b \neq 0$  by our assumptions). Replacing  $x_1, y_2$  by  $x_1 - ax_2, y_2 + ay_1$  we can assume that  $a = 0$ . Then replace  $x_1, \dots, y_5$  by  $(1/b)x_1, (1/b^2)x_2, (1/b^3)x_3, b^3x_4, bx_5, by_1, b^2y_2, b^3y_3, (1/b^3)y_4, (1/b)y_5$  we can assume that  $b = 1$ . Thus

$$x_1y_3 = x_4. \quad (16)$$

This leaves us with the following triples.

$$\begin{aligned} (x_1y_4, y_5) &= a, & (y_1y_4, y_5) &= c, & (y_3y_4, y_5) &= e, \\ (x_2y_4, y_5) &= b, & (y_2y_4, y_5) &= d, & (x_3y_4, y_5) &= r, \end{aligned}$$

where  $r \neq 0$ . We refine our basis further such that  $a = b = c = d = e = 0$ . Replacing  $y_5, y_1, x_3, y_3, x_2, y_4$  by  $y_5 + cy_2, y_1 + dx_4, x_3 + bx_4, y_3 - (e/r)x_3 - b(e/r)x_4, x_2 + ax_4 - cx_5, y_4 - ay_2 - by_3 + dx_1$ . Thus  $L$  has a presentation of the form  $\mathcal{P}_{10}^{(2,4)}(r)$  as described in the next proposition.

**Proposition 3.4** *Let  $L$  be a nilpotent SAA of dimension 10 with an isotropic centre of dimension 2 that is of class 7 and has the further properties that  $L^4L^3 = L^7$  and  $\dim UL^2 = 2$ . This algebra can be given by a presentation of the form*

$$\mathcal{P}_{10}^{(2,4)}(r) : (x_3y_4, y_5) = r, (x_2y_3, y_5) = 1, (x_1y_2, y_5) = 1, (x_1y_3, y_4) = 1, (y_1y_2, y_4) = 1,$$

where  $r \neq 0$ . Furthermore two such presentations  $\mathcal{P}_{10}^{(2,4)}(r)$  and  $\mathcal{P}_{10}^{(2,4)}(s)$  describe the same algebra if and only if  $s/r \in (\mathbb{F}^*)^{11}$ . Conversely any algebra with such a presentation has the properties stated.

**Proof** We have already seen that any such algebra has such a presentation. Direct calculations show that an algebra with a presentation  $\mathcal{P}_{10}^{(2,4)}(r)$  has the properties stated. We turn to the isomorphism property. To see that the condition is sufficient, suppose we have an algebra  $L$  with a presentation  $\mathcal{P}_{10}^{(2,4)}(r)$  with respect to some standard basis  $x_1, y_1, \dots, x_5, y_5$ . Suppose that  $s/r = a^{11}$  for some  $a \in \mathbb{F}^*$ . Consider a new standard basis  $\tilde{x}_1 = ax_1$ ,  $\tilde{y}_1 = (1/a)y_1$ ,  $\tilde{x}_2 = a^3x_2$ ,  $\tilde{y}_2 = (1/a^3)y_2$ ,  $\tilde{x}_3 = a^5x_3$ ,  $\tilde{y}_3 = (1/a^5)y_3$ ,  $\tilde{x}_4 = (1/a^4)x_4$ ,  $\tilde{y}_4 = a^4y_4$ ,  $\tilde{x}_5 = (1/a^2)x_5$ ,  $\tilde{y}_5 = a^2y_5$ . Calculations show that  $L$  has then presentation  $\mathcal{P}_{10}^{(2,4)}(s)$  with respect to the new basis. It only remains now to see that the conditions is also necessary. Consider an algebra  $L$  with presentation  $\mathcal{P}_{10}^{(2,4)}(r)$  with respect to some standard basis  $x_1, y_1, \dots, x_5, y_5$ . Take some arbitrary new standard basis  $\tilde{x}_1, \tilde{y}_1, \dots, \tilde{x}_5, \tilde{y}_5$  such that  $L$  satisfies the presentation  $\mathcal{P}_{10}^{(2,4)}(s)$  with respect to the new basis. Using the fact that we have an ascending chain of characteristic ideals we know that

