The local nilpotence theorem for 4-Engel groups revisited

Gunnar Traustason
Department of Mathematical Sciences
University of Bath
Bath BA2 7AY, UK

Abstract

The proof of the local nilpotence theorem for 4-Engel groups was completed by G. Havas and M. Vaughan-Lee in 2005. The complete proof on the other hand is spread over several articles and the aim of this paper is to give a complete coherent linear version. In the process we are also able to make a few simplifications and in particular we are able to merge two of the key steps into one.

0 Introduction and the outline of the proof

The aim of this article is to present a complete proof of the local nilpotence theorem for 4-Engel groups in one paper. The proof is long and technical and is spread over a number of articles and we aim here to give a coherent linear version. In the process, we are able to make a few simplifications and in particular, two main steps have been merged into one. Unlike the existing version, the account here does not depend on machine calculations and the nilpotent quotient algorithm. All the needed hand calculations, most of these very short although they are quite numerous, are given in appendices. In this way the reader has a choice whether he wishes to follow the calculations by hand or do them by a machine.

The outline of the paper is as follows. In Section 1 we include some preliminary material that will be needed later on. In particular we prove there the well known fact that the variety of n-Engel groups has the radical property with respect to the Hirsch-Plotkin radical. This is to say that if G is a n-Engel group and $R(G)$ is the Hirsch-Plotkin radical of G then the Hirsch-Plotkin radical of the quotient $G/R(G)$ is trivial. This will be crucial later on. We also show that to prove that a 4-Engel group is locally nilpotent, it suffices to show that all 3-generator subgroups are nilpotent. The largest nilpotent 3-generator 4-Engel group has been computed by W. Nickel [7] and it is quite huge with nilpotence class 9 and Hirsch length 88. To tackle the 3-generator groups therefore seems a daunting task. In fact the proof avoids this and instead works through establishing the nilpotence of much smaller groups: the free 2-generator 4-Engel group, a certain specific 3-generator group of class 4 and the subgroups generated by 3-conjugates.

In Section 2 we deal with the first step. We show that any subgroup of a 4-Engel group, that is generated by two conjugates, is nilpotent. This was first proved by the author for torsion groups [8] and then later for torsion-free groups [5] by P. Longobardi and M. Maj. The general result was established by Vaughan-Lee who gave a computer proof, using the Knuth-Bendix algorithm, (unpublished). We will present a machine free proof [9] that appeared around the same time. Building on this result we show that the 2-elements in any 4-Engel group form a locally finite subgroup. We also show that in any 4-Engel p-group, G^p is locally finite. As groups of exponent 3 are locally finite, it follows in particular that any 4-Engel 3-group is locally finite.

In Section 3 we build on Section 2 to deal with one of the main ingredients of the proof by showing that any 2-generator 4-Engel group is nilpotent [10]. The structure of the free 2-generator group can then be determined using the nilpotent quotient algorithm (see for example [7]). We obtain all the needed facts by hand calculations in Appendix A. One should note here one of the many subtleties of the proof. We do not deduce the fact that the 2-elements form a subgroup of a 4-Engel group by showing first that all 2-generator subgroups are nilpotent. On the contrary the proof of the nilpotence of the 2-generator groups depends on the previously established fact that the 2-elements form a subgroup N. Then G/N is a 2-generator group without elements of order 2 and this fact we need in showing that G/N is nilpotent.

One immediate corollary of the nilpotence of 2-generator groups is that the torsion-elements of any 4-Engel group G form a subgroup that is a direct product of p-groups. Now let $R(G)$ be the Hirsch-Plotkin radical of G. To show that G is locally nilpotent is equivalent to show that $E = G/R(G)$ is trivial and as G has the radical property with respect to the Hirsch-Plotkin radical, it suffices to show that E is locally nilpotent. As the torsion subgroup of E is a direct product of p-groups, and as the 4-Engel 2-groups and the 4-Engel 3-groups are locally finite, we have that E is $\{2, 3\}$-torsion free. This is a property that the following sections depend heavily on.

In Section 4 we deal with a special 3-generator 4-Engel group $T = \langle a, b, c\rangle$. T is the largest 3-generator 4-Engel group where a and b commute and where the subgroups $\langle a, c\rangle$, $\langle b, c\rangle$ are nilpotent of classes 2 and 3 respectively. The nilpotence of this group was first established by G. Havas and M. Vaughan-Lee [2]. Their proof is a computer proof using the Knuth-Bendix algorithm. The proof that we present here and appeared at the same time is a short machine free proof from [11]. The structure is then further determined in Appendix B when T is $\{2, 3\}$-torsion free.

Let G be a 4-Engel group and let $E = G/R(G)$ where $R(G)$ is the Hirsch-Plotkin radical. The remainder of the paper is about establishing that E is locally nilpotent. In Section 5, we show that all subgroups of E generated by three conjugates are nilpotent of class at most 5. We are here able to merge into one, two main steps from [13] and [2]. The first provided by M. Vaughan-Lee in 1997 and the 2nd by G. Havas and M. Vaughan-Lee in 2005. The first paper dealt with groups of exponent 5 and the latter with $\{2, 3, 5\}$-torsion free groups. Our proof uses mostly arguments from these two papers although our version does not rely on the nilpotent quotient algorithm and all the relevant calculations are done by hand in Appendix C.

In Section 6, we show that for any $a, x \in E$ we have that $[a, x, x, x]$ is a left 3-Engel element in E. Our proof here is a simplified version of the proof given in [2] and [13].

In Section 7, we then show that all the left 3-Engel elements of E belong to the Hirsch-Plotkin radical [2,13]. As the Hirsch-Plotkin radical of E is trivial, it follows from Section 6 that E is a 3-Engel group and thus nilpotent by a well know result of Heineken [3] (see also [6] and [12] for shorter proofs). Hence $E = G/R(G)$ is trivial and thus $G = R(G)$ is locally nilpotent.

1 Some preliminaries

Definition A group G is said to be *restrained* if

$$\langle a\rangle^{\langle b\rangle}$$

is finitely generated for all $a, b \in G$.

Notice that in every n-Engel group $\langle a\rangle^{\langle b\rangle}$ is generated by $a, a^b, a^{b^2}, \ldots a^{b^{n-1}}$ so every n-Engel group is restrained. This property was introduced by Kim and Rhemtulla [4]. The next lemma appeared in their paper. We include the short proof to aid the reader.

Lemma 1.1 *Let G be a finitely generated restrained group. If H is a normal subgroup of G such that G/H is cyclic, then H is finitely generated.*

Proof As G/H is cyclic, we have that $G = \langle H, g\rangle$ for some $g \in G$. As G is finitely generated we then have $G = \langle h_1, \ldots, h_r, g\rangle$ with $h_1, \ldots, h_r \in H$. Then

$$H = \langle h_1, \ldots, h_r\rangle^G \cdot \langle g^m\rangle$$

where m is the order of gH in G/H. So H is generated by g^m and

$$\langle h_1\rangle^{\langle g\rangle} \cup \cdots \cup \langle h_r\rangle^{\langle g\rangle}.$$

As G is restrained each subset in this union is finitely generated. Hence H is finitely generated. $\Box$

From this we get the following easy corollary.

Lemma 1.2 *Let G be a finitely generated restrained group. Then G' is finitely generated.*

Proof As G/G' is polycyclic, this follows from Lemma 1.1. $\Box$

Lemma 1.3 *Let G be an n-Engel group and let R be the Hirsch-Plotkin radical of G. Then the Hirsch-Plotkin radical of G/R is trivial.*

Proof Let S/R be the Hirsch-Plotkin radical of G/R. It suffices to show that S is locally nilpotent. Let H be a finitely generated subgroup of S. Then $H/(H \cap R) \cong HR/R$ is nilpotent and thus solvable of derived length, say m. As H is restrained we have by Lemma 1.2 that $H^{(m)}$ is a finitely generated subgroup of $H \cap R$ and thus nilpotent. Therefore H is solvable and thus nilpotent by a well known theorem of Gruenberg [1]. $\Box$

The next result turns out to be not needed but we include it here as it relevant in motivating the approach taken.

Proposition 1.4 *A 4-Engel group is locally nilpotent if and only if all its 3-generator subgroups are nilpotent.*

Proof One inclusion is obvious. Suppose now that G is a 4-Engel group with all its 3-generator subgroups nilpotent. Let $a, b, c \in G$. By our assumption $H = \langle a, b, c \rangle$ is nilpotent and one can read from a polycyclic presentation of the free nilpotent 3-generator 4-Engel group that $[[a, b, b, b], [a, b, b, b]^c] = 1$. It follows that $\langle [a, b, b, b] \rangle^G$ is abelian for all $a, b \in G$ and thus contained in the Hirsch-Plotkin radical R of G. It follows that G/R is a 3-Engel group and thus locally nilpotent by Heineken's Theorem. But by Lemma 1.3, the Hirsch-Plotkin radical of G/R is trivial. Hence $G/R = \{1\}$ and $G = R$ and thus locally nilpotent. $\Box$

2 Subgroups generated by two conjugates

In this section we prove the following main step.

Proposition 2.1 *Let G be a 4-Engel group and let $a, b \in G$. Then $\langle a, a^b \rangle$ is metabelian and nilpotent of class at most 4.*

Before we get into the proof we make a short useful remark about n-Engel groups. As

$$[[[[y, \underbrace{x], x] \cdots], x]}_{n} = [\underbrace{x^{-1}, [\cdots [x^{-1}, [x^{-1}}_{n}, y]]]]^{x^n},$$

it does not matter whether we use bracketing from the right or from the left in the definition. We will use both in the following calculations.

Lemma 2.2 *We have that $[a, a^b]$ and $[a, a^b]^{aa^b}$ commute with $[a, a^b]^a$ and $[a, a^b]^{a^b}$.*

Proof We have

$$\begin{aligned}
1 &= [a,[a,[a,[a,b]]]] \\
&= [a,([a,[a,b]]^{-a}\cdot[a,[a,b]])] \\
&= [a,[a,b]]^{-a}\cdot[a,[a,b]]^{a^2}\cdot[a,[a,b]]^{-a}\cdot[a,[a,b]] \\
&= [a,b]^{-a}[a,b]^{a^2}[a,b]^{-a^3}[a,b]^{a^2}[a,b]^{-a}[a,b]^{a^2}[a,b]^{-a}[a,b]
\end{aligned}$$

which implies that

$$[a,[a,b]]^{a^2}=[a,[a,b]]^{a}\cdot[a,[a,b]]^{-1}\cdot[a,[a,b]]^{a} \tag{1}$$
$$[a,b]^{a^3}=[a,b]^{a^2}[a,b]^{-a}[a,b]^{a^2}[a,b]^{-a}[a,b][a,b]^{-a}[a,b]^{a^2}. \tag{2}$$

Also

$$\begin{aligned}
1 &= [[[[b,a],a],a],a] \\
&= [[a,b][a,b]^{-a},a,a] \\
&= [[a,b]^{a}[a,b]^{-1}[a,b]^{a}[a,b]^{-a^2},a] \\
&= [a,b]^{a^2}[a,b]^{-a}[a,b][a,b]^{-a}[a,b]^{a^2}[a,b]^{-a}[a,b]^{a^2}[a,b]^{-a^3}
\end{aligned}$$

and therefore

$$[a,b]^{a^3}=[a,b]^{a^2}[a,b]^{-a}[a,b][a,b]^{-a}[a,b]^{a^2}[a,b]^{-a}[a,b]^{a^2}. \tag{3}$$

From (2) and (3) we have

$$[a,b]^{a^2}[a,b]^{-a}[a,b]=[a,b][a,b]^{-a}[a,b]^{a^2}. \tag{4}$$

But it is easy to see that this is equivalent to

$$[a,[a,b]]^{a}\cdot[a,[a,b]]=[a,[a,b]]\cdot[a,[a,b]]^{a}$$

which can also be written

$$[a,a^b]^{a}\cdot[a,a^b]=[a,a^b]\cdot[a,a^b]^{a}. \tag{5}$$

By symmetry we also have that $[a,a^b]$ commutes with $[a,a^b]^{a^b}$. It follows from this that $[a,a^b]^{a^b}$ commutes with $[a,a^b]^{aa^b}$ and that $[a,a^b]^{a}$ commutes with $[a,a^b]^{a^b a[a,a^b]}=[a,a^b]^{aa^b}$. $\square$

Lemma 2.3 *$\langle a, a^b\rangle'$ is generated by $[a, a^b]$, $[a, a^b]^a$, $[a, a^b]^{a^b}$ and $[a, a^b]^{aa^b}$.*

Proof Since $\langle a, a^b\rangle'$ is the normal closure of $[a, a^b]$ in $\langle a, a^b\rangle$, it is sufficient to show that the group generated by $[a, a^b]$, $[a, a^b]^a$, $[a, a^b]^{a^b}$ and $[a, a^b]^{aa^b}$ is normal in $\langle a, a^b\rangle$. From (1) and (5) we have $[a, a^b]^{a^2} = [a, a^b]^{2a}[a, a^b]^{-1}$ and then also $[a, a^b]^{a^{-1}} = [a, a^b]^{-a+2}$. By symmetry we have as well $[a, a^b]^{a^{2b}} = [a, a^b]^{2a^b}$ and $[a, a^b]^{a^{-b}} = [a, a^b]^{-a^b+2}$. We also have the following relations.

$$\begin{aligned}
[a, a^b]^{aa^{-b}} &= [a, a^{-b}]^{-1}[a, a^b]^{a^{-b}a}[a, a^{-b}] \\
&= [a, a^b]^{a^{-b}}[a, a^b]^{-a^b a}[a, a^b]^{2a}[a, a^b]^{-a^{-b}} \\
&= [a, a^b]^{a^{-b}}[a, a^b][a, a^b]^{-aa^b}[a, a^b]^{-1}[a, a^b]^{2a}[a, a^b]^{-a^{-b}},
\end{aligned}$$

$$[a, a^b]^{a^b a} = [a, a^b][a, a^b]^{aa^b}[a, a^b]^{-1},$$

$$\begin{aligned}
[a, a^b]^{a^b a^{-1}} &= [a^{-1}, a^b][a, a^b]^{a^{-1}a^b}[a^{-1}, a^b]^{-1} \\
&= [a, a^b]^{-a^{-1}}[a, a^b]^{-aa^b}[a, a^b]^{2a^b}[a, a^b]^{a^{-1}},
\end{aligned}$$

$$\begin{aligned}
[a, a^b]^{aa^b a} &= [a, a^b][a, a^b]^{a^2a^b}[a, a^b]^{-1} \\
&= [a, a^b][a, a^b]^{2aa^b}[a, a^b]^{-a^b}[a, a^b]^{-1},
\end{aligned}$$

$$\begin{aligned}
[a, a^b]^{aa^b a^{-1}} &= [a^{-1}, a^b][a, a^b]^{a^b}[a^{-1}, a^b]^{-1} \\
&= [a, a^b]^{-a^{-1}}[a, a^b]^{a^b}[a, a^b]^{a^{-1}},
\end{aligned}$$

$$\begin{aligned}
[a, a^b]^{aa^b a^b} &= [a, a^{2b}]^{-1}[a, a^b]^{a^{2b}a}[a, a^{2b}] \\
&= [a, a^b]^{-a^b-1}[a, a^b]^{2a^b a}[a, a^b]^{-a}[a, a^b]^{1+a^b} \\
&= [a, a^b]^{-a^b}[a, a^b]^{2aa^b}[a, a^b]^{-1}[a, a^b]^{-a}[a, a^b]^{1+a^b}.
\end{aligned}$$

From these equalities it is clear that $\langle [a, a^b], [a, a^b]^a, [a, a^b]^{a^b}, [a, a^b]^{aa^b}\rangle$ is normal in $\langle a, a^b\rangle$. $\square$

Let $x = [a, a^b]$ and $u = [x^a, x^{a^b}]$.

Lemma 2.4 *We have that $[x^a, x^{a^b}] = [x^{aa^b}, x]$. The group $\langle a, a^b\rangle''$ is cyclic generated by u.*

Proof It follows from Lemmas 2.2 and 2.3 that every element in $\langle a, a^b\rangle'$ can be written in the form tz with $t \in \langle x^{aa^b}, x\rangle$ and $z \in \langle x^a, x^{a^b}\rangle$, since Lemma 2.2 tells us that the elements in $\langle x^{aa^b}, x\rangle$ commute with the elements in $\langle x^a, x^{a^b}\rangle$. This fact will sometimes be used in the following calculations without mention.

