Normal right Engel subgroups of compact Hausdorff groups

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Let G be a finitely generated compact Hausdorff topological group and let H be a closed normal subgroup consisting of right Engel elements. We show that H belongs to some term of the upper central series of G.

1 Introduction

Let G be a group. An element $a \in G$ is said to be a right Engel element in G if for each $x \in G$ there exists a positive integer $n = n(x)$ such that

$$[a,_n x] = [[[a, \underbrace{x], \cdots x], x}_{n}] = 1.$$

The group G is said to be an Engel group if all its elements are right Engel elements. According to a classical result of Zorn, every finite Engel group is nilpotent [13] and more generally the right Engel elements of any group G, satisfying the maximal condition, belong to some common term of the upper central series of G [1]. Even when a group is finitely generated the situation is much more complicated in general as one can see from the well known examples of Golod [5], where we have an infinite 3-generator p-group all of whose 2-generator subgroups are finite. The situation is however well understood for a number of classes that include, apart from groups satisfying the maximal condition, the class of finitely generated solvable groups [2,6].

In [8] Medvedev proved a strong generalisation of Zorn's Theorem. His result can be expressed as saying that any compact Hausdorff Engel group that is finitely generated, as a topological group, is nilpotent. This generalised a theorem of Wilson and Zel'manov that proved this for profinite groups [11]. Recall that a compact Hausdorff group is profinite if the open subgroups form a base for the neighbourhood of identity.

In this paper we will prove the following generalisation of Medvedev's result.

Theorem. *Let G be a finitely generated compact Hausdorff group and let H be a closed normal subgroup of G consisting of right Engel elements. Then H belongs to some term of the upper central series of G.*

This is a paper by an algebraist and is written with algebraists in mind. The approach, that is modelled on Medvedev's paper, uses some topologial arguments that are mostly quite elementary. We end this section by summarising some topological properties that we will be using later on.

For a subset A of a topological space we will denote by $\overline{A}$, the topological closure of A. Let $\phi : G \to H$ be a continuous map between two compact Hausdorff groups. Then for any subset A of G, we have that $\phi(\overline{A}) = \overline{\phi(A)}$. In particular if H is a subgroup of G then $\overline{H}\cdot\overline{H} = \overline{H \cdot H} = \overline{H}$ and $\overline{H}^{-1} = \overline{H^{-1}} = \overline{H}$. This shows that $\overline{H}$ is also a subgroup of G. Like Medvedev, we will be making use of Baire Category Theorem. We state here the version for compact spaces.

Baire Category Theorem. *Let X be a compact Hausdorff space and suppose that (A_n) is a countable collection of closed subsets of X such that $\bigcup A_n$ has non-empty interior. Then for some $n \in \mathbb{N}$ we have that A_n has a non-empty interior.*

For a proof see for example [9, Theorem 7.2, page 294]. The only topological fact that we will be using and is not elementary is the fact that every compact Hausdorff group is a subcartesian product of closed subgroups of unitary matrix groups [7, Corollary 22.14, p. 345]. This fact uses a highly developed representation theory of compact Hausdorff groups. As every finitely generated linear group is residually finite, and as the property of being residually finite is inherited when taking cartesian products and subgroups, it follows that any finitely generated subgroup of a compact Hausdorff group is residually finite.

Although our proof is modelled on [8], it differs in many respects as we are aiming for a more general result. Like in that proof we make use of Lie ring methods as well as topological arguments and we will use a deep result of Zel'manov on Lie rings. In order to apply Zel'manov's result to our more general setting we will use a construction from [3,4].

2 Proof of the Theorem

Let $G = \overline{\langle x_1, x_2, \ldots, x_r \rangle}$ be a finitely generated compact Hausdorff topological group and let H be a closed normal subgroup of G consisting of right Engel elements in G. The aim is to show that $H \leq Z_m(G)$ for some positive integer m. As a first step we show that it suffices to show that H is hypercentral in G, that is to say that each element in H belong to some term of the upper central series of G.