$$\begin{aligned}\tilde{y}_1 &= ay_1 + \beta_{11}x_1 + \dots + \beta_{15}x_5, \\ \tilde{y}_2 &= by_2 + \alpha_{21}y_1 + \beta_{21}x_1 + \dots + \beta_{25}x_5, \\ \tilde{y}_3 &= cy_3 + \alpha_{32}y_2 + \alpha_{31}y_1 + \beta_{31}x_1 + \dots + \beta_{35}x_5, \\ \tilde{y}_4 &= dy_4 + \alpha_{43}y_3 + \alpha_{42}y_2 + \alpha_{41}y_1 + \beta_{41}x_1 + \dots + \beta_{45}x_5, \\ \tilde{y}_5 &= ey_5 + \alpha_{54}y_4 + \dots + \alpha_{51}y_1 + \beta_{51}x_1 + \dots + \beta_{55}x_5, \\ \tilde{x}_1 &= (1/a)x_1 + \gamma_{12}x_2 + \dots + \gamma_{15}x_5, \\ \tilde{x}_2 &= (1/b)x_2 + \gamma_{23}x_3 + \gamma_{24}x_4 + \gamma_{25}x_5, \\ \tilde{x}_3 &= (1/c)x_3 + \gamma_{34}x_4 + \gamma_{35}x_5, \\ \tilde{x}_4 &= (1/d)x_4 + \gamma_{45}x_5, \\ \tilde{x}_5 &= (1/e)x_5,\end{aligned}$$

for some  $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, a, b, c, d, e \in \mathbb{F}$  where  $a, b, c, d, e \neq 0$ . Direct calculations show that

$$\begin{aligned}1 &= (\tilde{x}_1\tilde{y}_2, \tilde{y}_5) = be/a \Rightarrow e = a/b \\ 1 &= (\tilde{y}_1\tilde{y}_2, \tilde{y}_4) = abd \Rightarrow d = 1/(ab) \\ 1 &= (\tilde{x}_1\tilde{y}_3, \tilde{y}_4) = cd/a \Rightarrow c = a^2b \\ 1 &= (\tilde{x}_2\tilde{y}_3, \tilde{y}_5) = ce/b \Rightarrow b = a^3.\end{aligned}$$

Thus  $b = a^3$ ,  $c = a^5$ ,  $d = 1/a^4$ ,  $e = 1/a^2$  and it follows that

$$s = (\tilde{x}_3\tilde{y}_4, \tilde{y}_5) = (de/c)r = (1/a)^{11}r.$$

Hence  $s/r \in (\mathbb{F}^*)^{11}$ .  $\square$

**Remarks.** It follows that if  $(\mathbb{F}^*)^{11} = \mathbb{F}^*$  then there is only one algebra of this type. This includes any algebraically closed field and  $\mathbb{R}$ . If  $\mathbb{F}$  is a finite field of order  $p^n$ , then the number of algebras is either 11 or 1 according to whether 11 divides  $p^n - 1$  or not. Notice also that there are infinitely many algebras over  $\mathbb{Q}$ .

### 3.2.2 Algebras where $L^4L^3 \neq L^7$ and $L^5L^2 \leq L^4L^3$

Here we can pick our basis such that

$$L^4L^3 = \mathbb{F}x_5 + \mathbb{F}x_3.$$

Notice also that as before  $U = \{x \in L^4 : xL^3 \leq L^5L^2\}$  and thus again  $x_1y_2 \in \mathbb{F}x_5$ . Notice also that

$$(L^4L^3)^\perp = \mathbb{F}x_5 + \cdots + \mathbb{F}x_1 + \mathbb{F}y_1 + \mathbb{F}y_2 + \mathbb{F}y_4.$$

Then  $L^5(L^4L^3)^\perp = (\mathbb{F}x_3 + \mathbb{F}x_2)y_4 = L^5L^2$ . Consider the characteristic subspace

$$V = \{x \in L^5 : x(L^4L^3)^\perp = 0\}.$$

Here  $x_3y_4 \neq 0$  and thus  $V$  is the kernel of a surjective linear map  $L^5 \rightarrow L^5L^2$ ,  $x \mapsto xy_4$  and has codimension 1 in  $L^5$ . We pick our standard basis such that

$$V = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_2.$$

In particular

$$x_2y_4 = 0. \tag{17}$$

Here we have again  $UL^2 = \mathbb{F}x_5 + \mathbb{F}x_1y_3$  and thus either the dimension of  $UL^2$  is 1 or 2. We consider these cases separately.