Let $y = [a, [a, bx]]$. From Lemma 2.2, we have that y commutes with y^a. It then follows from the 4-Engel law that

$$1 = [a, [a, y]] = y^{a^2} y^{-a} y y^{-a}.$$

We next expand y.

$$\begin{aligned}
y &= [a, ([a, x][a, b][[a, b], x])] \\
&= [a, (x^{-a+1}[a, b]x^{-a^{-1}a^b+1})] \\
&= x^{-a}x^{a^{-1}a^b a}[a, b]^{-a}x^{-a+a^2}x^{-a+1}[a, b]x^{-a^{-1}a^b+1} \\
&= x^{-a}x^{a^b[a,a^b]^{-1}}[a, [a, b]]x^{(a-2)a^b+1} \\
&= x^{-a+a^b+1+aa^b-2a^b+1} \\
&= x^{-a+1+aa^b-a^b+1}.
\end{aligned}$$

We then have

$$\begin{aligned}
y^{a^2} &= x^{-a^3+a^2+aa^ba^2-a^ba^2+a^2} \\
&= x^{-a^3+a^2+(a-1)a^2a^b[a^2,a^b]^{-1}+a^2} \\
&= x^{-a^3+a^2+(a^3-a^2)a^b[a,a^b]^{-(a+1)}+a^2} \\
&= x^{2+(a-1)a^b+a-2},
\end{aligned}$$

and

$$\begin{aligned}
y^{-a} &= x^{-a+a^ba-aa^ba-a+a^2} \\
&= x^{-a+1+aa^b-a^2a^b-1-a+a^2} \\
&= x^{-a+1+(-a+1)a^b+a-2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
1 &= y^{a^2-a+1-a} \\
&= x^{2+aa^b-a^b+a-2}x^{-a+1-aa^b+a^b+a-2}
\end{aligned}$$

$$\begin{aligned}
& x^{-a+1+aa^b-a^b+1}x^{-a+1-aa^b+a^b+a-2} \\
= \; & x^{2+aa^b-2+1-aa^b-2+1+aa^b+1+1-aa^b-2} \\
& x^{-a^b+a-a+a^b+a-a-a^b-a+a^b+a} \\
= \; & x^{2+aa^b-1}x^{-aa^b-1+aa^b+1}x^{-a^b-a+a^b+a}x^{1-aa^b-2}.
\end{aligned}$$

Conjugation with x^{2+aa^b-1} gives

$$1 = [x^{aa^b}, x][x^{a^b}, x^a].$$

Therefore $[x^a, x^{a^b}] = [x^{aa^b}, x]$ and by Lemma 2.2 and Lemma 2.3 we have that $\langle a, a^b\rangle''$ is the normal closure of u in $\langle a, a^b\rangle'$. Since $u = [x^{aa^b}, x] = [x^{a^b}, x^a]$. We have by Lemma 2.2 that u commutes with all elements in $\langle a, a^b\rangle'$ and thus $\langle a, a^b\rangle'' = \langle u\rangle$. $\Box$

Proof of Proposition 2.1 By Lemma 2.4 we have that $\langle a, a^b\rangle$ is soluble. By Gruenberg's Theorem it thus follows that $\langle a, a^b\rangle$ is nilpotent. To show that the class is 4, we can assume that $\gamma_6(\langle a, a^b\rangle) = \{1\}$. As $[a, a^b]$ commutes with $[a, a^b]^a$, we have

$$1 = [a^b, a, a, [a^b, a]] = [a^b, a, a, a^b, a]$$

and by symmetry, $[a^b, a, a, a^b]$ commutes with a^b. From the 4-Engel identity we know that $\gamma_4(\langle a, a^b\rangle) = \langle [a^b, a, a, a^b]\rangle\gamma_5(\langle a, a^b\rangle)$. Thus $\gamma_5(\langle a, a^b\rangle$ is generated by $[a^b, a, a, a^b, a]$ and $[a^b, a, a, a^b, a^b]$. As we have seen that both these are trivial, it follows that the class of $\langle a, a^b\rangle$ is at most 4. In particular it follows that $\langle a, a^b\rangle$ is metabelian. $\Box$

We use Proposition 2.1 to prove two corollaries that will be essential later on. First we need a lemma.

Lemma 2.5 *Let G be a 4-Engel group. Suppose that $a^{p^i} = 1$ where p is a prime and either $i \geq 2$ and p is odd or $i \geq 3$ and $p = 2$. Then*

$$[a^{p^{i-1}}, a^{p^{i-1}b}] = 1$$

for all $b \in G$.

Proof First we deal with the case when $p = 2$. From the Engel-4 identity we have that $[a, a^b]^{(a-1)^2} = 1$. It follows that

$$[a, a^b]^{(a^m-1)^2} = 1$$

for all $m \in \mathbb{N}$ since $(a-1)^2|(a^m-1)^2$ in $\mathbb{Z}\langle a\rangle$. (We are repeatedly using the fact that $\langle a, a^b\rangle$ is metabelian). It follows that

$$[a, a^b]^{1+a^{2m}} = [a, a^b]^{1+(2a^m-1)} = [a, a^b]^{2a^m} \tag{6}$$

for all $m \in \mathbb{N}$. Let $m = 2^{i-3}$. Since $a^{8m} = 1$, we have

$$\begin{aligned} 1 &= [a^{8m}, a^b] \\ &= [a^{4m}, a^b]^{a^{4m}+1} \\ &= [a^{4m}, a^b]^{2a^{2m}} \qquad (by(6)). \end{aligned}$$

This implies that $[a^{4m}, a^b]^2 = 1$. But then

$$\begin{aligned} [a^{4m}, [a^{4m}, b]] &= [a^{4m}, a^{4mb}] \\ &= [a^{4m}, a^{2mb}]^{1+a^{2mb}} \\ &= [a^{4m}, a^{2mb}]^{2a^{mb}} \qquad \text{(by (6))} \\ &= [a^{4m}, a^b]^{2a^{mb}(1+a^b+\cdots+a^{(2m-1)b})} \\ &= 1. \end{aligned}$$

Now we deal with the odd case. Let $m = p^{i-1}$ and $q = pm$. From the 4-Engel identity we have

$$1 = [a^q, [a, a^b]] = [a, [a, a^b]]^q = [a, a^b]^{-qa+q}. \tag{7}$$

Then, using the fact that $[a, a^b]^{a^2} = [a, a^b]^{2a-1}$, we have

$$\begin{aligned} 1 &= [a^q, a^b] \\ &= [a, a^b]^{a^{q-1}+a^{q-2}+\cdots+a+1} \\ &= [a, a^b]^{(q-1)a-(q-2)+(q-2)a-(q-3)+\cdots+2a-1+a+1} \\ &= [a, a^b]^{\frac{q(q-1)}{2}a-q(\frac{q-1}{2}-1)} \\ &= [a, a^b]^q \qquad \text{(by (7) since } q \text{ is odd.)} \end{aligned}$$

It follows that

$$\begin{aligned} [a^m, [a^m, b]] &= [a^m, a^{mb}] \\ &= [a, a^b]^{(a^{m-1}+\cdots+1)(a^{(m-1)b}+\cdots+1)} \\ &= [a, a^b]^{(\frac{m(m-1)}{2}a-m(\frac{m-1}{2}-1))(\frac{m(m-1)}{2}a^b-m(\frac{m-1}{2}-1))}. \end{aligned}$$

From above we have that $[a, a^b]^{pm} = 1$. Since pm divides m^2 it follows from the calculations above that $[a^m, [a^m, b]] = 1$. $\Box$

From this we can deduce the following main ingredient towards the proof of the local nilpotence result.

Proposition 2.6 *Let G be a 4-Engel p-group. If $p = 2$ or $p = 3$ then G is locally finite. For $p \geq 5$ we have that G^p is locally finite.*

Proof Let R be the Hirsch-Plotkin radical of G and consider $H = G/R$. We need to show that $H^p = \{1\}$. First assume that p is an odd prime number. Let $a \in G$. We claim that the order of $\bar{a} = aR \in H$ divides p. We argue by contradiction and suppose that $\bar{a}$ has order p^i for some $i \geq 2$. By Lemma 2.5 $\langle \bar{a}^{p^{i-1}} \rangle^H$ is abelian and thus contained in the Hirsch-Plotkin radical of H. But we know from Lemma 1.3 that this radical is trivial. Hence we get the contradiction that $\bar{a}^{p^{i-1}} = 1$. In the case when $p = 3$ we know that groups of exponent 3 are locally finite so we conclude, using Lemma 1.3, that all 4-Engel 3-groups are locally finite. This leaves us with the case $p = 2$. Arguing in the same manner as for the odd case we see from the last lemma that G/R has exponent dividing 4. As groups of exponent 4 are known to be locally finite it follows from this and Lemma 1.3 that G is locally finite. $\Box$

Proposition 2.7 *Let G be a 4-Engel group. The 2-elements form a subgroup of G.*

Proof Suppose that $a^{2^i} = 1$. We then have

$$(ab)^{2^i} = a^{-2^i}(ab)^{2i} = b^{a^{2^i-1}} b^{a^{2^i}-2} \cdots b^a b.$$

Since $\langle u, u^x \rangle$ is nilpotent for all $u, x \in G$, we have that $u^x u$ is a 2-element whenever u is a 2-element. Therefore $b^a b$ is a 2-element and then also $b^{a^3} b^{a^2} b^a b = (b^a b)^{a^2}(b^a b)$. By induction we get that

$$b^{a^{2^i-1}} b^{a^{2^i-2}} \cdots b^a b = (ab)^{2^i}$$

is a 2-element and hence ab is a 2-element. $\Box$

3 4-Engel groups of rank 2

In this section we prove.

Proposition 3.1 *All 2-generator 4-Engel groups are nilpotent.*

Let $G = \langle x, y\rangle$ be a 2-generator 4-Engel group. We want to show that G is nilpotent. If R is the Hirsch-Plotkin radical of G this is the same as proving that G/R is trivial. By Lemma 1.3 we know that the Hirsch-Plotkin radical of G/R is trivial. By replacing G by G/R we can thus without loss of generality assume that G has a trivial Hirsch-Plotkin radical. It follows by Gruenberg's Theorem that there are no normal locally-solvable subgroups. We prove in few steps that G must then be trivial. By the fact, established in last section, that the 2-elements form a locally finite subgroup, we know that G has no elements of order 2.

We will make use of Proposition 2.1 that tells us that a subgroup generated by two conjugates is nilpotent. In particular the subgroup $\langle xy^{-1}, y^{-1}x\rangle$ of G is nilpotent. The aim is to show that this subgroup is trivial and as it is nilpotent it suffices to show that the centre is trivial. Let a be an arbitrary element of the centre. Then $a^x = a^y$ and $a^{x^{-1}} = a^{y^{-1}}$. If we let $c = a^{x^{-1}}$ then $c^x = c^y = a$ and $c^{x^2} = c^{y^2}$.

Lemma 3.2 *The subgroups $\langle x, x^c\rangle$ and $\langle y, y^c\rangle$ are nilpotent of class at most 3 and furthermore $[[c, x, x], [c, x]] = [[c, y, y], [c, y]] = 1$.*

Proof From Proposition 2.1 we know that the subgroups $\langle x, x^c\rangle$ and $\langle y, y^c\rangle$ are nilpotent of class at most 4. We also know that any commutator in x and x^c with three occurrences of either x or x^c must be trivial. From this and the choice of the element c we have that $[[c, x, x], [c, x], [c, x]] = [[c, y, y], [c, y], [c, y]]$ and that this element commutes with x and y and is thus in the centre of G. But then it must be the identity. Expanding we get

$$\begin{aligned} 1 &= [x^{-c}x, x, x^{-c}x, x^{-c}x] \\ &= [x^{-c}, x, x^{-c}, x][x^{-c}, x, x, x^{-c}] \\ &= [x^c, x, x^c, x]^2. \end{aligned}$$

As G has no element of order 2 it follows that $\langle x, x^c\rangle$ and likewise $\langle y, y^c\rangle$ are nilpotent of class at most 3. Next consider the element $[[c, x, x], [c, x]] = [[c, y, y], [c, y]]$. From what we have just proved this element commutes with both x and y and is thus in the centre of G and must therefore be the identity. $\square$

Lemma 3.3 *The subgroups $\langle cx, xc\rangle$ and $\langle cy, yc\rangle$ are nilpotent of class at most 3. Furthermore $[x,c]^{x^{-1}}$ commutes with all elements in $\langle [x,c]\rangle^{\langle c\rangle}$.*

Proof Let $d = [xc, cx, (cx)^{-1}xc, (cx)^{-1}xc]$. As the conjugates xc, cx generate a subgroup that is nilpotent of class at most 4 we know that this element commutes with xc and cx. By Lemma 3.2 we know that $[x,c,x]$ commutes with $[x,c]$. Thus

$$\begin{aligned} d &= [cx[x,c], cx, [x,c], [x,c]] \\ &= [[x,c,cx,[x,c],[x,c]] \\ &= [[x,c,x][x,c,c]^x, [x,c],[x,c]] \\ &= [[x,c,c]^x, [x,c], [x,c]]. \end{aligned}$$

But this element is in $\langle c, c^x, c^{x^2}\rangle$ and by our choice of c it follows that

$$d = [[y,c,c]^y, [y,c],[y,c]] = [yc, cy, (cy)^{-1}yc, (cy)^{-1}yc]$$

and d commutes with xc, cx, yc and cy. Thus

$$d^x = d^y = d^{c^{-1}} \text{ and } d^{x^{-1}} = d^{y^{-1}} = d^c.$$

Then also

$$d^{x^2} = d^{c^{-1}x} = d^{xc^{-x}} = d^{yc^{-y}} = d^{c^{-1}y} = d^{y^2}.$$

As $\langle d\rangle^{\langle x\rangle}$ and $\langle d\rangle^{\langle y\rangle}$ are generated by $\langle d^{x^{-1}}, d, d^x, d^{x^2}\rangle$ and $\langle d^{y^{-1}}, d, d^y, d^{y^2}\rangle$ respectively, it follows that

$$d^{x^i} = d^{y^i}$$

for all $i \in Z$. In particular $[d, z_1, z_2, z_3, z_4] = [d, x, x, x, x] = 1$ for all $z_1, z_2, z_3, z_4 \in \langle x, y\rangle$ and d is in the 4-th center of G. Hence $d = 1$. As in the proof of Lemma 3.2, we use the fact that every commutator in the conjugates xc and cx with 3 occurrences of either xc or cx is trivial. It follows that

$$1 = d = [xc, cx, (cx)^{-1}, xc][xc, cx, xc, (cx)^{-1}] = [xc, cx, cx, xc]^{-2},$$

and as there are no elements of order 2 we conclude that $\langle xc, cx\rangle$ and likewise $\langle yc, cy\rangle$ are nilpotent of class at most 3. Next consider the element

$$\begin{aligned} e &= [xc, cx, (cx)^{-1}xc] \\ &= [x, c, cx, [x,c]] \end{aligned}$$

$$
\begin{aligned}
&= [[x,c,x][x,c,c]^x,[x,c]] \\
&= [[x,c,c]^x,[x,c]] \\
&= [[y,c,c]^y,[y,c]] \\
&= [yc,cy,(cy)^{-1}yc].
\end{aligned}
$$

By what we have just seen, this element commutes with xc, cx, yc, cy and the same argument as we used previously shows that $e = 1$. So $[x,c]$ commutes with $[x,c,c]^x$, that is to say $[x,c]^{-x}[x,c]^{cx}$ and by Lemma 3.2, $[x,c]$ commutes then with $[x,c]^{cx}$ or equivalently $[x,c]^c$ commutes with $[x,c]^{x^{-1}}$. Replacing c by c^{-1} it follows that $[x,c^{-1}]^{c^{-1}}$ commutes with $[x,c^{-1}]^{x^{-1}}$ or equivalently $[c,x]^{c^{-2}}$ commutes with $[c,x]^{c^{-1}x^{-1}}$. Conjugating by cx we get that $[c,x]^{c^{-1}x}$ commutes with $[c,x]^{[c,x]} = [c,x]$ and $[x,c]^{x^{-1}}$ commutes with $[x,c]^{c^{-1}}$. From this and Lemma 3.2 we now know that $[x,c]^{x^{-1}}$ commutes with $[x,c]$, $[x,c]^c$ and $[x,c]^{c^{-1}}$. Let $u = [c,x]^{c^{-1}}$ and $v = [c,x]^{x^{-1}}$. Then v commutes with u, $[u,c]$ and $[u,c,c]$ and thus

$$
\begin{aligned}
1 &= [u,vc,vc,vc,vc] \\
&= [u,c,vc,vc,vc] \\
&= [u,c,c,vc,vc] \\
&= [u,c,c,c,vc] \\
&= [u,c,c,c,v]^c.
\end{aligned}
$$

As $\langle [x,c]\rangle^{\langle c\rangle}$ is generated by u, u^c, u^{c^2} and u^{c^3}, this concludes the proof of the lemma. □

Lemma 3.4 $Z(\langle c, c^x, c^{x^2}\rangle) = \{1\}$.