Lemma 2.1 *Suppose that for each $h \in H$ there exists a positive integer $m(h)$ such that $h \in Z_{m(h)}(G)$. Then there exists a positive integer m such that $H \leq Z_m(G)$.*

Proof For each n-tuple $(g_1, \ldots, g_n) \in \underbrace{G \times \cdots \times G}_{n}$ we consider the continuous map $f_{(g_1,\ldots,g_n)} : H \to H,\ h \mapsto [h, g_1, \ldots, g_n]$. Let $H_n = H \cap Z_n(G)$. Notice that H_n is the intersection of the closed subsets $f^{-1}_{(g_1,\ldots,g_n)}(1)$, where $(g_1, \ldots, g_n)$ runs through $\underbrace{G \times \cdots \times G}_{n}$, and thus a closed subgroup of H. By our assumptions we know that

$$H = \bigcup_{i=1}^{\infty} H_i.$$

It thus follows from the Baire Category Theorem that, for some positive integer n, H_n contains an open subset of the form hU where U is an open neighbourhood in H around 1. In particular $h \in H_n$ and as H_n is a subgroup we get that $U = h^{-1} \cdot hU \subseteq H_n$. Now $H = \bigcup_{a \in H} Ua$ and as H is compact we have $H = Ua_1 \cup \cdots \cup Ua_l$ for some finitely many elements $a_1, \ldots, a_l \in H$. It follows that $H = H_n a_1 \cup \cdots \cup H_n a_l$ and thus $[H : H \cap Z_n(G)] = l$. Hence $H = H \cap Z_{n+l}(G)$ and thus $H \leq Z_{n+l}(G)$. $\square$

Our problem has thus been reduced to the local problem of showing that h is in some term of the upper central series of G for all $h \in H$.

Let $D = \langle x_1, \ldots, x_r \rangle$. By adding h as a generator, we can assume without loss of generality that $h \in D$. We let $E = \langle h \rangle^D$. As the map $\phi : G \times G \to G,\ (a, b) \mapsto [a, b]$ is a continuous map between compact Hausdorff spaces, it follows that $\phi(\overline{E}, G) = \phi(\overline{E}, \overline{D}) = \overline{\phi(E, D)} \leq \overline{E}$. Hence $\overline{E}$ is normal in G. Thus $\overline{E}$ is the smallest closed subgroup of H that contains h and is normal in G. The function

$$\phi : \overline{E} \times \underbrace{G \times \cdots G}_{m} \to \overline{E}$$

that maps $(h, g_1, \ldots, g_m)$ to $[h, g_1, \ldots, g_m]$ is a continuous map between compact Hausdorff spaces. Thus

$$\phi(\overline{E}, \underbrace{G, \ldots, G}_{m}) = \overline{\phi(E, \underbrace{D, \ldots, D}_{m})}.$$

It follows that $\overline{E} \leq Z_m(G)$ if and only if $E \leq Z_m(D)$.

As we said in the introduction, the compact group G is a subcartesian product of linear groups. As every finitely generated linear group is residually finite, it follows that D is residually finite.

Consider now the chains $(E_i)_{i=0}^{\infty}$ and $(D_i)_{i=0}^{\infty}$ where $E_i = [E,_i D]$ and $D_i = \gamma_i(D)$. We use these chains to construct an associated Lie ring like was done [3,4]. First we consider the abelian groups

$$A = A_0 \oplus A_1 \oplus \cdots \quad \text{and} \quad L = L_1 \oplus L_2 \oplus \cdots$$

where $A_i = E_i/E_{i+1}$ and $L_i = D_i/D_{i+1}$. We let L be the associated Lie ring of D and consider A as an abelian Lie ring. We furthermore define an action from L on A as follows. First, for $\bar{a} = aE_{i+1} \in A_i$ and $\bar{d} = dD_{j+1} \in L_j$, we let

$$\bar{a} \cdot \bar{d} = [a, d]E_{i+j+1} \in A_{i+j+1}.$$

Then we extend linearly to get a semidirect product $T = A \ltimes L$. The aim is now to show that $AL^m = 0$ for some non-negative integer m. This would imply that $[E,_m D] = [E,_{m+1} D]$. Let us see why this is sufficient. Let N be a normal subgroup of D that is of finite index. In finite groups we know that the right Engel elements belong to some term of the upper central series and thus by what we have said above it would follow that $[E,_m D] \leq N$. As D is residually finite, it follows then that $[E,_m D] = 1$.