#### I. Algebras where $UL^2$ is 1-dimensional

Notice that

$$V^\perp = \mathbb{F}x_5 + \cdots + \mathbb{F}x_1 + \mathbb{F}y_1 + \mathbb{F}y_3,$$

and that  $UV^\perp = \mathbb{F}x_5 = L^5L^2$ . Let

$$W = \{x \in U : xV^\perp = 0\}.$$

Here  $x_2y_3 \neq 0$  and  $W$  is the kernel of the surjective linear map  $U \rightarrow L^5L^2$ ,  $x \mapsto xy_3$ . We choose our standard basis further such that

$$W = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_1.$$

In particular

$$x_1y_3 = 0. \quad (18)$$

Next look at  $L^4V^\perp = \mathbb{F}x_5 + \mathbb{F}y_1y_3$ . Notice that  $y_1y_3 \in V$  and that  $(y_1y_3, y_2) \neq 0$  (as  $\mathbb{F}y_1y_2 + \mathbb{F}x_5 = \mathbb{F}x_3 + \mathbb{F}x_5$ ). We choose our basis further such that

$$L^4V^\perp = \mathbb{F}x_5 + \mathbb{F}x_2.$$

In particular

$$(y_1y_3, y_4) = 0. \quad (19)$$

Now consider the characteristic subspace

$$T = L^4V^\perp + L^4L^3 = \mathbb{F}x_5 + \mathbb{F}x_3 + \mathbb{F}x_2.$$

Notice that  $T^\perp = L^4 + \mathbb{F}y_4$  and  $WT^\perp = \mathbb{F}x_3y_4 + \mathbb{F}x_1y_4 = L^5L^2$ . Let

$$R = \{x \in W : xT^\perp = 0\}.$$

We have  $x_3y_4 \neq 0$  and  $R$  is the kernel of the surjective linear map  $W \rightarrow L^5L^2$ ,  $x \mapsto xy_4$ . We now refine our basis further such that

$$R = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_1.$$

In particular  $x_1y_4 = 0$ . We have thus got a basis where  $x_1y_3 = x_1y_4 = x_2y_4 = 0$  and where  $(y_1y_3, y_4) = 0$ . It is not difficult that we can furthermore assume that

$$\begin{aligned} x_3y_4 = x_5, \quad x_2y_4 = 0, \quad x_1y_2 = x_5, \\ x_1y_3 = 0, \quad x_1y_4 = 0, \quad y_1y_2 = x_3, \quad (y_1y_3, y_4) = 0. \end{aligned}$$

We still need to consider the following triples.

$$\begin{aligned} (y_3y_4, y_5) = a, \quad (y_1y_4, y_5) = c, \quad (y_2y_4, y_5) = f, \\ (y_1y_3, y_5) = b, \quad (y_2y_3, y_4) = d, \quad (y_2y_3, y_5) = e, \\ (x_2y_3, y_5) = r, \end{aligned}$$

where  $r \neq 0$ . Replacing  $x_1, x_2, y_1, y_2, y_3, y_4, y_5$  by  $x_1 + dx_4, x_2 - bx_5, y_1 - cx_3, y_2 - (1/r)(c+e)x_2 - fx_3 + (b/r)(c+e)x_5, y_3 - cx_1 - fx_2 - (a+bd)x_3 - cdx_4 + bfx_5, y_4 - dy_1, y_5 + by_2$  we can assume that  $a = b = c = d = e = f = 0$ . It follows that  $L$  has a presentation of the form  $\mathcal{P}_{10}^{(2,5)}(r)$  as in the following proposition.