Proof First notice that if $u \in C_G(\langle c, c^x, c^{x^2}\rangle)$ then

$$
\begin{aligned}
1 &= [c,ux,ux,ux,ux] = [c,x,ux,ux,ux] = [c,x,x,ux,ux] \\
&= [c,x,x,x,ux] = [c,x,x,x,u]^x
\end{aligned}
$$

and thus u commutes with c^{x^3}. As $\langle c\rangle^{\langle x\rangle}$ is generated by c, c^x, c^{x^2}, c^{x^3} this shows that $C_G(\langle c, c^x, c^{x^2}\rangle) = C_G(\langle c\rangle^{\langle x\rangle})$ and similarly $C_G(\langle c, c^y, c^{y^2}\rangle) = C_G(\langle c\rangle^{\langle y\rangle})$. As $c^x = c^y$ and $c^{x^2} = c^{y^2}$ it follows that

$$
C_G(\langle c, c^x, c^{x^2}\rangle) = C_G(\langle c\rangle^{\langle x\rangle}) = C_G(\langle c\rangle^{\langle y\rangle})
$$

which is normalised by both x and y and therefore normal in G. Thus

$$C_G(c, c^x, c^{x^2}) = C_G(\langle c\rangle^G).$$

In particular $Z(\langle c, c^x, c^{x^2}\rangle) \leq Z(\langle c\rangle)^G)$. As the latter is an abelian normal subgroup of G it must be trivial. $\Box$

Lemma 3.5 *The subgroup $\langle c, c^x\rangle$ is nilpotent of class at most 3. Furthermore $[x, c, c, c] = 1$.*

Proof By Lemma 3.3 we have that $[x, c, c, c]^x$ commutes with $[x, c]$ and thus

$$[x, c, c, c]^{xc} = [x, c, c, c]^{cx[x,c]} = [x, c, c, c]^{x[x,c]} = [x, c, c, c]^x$$

and c commutes with $[x, c, c, c]^x = [c^{-x^2}, c^x, c^x]$. Replacing c by c^{-1} we see that c commutes also with $[c^{x^2}, c^x, c^x]$. As

$$[c^{-x^2}, c^x, c^x] = [c^{x^2}, c^x, c^x]^{-1}[c^{x^2}, c^x, c^{x^2}, c^x],$$

it follows that $[c^{x^2}, c^x, c^{x^2}, c^x]$ commutes with c. As this element commutes also with c^x and c^{x^2}, it follows from Lemma 3.4 that $[c^{x^2}, c^x, c^x, c^{x^2}] = 1$. Hence the subgroup $\langle c^x, c^{x^2}\rangle$ is nilpotent of class at most 3 and then of course the same is true for the subgroup $\langle c, c^x\rangle$. We have therefore seen that the element $[c^{x^2}, c^x, c^x]$ commutes with c and c^x and c^{x^2} and again Lemma 3.4 implies that $[c^{x^2}, c^x, c^x] = 1$ and conjugation by x^{-1} gives the second statement of the Lemma. $\Box$

Lemma 3.6 *The element c is trivial.*

Proof From Lemma 3.3 and Lemma 3.5 we know that $[x, c, c]$ commutes with c and that $[x, c, c]^x$ commutes with $[x, c]$. Thus

$$[x, c, c]^{xc} = [x, c, c]^{cx[x,c]} = [x, c, c]^{x[x,c]} = [x, c, c]^x,$$

and c commutes with

$$[x, c, c]^x = [c^{-x^2}c^x, c^x] = [c^{-x^2}, c^x][c^{-x^2}, c^x, c^x] = [c^{-x^2}, c^x].$$

Replacing c by c^{-1} we see that c commutes with $[c^{x^2}, c^x]^{-1}$. As

$$[c^{-x^2}, c^x] = [c^{x^2}, c^x]^{-1}[c^{x^2}, c^x, c^{x^2}]$$

it follows that $[c^{x^2}, c^x, c^{x^2}]$ commutes with c and by Lemma 3.5 it commutes also with c^x and c^{x^2}. By Lemma 3.4 this element is then trivial. Thus $[c^{x^2}, c^x]$ commutes with c^{x^2} and c but by the second part of Lemma 3.5 it also commutes with c^x. It follows again from Lemma 3.4 that $[c^{x^2}, c^x] = 1$ and conjugation by x^{-1} gives that $[x, c, c] = 1$. From Lemma 3.2 we have also that $[c, x]$ commutes with $[c, x, x]$. So $[c, x]$ commutes with c, c^x and c^{x^2} and Lemma 3.4 implies that $[c, x] = 1$. As $[c, x] = [c, y]$ this implies that c is in the centre of G and thus trivial. $\square$

Proof of Proposition 3.1 From the previous lemmas we have seen that the centre of $\langle xy^{-1}, y^{-1}x\rangle$ is trivial. But this group is nilpotent because it is generated by two conjugates. It follows that it must also be trivial. Thus $x = y$ and G is cyclic and thus trivial as the Hirsch-Plotkin radical of G is trivial. $\square$

From this and Proposition 2.6 we get the following easy corollaries.

Proposition 3.7 *Let G be a 4-Engel group. The torsion elements form a subgroup that is a direct product of p-groups. If R is the Hirsch-Plotkin radical of G then the torsion subgroup of G/R is a direct product of groups of prime exponent $p \neq 2, 3$.*

Lemma 3.8 *Let G be a 4-Engel group and $a, b \in G$. Then $\langle a\rangle^{\langle b\rangle}$ is nilpotent of class at most 3.*

The lemma can be read off from the polycyclic structure of the free two-generator 4-Engel group (see Appendix A or [7]) that we now know is nilpotent. Reducing the bound 4 to the correct bound 3 turned out to be significant for finishing the proof of the local nilpotence theorem.

Remarks. By Lemma 1.3, it suffices to show that $E = G/R$ is locally nilpotent. This will be shown in the later sections and the proof relies heavily on the fact that E is $\{2, 3\}$-torsion free. The fact that any 2-generator 4-Engel group is nilpotent has another subtle implication. Suppose that we want to show that some two elements $a, b \in E$ commute. It then suffices to show that some a^n and b^m commute for some $\{2, 3\}$-numbers n, m. The reason for this is that $H = \langle a, b\rangle$ is nilpotent and $\gamma_2(H)/\gamma_3(H)$ is a $\{2, 3\}$-group. Hence $\gamma_2(H)$ is a $\{2, 3\}$-group and then trivial as E has no elements of order 2 or 3. This simple application will be used frequently in the later sections.

4 A specific 4-Engel group of rank 3

We saw earlier on that in order to prove that 4-Engel groups are locally nilpotent it suffices to show that all three-generator 4-Engel groups are nilpotent. In fact this is not proved directly. Instead one obtains the weaker result that any subgroup generated by three conjugates is nilpotent. This result will be proved in Section 5. We say that a three-generator group $G = \langle a, b, c\rangle$ is of type (r, s, t) if

$$\begin{aligned} &\langle a, b\rangle \text{ is nilpotent of class at most } r \\ &\langle a, c\rangle \text{ is nilpotent of class at most } s \\ &\langle b, c\rangle \text{ is nilpotent of class at most } t. \end{aligned}$$

By Lemma 3.8, every 4-Engel group generated by three conjugates is of type $(3, 3, 3)$. The way the proof works, roughly speaking, is by induction on the complexity of the type. The main result of this section provides the induction basis but is also another important tool that will be used throughout.

Proposition 4.1 *If $G = \langle a, b, c\rangle$ is a 4-Engel group of type $(1, 2, 3)$, then G is nilpotent. If G is $\{2, 3\}$-torsion free, then the class is at most 4. If G is furthermore of type $(1, 2, 2)$ and 2-torsion free, then the class of G is at most 3.*

The proof uses Proposition 3.1 as a main tool. In particular, we need the following lemma.

Lemma 4.2 *Let G be any 4-Engel group without 2-elements and let $x, y \in G$.*

(1) If $[y, x, x] = 1$ then $\langle x\rangle^{\langle y\rangle}$ is abelian.
(2) If $[y, x, x, x] = 1$ then $\langle x\rangle^{\langle y\rangle}$ is nilpotent of class at most 2.

Furthermore if $c = x^y, d = x^{y^2}, e = x^{y^{-1}}$, then

$$[c, x, c]^8 = [c, x, x]^{-34}[d, x, x]^6[e, x, x]^{-30}.$$

Proof. We know from last section that the group $\langle x, y\rangle$ is nilpotent. All the statements can now be read from a polycyclic presentation of the free 2-generator 4-Engel group given in Appendix A. In fact part (1) follows also directly from the next lemma. □

Lemma 4.3 *Let G be any 4-Engel group and $x, y, z \in G$. If z commutes with x, x^y and $x^{y^{-1}}$ then z commutes with all elements in $\langle x\rangle^{\langle y\rangle}$.*

Proof Let $u = x^{y^{-1}}$. Then z commutes with u, u^y and u^{y^2} and thus with $u, [u, y], [u, y, y]$. As G is 4-Engel, we have

$$\begin{aligned} 1 &= [u, zy, zy, zy, zy] \\ &= [u, y, zy, zy, zy] \\ &= [u, y, y, zy, zy] \\ &= [u, y, y, y, zy] \\ &= [u, y, y, y, z]^y. \end{aligned}$$

Thus z commutes also with u^{y^3} and as $\langle x\rangle^{\langle y\rangle} = \langle u\rangle^{\langle y\rangle}$ is generated by u, u^y, u^{y^2} and u^{y^3}, the result follows. $\Box$

We now prove that the group $T = \langle a, b, c\rangle$ is nilpotent. By Lemma 1.3, we can without loss of generality assume that T has trivial Hirsch-Plotkin radical. Thus in particular T has a trivial centre and no elements of order 2 or 3. As well as the lemmas above the following lemma is going to play a key role in the proof.

Lemma 4.4 *Let u be an element in T. If*

$$H = \langle u\rangle^{\langle [c,a]\rangle} \leq C_T(\langle b, c\rangle)$$

then $u = 1$.

Proof First consider any $h \in H$ that commutes with $[c, a]$. Then h commutes with c, $[c, a]$ and b. We next show that the same holds for the elements $[h, a], [h, a, a]$ and $[h, a, a, a]$. Firstly, as a commutes with b and $[c, a]$ it is clear that these elements commute with b and $[c, a]$. We show by an inductive argument that they also commute with c. But

$$\begin{aligned} [h, a]^c &= [h, a[a, c]] \\ &= [h, [a, c]][h, a]^{[a,c]} \\ &= [h, a], \end{aligned}$$

and thus $[h, a]$ commutes with c. In fact this argument shows that if v in T commutes with c and $[c, a]$ then the same is true for $[v, a]$. Thus the elements $h, [h, a], [h, a, a]$ and $[h, a, a, a]$ all commute with c and b. As T is 4-Engel it follows that $[h, a, a, a] \in Z(T)$ and thus trivial. But then $[h, a, a]$ is in the centre and also trivial. Continuing like this, we see that $h = 1$.

We use this now to deduce that $u = 1$. As $[u, [c, a], [c, a], [c, a]]$ commutes with $[c, a]$ it is trivial by the previous paragraph. But then $[u, [c, a], [c, a]]$ commutes with $[c, a]$ and is also trivial. Continuing like this we see that $u = 1$. $\Box$

Proof of the nilpotence of T. We divide the rest of the proof into few steps.

Step 1. We show that $[c, b, b] = 1$.

The 2-generator subgroup $\langle ac, b\rangle$ is nilpotent and

$$[ac, b, b, b] = [c, b, b, b] = 1.$$

It follows from Lemma 4.2 that $\langle b\rangle^{\langle ac\rangle}$ is nilpotent of class at most two. In particular $[ac, b, b, ac]$ commutes with b. Then

$$[ac, b, b, ac] = [c, b, b, ac] = [c, b, b, a][c, b, b, a, c],$$

and as $[c, b, b, a]$ commutes with b (as $[c, b, b]$ and a do) we conclude that

$$[c, b, b, a, c, b] = 1. \tag{8}$$

The Hall-Witt identity gives us

$$\begin{aligned} 1 &= [[c, b, b], c^{-1}, a]^c[c, a^{-1}, [c, b, b]]^a[a, [c, b, b]^{-1}, c]^{[c,b,b]} \\ &= [[c, a]^{-1}, [c, b, b]]^a[c, b, b, a, c]. \end{aligned}$$

By (8) the latter commutator commutes with b. Thus $[[c, b, b], [c, a]^{-1}]$ commutes with b and replacing a by a^{-1} we see that $[[c, b, b], [c, a]]$ also commutes with b. We have now seen that

$$[c, b, b], [c, b, b]^{[c,a]}, [c, b, b]^{[c,a]^{-1}}$$

all commute with b and by Lemma 4.3 we can then deduce that all the elements in $H = \langle [c, b, b]\rangle^{\langle [c,a]\rangle}$ commute with b. These elements also clearly commute with c. Now Lemma 4.4 gives that $[c, b, b] = 1$.

Step 2. We show that $[b, c, c] = 1$.

By Step 1 we have that

$$[ac, b, b] = [c, b, b] = 1.$$

It follows from Lemma 4.2 that $\langle b\rangle^{\langle ac\rangle}$ is abelian. In particular it follows that the element

$$[ac, b, ac] = [c, b, ac] = [c, b, c][c, b, a][c, b, a, c]$$

commutes with b and as $[c, b, a], [c, b, c]$ commute now with b, we conclude that

$$[c, b, a, c, b] = 1. \tag{9}$$

We again use the Hall-Witt identity. This time we have

$$1 \;=\; [[c, b], a, c]^{a^{-1}}[a^{-1}, c^{-1}, [c, b]]^c[c, [c, b]^{-1}, a^{-1}]^{[c,b]}.$$

By (9) the first commutator commutes with b and it is clear that the last one also commutes with b as $[b, c, c], a$ and $[c, b]$ all commute with b. It follows that the second commutator also commutes with b. That is

$$[[c, b]^c, [c, a]^{-1}] \text{ commutes with } b. \tag{10}$$

And replacing a with a^{-1} we see that $[[c, b]^c, [c, a]]$ also commutes with b. This means that for $u = [c, b][c, b, c]$ we have that $[u, [c, a]]$ commutes with b. Replacing c, a by c^{-1}, a^{-1} we see that for $v = [c, b]^{-1}[c, b, c]^2$, $[v, [c, a]]$ commutes with b. Then

$$[uv, [c, a]] = [u, [c, a]]^v[v, [c, a]]$$

commutes with b as v also commutes with b by Step 1. But $uv = [b, c, c]^{-3}$. So $[b, c, c]^{-3}, [[b, c, c]^{-3}, [c, a]]$ and $[[b, c, c]^{-3}, [c, a]^{-1}]$ all commute with b. Lemma 4.3 and Lemma 4.4 give just as in the proof of Step 1 that $[b, c, c]^3 = 1$. As T has no elements of order 3 it follows that $[b, c, c] = 1$.

Step 3. We show that $T = \{1\}$.

Now by Step 1 and Step 2, $[b, c]$ commutes with b and c. By (10) we have as before that all elements in

$$\langle[b, c]\rangle^{\langle[c,a]\rangle}$$

commute with b and c and we conclude as before, using Lemmas 4.3 and 4.4, that $[c, b] = 1$.

Now b is in the centre of T and thus trivial. Then $T = \langle a, c\rangle$ which was nilpotent by our assumption. Hence T is trivial. In Appendix B we establish the details about the nilpotence classes. $\square$

5 Subgroups generated by three conjugates

Let G be a 4-Engel group and let $E = G/R(G)$, where $R(G)$ is the Hirsch-Plotkin radical of G. To complete the proof of the local nilpotence theorem, it suffices to show that E is locally nilpotent. By the facts that have been previously established we know that E is $\{2, 3\}$-torsion free. We will prove the following main step.

Proposition 5.1 *Let $a, x, y \in E$. The subgroup $\langle a, a^x, a^y\rangle$ is nilpotent of class at most 5 and $\langle a\rangle^{\langle a, a^x, a^y\rangle}$ is nilpotent of class at most 2.*

The proof is very technical and consists of few steps.

Step 1. $\langle a, a^{a^{a^x}}, a^{a^y}\rangle$ and $\langle a^{a^{a^x}}, a, (a^{a^{a^x}})^{a^y}\rangle$ are nilpotent.

Step 2. $\langle a, a^{a^{a^x}}, a^y\rangle$ is nilpotent.

Step 3. $\langle a, a^{a^x}, a^{a^y}\rangle$ and $\langle a^{a^x}, a, (a^{a^x})^{a^y}\rangle$ are nilpotent.

Step 4. $\langle a, a^{a^x}, a^y\rangle$ is nilpotent.

Step 5. $\langle a, a^x, a^y\rangle$ is nilpotent.

Notice that the groups in Step 1 are of type $(1, 2, 3)$ and therefore nilpotent by the main result of last section. The group in Step 2 is of type $(1, 3, 3)$, those in Step 3 of type $(2, 2, 3)$, the one in Step 4 of type $(2, 3, 3)$ and finally the one in Step 5 of type $(3, 3, 3)$. The proof of each step consists of careful commutator calculus building on the previous steps to obtain nilpotence. When the nilpotence has been established one can get a precise information about the structure using either machine or hand calculations. All the relevant hand calculations are given in Appendix C.

5.1 The subgroup $\langle a, a^{a^{a^x}}, a^y\rangle$

Let $b = a^{a^{a^x}}$ and $c = a^y$. We show in this section that $\langle a, b, c\rangle$ is nilpotent of class at most 4.

Step 1. We show that $\langle [c, a, a], [c, a, c], b\rangle$ is nilpotent of class at most 2.