The main tool for proving the Lie ring claim will be a deep result of Zel'manov [12]. This result is a stronger version of a key result for his solution to the Restricted Burnside Problem and is stated in a quantitative form as Proposition 2 in [12].

Theorem (Zel'manov). *Let $F = \langle f_1, \dots, f_r \rangle$ be a multigraded Lie ring and suppose the following conditions hold. Firstly suppose there is some positive integer l where*

(a) $\sum_{\sigma \in S_l} y y_{\sigma(1)} \cdots y_{\sigma(l)} = 0$ *for all* $y, y_1, \dots, y_l \in F$.

Secondly, for each simple product f in $f_1, \dots, f_r$, there exists a positive integer $k = k(f)$ such that

(b) $y f^k = 0$ *for all* $y \in F$.

Then F is nilpotent.

It is not difficult to see that any multigraded Lie ring satisfying condition (b) has the radical property. This result is also used in [8] but stated there without a proof. As a proof doesn't seem to exist in the literature, we include one here for a completion. The proof is standard and very similar for example to the proof of a corresponding statement for $(p-1)$-Engel Lie rings of characteristic p (see Corollary 1.1.13 in [10]).

Proposition 2.2 *Let $F = \langle f_1, \dots, f_r \rangle$ be a multigraded Lie ring where for each simple product f in $f_1, \dots, f_r$, there exists an integer $k = k(f)$ such that $y f^{k(f)} = 0$ for all $y \in F$. Suppose that I and J are multigraded ideals where $I \subseteq J$ and where $I, J/I$ are locally nilpotent. Then J is locally nilpotent.*

Proof Let M be a finitely generated subalgebra of J. Without loss of generality, we can suppose that M is multigraded. Now $M/M\cap I \simeq M+I/I$ is a finitely generated subgroup of J/I and thus nilpotent. It follows that we get a central chain in M

$$M \cap I = M_0 < M_1 < \cdots < M_s = M$$

where $M_i = M_{i-1} + \mathbb{Z}y_i$ for some simple Lie products $y_1, \ldots, y_s$ in $f_1, \ldots, f_r$. We show by induction that M_i is locally nilpotent for $i = 1, \ldots, s$. As $M_0 \leq I$, this is clear for $i = 0$. Now let $1 \leq i \leq s$ and suppose M_{i-1} is locally nilpotent. To show that M_i is locally nilpotent, it suffices to show that if $c_1, \ldots, c_p \in M_{i-1}$ then $N = \langle y_i, c_1, \ldots, c_p\rangle$ is nilpotent. Suppose $k(y_i) = k$ and consider

$$P = \langle c_j y_i^l : \ j = 1, \ldots, p, \ l = 0, \ldots, k-1\rangle.$$

Then $P \leq M_{i-1}$ and $P \trianglelefteq N$. By the induction hypothesis P is nilpotent of class say q. Now every product of weight $qk+1$ in $y_i, c_1, \ldots, c_p$ must have at least $q+1$ occurrences from $\{c_1, \ldots, c_p\}$ if we are to avoid k consecutive occurrences of y_i. As P is nilpotent of class q, it then follows that all such products are 0 and thus N is nilpotent of class at most qk. This finishes the proof of the inductive step and thus M is locally nilpotent. As M is finitely generated it is then nilpotent. We have thus shown that J is locally nilpotent. □

Remark. It follows that for any graded Lie ring L, satisfying the conditions of the Proposition, we have a unique maximal locally nilpotent ideal $R(L)$ and $L/R(L)$ has no non-trivial locally nilpotent ideal.

If V is an open neighbourhood of 1 in a topological group, then $V \cap V^{-1}$ is an open neighbourhood that is closed under taking inverses. Throughout the paper all neighbourhoods of 1 will be assumed to be chosen such that they are closed under taking inverses.

Then next elementary lemma is a variant of a corresponding lemma from [8], with essentially the same proof.

Lemma 2.3 *Corresponding to each neighbourhood V of $1 \in \overline{E}$, there exists a positive integer $t = t(V)$ such that for each pair of elements $(a, d) \in E \times D$,* *there exist integers* $0 \leq k < m \leq t$ *such that*

$$[a,_k d] = v[a,_m d]$$

for some $v \in V$.