**Proposition 3.5** *Let  $L$  be a nilpotent SAA of dimension 10 with an isotropic centre of dimension 2 that is of class 7 and has the further properties that  $L^4L^3 \neq L^7$ ,  $L^5L^2 \leq L^4L^3$  and  $UL^2$  is 1-dimensional. The algebra can be given by a presentation of the form*

$$\mathcal{P}_{10}^{(2,5)}(r) : (x_2y_3, y_5) = r, (x_3y_4, y_5) = 1, (x_1y_2, y_5) = 1, (y_1y_2, y_3) = 1,$$

where  $r \neq 0$ . Furthermore two such presentations  $\mathcal{P}_{10}^{(2,5)}(r)$  and  $\mathcal{P}_{10}^{(2,5)}(s)$  describe the same algebra if and only if  $s/r \in (\mathbb{F}^*)^3$ . Conversely any algebra with such a presentation has the properties stated.

**Proof** We have already seen that any such algebra has a presentation of this form. Conversely, direct calculations show that any algebra with a presentation  $\mathcal{P}_{10}^{(2,5)}(r)$  satisfies the properties stated. We turn to the isomorphism property. To see that condition is sufficient, suppose we have an algebra  $L$  with presentation  $\mathcal{P}_{10}^{(2,5)}(r)$  with respect to some standard basis  $x_1, y_1, \dots, x_5, y_5$ . Suppose that  $s/r = a^3$  for some  $a \in \mathbb{F}^*$ . Consider a new standard basis  $\tilde{x}_1 = x_1, \tilde{y}_1 = y_1, \tilde{x}_2 = ax_2, \tilde{y}_2 = (1/a)y_2, \tilde{x}_3 = (1/a)x_3, \tilde{y}_3 = ay_3, \tilde{x}_4 = x_4, \tilde{y}_4 = y_4, \tilde{x}_5 = (1/a)x_5, \tilde{y}_5 = ay_5$ . Calculations show that  $L$  has then the presentation  $\mathcal{P}_{10}^{(2,5)}(s)$  with respect to the new basis. It only remains to see that the condition is also necessary. Consider an algebra  $L$  with presentation  $\mathcal{P}_{10}^{(2,5)}(r)$  with respect to some standard basis  $x_1, y_1, \dots, x_5, y_5$ . Take some arbitrary new standard basis  $\tilde{x}_1, \tilde{y}_1, \dots, \tilde{x}_5, \tilde{y}_5$  such that  $L$  also satisfies presentation  $\mathcal{P}_{10}^{(2,5)}(s)$  with respect to the new basis. Using the fact that we have an ascending chain of characteristic ideals as well as the fact that  $L^4L^3 = \mathbb{F}x_5 + \mathbb{F}x_3$ ,  $L^4V^\perp = \mathbb{F}x_5 + \mathbb{F}x_2$ ,  $R = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_1$ ,  $W^\perp = \mathbb{F}y_2 + U$ ,  $(R + L^4V^\perp)^\perp = \mathbb{F}y_3 + U$  and  $T^\perp = \mathbb{F}y_4 + \mathbb{F}y_1 + U$  are characteristic subspaces, we know that

$$\begin{aligned} \tilde{y}_1 &= (1/a)y_1 + \beta_{11}x_1 + \dots + \beta_{15}x_5, \\ \tilde{y}_2 &= (1/b)y_2 + \beta_{21}x_1 + \dots + \beta_{25}x_5, \\ \tilde{y}_3 &= (1/c)y_3 + \beta_{31}x_1 + \dots + \beta_{35}x_5, \\ \tilde{y}_4 &= (1/d)y_4 + \alpha_{41}y_1 + \beta_{41}x_1 + \dots + \beta_{45}x_5, \\ \tilde{y}_5 &= (1/e)y_5 + \alpha_{54}y_4 + \dots + \alpha_{51}y_1 + \beta_{51}x_1 + \dots + \beta_{55}x_5, \\ \tilde{x}_1 &= ax_1 + \gamma_{14}x_4 + \gamma_{15}x_5, \\ \tilde{x}_2 &= bx_2 + \gamma_{25}x_5, \\ \tilde{x}_3 &= cx_3 + \gamma_{35}x_5, \\ \tilde{x}_4 &= dx_4 + \gamma_{45}x_5, \\ \tilde{x}_5 &= ex_5, \end{aligned}$$