Let $A_1(x) = \langle a\rangle^{\langle x\rangle}$ and $A_1(y) = \langle a\rangle^{\langle y\rangle}$. By Lemma 3.8, we know that these subgroups are nilpotent of class at most 3. Then let

$$\begin{array}{cc} A_2(x) = \langle a\rangle^{A_1(x)}, & A_3(x) = \langle a\rangle^{A_2(x)}, \\ A_2(y) = \langle a\rangle^{A_1(y)}, & A_3(y) = \langle a\rangle^{A_2(y)}. \end{array}$$

Notice that $A_2(x), A_2(y)$ are nilpotent of class at most 2, whereas $A_3(x), A_3(y)$ are abelian. Let $d = a^{a^x} \in A_2(x)$. Notice that $a^{a^c} \in A_3(y)$ and a thus commutes with a^{a^c} and a^d. Using the fact that $\langle a, c\rangle \subseteq A_1(y)$ is nilpotent of class at most 3, we see from this that

$$\begin{aligned} [c, a, a, b, b] &= [c, a^{-1}, a^{-1}, a[a, d], a[a, d]] \\ &= [a^{a^c} a^{-1}, a[a, d], a[a, d]] \\ &= [a^{a^c}, [a, d], [a, d]]. \end{aligned}$$

As $a, a^{a^c} \in A_3(y)$ and $a, d \in A_2(x)$, we have that $\langle a, a^{a^c}, d\rangle$ is of type $(1, 2, 3)$. (Notice that a^{a^c}, d are conjugates and thus $\langle a^{a^c}, d\rangle$ nilpotent of class at most 3). Thus, by Proposition 4.1, $\langle a, a^{a^c}, d\rangle$ is nilpotent of class at most 4. This implies that

$$[c, a, a, b, b] \in \gamma_5(\langle a, a^{a^c}, d\rangle) = \{1\}.$$

Also

$$\begin{aligned} [c, a, a, b, [c, a, a]] &= [a^{-1}a^{a^c}, a^d, a^{-1}a^{a^c}] \\ &= [a, a^c, a^d, [a, a^c]]. \end{aligned}$$

As $a, a^d \in A_3(x)$ and $a, a^c \in A_2(y)$ we have again that $\langle a, a^d, a^c\rangle$ is of type $(1, 2, 3)$ and thus

$$[c, a, a, b, [c, a, a]] \in \gamma_5(\langle a, a^d, a^c\rangle) = \{1\}.$$

By Lemma 4.2 $[c, a, c]^8 = [f, a, a]$ where $f = a^{6y^2} a^{-34y} a^{-30y^{-1}} \in A_1(y)$. Thus $a^f \in A_2(y)$ and $a^{a^f} \in A_3(y)$. As before we get $\langle a, a^{a^f}, d\rangle$ and $\langle a, a^d, a^f\rangle$ are of type $(1, 2, 3)$ and thus nilpotent of class at most 4. Hence

$$[[c, a, c]^8, b, b] = [a^{a^f}, [a, d], [a, d]] = 1$$

and

$$[[c,a,c]^8, b, [c,a,c]^8] = [a, a^f, a^d, [a, a^f]] = 1.$$

We have thus seen that for $M = \langle [c,a,c], b\rangle$, we have that $\gamma_3(M)/\gamma_4(M)$ is a 2-group and as M is nilpotent, being a 4-Engel group of rank 2, it follows that $\gamma_3(M)$ is a 2-group and thus trivial as E is 2-torsion free. Next, observe that

$$[c,a,c]^8[c,a,a] = [fc,a,a].$$

As $fc \in A_1(y), a^{fc} \in A_2(y)$ and $a^{a^{fc}} \in A_3(y)$, we once again have that $\langle a, a^d, a^{fc}\rangle$ is of type $(1,2,3)$ and thus nilpotent of class at most 4. Hence

$$[b, [c,a,c]^8[c,a,a], [c,a,c]^8[c,a,a]] = [a^d, [a, a^{fc}], [a, a^{fc}]] = \{1\}.$$

Let $H = \langle [c,a,a], [c,a,c], b\rangle$. We have seen that H is of type $(1,2,2)$ and thus nilpotent of class at most 3 by Proposition 4.1. But

$$1 = [b, [c,a,c]^8[c,a,a], [c,a,c]^8[c,a,a]] = [b, [c,a,c], [c,a,a]]^{16}.$$

Thus $\gamma_3(H)$ is a 2-group and then trivial as E is 2-torsion free.

Now let $c = a^y, d = a^{y^2}$ and $e = a^{y^{-1}}$.

Step 2. $\langle [c,a,a], [d,a,a], [e,a,a], b\rangle$ is nilpotent of class at most 2.

To see this notice first that by Lemma 3.8 we have that $\langle a,c,d,e\rangle$ is nilpotent of class at most 3. Therefore $\langle [c,a,a], [d,a,a], [e,a,a]\rangle$ is abelian. By Step 1, $\langle [a^y,a,a], b\rangle = \langle [c,a,a], b\rangle$ is nilpotent of class at most 2. The result of course still holds if we replace y by either y^2 or y^{-1}. Thus we also have that $\langle [d,a,a], b\rangle$ and $\langle [e,a,a], b\rangle$ are nilpotent of class at most 2. This means that $\langle [c,a,a], [d,a,a], b\rangle$ is of type $(1,2,2)$ and thus nilpotent of class at most 3. Next notice that $\langle a^{cd}, a\rangle \leq A_2(y)$ is nilpotent of class at most 2 and thus $\langle a, b, a^{cd}\rangle$ is of type $(1,2,3)$ and nilpotent of class at most 4. Hence

$$\begin{aligned}
[b, [c,a,a][d,a,a], [c,a,a][d,a,a]] &= [b, [cd,a,a], [cd,a,a]] \\
&= [b, [a^{cd},a], [a^{cd},a]] \\
&= 1.
\end{aligned}$$

As we had already seen that $\langle [c,a,a], [d,a,a], b\rangle$ is nilpotent of class at most 3 and $[c,a,a]$ commutes with $[d,a,a]$, this implies that

$$[b, [c,a,a], [d,a,a]]^2 = 1.$$

Hence $[b,[c,a,a],[d,a,a]] = [c,[d,a,a],[c,a,a] = 1$ as E is 2-torsion free. Similarly $[b,[c,a,a],[e,a,a]] = [b,[e,a,a],[c,a,a]] = 1$ and $[b,[d,a,a],[e,a,a]] = [b,[e,a,a],[d,a,a]] = 1$. We have thus shown above that all commutators of weight 3 in $[c,a,a],[d,a,a],[e,a,a],b$ are trivial.

Step 3. $\langle [c,b,b],[c,b,c],a\rangle$ is nilpotent of class at most 2.

Let $d = a^{a^x}$. Then $a = b^{d^{-1}}$. Notice that $c = a^y = b^u$ for some $u \in G$. Define

$$B_1(u) = \langle b\rangle^{\langle u\rangle}, B_2(u) = \langle b\rangle^{B_1(u)}, B_3(u) = \langle b\rangle^{B_2(u)}.$$

Notice that $b^c = b^{b^u} \in B_2(u)$ and $b^{b^c} = b^{b^{b^u}} \in B_3(u)$. In particular b commutes with b^{b^c}. Notice also that $\langle b,d\rangle \leq A_2(x)$ and thus nilpotent of class at most 2. In particular $[b,b^{d^{-1}}] = 1$. Thus both the groups $\langle b,b^{b^c},d\rangle$ and $\langle b,b^{d^{-1}},b^c\rangle$ are of type $(1,2,3)$ and therefore nilpotent of class at most 4. Thus

$$\begin{aligned}
[c,b,b,a,a] &= [c,b^{-1},b^{-1},a,a] \\
&= [b^{b^c}b^{-1}, b[b,d^{-1}], b[b,d^{-1}]] \\
&= [b^{b^c}, [b,d^{-1}], [b,d^{-1}]] \\
&= 1,
\end{aligned}$$

and

$$[c,b,b,a,[c,b,b]] = [b,b^c,b^{d^{-1}},[b,b^c]] = 1.$$

We use the fact that $[c,b,c]^8 = [f,b,b]$, where $f = b^{6u^2}b^{-34u}b^{-30u^{-1}} \in B_1(u)$. As above we see that the groups $\langle b,b^{b^f},d^{-1}\rangle, \langle b,b^{d^{-1}},b^f\rangle$ and $\langle b,b^{d^{-1}},b^{fc}\rangle$ are all of type $(1,2,3)$ and thus nilpotent of class at most 4. Using this we see as in Step 1 that $\langle [c,b,c],a\rangle$ is nilpotent of class at most 2 and then that $\langle [c,b,c],[c,b,b],a\rangle$ is nilpotent of class at most 2.

Let us summarise some of the properties that we established.

(1) $\langle a,b\rangle$ is abelian.
(2) $\langle a,c\rangle$ and $\langle b,c\rangle$ are nilpotent of class at most 3.
(3) $\langle [c,a,a],[c,a,c],b\rangle$ and $\langle [c,b,b],[c,b,c],a\rangle$ are nilpotent of class at most 2.

We will now show that these three properties imply that $G = \langle a,b,c\rangle$ is nilpotent. Notice that the properties are symmetrical in a and b. We let

$A = G/HP(G)$. To show that G is nilpotent is then the same as showing that A is trivial. Notice that $HP(A) = \{1\}$ and that A is $\{2,3\}$-torsion free. In particular we have that $Z(A) = \{1\}$. With some abuse of notation we will use a, b, c also for the homomorphic images of these elements in A. (So we are simply calculating modulo $HP(G)$). Notice that if $x \in A$ commutes with a, b, c then $x = 1$. We will use this repeatedly in the following two steps.

Step 4. $[c, a, a] = [c, b, b] = 1$.

By symmetry, it suffices to show that $[c, a, a] = 1$. As $[c, a, a]$ commutes with a, c by (2), it suffices to show that $[c, a, a, b] = 1$. From (3), we known that $[c, a, a, b, b] = 1$. Also a commutes with both $[c, a, a]$ and b and thus with $[c, a, a, b]$. It follows that we only need to show that $[c, a, a, b, c] = 1$.

Notice next that $\langle [c, a, a], c, b\rangle$ is of type $(1, 2, 3)$ by (2) and (3). Thus this subgroup is nilpotent of class 4. The element $[c, a, a, [c, b, b]]$ is then in $\gamma_4(\langle [c, a, a], c, b\rangle)$ and commutes with c and b. By symmetry it also commutes with a and as $Z(A) = \{1\}$ it follows that $[c, a, a, [c, b, b]] = 1$. Thus, using again the fact that $\langle [c, a, a], c, b\rangle$ is nilpotent of class at most 4,

$$1 = [c, a, a, [c, b, b]] = [c, a, a, b, c, b]^{-2}.$$

As A is 2-torsion free, it follows that $[c, a, a, b, c, b] = 1$.

Next consider the subgroup $\langle [b, c, c], c, a\rangle$. By (2) and (3), this is of type $(1, 2, 3)$ and thus nilpotent of class at most 4. Thus the element $[c, a, a, [b, c, c]] \in \gamma_4(\langle [b, c, c], c, a\rangle) \cap \gamma_4(\langle [c, a, a], c, b\rangle)$ commutes with a, b, c and is thus trivial. This gives

$$1 = [c, a, a, [b, c, c]] = [c, a, a, b, c, c].$$

We have now shown that $[c, a, a, b, c]$ commutes with b and c and to finish the proof of Step 4, it suffices to show that $[c, a, a, b, c, a] = 1$. For this we consider the subgroup $\langle [c, a, a, b], a, c\rangle$. We have seen above that $\langle [c, a, a], c, b\rangle$ is nilpotent of class at most 4 and this implies in particular that $[c, a, a, b, c, [c, a, a, b]] = 1$. We have also shown above that $[c, a, a, b, c, c] = 1$. Hence the group $\langle [c, a, a, b], a, c\rangle$ is of type $(1, 2, 3)$ and thus of class at most 4. Thus

$$[c, a, a, b, c, a, a] = [c, a, a, b, [c, a, a]] \stackrel{(3)}{=} 1$$

and

$$[c,a,a,b,c,a,c]^{-2} = [c,a,a,b,[a,c,c]] \overset{(3)}{=} 1.$$

As A is 2-torsion free, it follows that $[c,a,a,b,c,a]$ commutes with a and c. But we know also that b commutes with both a and $[c,a,a,b,c]$ and therefore $[c,a,a,b,c,a]$. Hence $[c,a,a,b,c,a]=1$. This finishes the proof of Step 4.

Step 5. $[a,c,c]=[b,c,c]=1$.

By symmetry, it suffices to show that $[a,c,c]=1$ and as this element commutes with a,c by (2), it suffices to show that $[a,c,c,b]=1$. By (3), we also know that $[a,c,c,b,b]=1$ and as a commutes with both $[a,c,c]$ and b, we also have $[a,c,c,b,a]=1$. It thus suffices to show that $[a,c,c,b,c]=1$. Now $\langle [a,c,c],c,b\rangle$ and $\langle [c,b,b],c,a\rangle$ are of type $(1,2,3)$ and thus nilpotent of class at most 4. As the element $[a,c,c,[c,b,b]]$ is in the intersection of $\gamma_4(\langle [a,c,c],c,b\rangle)$ and $\gamma_4(\langle [c,b,b],c,a\rangle)$, it commutes with a,b and c and is therefore trivial. Thus

$$1=[a,c,c,[c,b,b]]=[a,c,c,b,c,b]^{-2}.$$

Thus $[a,c,c,b,c,b]=1$ as there is no 2-torsion. Similarly $[a,c,c,[b,c,c]]$ is in the intersection of $\gamma_4(\langle [a,c,c],c,b\rangle)$ and $\gamma_4(\langle [b,c,c],c,a\rangle)$ and thus commutes with a,b,c. Therefore

$$1=[a,c,c,[b,c,c]]=[a,c,c,b,c,c]$$

and $[a,c,c,b,c]$ commutes with b and c. We finish the proof of Step 5 by showing that $[a,c,c,b,c,a]=1$. Notice that we know from above that $\langle [a,c,c],c,b\rangle$ is nilpotent of class at most 4 and thus in particular that $[a,c,c,b,c,[a,c,c,b]]=1$. We have also seen that $[a,c,c,b,c,c]=1$. Thus $\langle [a,c,c,b],a,c\rangle$ is of type $(1,2,3)$ and thus nilpotent of class at most 4. This implies that

$$[a,c,c,b,c,a,c]^{-2}=[a,c,c,b,[a,c,c]] \overset{(3)}{=} 1$$

and

$$[a,c,c,b,c,a,a]=[a,c,c,b,[c,a,a]] \overset{(3)}{=} 1.$$

Therefore $[a,c,c,b,c,a]$ commutes with a and c. As b commutes with both $[a,c,c,b,c]$ and a, and thus $[a,c,c,b,c,a]$, it follows that $[a,c,c,b,c,a]=1$.

The previous steps imply that $\langle a, b, c\rangle$ is of type $(1, 2, 2)$ and thus $A = G/HP(G)$ is nilpotent by Proposition 4.1 and thus trivial. This finishes the proof that G is nilpotent. On can now show that the class of G is at most 4. This can be done either with an aid of a machine using the nilpotent quotient algorithm or by hand calculations. In Appendix C this is done using hand calculations. In the next step we strengthen this result.

Step 6. $\gamma_4(\langle a, a^{a^{a^x}}, a^y\rangle) \leq Z(a, a^x, a^y)$.

Let $c = a^y$ and $b = a^{a^{a^x}}$. As we noted above $H = \langle a, b, c\rangle$ is nilpotent of class at most 4 and it is easy to see that $\gamma_4(H)$ is generated by $[c, a, a, b], [c, b, b, a], [c, a, c, b], [c, b, c, a]$ and $[c, b, a, c] = [c, a, b, c] = [a, c, b, c]^{-1}$. We also have that

$$[a, c, b, c]^{-2} = [a, [b, c, c]][a, c, c, b]^{-1} = [c, b, c, a][c, a, c, b].$$

Thus if $[c, b, c, a]$ and $[c, a, c, b]$ commute with a^x, it would follow that $[a, c, b, c]^2$ commutes with a^x. By the remark made after Lemma 3.8 it would follow then that $[a, c, b, c]$ commutes with a^x. This shows that it suffices to deal with the elements $[c, a, a, b], [c, a, c, b], [c, b, b, a]$ and $[c, b, c, a]$. The argument above also shows that it suffices to show that some 2-power of these elements commute with a^x.

Let $d = a^{y^2}$ and $e = a^{y^{-1}}$. First we notice that

$$[c, a, a, b] = [a^{a^{a^y}} a^{-1}, a^{a^{a^x}}] = [a^{a^{a^y}}, a^{a^{a^x}}]$$

is in $Z(\langle a, b, c\rangle)$ and thus commutes with a and a^y. By symmetry it is also centralised by a^x. Replacing y with y^2 or y^{-1} we see that $[d, a, a, b]$ and $[e, a, a, b]$ also commute with a^x. From Steps 1 and 2, we know that $\langle [c, a, c], b\rangle$ and $\langle [c, a, a], [d, a, a], [e, a, a], b\rangle$ are nilpotent of class at most 2. From Lemma 4.2 we have

$$[c, a, c, b]^8 = [c, a, a, b]^{-34}[d, a, a, b]^6[e, a, a, b]^{-30}$$

as $[c, a, a, b], [d, a, a, b]$ and $[e, a, a, b]$ all commute with a^x it follows that $[c, a, c, b]^8$ commutes with a^x. As we pointed out above, this then implies that $[c, a, c, b] \in Z(\langle a, a^x, a^y\rangle)$.

This leaves us with the commutators $[c, b, b, a]$ and $[c, b, c, a]$. Notice that

$$\begin{aligned} [c, b, b, a] &= [c^{z^{-1}}, a, a, a^{z^{-1}}]^z \\ [c, b, c, a] &= [c^{z^{-1}}, a, c^{z^{-1}}, a^{z^{-1}}]^z, \end{aligned}$$

where $z = a^{a^x}$. Now $a^{z^{-1}} = a[a, z^{-1}] = a[a, z]^{-1} = a^2a^{-z} = a^2b^{-1}$. As $\langle c^{z^{-1}}, a, b\rangle = \langle c, b, a\rangle^{z^{-1}}$ is nilpotent of class at most 4, it follows that

$$\begin{aligned} [c, b, b, a] &= [c^{z^{-1}}, a, a, b]^{-z} \\ [c, b, c, a] &= [c^{z^{-1}}, a, c^{z^{-1}}, b]^{-z}. \end{aligned}$$

commute with a and $c = a^y$. It suffices to show that $[c^{z^{-1}}, a, a, b]$ and $[c^{z^{-1}}, a, c^{z^{-1}}, b]$ commute with a^x (as then also $[c, b, b, a]$ and $[c, b, c, a]$ will commute with $z = a^{a^x}$ and $[c, b, b, a] = [c^{z^{-1}}, a, a, b]^{-1}$ and $[c, b, c, a] = [c^{z^{-1}}, a, c^{z^{-1}}, b]^{-1}$). Let $\tilde{c} = a^{yz^{-1}}$, $\tilde{d} = a^{(yz^{-1})^2}$ and $\tilde{e} = a^{zy^{-1}}$. We have seen above that $[a^{a^{a^y}}, a^{a^{a^x}}]$ commutes with a^x. By replacing y by yz^{-1}, it follows that

$$[\tilde{c}, a, a, b] = [c^{z^{-1}}, a, a, b] = [a^{a^{a^{yz^{-1}}}}, a^{a^{a^x}}]$$

commutes also with a^x. By replacing y by $(yz^{-1})^2$ and zy^{-1} we see as well that $[\tilde{d}, a, a, b]$ and $[\tilde{e}, a, a, b]$ are centralised by a^x. Now steps 1 and 2 imply that $\langle [\tilde{c}, a, \tilde{c}], b\rangle$ and $\langle [\tilde{c}, a, a], [\tilde{d}, a, a], [\tilde{e}, a, a], b\rangle$ are nilpotent of class at most 2. Then Lemma 4.2 implies that

$$[c^{z^{-1}}, a, c^{z^{-1}}, b]^8 = [\tilde{c}, a, \tilde{c}, b]^8 = [\tilde{c}, a, a, b]^{-34}[\tilde{d}, a, a, b]^6[\tilde{e}, a, a, b]^{-30}.$$

Thus $[c^{z^{-1}}, a, c^{z^{-1}}, b]^8$ commutes with a^x. This finishes the proof of Step 6.