Proof Pick a neighbourhood $V(2)$ of 1 in $\overline{E}$ such that $V(2)^2 \subseteq V$. As $\overline{E}$ is compact, one sees readily that there exist finitely many elements $h_1, \ldots, h_t \in \overline{E}$ such that

$$\overline{E} = \bigcup_{i=1}^{t} V(2)h_i.$$

Thus there must be some two elements among $a, [a, d], \ldots, [a,_t d]$ that belong to the same open set $V(2)h_i$, say $[a,_k d] = v_1 h_i$ and $[a,_m d] = v_2 h_i$ for some $0 \leq k < m \leq t$. Hence $[a,_k k] = v_1 v_2^{-1}[a,_m d]$. Where $v = v_1 v_2^{-1} \in V(2)^2 \subseteq V$. $\square$

The next proposition is also a variant of a similar result from [8].

Proposition 2.4 *Consider the Lie ring* L. *For each simple Lie product* $\bar{d}$ *in* $\bar{x_1}, \ldots, \bar{x_r}$, *we have that there exists a positive integer* $k_1(\bar{d})$ *such that* $a\bar{d}^{k_1(\bar{d})} = 0$ *for all* $a \in A$.

Proof Let $u \in D$ and consider the closed subsets

$$T_i(u) = \{x \in \overline{E} : \ [x,_i u] = 1\}.$$

As $\overline{E}$ consists of right Engel elements, we get

$$\overline{E} = \bigcup_{i=1}^{\infty} T_i(u)$$

and by the Baire Category Theorem, one of these, say T_s, contains an open neighbourhood in $\overline{E}$. As E is dense in $\overline{E}$, we have that T_s contains an open neighbourhood of the form Ve where $e \in E$ and V is a neighbourhood of 1 in $\overline{E}$. Thus

$$[ve,_s u] = 1 \tag{1}$$

for all $v \in V$. In particular $[e,_s u] = 1$ and then expanding (1), we get

$$1 = [v,_s u]c_1(v, u, e) \ \text{ for all } v \in V \tag{2}$$

where $c_1(v,u,e)$ is a product of simple commutators in v,u,e that are of higher multiweight than $[v,_s u]$. Now use Lemma 2.3 for some $z \in E$ and $d \in D$. Thus there exist integers k,m such that $0 \leq k < m \leq t = t(V)$ such that

$$v = [z,_k d][z,_m d]^{-1} \in V.$$

Thus from this and (2) we see that

$$1 = [z,_{t+s} d]c_2(z,d,e) \tag{3}$$

where $c_2(z,d,e)$ is a product of simple commutators in z,d,e of higher multiweight than $[z,_{t+s} d]$. Let z be any commutator of the form $[h, x_{i_1}, \ldots, x_{i_l}]$ with $i_1, i_2, \ldots, i_l \in \{1,\ldots,r\}$, and let d be any simple commutator in $x_1,\ldots,x_r$. We now see from (3) that

$$1 = [h, x_{i_1}, \ldots, x_{i_l},_{t+s} d]c(h, x_1, \ldots, x_r)$$

where $c(h,x_1,\ldots,x_r)$ is a product of simple commutators in $h,x_1,\ldots,x_r$ of higher multiweight than $[h, x_{i_1}, \ldots, x_{i_l},_{t+s} d]$. Now let $\overline{d}$ be the Lie product in $\overline{x_1},\ldots,\overline{x_r}$ that corresponds to d and a be the element in A that corresponds to $[h, x_{i_1}, \ldots, x_{i_l}]$. We see that if follows from the previous discussion that $a\bar{d}^{t+s} = 0$. Thus $a\bar{d}^{k_1} = 0$ for all $a \in A$ where $k_1 = t+s$. Notice that t,s only depend on $\bar{d}$ and not a. $\square$

Let $\tilde{T} = A \rtimes L/C_L(A)$. With a slight abuse of notation we will use also $\bar{x_1},\ldots,\bar{x_r}$ to denote their images in $L/C_L(A)$.