for some  $a, b, c, d, e, \alpha_{ij}, \beta_{ij}, \gamma_{ij}$  where  $a, b, c, d, e \neq 0$ . It follows that

$$\begin{aligned} 1 &= (\tilde{x}_3\tilde{y}_4, \tilde{y}_5) = c/(de) \\ 1 &= (\tilde{x}_1\tilde{y}_2, \tilde{y}_5) = a/(be) \\ 1 &= (\tilde{y}_1\tilde{y}_2, \tilde{y}_3) = 1/(abc). \end{aligned}$$

This gives  $c = 1/(ab)$ ,  $e = a/b$ ,  $d = 1/a^2$  and then

$$s = (\tilde{x}_2\tilde{y}_3, \tilde{y}_5) = br/(ce) = b^3r.$$

This finishes the proof.  $\square$

**Remarks.** Again we just get one algebra if  $(\mathbb{F}^*)^3 = \mathbb{F}^*$ . This includes all fields that are algebraically closed as well as  $\mathbb{R}$ . For a finite field of order  $p^n$  there are 3 algebras if  $3|p^n - 1$  but otherwise one. For  $\mathbb{Q}$  there are infinitely many algebras.

## II. Algebras where $UL^2$ is 2-dimensional

Recall that  $UL^2 = \mathbb{F}x_5 + \mathbb{F}x_1y_3$ ,  $L^4L^3 = \mathbb{F}x_5 + \mathbb{F}x_3$  and  $x_2y_4 = 0$ . It is not difficult to see that one can further refine the basis such that

$$x_3y_4 = \alpha x_5, \quad x_2y_3 = rx_5, \quad x_2y_4 = 0, \quad x_1y_2 = x_5, \quad x_1y_3 = x_4, \quad y_1y_2 = x_3, \quad (20)$$

where  $\alpha, r \neq 0$ . Replacing  $x_1, y_1, \dots, x_5, y_5$  by  $\alpha x_1, (1/\alpha)y_1, (1/\alpha^3)x_2, \alpha^3y_2, \alpha^2x_3, (1/\alpha^2)y_3, (1/\alpha)x_4, \alpha y_4, \alpha^4x_5, (1/\alpha^4)y_5$ , implies that we can furthermore assume that  $\alpha = 1$ . We have also the following triples to sort out.

$$\begin{aligned} (y_1y_3, y_4) &= a, & (y_2y_3, y_4) &= d, & (x_1y_4, y_5) &= g, \\ (y_1y_3, y_5) &= b, & (y_2y_3, y_5) &= e, & (y_3y_4, y_5) &= h. \\ (y_1y_4, y_5) &= c, & (y_2y_4, y_5) &= f. \end{aligned}$$

Let  $\gamma = c + e$ ,  $\beta = ae - h - bd$  and replace  $x_2, x_1, y_1, y_2, y_3, y_4, y_5$  by  $x_2 - ax_4 - bx_5, x_1 - (a + g)x_3 + dx_4 + \gamma x_5, y_1 - cx_3, y_2 - fx_3, y_3 - cx_1 - fx_2 + \beta x_3 + (af - cd)x_4 + bf x_5 + (a + g)y_1, y_4 - dy_1 + ay_2, y_5 + c\gamma x_3 - \gamma y_1 + by_2$ . Then we can assume that  $a = b = c = d = e = f = g = h = 0$  and  $L$  has a presentation  $\mathcal{P}_{10}^{(2,6)}(r)$  as in the following proposition.