5.2 The subgroups $\langle a, a^{a^x}, a^{a^y}\rangle$ and $\langle a, a^{a^x}, (a^{a^x})^{a^y}\rangle$

Let $b = a^{a^x}, c = a^{a^y}$. We wish to show that $G = \langle a, b, c\rangle$ is nilpotent of class at most 4. As any two conjugates generate a nilpotent group of class 2, we have that

$$\begin{aligned} \langle a, b\rangle, \langle a, c\rangle &\quad \text{are nilpotent of class at most 2} \\ \langle b, c\rangle &\quad \text{is nilpotent of class at most 3.} \end{aligned} \tag{11}$$

Also, $[c, a, b, a] = [c, a, [a, a^x], a] = [a(a^{a^{a^y}})^{-1}, [a, a^x], a]$, $[c, a, b, b] = [a(a^{a^{a^y}})^{-1}, [a, a^x], a^{a^x}]$ and $[c, a, b, [c, a]] = [a(a^{a^{a^y}})^{-1}, [a, a^x], a(a^{a^{a^y}})^{-1}]$ are in $\gamma_4(\langle a, a^{a^{a^y}}, a^x\rangle)$ and by symmetry we also have that $[b,a,c,a],[b,a,c,c],[b,a,c,[b,a]]$ are in $\gamma_4(\langle a, a^{a^{a^x}}, a^y\rangle)$. By Step 6 in 5.1 we thus have

$$\begin{aligned}
&[c,a,b,a],[c,a,b,b],[c,a,b,[c,a]] \in Z(\langle a, a^x, a^y\rangle)\\
&[b,a,c,a],[b,a,c,c],[b,a,c,[b,a]] \in Z(\langle a, a^x, a^y\rangle).
\end{aligned} \tag{12}$$

Like in 5.1 we now consider the group $A = G/HP(G)$. Thus the Hirsch-Plotkin radical of A is trivial and A has no elements of order 2 or 3. We want to show that A is trivial. We continue using a, b, c for the generators but we are now calculating modulo $HP(G)$. Thus

$$[c,a,b,a] = [c,a,b,b] = [b,a,c,a] = [b,a,c,c] = [b,a,c,[b,a]] = [c,a,b,[c,a]] = 1.$$

Now

$$\begin{aligned}
[b,a,c,b]^a &= [b,a,c,b[b,a]]\\
&= [b,a,c,[b,a]b]\\
&= [b,a,c,b][b,a,c,[b,a]]^b\\
&= [b,a,c,b].
\end{aligned}$$

The next aim is to show that $[b,a,c,b] = 1$ and as $Z(A) = \{1\}$ it suffices to show as well that $[b,a,c,b]$ commutes with b and c. Notice that $[b,a,c,b,b],[b,a,c,b,c] \in \gamma_4(\langle b, b^a, c\rangle)$. As b and a are conjugates we have $a^x = b^z$ for some z and thus $a = b^{a^{-x}} = b^{b^{-z}}$. It follows that

$$\begin{aligned}
b^a &= b[b, b^{b^{-z}}]\\
&= b[b,[b,b^z]]^{-1}\\
&= b(b^{-1}b^{b^{b^z}})^{-1}\\
&= b^2(b^{b^{b^z}})^{-1}.
\end{aligned}$$

Thus $\gamma_4(\langle b, b^a, c\rangle) = \gamma_4(\langle b, b^{b^{b^z}}, c\rangle)$ and by Step 6 from 5.1, it follows that $[b,a,c,b,b],[b,a,c,b,c]$ then commute with b, c and b^z and then also $a = b^{b^{-z}}$. Hence $[b,a,c,b,b],[b,a,c,b,c]$ are trivial that implies that $[b,a,c,b] \in Z(A)$ and thus trivial as well. From (12) we know that $[b,a,c,a]$ and $[b,a,c,c]$ are trivial and we have thus shown that $[b,a,c] \in Z(A)$ and thus trivial as well. As $[b,a]$ commutes also with a and b it follows that $[b,a] = 1$. Thus A is of type $(1,2,3)$ and thus nilpotent by Proposition 4.1. Hence $A = G/HP(G)$ is trivial and G is nilpotent. One can derive from the relations (11) and (12) that G is nilpotent of class at most 4. See Appendix C for details.

Similarly we can see that $H = \langle a^{a^x}, a, (a^{a^x})^{a^y} \rangle$ is nilpotent of class at most 4. To see this note that if we let $b = a^{a^x}$ and $c' = (a^{a^x})^{a^y}$, then we can write $a^x = b^z$ and $a^y = b^t$ for some z, t. Thus $a = b^{b^{-z}}$ and $c' = b^{b^t}$. Then $[c', b, a, b] = [b(b^{b^{b^t}})^{-1}, [b, b^{-z}], b]$, $[c', b, a, a] = [b(b^{b^{b^t}})^{-1}, [b, b^{-z}], b^{b^{-z}}]$ and $[c', b, a, [c', b]] = [b(b^{b^{b^t}})^{-1}, [b, b^{-z}], b(b^{b^{b^t}})^{-1}]$ are in $\gamma_4(\langle b, b^{b^{b^t}}, b^z \rangle)$. By Step 6 from 5.1, these then commute with $b^t = a^y, b^z = a^x$ as well as b and thus $a = b^{a^{-x}}$. Hence these are all in $Z(\langle a, a^x, a^y \rangle)$. As

$$[a, b] = [b, b^{-z}, b] = [b, b^z, b]^{-1} = b^{-1} b^{b^{b^z}},$$

we also have that $[a, b, c', b] = [b^{-1} b^{b^{b^z}}, [b, b^t], b]$, $[a, b, c', c'] = [b^{-1} b^{b^{b^z}}, [b, b^t], b^{b^t}]$ and $[a, b, c', [a, b]] = [b^{-1} b^{b^{b^z}}, [b, b^t], b^{-1} b^{b^{b^z}}]$ are all in $\gamma_4(\langle b, b^{b^{b^z}}, b^t \rangle)$. Again using Step 6 from 5.1, we see that these elements commute with b^z, b^t, b and thus a, a^x, a^y. Thus we have seen that

$$\begin{aligned} [c', b, a, b], [c', b, a, a], [c', b, a, [c', b]] &\in Z(\langle a, a^x, a^y \rangle) \\ [a, b, c', b], [a, b, c', c'], [a, b, c', [a, b]] &\in Z(\langle a, a^x, a^y \rangle). \end{aligned}$$

Notice also that $\langle b, a \rangle$ and $\langle b, c' \rangle = \langle b, b^{b^z} \rangle$ are nilpotent of class 2 whereas $\langle a, c' \rangle$ is nilpotent of class at most 3. Thus (c', b, a) corresponds to (c, a, b) in the argument above and the same argument shows that H is nilpotent of class at most 4.

5.3 The subgroup $\langle a, a^{a^x}, a^y \rangle$

We let $b = a^{a^x}$ and $c = a^y$ and consider $G = \langle a, b, c \rangle$. We will show that G is nilpotent of class at most 4. Clearly $\langle a, b \rangle$ is nilpotent of class at most 2 and $\langle a, c \rangle$, $\langle b, c \rangle$ are nilpotent of class at most 3. By results of Section 5.2 we have that $\langle a, b, a^c \rangle$ and $\langle a, b, b^c \rangle$ are nilpotent of class at most 4.

Now

$$\gamma_3(\langle [c, a, a], a, b \rangle) = \gamma_3(\langle a, a^{a^{a^y}}, [a, a^x] \rangle) \leq \gamma_4(\langle a, a^{a^{a^y}}, a^x \rangle)$$

By Step 6 of 5.1, it follows that $\gamma_3(\langle [c, a, a], a, b \rangle) \leq Z(a, a^x, a^y)$ and thus in particular in $Z(G)$. We have that $c = b^u$ and $a = b^v$ for some u and thus similarly

$$\gamma_3(\langle [c, b, b], a, b \rangle) = \gamma_3(\langle b, b^{b^{b^u}}, a \rangle)$$

and $\gamma_3(\langle [c,b,b],a,b\rangle) \leq Z(b,b^u,a) = Z(G)$. Next notice that

$$\gamma_4(\langle [a,b],a,c\rangle) = \gamma_4(\langle a, a^{a^{a^x}}, a^y)$$

and thus also a subgroup of $Z(G)$. Then pick u such that $a^x = b^u$ and $b^a = b^{b^{a^{-x}}} = b^{b^{b^{-u}}} = b^2(b^{b^{b^u}})^{-1}$ and

$$\gamma_4(\langle [a,b],b,c\rangle) = \gamma_4(\langle b, b^{b^{b^u}}, a^y\rangle$$

is centralised by b,c and $b^u = a^x$ and thus also $a = b^{a^{-x}}$. Hence $\gamma_4(\langle [a,b],b,c\rangle) \leq Z(G)$.

If $a = c^u$, then

$$\gamma_4(\langle [a,c,c],b,c\rangle) = \gamma_4(\langle c, c^{c^{c^u}}, b\rangle \leq Z(\langle c, c^u, b\rangle) = Z(G).$$

Also if $b = c^u$ then

$$\gamma_4(\langle [c,b,c],a,c\rangle) = \gamma_4(\langle c, c^{c^{c^u}}, a\rangle) =\leq Z(c,c^u,a) = Z(G).$$

We show that these properties imply that G is nilpotent. Let $A = G/HP(G)$. We will show that $A = \{1\}$. In steps 1 to 4 we will thus calculate modulo $HP(G)$. As before we have that $Z(A) = \{1\}$ and that A is $\{2,3\}$-torsion free. By the work above we now have the following properties.

(1) $\langle a,b\rangle$ is nilpotent of class at most 2,
(2) $\langle a,c\rangle, \langle b,c\rangle$ are nilpotent of class at most 3,
(3) $\langle a,b,a^c\rangle, \langle a,b,b^c\rangle$ are nilpotent of class at most 4,
(4) $\langle [c,a,a],a,b\rangle, \langle [c,b,b],a,b\rangle$ are nilpotent of class at most 2,
(5) $\langle [a,b],a,c\rangle, \langle [a,b],b,c\rangle$ are nilpotent of class at most 3,
(6) $\langle [c,a,c],b,c\rangle, \langle [c,b,c],a,c\rangle$ are nilpotent of class at most 3.

Notice that these relations are symmetrical in a and b. We will show that these imply that A is trivial.

Step 1. We show that $\langle [c,a,a],c,b\rangle$, $\langle [c,a,c],a,b\rangle$ are nilpotent of class at most 3. Also $\langle [c,a,a],[c,a,c],b\rangle$ is nilpotent of class at most 2.

By (2) and (4) we have that $\langle [c,a,a],c,b\rangle$ is of type $(1,2,3)$ and thus nilpotent of class at most 4 by Proposition 4.1. Thus

$$[[c,a,a],[c,b,b],b] = [[c,a,a],[c,b,b],c] = 1.$$

By symmetry in a and b we also have $[[c,a,a],[c,b,b],a] = 1$. Therefore $[[c,a,a],[c,b,b]]$ is central and thus trivial. By (6) we also have that $\langle [c,b,c],a,c\rangle$ is nilpotent of class at most 3 and thus $[[c,a,a],[c,b,c]] = 1$. From (2) and (4) it then also follows that $[c,a,a,c] = [c,a,a,b,b] = 1$. Using again the fact that $\langle [c,a,a],c,b\rangle$ has class at most 4 we obtain

$$1 = [c,a,a,[c,b,b]] = [c,a,a,b,c,b]^{-2}$$

that implies that $[c,a,a,b,c,b] = 1$. Also

$$1 = [c,a,a,[b,c,c]] = [c,a,a,b,c,c].$$

Notice that we also have by (3) that

$$[[c,a,a],b,[c,a,a]] \in \gamma_5(\langle a,b,a^c\rangle) = \{1\}.$$

This implies that $\langle [c,a,a],b,c\rangle$ is nilpotent of class at most 3. (Notice that as $[c,a,a,b]$ and c commute with $[c,a,a]$ it follows that $[c,a,a,b,c,[c,a,a]] = 1$.)

We will next show that $\langle [c,a,c],a,b\rangle$ is nilpotent of class at most 3. Notice that by (5) $[a,c,c,[a,b]] = 1$ and thus, using (1),

$$[a,c,c,b]^a = [a,c,c,[b,a]b] = [a,c,c,b]$$

and then also, as $[b,a]$ commutes with both $[a,c,c]$ and b,

$$[a,c,c,b,b]^a = [a,c,c,b,[b,a]b] = [a,c,c,b,b].$$

From (6) we know that $[a,c,c,b,b]$ also commutes with b and c. Thus $[a,c,c,b,b] = 1$. Next we see that

$$[a,c,c,b,[a,c,c]]^a = [a,c,c,[b,a]b,[a,c,c]] = [a,c,c,b,[a,c,c]]$$

and $[a,c,c,b,[a,c,c]]$ commutes with a. From (6) this element also commutes with b and c. Hence $[a,c,c,b,[a,c,c]] = 1$ and $\langle [a,c,c],b\rangle$ is nilpotent of class at most 3. Thus, by this and (1), we have that $\langle [a,c,c],c,b\rangle$ is of type $(1,2,2)$ and thus nilpotent of class at most 3 by Proposition 4.1.

For the final claim, we have already seen that $\langle [c,a,a],b\rangle$ and $\langle [c,a,c],b\rangle$ are nilpotent of class at most 2. Thus $\langle [c,a,a],[c,a,c],b\rangle$ is of type $(1,2,2)$ and thus nilpotent of class at most 3 by Proposition 4.1. In particular

$[c, a, a, b, [c, a, c]]$ commutes with b. To see that the class is in fact at most 2, we only need to show that $[c, a, a, b, [a, c, c]] = [b, [c, a, a], [a, c, c]]^{-1}$ is trivial. By (4) we know that $[c, a, a, b]$ commutes with a. We also saw above that $\langle [c, a, a], b, c\rangle$ is nilpotent of class at most 3 that implies that $\langle [c, a, a, b], c\rangle$ is nilpotent of class at most 2. Thus $\langle [c, a, a, b], a, c\rangle$ is of type $(1, 2, 3)$ and thus nilpotent of class at most 4 by Proposition 4.1. Hence $[c, a, a, b, [a, c, c]]$ commutes with both a and c. We saw above that this element also commutes with b so it is in $Z(A)$ and thus trivial. This finishes Step 1.

Step 2. One shows that $[c, a, a] = [c, b, b] = 1$.

By symmetry, it suffices to show that $[c, a, a] = 1$. We saw in the proof of Step 1 that $\langle [c, a, a, b], a, c\rangle$ is nilpotent of class at most 4. It therefore follows that

$$\begin{aligned} [c, a, a, b, c, a, c]^{-2} &= [c, a, a, b, [a, c, c]] \\ [c, a, a, b, c, a, a] &= [c, a, a, b, [c, a, a]] \end{aligned}$$

and that these two elements commute with a and c. By Step 1 we also have that $\langle [c, a, a], [a, c, c], b\rangle$ is nilpotent of class at most 2. Thus the two elements above also commute with b and are thus trivial. As A is 2-torsion free, it follows that $[c, a, a, b, c, a, c] = [c, a, a, b, c, a, a] = 1$. The next aim is to show that $[c, a, a, b, c, a] = 1$ by showing that it commutes also with b. We use the fact that $\langle [c, a, a, b], c, a\rangle$ is nilpotent of class at most 4. This implies that $[c, a, a, b, c]$ commutes with $[c, a, a, b, [c, a]]$. Therefore,using $[c, a, a, b, c, a, c] = 1$,

$$\begin{aligned} [c, a, a, b, c, a] &= [c, a, a, b, c]^{-1}[c, a, a, b, c]^{a} \\ &= [c, a, a, b, c]^{-1}[c, a, a, b, c^{a}] \\ &= [c, a, a, b, c]^{-1}[c, a, a, b, c[c, a]] \\ &= [c, a, a, b, [c, a]][c, a, a, b, c, [c, a]] \\ &= [c, a, a, b, [c, a]]. \end{aligned}$$

By (3) we have that $[c, a, a, b, [c, a], b] \in \gamma_5(\langle a, b, a^{c}\rangle) = \{1\}$, and so by the equation above, we see that $[c, a, a, b, c, a]$ commutes with b as claimed. Hence $[c, a, a, b, c, a] = 1$. By Step 1 we also have that $[c, a, a, b, c]$ commutes with b and c and thus $[c, a, a, b, c] = 1$. By (4) $[c, a, a, b]$ commutes with a and b that gives us that $[c, a, a, b] = 1$. Finally, we know from (2) that $[c, a, a]$ commutes with a and c and therefore $[c, a, a] = 1$.

Step 3. One shows that $[a, b] = 1$.