Corollary 2.5 *Consider the Lie ring $\tilde{T}$. For each simple product $\bar{u}$ in $\bar{x_1},\ldots,\bar{x_r},\bar{h}$ we have that there exists a positive integer $k(\bar{u})$ such that $y\bar{u}^{k(\bar{u})} = 0$ for all $y \in \tilde{T}$.*

Proof If $\bar{u}$ involves $\bar{h}$, then clearly $y\bar{u}^2 = 0$ for all $y \in \bar{T}$. Now suppose $\bar{u}$ is a Lie product in $\bar{x_1},\ldots,\bar{x_r}$. By Proposition 2.4 we know that there exists $k_1 = k_1(\bar{u})$ such $a\bar{u}^{k_1(\bar{u})} = 0$ for all $a \in A$. Now let $y \in L/C_L(A)$. For all $a \in A$, we have

$$a(y\bar{u}^{2k_1(\bar{u})-1}) = 0.$$

We thus see that $y\bar{u}^{k(\bar{u})} = 0$ for all $y \in \bar{T}$, where $k(\bar{u}) = \max\{2, k_1(\bar{u}), 2k_1(\bar{u}) - 1\}$. $\square$

Remark. Thus $\bar{T}$ satisfies the conditions of Proposition 2.2 and we know that $\bar{T}$ has a locally nilpotent radical $R(\bar{T})$ where $\bar{T}/R(\bar{T})$ has no non-trivial normal locally nilpotent ideal.

We now turn to showing that $AL^m = 0$ for some m. Again we are going to make use of the Baire category Theorem.

Consider the closed sets

$$T_n = \{(x, y) \in \bar{E} \times G : [x,_n y] = 1\},$$

$n = 1, 2, \ldots$. As $\bar{E}$ consists of right Engel elements we have

$$\overline{E} \times G = \bigcup_{n=1}^{\infty} T_n.$$

By the Baire category Theorem, at least one of the closed subsets, say T_{m_1}, contains an open neighbourhood. As $E \times D$ is dense in $\bar{E} \times G$, we get such a neighbourhood of the form $Ue \times Vd$ with $e \in E$ and $d \in D$. Thus

$$1 = [xe,_{m_1} yd]$$

for all $x \in U$ and $y \in V$. Let W be a neighbour hood of 1 in G such that $W^{m_1} \subseteq V$. By replacing V by $V \cap W \cap U$ we then have for all $v_0 \in V \cap \overline{E}$ and $v_1, \ldots, v_{m_1} \in V$ that

$$1 = [v_0 e,_{m_1} v_1 \cdots v_{m_1} d].$$

Using the fact that $[x,_{m_1} d] = 1$ and expanding using Hall's collection process, we get

$$1 = \prod_{\sigma \in \mathrm{S}_{m_1}} [v_0, v_{\sigma(1)}, \ldots, v_{\sigma(m_1)}] c \tag{4}$$

where c is a product of simple commutators in $v_0, \ldots, v_{m_1}, e, d$ of higher multiweight than $\prod_{\sigma \in \mathrm{S}_{m_1}} [v_0, v_{\sigma(1)}, \ldots, v_{\sigma(m_1)}]$. As a preparation for a consequence of this result we let $m_2 = 2m_1 - 1$ and we pick a neighbourhood $U(m_2)$ around 1 in G such that $U(m_2)^{4^{m_2}} \leq V$. We then replace V by $V \cap U(m_2)$.

We now turn to the consequence of (4). We first need to introduce some notation. For any positive integer n we let $\mathcal{C}_n = \{1, 2, \ldots, n\}$ and we let $\mathcal{P}(\mathcal{C}_n)$ be the powerset of $\mathcal{C}_n$. Then we let

$$\mathcal{R}_n = \{(S,T) \in \mathcal{P}(\mathcal{C}_n) \times \mathcal{P}(\mathcal{C}_n) : S \cup T = \{1, \ldots, n\} \text{ and } S \cap T = \emptyset\}.$$

We will use the following well known formula.