**Proposition 3.6** *Let  $L$  be a nilpotent SAA of dimension 10 with an isotropic centre of dimension 2 that is of class 7 and has the further properties that*

$L^4L^3 \neq L^7$ ,  $L^5L^2 \leq L^4L^3$  and  $UL^2$  is 2-dimensional. The algebra can be given by a presentation of the form

$$\mathcal{P}_{10}^{(2,6)}(r) : (x_2y_3, y_5) = r, (x_3y_4, y_5) = 1, (x_1y_2, y_5) = 1, (x_1y_3, y_4) = 1, (y_1y_2, y_3) = 1$$

where  $r \neq 0$ . Furthermore two such presentations  $\mathcal{P}_{10}^{(2,6)}(r)$  and  $\mathcal{P}_{10}^{(2,6)}(s)$  describe the same algebra if and only if  $s/r \in (\mathbb{F}^*)^{12}$ . Conversely any algebra with such a presentation has the properties stated.

**Proof** We have already seen that any such algebra has a presentation of this form. Conversely, direct calculations show that any algebra with a presentation  $\mathcal{P}_{10}^{(2,6)}(r)$  satisfies the properties stated. We turn to the isomorphism property. To see that condition is sufficient, suppose we have an algebra  $L$  with presentation  $\mathcal{P}_{10}^{(2,6)}(r)$  with respect to some standard basis  $x_1, y_1, \dots, x_5, y_5$ . Suppose that  $s/r = a^{12}$  for some  $a \in \mathbb{F}^*$ . Consider a new standard basis  $\tilde{x}_1 = (1/a)x_1$ ,  $\tilde{y}_1 = ay_1$ ,  $\tilde{x}_2 = a^4x_2$ ,  $\tilde{y}_2 = (1/a^4)y_2$ ,  $\tilde{x}_3 = (1/a^3)x_3$ ,  $\tilde{y}_3 = a^3y_3$ ,  $\tilde{x}_4 = a^2x_4$ ,  $\tilde{y}_4 = (1/a^2)y_4$ ,  $\tilde{x}_5 = (1/a^5)x_5$ ,  $\tilde{y}_5 = a^5y_5$ . Calculations show that  $L$  has then the presentation  $\mathcal{P}_{10}^{(2,6)}(s)$  with respect to the new basis. It only remains to see that the condition is also necessary. Consider an algebra  $L$  with presentation  $\mathcal{P}_{10}^{(2,6)}(r)$  with respect to some standard basis  $x_1, y_1, \dots, x_5, y_5$ . Take some arbitrary new standard basis  $\tilde{x}_1, \tilde{y}_1, \dots, \tilde{x}_5, \tilde{y}_5$  such that  $L$  also satisfies presentation  $\mathcal{P}_{10}^{(2,6)}(s)$  with respect to the new basis. Using the fact that we have an ascending chain of characteristic ideals as well as the fact that  $L^4L^3 = \mathbb{F}x_5 + \mathbb{F}x_3$ ,  $V = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_2$ , are characteristic subspaces, we know that

$$\begin{aligned} \tilde{y}_1 &= (1/a)y_1 + \beta_{11}x_1 + \dots + \beta_{15}x_5, \\ \tilde{y}_2 &= (1/b)y_2 + \alpha_{21}y_1 + \beta_{21}x_1 + \dots + \beta_{25}x_5, \\ \tilde{y}_3 &= (1/c)y_3 + \alpha_{31}y_1 + \beta_{31}x_1 + \dots + \beta_{35}x_5, \\ \tilde{y}_4 &= (1/d)y_4 + \alpha_{42}y_2 + \alpha_{41}y_1 + \beta_{41}x_1 + \dots + \beta_{45}x_5, \\ \tilde{y}_5 &= (1/e)y_5 + \alpha_{54}y_4 + \dots + \alpha_{51}y_1 + \beta_{51}x_1 + \dots + \beta_{55}x_5, \\ \tilde{x}_1 &= ax_1 + \gamma_{12}x_2 + \dots + \gamma_{15}x_5, \\ \tilde{x}_2 &= bx_2 + \gamma_{24}x_4 + \gamma_{25}x_5, \\ \tilde{x}_3 &= cx_3 + \gamma_{35}x_5, \\ \tilde{x}_4 &= dx_4 + \gamma_{45}x_5, \\ \tilde{x}_5 &= ex_5, \end{aligned}$$

for some  $a, b, c, d, e, \alpha_{ij}, \beta_{ij}, \gamma_{ij}$  where  $a, b, c, d, e \neq 0$ . It follows that

$$\begin{aligned} 1 &= (\tilde{x}_3 \tilde{y}_4, \tilde{y}_5) = c/(de) \\ 1 &= (\tilde{x}_1 \tilde{y}_2, \tilde{y}_5) = a/(be) \\ 1 &= (\tilde{y}_1 \tilde{y}_2, \tilde{y}_3) = 1/(abc) \\ 1 &= (\tilde{x}_1 \tilde{y}_3, \tilde{y}_4) = a/(cd). \end{aligned}$$