We know by (5) that $\langle [a, b], a, c \rangle$ has class at most 3, and so

$$[a, b, c, c, a] = [a, b, c, c, c] = 1.$$

We also know that $\langle [a, b], b, c \rangle$ has class at most 3 and so $[a, b, c, c, b] = 1$. Hence $[a, b, c, c] = 1$. Using again the fact that $\langle [a, b], b, c \rangle$, $\langle [a, b], a, c \rangle$ are nilpotent of class at most 3, we see that $[[a, b], c, [a, b]]$ is central and thus trivial.

Now let x, y, z be any elements of G. Then

$$xyz = yzx[x, yz] = zxy[y, zx][x, yz] = xyz[z, xy][y, zx][x, yz]$$

so

$$[z, xy][y, zx][x, yz] = 1.$$

Substituting $[a, b, c]$ for x and substituting a, b for y, z respectively, we obtain

$$\begin{aligned} 1 &= [b, [a, b, c]a][a, b[a, b, c]][a, b, c, ab] \\ &= [b, a][a, b, c, b]^{-a}[a, b, c, a]^{-1}[a, b][a, b, [a, b, c]][a, b, c, b][a, b, c, a]^{b}. \end{aligned}$$

Now, we showed above that $[[a, b], [a, b, c]] = 1$, and since $[a, b]$ commutes with a and b, we see that $[a, b]$ commutes with $[a, b, c, a]$ and $[a, b, c, b]$. So

$$\begin{aligned} 1 &= [a, b, c, b]^{-a}[a, b, c, a]^{-1}[a, b, c, b][a, b, c, a]^{b} \\ &= [a, b, c, b, a]^{-1}[a, b, c, b]^{-1}[a, b, c, a]^{-1}[a, b, c, b][a, b, c, a][a, b, c, a, b] \\ &= [a, b, c, b, a]^{-1}[[a, b, c, b], [a, b, c, a]][a, b, c, a, b], \end{aligned}$$

which is equivalent to

$$[[a, b, c, b], [a, b, c, a]] = [a, b, c, b, a][a, b, c, a, b]^{-1}.$$

Now $[[a, b, c, b], [a, b, c, a]]$ commutes with c since $\langle [a, b], a, c \rangle$ and $\langle [a, b], b, c \rangle$ have class at most 3. Thus $[a, b, c, b, a][a, b, c, a, b]^{-1}$ commutes with c. Also, since $\langle [a, b], b, c \rangle$ has class at most 3,

$$[a, b, c, b] = [a, b, [c, b]] = [a, b, b^{-c}b],$$

and so

$$[a,b,c,b,a,a],[a,b,c,b,a,b] \in \gamma_5(\langle a,b,b^c\rangle) = \{1\}.$$

So $[a,b,c,b,a]$ is centralised by a and b. Similarly $[a,b,c,a,b]$ is centralised by a and b. Hence $[a,b,c,b,a][a,b,c,a,b]^{-1}$ is centralised by a,b,c and thus trivial.

Now consider the subgroup $\langle [a,b,c,a],c,b\rangle$. Since $\langle [a,b],a,c\rangle$ has class at most 3, we see that $[a,b,c,a]$ commutes with c. Also, using again the fact that $\langle [a,b],a,c\rangle$ has class at most 3, we have

$$[a,b,c,a] = [a,b,[c,a]] = [a,b,a^{-c}a]$$

and from $\gamma_5(\langle a,b,a^c\rangle = \{1\}$ it then follows that $\langle [a,b,c,a],b\rangle$ has class at most 2. Thus $\langle [a,b,c,a],c,b\rangle$ is of type $(1,2,3)$ and thus nilpotent of class at most 4. So

$$[a,b,c,a,b,c,c,b] = [a,b,c,a,b,c,c,c] = 1.$$

By symmetry between a and b, $\langle [a,b,c,b],c,a\rangle$ and has class at most 4, which implies that $[a,b,c,b,a,c,c,a] = 1$. But then the fact that $[a,b,c,b,a] = [a,b,c,a,b]$ implies that $[a,b,c,a,b,c,c]$ is central and thus trivial.

By Step 2 we know that $[c,b,b] = 1$. We know also that $[a,b,c,a,c] = 1$ and that $[a,b,c,a,b,b] \in \gamma_5(\langle a,b,a^c\rangle = \{1\}$, so (using again the fact that $\langle [a,b,c,a],c,b\rangle$ is of class at most 4)

$$1 = [a,b,c,a,[c,b,b]] = [a,b,c,a,b,c,b]^{-2}$$

and hence $[a,b,c,a,b,c,b] = 1$. By symmetry in a and b, we also have $[a,b,c,a,b,c,a] = [a,b,c,b,a,c,a] = 1$. Hence $[a,b,c,a,b,c] = [a,b,c,b,a,c]$ is central and thus trivial. Since $[a,b,c,a,b] = [a,b,[c,a],b]$ commutes with a,b by (3) we see then that $[a,b,c,a,b]$ is central and thus trivial. By (5), $[a,b,c,a,a] = [a,b,c,a,c] = 1$ that implies that $[a,b,c,a] = 1$. By symmetry in a and b, $[a,b,c,b] = 1$. We showed above that $[a,b,c,c] = 1$ and so $[a,b,c] = 1$. Since $\langle a,b\rangle$ is nilpotent of class at most 2, it follows that $[a,b]$ also commutes with a and b and thus $[a,b] = 1$ as required.

Step 4. We show that $[c,a,c] = [c,b,c] = 1$.

By symmetry it suffices to deal with $[a,c,c]$. To see that $[a,c,c] = 1$, it suffices to show that $[a, c, c, b] = 1$ as then $[a, c, c] \in Z(A) = \{1\}$. As $[a, c, c]$ and b commute with a, it follows that $[a, c, c, b]$ commutes with a. By Step 1 we also know that $[a, c, c, b]$ commutes with b. It thus remains to see that $[a, c, c, b, c] = 1$. By (6) this element commutes with b and c and our task is reduced to showing that $[a, c, c, b, c, a] = 1$. As $[a, c, c, b, c]$ and a commute with b we know that $[a, c, c, b, c, a]$ commutes with b. We finish the proof by showing that $[a, c, c, b, c, a]$ also commutes with a and c. Now $[a, c, c, b]$ commutes with a an by (6) we have that $\langle [a, c, c, b], c\rangle$ is nilpotent of class at most 2. Thus $\langle [a, c, c, b], a, c\rangle$ is of type $(1, 2, 3)$ and thus nilpotent of class at most 4. Thus by Step 1

$$[a, c, c, b, c, a, c]^{-2} = [a, c, c, b, [a, c, c]] = 1$$

and

$$[a, c, c, b, c, a, a] = [a, c, c, b, [c, a, a]] = 1.$$

As A is 2-torsion free, this shows that $[a, c, c, b, c, a]$ commutes with c and a as required. This completes Step 4.

So $\langle a, b, c\rangle$ is of type $(1, 2, 2)$ and thus nilpotent. Hence $A = G/HP(G)$ is trivial and thus G is nilpotent.

Notice that we have $[c, a, a, a^{-1}b, a^{-1}b] = [a^{-1}a^{a^{a^y}}, [a, a^x], [a, a^x]]$ is in $\gamma_5(\langle a, a^{a^{a^y}}, a^x\rangle)$. By Section 5.1 we know that $\langle a, a^{a^{a^y}}, a^x\rangle$ is nilpotent of class at most 4. We thus have the following relations for G.

$$\begin{array}{ll} [c, a, a, a^{-1}b, a^{-1}b] = 1 & \\ \langle a, b\rangle \text{ is nilpotent of class at most 2} & (*) \\ \langle u, v^w\rangle \text{ is nilpotent of class at most 3 for all } u, v, w \in \{a, b, c\}. & \end{array}$$

Notice that the last two relations in (*) are symmetric in a, b. We will now show that G is nilpotent of class at most 4. The proof of this is quite subtle and is done in two steps. First we use (*) to show that G is nilpotent of class at most 5. One can show this using either the nilpotent quotient algorithm or by hand. The lengthy but straightforward calculations are done by hand in Appendix C. We want to strengthen this and show that the class is at most 4. We first show that all commutators of weights $(1, 3, 1), (3, 1, 1)$ and $(1, 1, 3)$ are trivial. It follows that the first relation in (*) becomes $[c, a, a, b, b] = 1$. One then sees that all commutators of weight $(2, 2, 1)$ are trivial. The routine calculations can again be done using the nilpotent quotient algorithm or by hand as is done in Appendix C.

It remains to deal with commutators of weights $(1,2,2)$ and $(2,1,2)$. For this part we will need the following claim that is proved in Appendix C.

Claim *Let $H = \langle \bar{a}, \bar{b}, \bar{c} \rangle$ be a nilpotent 4-Engel group where $\langle \bar{a}, \bar{b} \rangle$, $\langle \bar{a}, \bar{c} \rangle$ are nilpotent of class at most 2 and where $\langle u, v^w \rangle$ is nilpotent of class at most 3 for all $u, v, w \in \{\bar{a}, \bar{b}, \bar{c}\}$. Then all commutators of multiweight $(2,1,2)$ in $\bar{a}, \bar{b}, \bar{c}$ are trivial.*

We now proceed with the commutators of weight $(2,1,2)$ and $(1,2,2)$. As all commutators of weight $(2,2,1)$ are trivial we have

$$[a^y, a, a, b, b] = [a^y, a, a, b, [a^y, a, a]] = 1.$$

Notice that this identity holds not only modulo $\gamma_6(\langle a, b, c\rangle)$ as we have shown that $\gamma_6(\langle a, b, c\rangle)$ is trivial. The reason why we needed to establish this fact first will now become clear. The displayed relations remain true if y is replaced by y^2 or y^{-1}. Thus if we let $d = a^{y^2}$ and $e = a^{y^{-1}}$ then $\langle [d, a, a], b\rangle$ and $\langle [e, a, a], b\rangle$ are also nilpotent of class at most 2. We continue arguing like in Step 2 of 5.1. Having already shown that $\langle a, a^{a^x}, a^y\rangle$ is nilpotent of class at most 5, we have in particular that $\langle a, a^{a^x}, a^{cd}\rangle$ is nilpotent of class at most 5. Let $\bar{c} = a^{cd}$ then $\langle a, b, \bar{c}\rangle$ has the further property that $\langle a, \bar{c}\rangle$ has class at most 2. By the claim above it follows that all commutators of weight $(2,1,2)$ in a, b and $\bar{c}$ are trivial. Thus

$$[b, [c, a, a][d, a, a], [c, a, a][d, a, a]] = [b, [cd, a, a], [cd, a, a]] = [b, [\bar{c}, a], [\bar{c}, a]] = 1.$$

Now $\langle [c, a, a], [d, a, a], b\rangle$ is of type $(1,2,2)$ and thus nilpotent of class at most 3. From $[b, [c, a, a][d, a, a], [c, a, a][d, a, a]] = 1$ we thus get that

$$[b, [c, a, a], [d, a, a]]^2 = 1.$$

that implies that $[b, [c, a, a], [d, a, a]] = 1$ as E is 2-torsion free. Similarly $[b, [d, a, a], [c, a, a]] = 1$, $[b, [d, a, a], [e, a, a]] = [b, [e, a, a], [c, a, a]] = 1$ and $[b, [e, a, a], [c, a, a]] = [b, [c, a, a], [e, a, a]] = 1$. Thus $\langle [c, a, a], [d, a, a], [e, a, a], b\rangle$ is nilpotent of class at most 2. Now using Lemma 4.2 we see that

$$[c, a, c, b]^8 = [c, a, a, b]^{-34}[d, a, a, b]^6[e, a, a, b]^{-30}$$

and as $[c,a,a,b],[d,a,a,b],[e,a,a,b]$ commute with b and a, it follows that $[c,a,c,b,b] = [c,a,c,b,a] = 1$. Notice that $c = a^y = b^u$ for some u. We know that $\langle [c,b,b],a\rangle$ is nilpotent of class at most 2 and by replacing u by u^2 or u^{-1} we see that it follows that $\langle [d,b,b],a\rangle$ and $\langle [e,b,b],a\rangle$ are also nilpotent of class at most 2, where $d = b^{u^2}$ and $e = b^{u^{-1}}$. We now argue similarly as above. Let $\bar{c} = b^{cd}$. We have that $\langle a,b,\bar{c}\rangle$ is nilpotent of class at most 5, since $\bar{c}$ is a conjugate of a and $\langle a, a^{a^x}, a^y\rangle$ is nilpotent of class at most 5. From the fact that $\langle b,\bar{c}\rangle$ is nilpotent of class at most 2 we see again that it follows from the claim above that all commutators of weight $(1,2,2)$ in $a,b,\bar{c}$ are trivial. Thus

$$[a,[c,b,b][d,b,b],[c,b,b][d,b,b]] = [a,[\bar{c},b],[\bar{c},b]] = 1$$

and $\langle [c,b,b],[d,b,b],[e,b,b],a\rangle$ is nilpotent of class at most 2 as before. Using Lemma 4.2 again we see that it follows that

$$[c,b,c,a]^8 = [c,b,b,a]^{-34}[d,b,b,a]^6[e,b,b,a]^{-30}$$

and the fact that $[c,b,b,a],[d,b,b,a],[e,b,b,a]$ commute with a and b implies hat $[c,b,c,a,b] = [c,b,c,a,a] = 1$.

Together with the relations (*) we have thus obtained the extra relations

$$[c,a,c,b,b] = [c,a,c,b,a] = [c,b,c,a,b] = [c,b,c,a,a] = 1.$$

One can now see that it follows from this that all commutators of weights $(2,1,2)$ and $(1,2,2)$ are trivial. The routine calculations are given in Appendix C. Thus G is nilpotent of class at most 4.

5.4 The subgroup $\langle a, a^x, a^y\rangle$

We let $G = \langle a,b,c\rangle$ where $b = a^x$ and $c = a^y$. We want to show that G is nilpotent. Using the results in Section 5.3 we may assume that $\langle a,b,a^c\rangle$ is nilpotent of class at most 4, with similar results for any permutations of a,b,c.

Step 1. We show that the subgroup $H = \langle a,[a,b,b],c\rangle$ is nilpotent of class at most 3.

We have

$$[[b,a,a],[c,a,a]] = [[b,a,a],[c,a,c]] = 1$$

since both these elements lie in $\gamma_5(\langle a, c, a^b\rangle(= \{1\}$. Using Lemma 4.2 (with b and c interchanged) this implies

$$[[b, a, b]^8, [c, a, a]] = [[b, a, b]^8, [c, a, c]] = 1.$$

By the remark made after Lemma 3.8, we thus have

$$[[b, a, b], [c, a, a]] = [[b, a, b], [c, a, c]] = 1.$$

This implies that $[c, a, a]$ and $[c, a, c]$ are in $Z(H)$. Also

$$[[a, b, b], c, [a, b, b]] = [[a, b, b], c, c, c] = [[a, b, b], c, c, [a, b, b]] = 1$$

since these elements lie in $\gamma_5(\langle b, c, b^a\rangle) = \{1\}$. Thus it follows that $H/Z(H) = \langle aZ(H), [a, b, b]Z(H), cZ(H)\rangle$ is of type $(1, 2, 3)$ and thus nilpotent of class at most 4. Hence H is nilpotent of class at most 5. We will see that the class is in fact going to be at most 3. For this we need one more relation. We have

$$[[b, a, a], [c, a], c] \in \gamma_5(\langle a, c, a^b\rangle) = \{1\}$$

and we use Lemma 4.2 to show that this implies that

$$[[a, b, b], [c, a], c] = 1.$$

We have that $b = a^x$, and we let $d = a^{x^2}, e = a^{x^{-1}}$. By Lemma 3.8 the subgroup $\langle a, b, d, e\rangle$ is nilpotent of class at most 3. Notice also that if u is one of b, d, e then $\gamma_3(\langle [u, a, a], [c, a]\rangle) \leq \gamma_5(\langle a, u, a^c\rangle = \{1\}$. Also, if u, v are two of b, d, e then

$$[[c, a], [u, a, a][v, a, a], [u, a, a][v, a, a]] \in \gamma_5(\langle a, a^{uv}, a^c\rangle = \{1\}$$

(notice that $\gamma_5(\langle a, a^{uv}, a^c\rangle) = \{1\}$ as $\gamma_5(\langle a, a^x, a^{a^y}\rangle = \{1\}$ and as x is arbitrary we can replace it with uv). Now $\langle [u, a, a], [v, a, a], [c, a]\rangle$ is of type $(1, 2, 2)$ and thus nilpotent of class at most 3. Then the last displayed equation shows that

$$[[c, a], [u, a, a], [v, a, a]]^2 = 1$$

and thus $[[c, a], [u, a, a], [v, a, a]] = 1$. Hence $\langle [b, a, a], [d, a, a], [e, a, a], [c, a]\rangle$ is nilpotent of class at most 2 and Lemma 4.2 implies that

$$[[b, a, b]^8, [c, a]] = [[b, a, a], [c, a]]^{-34}[[d, a, a], [c, a]]^6[[e, a, a], [c, a]]^{-30}.$$

We have showed above that $[b, a, a, [c, a]]$, $[d, a, a, [c, a]]$ and $[e, a, a, [c, a]]$ commute with c. It follows that $[[b, a, b]^8, [c, a], c] = 1$ and using the relations we have obtained for $\langle [b, a, b], a, c \rangle$, one can show that $[b, a, b, [c, a], c]^8 = 1$ (see Appendix C) and thus $[[a, b, b], [c, a], c] = 1$ as claimed. Adding this relation to H we can show (see Appendix C) that H is nilpotent of class at most 3.

Step 2. We show that $[a, b, b] \in Z_3(G)$.