$$[z, [y, x_1, \ldots, x_n]] = (\prod_{(S,T)\in\mathcal{R}_n} [z, \overline{x}_T, y, x_S]^{(-1)^{|T|}}) \cdot c$$

where c is of higher multiweight than $(1, 1, \ldots, 1)$ in $z, y, x_1, \ldots, x_n$ and where, for $S = \{i_1, \ldots, i_k\}$, $i_1 < i_2 < \ldots < i_k$, and $T = \{j_1, \ldots, j_m\}$, $j_1 < j_2 < \ldots < j_m$, we let

$$[z, \bar{x}_T, y, x_S] = [z, x_{j_k}, x_{j_{k-1}}, \ldots, x_{j_1}, y, x_{i_1}, x_{i_2}, \ldots, x_{i_m}].$$

For each $\sigma \in \mathrm{S}_n$ we also use $[z, \bar{x}_T, y, x_S]^\sigma$ for

$$[z, x_{\sigma(j_k)}, x_{\sigma(j_{k-1})}, \ldots, x_{\sigma(j_1)}, y, x_{\sigma(i_1)}, \ldots, x_{\sigma(i_m)}].$$

We then have

$$\prod_{\sigma\in S_{m_2}} [v_0, [v, v_{\sigma(1)}, \cdots, v_{\sigma(m_2)}]] = (\prod_{(S,T)\in\mathcal{R}_{m_2}} \prod_{\sigma\in S_{m_2}} [v_0, \bar{v}_T, v, v_S]^{\sigma(-1)^{|T|}}) \cdot c_1$$

where c_1 is a product of commutators of higher multiweight than $(1, 1, \ldots, 1)$ in $v_0, v, v_1, \ldots, v_{m_2}$. As either $|T|$ or $|S|$ is greater than or equal to m_1 it then follows from (4) and our choice of V that

$$\prod_{\sigma\in S_{m_2}} [v_0, [v, v_{\sigma(1)}, \cdots, v_{\sigma(m_2)}]] = c_2 \tag{5}$$

for all $v_0 \in V \cap \bar{E}$ and all $v, v_1, \ldots, v_{m_2} \in V$ and where c_2 is a product of commutators of higher multiweight than $(1, 1, \ldots, 1, 0, 0)$ in $v_0, v_1, \ldots, v_{m_2}, e, d$.

Before stating next result, we will need some extra notion and for that we first need the following elementary lemma.

Lemma 2.6 *There exist finitely many $d_1, \ldots, d_t \in D$ such that*

$$G = \bigcup_{i=1}^{t} V d_i.$$

Proof. Let $V(2)$ be an open neighbourhood of 1 in G such that $V(2)^2 \subseteq V$. As G is compact we have

$$G = \bigcup_{i=1}^{t} V(2)g_i$$

for some $g_1, \dots, g_t \in G$. Since D is dense in G we get in fact $d_i \in V(2)g_i \cap D$ for $i = 1, \dots, t$. Hence

$$V(2)g_i \subseteq V(2)V(2)^{-1}d_i \subseteq Vd_i$$

and $G = \bigcup_{i=1}^{t} Vd_i$. $\Box$

For each $g \in G$ we can thus choose an element $d(g) \in \{d_1, \dots, d_t\}$ such that $g \in Vd(g)$. Let then

$$\Delta(g) = \{d([g, x_{i_1}, \dots, x_{i_s}]) : \, i_1, \dots, i_s \in \{1, \dots, r\} \text{ and } s \geq 0\} \subseteq \{d_1, \dots, d_t\}.$$

For each $[h, x_{i_1}, \dots, x_{i_s}]$, we let $\overline{[h, x_{i_1}, \dots, x_{i_s}]}$ be the corresponding Lie element of $\tilde{T}$ in $\bar{h}, \bar{x}_{i_1}, \dots, \bar{x}_{i_s}$. Let

$$I_j = \langle \overline{[h, x_{i_1}, \dots, x_{i_s}]} : \, |\Delta([h, x_{i_1}, \dots, x_{i_s}])| \leq j \rangle.$$

This gives us an ascending sequence of multigraded ideals in $\tilde{T}$

$$\{0\} = I_0 \leq I_1 \leq \cdots \leq I_t = A.$$

We will show that there exists a positive integer m_3 such that

$$I_j L^{m_3} \leq I_{j-1}, \; j = 1, \dots, t.$$

From this it then follows that $AL^m = \{0\}$ where $m = tm_3$.