This gives  $b = (1/a^4)$ ,  $c = a^3$ ,  $d = (1/a^2)$ ,  $e = a^5$  and then

$$s = (\tilde{x}_2 \tilde{y}_3, \tilde{y}_5) = br/(ce) = (1/a^{12})r.$$

This finishes the proof.  $\square$

**Remarks.** The number of algebras depends thus again on the underlying field. In particular there is one algebra over the field  $\mathbb{C}$ , two algebras over  $\mathbb{R}$  and infinitely many over  $\mathbb{Q}$ . When  $\mathbb{F}$  is a finite field of order  $p^n$  then the number can be 12, 6, 3, 4, 2 or 1 depending on what the value of  $p^n$  is modulo 12.

### 3.2.3 Algebras where $L^4L^3 \neq L^7$ and $L^5L^2 \not\subseteq L^4L^3$

First we pick our standard basis such that

$$L^4L^3 \cap L^7 = \mathbb{F}x_4, \quad L^4L^3 = \mathbb{F}x_4 + \mathbb{F}x_3.$$

Thus in particular  $\mathbb{F}x_1y_2 = \mathbb{F}x_4$ . From this one sees that  $U(L^5L^2)^\perp = L^7 + \mathbb{F}x_1y_4$  where  $(x_1y_4, y_2) = -(x_1y_2, y_4) \neq 0$ . We further refine our basis such that

$$U \cdot (L^5L^2)^\perp = L^7 + \mathbb{F}x_2 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_2.$$

Then notice that  $(U(L^5L^2)^\perp)L = L^5L^2 + \mathbb{F}x_2y_5$ , where  $(x_2y_5, y_3) = -(x_2y_3, y_5) \neq 0$ . We refine our basis further such that

$$(U(L^5L^2)^\perp)L = L^5L^2 + \mathbb{F}x_3 = \mathbb{F}x_5 + \mathbb{F}x_3.$$

Notice that in particular  $x_2x_5 \in \mathbb{F}x_5 + \mathbb{F}x_3$  and thus  $(x_2x_4, x_5) = -(x_2x_5, x_4) = 0$ . We have as well  $(x_2y_4, y_3) = -(x_2y_3, y_4) = 0$  and thus

$$x_2y_4 = 0. \tag{21}$$

We also have

$$\mathbb{F}x_3 = (\mathbb{F}x_4 + \mathbb{F}x_3) \cap (\mathbb{F}x_5 + \mathbb{F}x_3) = (L^4L^3) \cap ((U(L^5L^2)^\perp)L).$$

Next consider the characteristic subspace

$$V = \{x \in L^4 : xL^3 \leq (L^4L^3) \cap ((U(L^5L^2)^\perp)L)\}.$$

Notice that  $x_1y_2 \neq 0$  and thus  $V$  is the kernel of a surjective linear map  $L^4 \rightarrow L^4L^3/\mathbb{F}x_3$ ,  $x \mapsto xy_2 + \mathbb{F}x_3$  and thus of codimension 1 in  $L^4$ . Notice also that  $L^5 \leq V$ . We refine our basis further such that

$$V = L^5 + \mathbb{F}y_1 = \mathbb{F}x_5 + \dots + \mathbb{F}x_2 + \mathbb{F}y_1.$$