As $\langle a, b \rangle$ is nilpotent of class at most 3, we need to show that $[a, b, b, c] \in Z_2(G)$ or equivalently that $[a, b, b, c, b], [a, b, b, c, a], [a, b, b, c, c] \in Z(G)$. By Step 1 we know that $\langle [a, b, b], c, a \rangle$ is nilpotent of class at most 3 and thus $[a, b, b, c, a], [a, b, b, c, c]$ commute with a, c. Also

$$[a, b, b, c, a, b] = [a, b, b, [c, a], b] \in \gamma_5(\langle b, a, a^c \rangle) = \{1\}$$

and

$$[a, b, b, c, c, b] \in \gamma_5(\langle b, c, b^a \rangle) = \{1\}.$$

Thus $[a, b, b, c, c], [a, b, b, c, a] \in Z(G)$ and we are only left with showing that $[a, b, b, c, b] \in Z(G)$. As $\langle b, c, b^a \rangle$ has class at most 4 we have $[a, b, b, c, b, b] = [a, b, b, c, b, c] = 1$ and as $\langle a, b, b^c \rangle$ is nilpotent of class at most 4 it follows that

$$[a, b, b, c, b, a] = [a, b, b, [c, b], a] = 1$$

(notice that $[a, b, b, c, b] = [a, b, b, [c, b]]$ follows from the fact that $\langle c, b, b^a \rangle$ is nilpotent of class at most 4). This finishes the proof of Step 2.

Notice that by symmetry we now have that $[b, a, a]$, $[a, b, b]$, $[c, a, a]$, $[a, c, c]$,$[c, b, b]$ and $[b, c, c]$ are all in $Z_3(G)$.

Step 3. G is nilpotent.

We let $A = G/HP(G)$ and we will show that A is trivial. Throughout Step 3 we are thus calculating modulo $HP(G)$. As before $Z(A) = \{1\}$ and A has no elements of order 2 or 3. From Step 2 we have that $[b, a, a]$, $[a, b, b]$, $[a, c, c]$, $[c, a, a]$, $[b, c, c]$, $[c, b, b]$ are all trivial. We show that A is abelian as this will imply that A is trivial. By symmetry it suffices to show that $[a, b] = 1$.

We know that the subgroups $\langle b, b^a, c \rangle$ and $\langle a, a^b, c \rangle$ are nilpotent of class at most 4. This implies that $\gamma_4(\langle [a,b],c\rangle) \leq Z(A)$ and thus trivial. The next step is to show that $[c,[a,b],[a,b]]$ is trivial. We know that this element commutes with c and as $[c,[a,b],[a,b]] = [c,[b,a],[b,a]]$ it suffices by symmetry to show that $[c,[a,b],[a,b]]$ commutes with a. But this follows from

$$\begin{aligned} 1 &= [c,a^b,a^b,a^b] \\ &= [c,a[a,b],a[a,b],a[a,b]] \\ &= [c,[a,b],[a,b],a]^3 \end{aligned}$$

where in the last identity we have used the fact that $\langle c,a,[a,b]\rangle$ is nilpotent of class at most 4 as well as $[c,a,a] = [b,a,a] = [c,[a,b],[a,b],[a,b]] = 1$. Thus

$$[a,b,c,[a,b]] = 1.$$

Next notice that $\langle a,a^b,c\rangle$ is nilpotent of class at most 4 and thus

$$\begin{aligned} 1 &= [c,a^b,a^b,c] \\ &= [c,a[a,b],a[a,b],c] \\ &= [c,a,[a,b],c]^2. \end{aligned}$$

Hence $[c,a,[a,b],c] = 1$. Notice that we have used in the calculations the facts that $[c,a,a] = [c,[a,b],[a,b]] = 1$. As $[[c,a],[a,b]] = [[c,a],[b,a]]^{-1}$ we have by symmetry that $[[c,a],[b,a]]$ commutes also with b. But both $[c,a]$ and $[b,a]$ commute with a. Hence $[[c,a],[b,a]] \in Z(A)$ and thus trivial. By symmetry we thus have that $[a,b],[b,c],[c,a]$ commute with each other. The next step is to show that $[a,b,c]$ commutes with a and b. As $[a,b,c,[a,b]] = 1$, we have $[b,a,c] = [a,b,c]^{-1}$ and it thus suffices by symmetry in a and b to show that $[a,b,c]$ commutes with a. This follows from

$$\begin{aligned} 1 &= [a,b,c]^a \\ &= [a,b,[c,a]a] \\ &= [a,b,c][a,b,[c,a]]^a \\ &= [a,b,c]. \end{aligned}$$

Thus $[a,b,c]$ commutes with both a and b. As $[a,c]$ and $[b,c]$ both commutes with $[a,b]$ and c we then have

$$[a,b,c,c]^a = [a,b,c,[c,a]c] = [a,b,c,c]$$

and likewise $[a, b, c, c]$ commutes with b. As we have seen above ($\gamma_4(\langle [a, b], c\rangle) = \{1\}$) that $[a, b, c, c, c] = 1$ we therefore have that $[a, b, c, c] \in Z(A)$ and thus trivial. We have then shown that $[a, b, c]$ commutes with a, b, c and is thus also trivial. Finally $[a, b]$ commutes also with a, b and hence $[a, b] = 1$. This finishes the proof of the nilpotence of G. In Appendix C we will show that the class is at most 5 and that $\langle a\rangle^{\langle a, a^x, a^y\rangle}$ is nilpotent of class at most 2. This finishes the proof of Proposition 5.1.

6 Reduction to left 3-Engel elements

In this section we prove the following result.

Proposition 6.1 *Let $a, x, c \in E$. Then*

$$[c, [x, a, a, a], [x, a, a, a], [x, a, a, a]] = 1.$$

As we know that every $\{2, 3\}$ element is in the Hirsch-Ploktin radical, we know that E is $\{2, 3\}$ torsion-free. By Proposition 5.1, we know that any subgroup of E generated by 3 conjugates a, a^u, a^v is nilpotent of class at most 5 and that the $\langle a\rangle^{\langle a, a^u, a^v\rangle}$ is nilpotent of class at most 2.

As $[x, a^{-1}, a^{-1}, a^{-1}] = [x, a^{-1}, a^{-1}, a^{-1}]^{a^3} = [[a, [a, [a, [a, x]]] = a^{-1}a^{a^{a^x}}$, we want to show that

$$[c, a^{-1}b, a^{-1}b, a^{-1}b] = 1$$

where $b = a^{a^{a^x}} = a[a, a^{a^x}]$. Now $a, b \in \langle a\rangle^{\langle a, a^x\rangle}$ and $a^{-1}b = [a, a^{a^x}] \in \gamma_2(\langle a\rangle^{\langle a, a^x\rangle})$. Thus

$$[c^r, u, v, w, z] = [c^r, u, a^{-1}b, v] = [c^r, u, v, a^{-1}b] = 1$$

for all $u, v, w, z \in \langle a\rangle \cup \langle b\rangle$. This is the case as these commutators are in $\gamma_3(\langle a\rangle^{\langle a, a^x, a^{c^r}\rangle})$ or $\gamma_3(\langle a\rangle^{\langle a, a^x, b^{c^r}\rangle})$, depending on whether $u \in \langle a\rangle$ or $u \in \langle b\rangle$. These identities will be use throughout in the following proof. Let $H = \langle a, b, c\rangle$ and $K = \langle a, b, [c, a], [c, b]\rangle$. We will first show that K is nilpotent of class at most 4 and then use this fact to prove Proposition 6.1.

Step 1. $[\gamma_3(\langle a\rangle^{\langle a, c\rangle}), K] = \{1\}$

To see this notice first that from the presentation of $\langle c, a\rangle$ (see Appendix A) we have that

$$\gamma_3(\langle a\rangle^{\langle a,c\rangle})^2 \leq \langle [c,a,a,c,a,c],[c,a,c,a,a],[c,a,a,a]\rangle.$$

By the remark made after Lemma 3.8, it therefore suffices to show that these three elements commute with all the elements in K. Let u be one of these three generators and let $k \in K$. In fact it follows from the same remark that it suffices to see that some $\{2,3\}$-number m we have that u^m commutes with k. Now

$$\begin{aligned} [c,a,c,a,a] &= [c,a,a,a]^{-2}[c^2,a,a,a] \\ [c,a,a,c,a,c]^{-2} &= [c,a,a,a]^3[c^2,a,a,a]^{-1}[c^{-1},a,a,a]. \end{aligned}$$

As we remarked above the right hand sides commute with a and b. Thus the three generators all commute with a and b and therefore, by the discussion above, all the elements in $\gamma_3(\langle a\rangle^{\langle a,c\rangle})$ commute with a and b. As $\gamma_3(\langle a\rangle^{\langle a,c\rangle})$ is normalised by c, we then also have $u^{[c,a]} = u^{c^{-1}a^{-1}ca} = u^{c^{-1}c} = u$ and similarly $u^{[c,b]} = u$.

Step 2. $[\gamma_2(\langle a\rangle^{\langle a,c\rangle}),K,K] = \{1\}$.

Let $N = \langle [c,a,a,c,c][c,a,c,a,c][c,a,c,c,a]\rangle\gamma_3(\langle a\rangle^{\langle a,c\rangle})$. From the presentation for $\langle a,c\rangle$ we know that $N^2 \leq \gamma_3(\langle a\rangle^{\langle a,c\rangle})$. Again, by the remark make after Lemma 3.8, it follows that $[N,K] = \{1\}$. By Step 1 it thus suffices to show that all the elements in $\gamma_2(\langle a\rangle^{\langle a,c\rangle})$ commute with the elements in K modulo N. From the presentation of $\langle a,c\rangle$, we can read that $\gamma_2(\langle a\rangle^{\langle a,c\rangle})$ is generated modulo N by $[c,a,a],[c,a,c,a]$ and $[c,a,c,a,c]$ and also that modulo N we have

$$\begin{aligned} [c,a,c,a] &= [c,a,a]^{-2}[c^2,a,a] \\ [c,a,c,a,c]^2 &= [c,a,a][c^2,a,a]^{-1}[c^{-1},a,a]. \end{aligned}$$

As $[c^r,a,a]$ commutes with $a^{-1}b$ for all integers r it follows that for any $u \in \gamma_2(\langle a\rangle^{\langle a,c\rangle}$, we have that $u^a = u^b$. Also $\gamma_2(\langle a\rangle^{\langle a,c\rangle})$ is normalised by c. As $[u,a],[u,[c,a]] \in N$ it follows that modulo N we have

$$u^b = u^a = u, \quad u^{[c,b]} = u^{c^{-1}b^{-1}cb} = u^{c^{-1}a^{-1}ca} = u.$$

Step 3. K is nilpotent of class at most 4.

By symmetry we also have that $[\gamma_2(\langle b\rangle^{\langle c,b\rangle}),K,K]=\{1\}$. Thus $[c,a,a],[c,b,b]\in Z_2(K)$. We calculate modulo $Z_2(K)$ and can assume that this subgroup is trivial. The aim is then to show that K is nilpotent of class at most 2. We have then that a commutes with a^c and b and that b commutes with b^c and a. Hence

$$K=\langle a,b,a^c,b^c\rangle=\langle a,b^c\rangle\times\langle b,a^c\rangle.$$

Now $[c,b]$ and a commute with b and thus $[c,b,a]$ commutes with ab^{-1} and b and thus a. Hence $[b^{-c},a^{-1},a^{-1}]=1$. Also $[c^{-1},a,b]^c$ commutes with a^c and $(a^{-1}b)^c$ and thus b^c. As a result we also have $[a^{-1},b^{-c},b^{-c}]=1$ and $\langle a,b^c\rangle$ is nilpotent of class at most 2. Similarly $\langle a^c,b\rangle$ is nilpotent of class at most 2 and thus $\gamma_3(K)\leq Z_2(K)$. This finishes the proof of Step 3.

We are now ready to prove Proposition 6.1 using the fact that K is nilpotent of class at most 4. We have

$$\begin{aligned}
[c,a^{-1}b,a^{-1}b,a^{-1}b] &= [[c,b][c,a^{-1}][c,a^{-1},b],a^{-1}b,a^{-1}b]\\
&= [[c,b][c,a^{-1}],a^{-1}b,a^{-1}b]^{[c,a^{-1},b]} \quad (\text{as } [c,a^{-1},b,a^{-1}b]=1])\\
&= [[c,b][c,a^{-1}],a^{-1}b,a^{-1}b]\\
&= [[d,b][d,a^{-1}][d,a^{-1},b],a^{-1}b] \quad (\text{where } d=[c,b][c,a^{-1}])\\
&= [[d,b][d,a^{-1}],a^{-1}b] \quad (\text{as } [d,a^{-1},b,a^{-1}b]=1).
\end{aligned}$$

Thus it suffices to show that $[d,b]$ and $[d,a^{-1}]$ commute with $a^{-1}b$. Firstly

$$[d,b]=[c,b,b][c,b,b,[c,a^{-1}]][c,a^{-1},b].$$

We know that $[c,b,b],[c,a^{-1},b]$ commute with $a^{-1}b$ and by Step 2 above we also know that $[c,b,b],[c,a^{-1}]]\in[\gamma_2(\langle b\rangle^{\langle c,b\rangle}),K]$ commutes with a and b. Hence $[d,b]$ commutes with $a^{-1}b$.

We want to deal similarly with $[d,a^{-1}]$. Notice first that

$$\begin{aligned}
[c,ba^{-1}] &= [c,a^{-1}][c,b][c,b,a^{-1}]\\
[c,a^{-1}b] &= [c,b][c,a^{-1}][c,a^{-1},b].
\end{aligned}$$

As a commutes with b this implies that

$$d=[c,b][c,a^{-1}]=[c,a^{-1}][c,b]t$$

where $t = [c, b, a^{-1}][c, a^{-1}, b]^{-1}$. Then

$$[d, a^{-1}] = [[c, a^{-1}][c, b], a^{-1}]^t [t, a^{-1}]$$

and as t, a^{-1} commute with $a^{-1}b$ it remains to see that $[[c, a^{-1}][c, b], a^{-1}]$ commmutes with $a^{-1}b$. However

$$[[c, a^{-1}][c, b], a^{-1}] = [c, a^{-1}, a^{-1}][c, a^{-1}, a^{-1}, [c, b]][c, b, a^{-1}].$$

Again $[c, a^{-1}, a^{-1}], [c, b, a^{-1}]$ commute with $a^{-1}b$ as well as $[c, a^{-1}, a^{-1}, [c, b]] \in [\gamma_2(\langle a\rangle^{\langle a,c\rangle}), K]$ by Step 2. This finishes the proof of Proposition 6.1.

7 Completion of the proof

It remains now only to see that all the left 3-Engel elements of E are in the Hirsch-Plotkin radical of E as then by last section E will be a 3-Engel group and thus locally nilpotent by Heineken's Theorem.

Proposition 7.1 *Any left* 3*-Engel element of E is contained in the Hirsch-Plotkin radical.*

Proof It is well known and easy to see that a is a left 3-Engel element in E if and only if any two conjugates of a generate a subgroup that is nilpotent of class at most 2. We need to show that $\langle a\rangle^E$ is locally nilpotent. Let $a_1, \ldots, a_k$ be conjugates of a, and let $H = \langle a_1, \ldots, a_k\rangle$. We will show by induction that H is nilpotent of class at most k and that the normal closure of a_i in H is abelian for $i = 1, \ldots, k$.

The case $k = 2$ holds by the assumption that a is a left 3-Engel element. Now suppose that $k = 3$. Here a_1 commutes with $a_1^{a_2}$ and thus $\langle a_1, a_1^{a_2}, a_3\rangle$ is of type $(1, 2, 2)$ and thus nilpotent of class at most 3 by Proposition 4.1 (we need here the fact that H is 2-torsion free). In particular any simple commutator in $a_1, a_1^{a_2}, a_3$ with two occurrences of a_3 is trivial that implies that

$$[a_1, a_2, a_3, a_3] = [a_1^{-a_2} a_1, a_3, a_3] = 1.$$

Similarly $[a_i, a_j, a_k, a_k] = 1$ for all $i, j, k \in \{1, 2, 3, \}$. It follows that H satisfies the relations:

$$\begin{array}{ll} [a_i, a_j, a_j] = 1 & \text{for all } i, j \in \{1, 2, 3\} \\ [a_i, a_j, a_k, a_k] = 1 & \text{for all } i, j, k \in \{1, 2, 3\}. \end{array}$$

Since H is generated by three conjugates, we know from last section that H is nilpotent. Let us see that the class is at most 3. We can then assume that $\gamma_5(H)$ is trivial. We need to show that any simple commutator of weight 4 in a_1, a_2, a_3, a_4 is trivial. By symmetry we only need to deal with commutators of weight $(2, 1, 1)$ in a_1, a_2, a_3. And the only commutator that we need to consider is $[a_1, a_2, a_3, a_1]$. However (using $[a_2, a_3, a_1, a_1] = 1$)

$$1 = [a_2, [a_3, a_1, a_1]] = [a_2, a_1, a_3, a_1]^{-2} = [a_1, a_2, a_3, a_1]^2$$

and as G has no elements of order 2 it follows that $[a_1, a_2, a_3, a_1] = 1$.