We now consider the Lie ring

$$T_q = I_q/I_{q-1} \rtimes L/C_L(I_q/I_{q-1})$$

where the action from L on I_q/I_{q-1} is the natural induced action. The aim is now to show that T_q is nilpotent of class m_3 where m_3 only depends on E and D.

From now on we will be working in T_q and with a slight abuse of notation, in order to make it not overcomplicated, we will also use $\bar{x}_1, \ldots, \bar{x}_r$ for the images of these elements in $L/C_L(I_q/I_{q-1})$. For every commutator $[x_{i_1}, \ldots, x_{i_s}]$ we will use the notation $\overline{[x_{i_1}, \ldots, x_{i_s}]}$ to denote the corresponding simple Lie product $\bar{x}_{i_1} \cdots \bar{x}_{i_s}$ in T_q. For each integer i such that $1 \le i \le t$, we let

$$X_i = \langle \overline{[x_{i_1}, \ldots, x_{i_s}]} : |\Delta([x_{i_1}, \ldots, x_{i_s}])| \le i \rangle.$$

Let $I = I_q/I_{q-1}$. We then have a chain of ideals

$$\{0\} \le I + X_1 \le I + X_2 \le \cdots \le I + X_t = T_q$$

in T_q.

Lemma 2.7 *For $i = 2, \ldots, t$ we have that the Lie ring $M_i = (I + X_i)/(I + X_{i-1})$ satisfies the linearised identity*

$$\sum_{\sigma \in S_{m_2}} y y_{\sigma(1)} \cdots y_{\sigma(m_2)} = 0$$

for all $y, y_1, \ldots, y_{m_2} \in M_i$.

Proof Let $a = [h, x_{i_1}, \ldots, x_{i_s}] \in I_q$ and $y, y_1, \ldots, y_{m_2}$ be simple commutators in $x_1, \ldots, x_r$ such that $|\Delta(y)| \le i$ and $|\Delta(y_j)| \le i$ for $j = 1, \ldots, m_2$. Consider the corresponding Lie elements $\bar{a}, \bar{y}, \bar{y}_1, \ldots, \bar{y}_{m_2} \in M_i$. It suffices to show that

$$\sum_{\sigma \in \mathrm{S}_{m_2}} \bar{a}(\bar{y}\bar{y}_{\sigma(1)} \cdots \bar{y}_{\sigma(m_2)}) = 0. \tag{6}$$

Notice first that if $\Delta([a, x_1]), \ldots, \Delta([a, x_r]) \le q - 1$ then $\bar{a}\bar{x}_1 = \ldots = \bar{a}\bar{x}_r = 0$ and (6) clearly holds. We thus can assume that $\Delta([a, x_j]) = \Delta(a)$ for some $1 \le j \le r$. This means that $d(a) = d(b)$ for some $b = [a, x_j, x_{j_1}, \ldots, x_{j_s}]$ where $j_1, \ldots, j_s$ are some elements in $\{1, \ldots, r\}$. In particular $ab^{-1} \in V$ and b is of higher multiweight in $h, x_1, \ldots, x_r$ than a. Similarly if we have $\Delta([y_j, x_1]), \ldots, \Delta([y_j, x_r]) \le i - 1$, then $\bar{y}_j\bar{x}_1 = \ldots = \bar{y}_j\bar{x}_r = 0$ and thus clearly

$$\sum_{\sigma \in \mathrm{S}_{m_2}} \bar{y}\bar{y}_{\sigma(1)} \cdots \bar{y}_{\sigma(m_2)} = 0.$$

Thus by the same argument as above we can assume that we have a simple commutator z_j of higher multiweight than y_j in $x_1, \ldots, x_r$ such that

$y_jz_j^{-1} \in V$. Similarly we can assume that there exists a simple commutator z of higher multiweight than y such that $yz^{-1} \in V$. We can now apply (5) for $v_0 = ab^{-1}$, $v = yz^{-1}$, $v_1 = y_1z_1^{-1}, \ldots, v_{m_2} = y_{m_2}z_{m_2}^{-1}$ to see that (6) holds. $\square$

Now let m_3 be the largest of the integers $m_2, m_1, 3m_1 - 1$.