It follows in particular that

$$\mathbb{F}y_1y_2 = \mathbb{F}x_3 \tag{22}$$

Notice that

$$(U(L^5L^2)^\perp)^\perp = \mathbb{F}x_5 + \dots + \mathbb{F}x_1 + \mathbb{F}y_1 + \mathbb{F}y_3$$

and then  $U \cdot (U(L^5L^2)^\perp)^\perp = L^5L^2 + \mathbb{F}x_1y_3$ . Now  $(x_1y_3, y_2) = -(x_1y_2, y_3) = 0$  but also  $x_1y_4 \in U(L^5L^2)^\perp = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_2$  and thus  $(x_1y_3, y_4) = -(x_1y_4, y_3) = 0$ . It follows that  $x_1y_3 \in \mathbb{F}x_5 = L^5L^2$ . It follows that  $U \cdot (U(L^5L^2)^\perp)^\perp = L^5L^2$ . Consider the characteristic subspace

$$R = \{x \in U : x(U(L^5L^2)^\perp)^\perp = 0\}.$$

We have that  $x_2y_3 \neq 0$  and thus  $R$  is the kernel of the surjective linear map  $U \rightarrow L^5L^2$ ,  $x \mapsto xy_3$ . Thus  $R$  is of codimension 1 in  $U$  and contains  $L^6$ . Now choose our basis further such that

$$R = L^6 + \mathbb{F}x_1 = \mathbb{F}x_5 + \mathbb{F}x_4 + \mathbb{F}x_3 + \mathbb{F}x_1.$$

In particular

$$x_1y_3 = 0. \tag{23}$$

Next consider  $L^4 \cdot (U(L^5L^2)^\perp)^\perp = L^5L^2 + \mathbb{F}y_1y_3$ . Notice that  $(y_1y_3, y_2) = -(y_1y_2, y_3) \neq 0$ . We can refine our basis further such that

$$L^4 \cdot (U(L^5L^2)^\perp)^\perp = \mathbb{F}x_5 + \mathbb{F}x_2.$$

In particular

$$(y_1y_3, y_4) = 0. \tag{24}$$

It is not difficult to see that we can now choose our basis such that

$$\begin{aligned} x_3y_4 = x_5, \quad x_2y_3 = rx_5, \quad x_2y_4 = 0, \\ x_1y_2 = x_4, \quad x_1y_3 = 0, \quad y_1y_2 = x_3, \quad (y_1y_3, y_4) = 0, \end{aligned}$$

where  $r \neq 0$ . Replacing  $x_1, y_1, \dots, x_5, y_5$  by  $(1/r)x_1, ry_1, rx_2, (1/r)y_2, x_3, y_3, (1/r^2)x_4, r^2y_4, r^2x_5, (1/r^2)y_5$ , we see that we can furthermore assume that  $r = 1$ . We are now only left with the triples

$$\begin{aligned} (x_1y_4, y_5) = a, \quad (y_1y_4, y_5) = d, \quad (y_2y_4, y_5) = g, \quad (y_3y_4, y_5) = h. \\ (y_1y_3, y_5) = c, \quad (y_2y_3, y_4) = e, \quad (y_2y_3, y_5) = f, \end{aligned}$$

We now show that we can further refine the basis such that all these values are zero. By replacing  $x_2, x_1, y_1, y_2, y_3, y_4, y_5$  by  $x_2 - cx_5, x_1 + (c - a)x_3 + (e + d)x_4 + fx_5, y_1 - dx_3, y_2 - gx_3, y_3 - dx_1 - gx_2 - (h + ce)x_3 + (cg - df)x_5 + (a - c)y_1, y_4 + d(d + e)x_3 - (e + d)y_1, y_5 - fy_1 + cy_2$ , we have thus arrived at a unique presentation.

**Proposition 3.7** *There is a unique nilpotent SAA of dimension 10 with an isotropic centre of dimension 2 that is of class 7 and has the further properties that  $L^4L^3 \neq L^7$ ,  $L^5L^2 \not\leq L^4L^3$ . The algebra can be given by the presentation*

$$\mathcal{P}_{10}^{(2,7)} : (x_2y_3, y_5) = 1, (x_3y_4, y_5) = 1, (x_1y_2, y_4) = 1, (y_1y_2, y_3) = 1$$

**Proof** We have already seen that any such algebra must have a presentation of this form and conversely direct calculations show that the algebra with the given presentation satisfies all the properties stated.  $\square$

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