Lemma 2.8 *The algebra $M_1 = I + X_1$ satisfies the identity*

$$\sum_{\sigma(m)} yy_{\sigma(1)} \cdots y_{\sigma(m)} = 0$$

for all $y, y_1, \ldots, y_m \in M_1$

Proof The proof is similar to the proof of Lemma 2.7. We let $a = [h, x_{i_1}, \ldots, x_{i_s}]$ be such that $\bar{a} \in I_q$ and we let $y_1, \ldots, y_{m_1}$ be simple commutators in $x_1, \ldots, x_r$ such that $|\Delta(y_j)| = 1$ for $j = 1, \ldots, m_1$. We first show that

$$\sum_{\sigma \in \mathrm{S}_{m_1}} \bar{a}\bar{y}_{\sigma(1)} \cdots \bar{y}_{\sigma(m_1)} = 0. \tag{7}$$

As in the proof of Lemma 2.7 we can assume that there exists a simple commutator b of multiweight higher than a such that $ab^{-1} \in V$. For each $j = 1, \ldots, m_1$ we have $\Delta([y_j, x_1] = \Delta(y_j)$ (the value of Δ is this time minimal) and thus as before we get z_j of higher multiweight than y_j such that $y_jz_j^{-1} \in V$. We now apply (4) for $v_0 = ab^{-1}, v_1 = y_1z_1^{-1}, \ldots, v_{m_1} = y_{m_1}z_{m_1}^{-1}$ and we see that (7) holds. From this we can now finish the proof. Notice first that it follows from (7) that (recall that $m_2 = 2m_1 = 1$)

$$\sum_{\sigma \in \mathrm{S}_{m_2}} z(yy_{\sigma(1)} \cdots y_{\sigma(m_2)}) = 0$$

for all $z \in I$ and $y_1, \ldots, y_{m_2} \in L_1$. Thus

$$\sum_{\sigma \in \mathrm{S}_{m_2}} yy_{\sigma(1)} \cdots y_{\sigma(m_2)} = 0$$

for all $y, y_1, \ldots, y_{m_2} \in X_1$. Finally suppose that we take some $y \in X_1$ and $y_1, \ldots, y_{3m_1-1}$ be $b, z_1, \ldots, z_{3n-2}$ where $b \in I$ and $z_1, \ldots, z_{3n-2} \in X_1$. Then notice that for $m_3 = 3m_1 - 1$ we have

$$\sum_{\sigma \in \mathrm{S}_{m_3}} y y_{\sigma(1)} \cdots y_{\sigma(m_3)} = \sum_{k=0}^{3m_1-2} \sum_{\sigma \in \mathrm{S}_{3m_1-2}} y z_{\sigma(1)} \cdots z_{\sigma(k)} b z_{\sigma(k+1)} \cdots z_{\sigma(3m_1-2)}.$$

Notice that this is zero as in each summand either $k \geq m_2 = 2m_1 - 1$ or $(3m_1 - 2) - k \geq m_1$. This deals with all possible situations and thus the lemma follows. $\Box$

Take now some arbitrary $\bar{a} \in I$ and consider the subring $M = \langle \bar{a}, \bar{x}_1, \ldots, \bar{x}_r \rangle$ of T_q. Notice that M satisfies condition (b) of Zel'manov's Theorem where $k(\bar{u})$ is like in Corollary 2.5. The reader will readily convince himself that the same values of k that worked for $\tilde{T}$ work in T_q. Replacing I by the ideal in M generated by $\bar{a}$, notice also that sections $M_1, \ldots, M_t$ of M satisfy the linearised m_3-Engel identity. It thus follows by Zelmanov's Theorem that $M_1, \ldots, M_t$ are nilpotent. By Proposition 2.2 we then have that M is nilpotent. Notice also that m_3 and the values of k do not depend on $\bar{a}$ but only E and D. Furthermore what the ideals $I + X_j$, $j = 1, \ldots, t$, thought of as ideals consisting of elements in variables $\bar{a}, \bar{x}_1, \ldots, \bar{x}_r$, are does also only depend on E and D. Thus we see that the class of M only depends on E and D. If this class is m_4 then we get in particular that $\bar{a}L^{m_4} = 0$ that implies that

$$I_q L^{m_4} \leq I_{q-1}$$

as we wanted to see. Thus $AL^m = 0$ where $m = tm_4$. This finishes the proof of our Theorem.

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