Now suppose that $k \geq 3$. Let $u = [a_1, a_2, \ldots, a_{k-2}]$. The subgroup $\langle a_{k-1}, a_{k-1}^u, a_k\rangle$ is then of type $(1, 2, 2)$ and thus nilpotent of class at most 3. By this and fact that any two conjugates generate a subgroup of class at most 2, it follows that

$$[a_1, a_2, \ldots, a_{k-1}, a_k, a_k] = [a_{k-1}^{-u} a_{k-1}, a_k, a_k] = 1$$

and

$$[a_k,_3 [a_1, a_2, \ldots, a_{k-1}]] = [a_k,_3\, a_{k-1}^{-u} a_{k-1}] = 1.$$

We thus have the following identities which hold for any conjugates $a_1, a_2, \ldots, a_k$ of a and for any $k \geq 3$.

$$\begin{aligned} [a_1, a_2, \ldots, a_{k-1}, a_k, a_k] &= 1, \\ [a_k, [a_1, a_2, \ldots, a_{k-1}], [a_1, a_2, \ldots, a_{k-1}], [a_1, a_2, \ldots, a_{k-1}]] &= 1. \end{aligned} \tag{13}$$

We now proceed with the induction step. Let $k \geq 4$ and suppose that the result is true for all smaller values of k. We first show that if $1 \leq r \leq k$, then

$$[[a_1, a_2, \ldots, a_r], [a_1, a_k, a_{k-1}, \ldots, a_{r+1}]] = [a_1, a_2, \ldots, a_k, a_1]^{(-1)^{k-r}}. \tag{14}$$

This is obvious when $r = k$. Now consider the case $r = k - 1$. Let $u = [a_1, \ldots, a_{k-1}]$. By the induction hypothesis and (14) we have that $\langle a_1, u, a_k\rangle$ is of type $(1, 2, 3)$ and thus nilpotent of class at most 4 by Proposition 4.1. Using the fact that u commutes with a_1 and the first identity in (14) one sees easily that all commutators of weight $(2, 1, 1)$ and $(1, 1, 2)$ in a_1, u, a_k are trivial. The only commutators that one needs to consider are $[u, a_k, a_1, a_1]$ and $[u, a_k, a_1, a_k]$ but as $[u, [a_k, a_1, a_1]] = [u, [a_1, a_k, a_k]] = 1$ we get by expanding these that

$$\begin{aligned} 1 &= [u, a_k, a_1, a_1], \\ 1 &= [u, a_k, a_1, a_k]. \end{aligned}$$

From this one sees that $[u, [a_1, a_k]] = [u, a_k, a_1]^{-1}$ that gives us the identity (15) when $r = k-1$. This argument also tells us that

$$[[a_1, a_k, \ldots, a_3], [a_1, a_2]] = [a_1, a_k, \ldots, a_2, a_1]^{-1}$$

and thus

$$[a_1, [a_1, a_k, \ldots, a_2]] = [[a_1, a_2], [a_1, a_k, \ldots, a_3]]^{-1}$$

that shows us that the case $r = 1$ follows if it holds for $r = 2$. To establish (15) it is thus sufficient to show that if $2 \leq r \leq k-2$, then

$$[[a_1, a_2, \ldots, a_r], [a_1, a_k, \ldots, a_{r+1}]] = [[a_1, a_2, \ldots, a_{r+1}], [a_1, a_k, \ldots, a_{r+2}]]^{-1}.$$

Let $u = [a_1, a_2, \ldots, a_r]$ and $v = [a_1, a_k, \ldots, a_{r+2}]$. By the induction hypothesis we have that u and v commute and that $\langle u, a_{r+1}\rangle$, $\langle v, a_{r+1}\rangle$ are nilpotent of class at most 2. Thus $\langle u, v, a_{r+1}\rangle$ is of type $(1, 2, 2)$ and thus nilpotent of class at most 3. Thus $[u, [v, a_{r+1}]] = [u, a_{r+1}, v]^{-1}$ as was required. This establishes (15).

We want to show that H is nilpotent of class at most k. We establish this in two steps. First we show that H is nilpotent (of class at most $k+1$) and then that $\gamma_{k+1}(H)$ is a $\{2, 3\}$ group and thus trivial. We turn to the first step. Consider a commutator $c = [b_1, b_2, \ldots, b_{k+1}]$ where $b_1, \ldots, b_{k+1}$ lie in $\{a_1, \ldots, a_k\}$. We want to show that $c \in Z(H)$. By induction $c = 1$ unless $\{b_1, \ldots, b_k\} = \{a_1, \ldots, a_k\}$. Also by (14) we have that $c = 1$ if $b_k = b_{k+1}$. So there is no loss of generality in assuming that $b_{k+1} = a_1, b_k = a_k$ and that $\{b_1, \ldots, b_{k-1}\} = \{a_1, \ldots, a_{k-1}\}$. Then, using the inductive hypothesis, we see that $[b_1, b_2, \ldots, b_{k-1}]$ can be expressed as a product $u_1 u_2 \cdots u_r$ where each u_i is a commutator of the form $[a_1, a_{\sigma(2)}, a_{\sigma(3)}, \cdots, a_{\sigma(k-1)}]$ for some permutation σ of $\{2, 3, \ldots, k-1\}$. So

$$c = [b_1, \ldots, b_{k+1}] = [u_1 \cdots u_r, a_k, a_1] = [\prod_{i=1}^{r} [u_i, a_k]^{u_{i+1} u_{i+2} \cdots u_r}, a_1].$$

Now the inductive hypothesis implies that $u_1, u_2, \ldots, u_r$ commute with a_1. So c is the product of conjugates of the commutators $[u_1, a_k, a_1], \ldots, [u_r, a_k, a_1]$. To show that $c \in Z(H)$ it thus clearly suffices to show that $d = [a_1, a_2, \ldots, a_k, a_1] \in Z(H)$.

So consider $d = [a_1, a_2, \ldots, a_k, a_1, a_i]$, where $1 \leq i \leq k$. If $i = 1$ then $d = 1$ by (14). If $i = k$, let $u = [a_1, a_2, \ldots, a_{k-1}]$. Then, using the induction hypothesis, $\langle a_1, u, a_k \rangle$ is of type $(1, 2, 3)$ and thus nilpotent of class at most 4. Thus

$$1 = [u, [a_1, a_k, a_k]] = [u, a_k, a_1, a_k]^{-2}.$$

This implies that $[u, a_k, a_1, a_k] = 1$ and thus $d = 1$ when $i = k$.

Now let $1 < i < k$. To show that $d = 1$, it suffices by (15) to show that $[u, a_i, v, a_i] = 1$ where $u = [a_1, a_2, \ldots, a_{i-1}]$ and $v = [a_1, a_k, a_{k-1}, \ldots, a_{i+1}]$. Now by the induction hypothesis $\langle u, v, a_i \rangle$ is of type $(1, 2, 3)$ and thus nilpotent of class at most 4. Thus again

$$1 = [u, [v, a_i, a_i]] = [u, a_i, v, a_i]^{-2}$$

that implies that $[a_1, a_2, \ldots, a_k, a_1]$ commutes with a_i. This finishes the proof that H is nilpotent of class at most $k + 1$. To show that the class is actually k it suffices to show that $[a_1, a_2, \ldots, a_k, a_1] = 1$ since by the argument above, this will imply that $[b_1, b_2, \ldots, b_{k+1}] = 1$ for all $b_1, \ldots, b_{k+1} \in \{a_1, \ldots, a_k\}$.

In order to achieve this we first will show that

$$[a_1, a_2, \ldots, a_{k-3}, a_{\sigma(k-2)}, a_{\sigma(k-1)}, a_{\sigma(k)}, a_1] = [a_1, a_2, \ldots, a_k, a_1]$$

for all permutations σ of $\{k-2, k-1, k\}$. By the induction hypothesis we have that $[a_1, a_2, \ldots, a_{k-3}, a_{k-2}^{-1}, a_{k-2}^{a_{k-1}}] = 1$ and that the elements $[a_1, a_2, \ldots, a_{k-3}, a_{k-2}^{-1}]$ and $[a_1, a_2, \ldots, a_{k-3}, a_{k-2}^{a_{k-1}}]$ commute with $[a_k, a_1]$. Thus

$$[a_1, a_2, \ldots, a_{k-3}, [a_{k-2}, a_{k-1}], a_k, a_1] = [a_1, a_2, \ldots, a_{k-3}, a_{k-2}^{-1} a_{k-2}^{a_{k-1}}, [a_k, a_1]] = 1.$$

Similarly, we have that $[a_1, a_2, \ldots, a_{k-3}, a_k]$, $[a_1, a_2, \ldots, a_{k-3}, a_k^{-[a_{k-2}, a_{k-2}]}]$ and $[a_1, a_2, \ldots, a_{k-3}, a_k^{-[a_{k-2}, a_{k-1}]}, a_k]$ commute with a_1 by the induction hypothesis, and thus that

$$\begin{aligned} 1 &= [a_1, a_2, \ldots, a_{k-3}, [a_{k-2}, a_{k-1}, a_k], a_1] \\ &= [a_1, \ldots, a_{k-3}, [a_{k-2}, a_{k-1}], a_k, a_1][a_1, \ldots, a_{k-3}, a_k, [a_{k-1}, a_{k-2}], a_1]^{-1}. \end{aligned}$$

By the previous part this implies that $[a_1, \ldots, a_{k-3}, a_k, [a_{k-1}, a_{k-2}], a_1] = 1$. This gives us that $[a_1, \ldots, a_{k-3}, a_{\sigma(k-2)}, a_{\sigma(k-1)}, a_{\sigma(k)}, a_1]$ is equal to $[a_1, a_2, \ldots, a_k, a_1]$ for all the permutations σ of $\{k-2, k-1, k\}$ as we wished to show.

Now let $u = x_{k-2}x_{k-1}x_kx_{k+1}$ and consider the following instance of the 4-Engel identity:

$$[x_1, x_2, \ldots, x_{k-3}, u, u, u, u] = 1.$$

If we expand the commutator $[x_1, x_2, \ldots, x_{k-3}, u, u, u, u]$ and pick out the terms which involve all the variables $x_1, \ldots, x_{k+1}$, then we obtain the identity

$$\prod_{\sigma}[x_1, x_2, \ldots, a_{k-3}, x_{\sigma(k-2)}, x_{\sigma(k-1)}, x_{\sigma(k)}, x_{\sigma(k+1)}] \in \gamma_{k+2}(\langle x_1, \ldots, x_{k+1}\rangle),$$

where the product ranges over all permutations of $\{k-2, k-1, k, k+1\}$. If we substitute a_i for x_i for $i = 1, \ldots, k$ and we substitute a_1 for x_{k+1}, and if we use the fact that H has class at most $k+1$, then we obtain

$$\prod_{\sigma}[a_1, a_2, \ldots, a_{k-3}, a_{\sigma(k-2)}, a_{\sigma(k-1)}, a_{\sigma(k)}, a_{\sigma(1)}] = 1,$$

where the product ranges over all permutations of $\{k-2, k-1, k, 1\}$. By the induction hypothesis all the factors where $\sigma(1) \neq 1$ is trivial and by the analysis above we know that all the remaining six factors are equal to $[a_1, \ldots, a_k, a_1]$. Hence $[a_1, \ldots, a_k, a_1]^6 = 1$ and as there are no elements in G of order 2 or 3, it follows that $[a_1, a_2, \ldots, a_k, a_1] = 1$ and required. This finishes the proof of Proposition 7.1. $\Box$.

Appendix A. 4-Engel groups of rank 2

Let $G = E(2, 4)$ be the free 4-Engel group of rank 2. We have already seen that this group is nilpotent. The structure of the group can then be determined using the nilpotent quotient algorithm. This has been done in [7] for example and there a consistent power commutator presentation is given. (Notice however that there is one misprint in the presentation and $[a_6, a_2]$ should be $a_9^{-2}a_{11}a_{12}^{-6}$). Such a presentation is certainly best obtained with an aid of a computer. We show however in this section that it is not that difficult to obtain the relevant relations by hand. The class of $E(2, 4)$ is only 6 and the 4-Engel identity only implies extra relations in $\gamma_5(G)$. Then $\gamma_5(G)/\gamma_6(G)$ is of rank 3 and $\gamma_6(G)$ is cyclic. The fact that the presentation that one obtains is consistent is not needed for the proof as we only need to show that the given relations hold in $E(2, 4)$. In part I we obtain the consequences of the 4-Engel identity in $\gamma_5(G)$ and then we use these and the Hall-Witt identity to obtain the presentation in part II.

I. Identities in $\gamma_5(G)$

Let x and y be the free generators of $G = E(2,4)$. We first consider relations modulo $\gamma_6(G)$. Let $c(y,x) = [y,x,y,y,x][y,x,x,y,y][y,x,y,x,y]$. Expanding $1 = [y,_4 y^n x]$ and then letting $n = 1$ and $n = -1$ gives

$$\begin{aligned} c(y,x) &\equiv c(x,y) \pmod{\gamma_6(G)} \\ c(x,y)^2 &\equiv 1 \pmod{\gamma_6(G)}. \end{aligned}$$

Let $a(x,y) = [x,y,y,x,x]$. It is now easy to see that $\gamma_5(G)$ is generated modulo $\gamma_6(G)$ by $c(y,x), a(y,x)$ and $a(x,y)$.

We next turn to relations modulo $\gamma_7(G)$. Notice first that

$$1 = [y^2,_4 x] = [y,_4 x]^2[y,x,y,_3 x] = [a(x,y),x]^{-1}.$$

Thus

$$\begin{aligned} [a(x,y),x] &\equiv 1 \pmod{\gamma_7(G)} \\ [a(y,x),y] &\equiv 1 \pmod{\gamma_7(G)}. \end{aligned}$$

Expansion of $1 = [y,_4 y^n x]$ gives us that modulo $\gamma_7(G)$

$$c(y,x)^{n^2} c(x,y)^{-n}[a(y,x),x]^{n^2-\binom{n}{2}}[a(x,y),y]^{2\binom{n}{2}-2n^2}[c,x]^{n^2+n+\binom{n}{2}}[c,y]^{n^3+n\binom{n}{2}+\binom{n}{2}} = 1. \tag{15}$$

Putting $n = 1$,$n = -1$ and $n = 2$ and comparing, shows that $[c,y] = 1$ and by symmetry $[c,x] = 1$. It follows that $\gamma_6(G)$ is generated by $[a(x,y),y]$ and $[a(y,x),y]$ modulo $\gamma_7(G)$.

Next we expand $[uv,_4 w]$ modulo $\gamma_7(\langle u,v,w\rangle)$. As $[u,_4 w] = [v,_4 w] = 1$, we get that $[u,w,v,_3 w] = 1$. Putting $u = v = y$ and $w = yx$, this gives (using $[c,x] = [c,y] = 1$) that

$$1 = [y,x,y,_3 yx] = [a(x,y),y][a(y,x),x].$$

Let $d = [a(y,x),x]$. Now

$$[a(x,y),y] = d^{-1}$$

and $\gamma_6(G)$ is generated modulo $\gamma_7(G)$ by d. In order to show that G is of class 6 it suffices to show that $[d,x],[d,y] \in \gamma_8(G)$.

Expanding $[uvw,_4 x] = 1$ modulo $\gamma_8(\langle u, v, w, x\rangle)$ and picking the term that involves all of u, v, w, x, we see that $1 = [u, x, v, w,_3 x]$ is in $\gamma_8(\langle u, v, w, x\rangle)$. In particular for $u = v = w = y$, we get (modulo $\gamma_8(G)$)

$$1 = [y, x, y, y, x, x, x] = [c(y, x)a(y, x)^{-2}, x, x] = [d, x]^{-2}.$$

Thus by symmetry $[d, y]^2 \in \gamma_8(G)$. Now using (16) for $n = 1$ and $n = -1$ gives that modulo $\gamma_8(G)$

$$1 \equiv [c(y, x)^2[a(y, x), x][a(x, y), y]^{-2}, x] = [d, x]^3.$$

Hence $[d, x] = [d, y] = 1$ and G is nilpotent of class at most 6.

II. Presentation

To summarise the relations from part I, we have that $\gamma_5(G)$ is generated by

$$\begin{aligned}
&a(y, x) = [y, x, x, y, y]\\
&a(x, y) = [x, y, y, x, x]\\
&c(y, x) = [y, x, y, y, x][y, x, y, x, y][y, x, x, y, y]\\
&d = [a(y, x), x]
\end{aligned}$$

and we have the relations (using (16) for $n = 1$ and $n = -1$)

$$\begin{aligned}
[a(y, x), x] &= d\\
[a(x, y), y] &= d^{-1}\\
[a(x, y), x] &= 1\\
[a(y, x), y] &= 1\\
c(x, y) &= c(y, x)d^3\\
c(y, x)^2 &= d^{-3}\\
[c(y, x), x] &= 1\\
[c(y, x), y] &= 1.
\end{aligned} \tag{16}$$

From these we see that all those commutators of weight 6 in x, y, where the multiweight is not $(3, 3)$, are trivial. Also

$$\begin{aligned}
[y, x, y, y, x, x] &= d^{-2}\\
[y, x, x, x, y, y] &= d^{-2}
\end{aligned}$$

$$
\begin{array}{rcl}
[y,x,y,x,y,x] & = & d \\
[y,x,x,y,y,x] & = & d \\
[y,x,y,x,x,y] & = & d \\
[y,x,x,y,x,y] & = & d \\
[y,x,y,x,[y,x]] & = & 1 \\
[y,x,y,[y,x,x]] & = & d^{-3}.
\end{array} \tag{17}
$$

Calculating modulo $\gamma_6(G)$, we have

$$
\begin{array}{rcl}
1 & \equiv & [y,x,y^{-1},x]^y[y,x^{-1},[y,x]]^x[x,[y,x]^{-1},y]^{[y,x]} \\
 & \equiv & [y,x,y,x]^{-1}[y,x,x,y][y,x,y,y,x][y,x,y,x,y]^{-1}[y,x,x,[y,x]].
\end{array}
$$

From this we get

$$
[y,x,x,y,x] = [y,x,y,x,x][y,x,y,x,y,x][y,x,y,y,x,x]^{-1}
$$

and thus

$$
\begin{array}{rcl}
[y,x,y,x,x] & = & a(x,y)^{-1} \\
[y,x,x,y,x] & = & a(x,y)^{-1}d^3 \\
[y,x,x,x,y] & = & c(y,x)^{-1}a(x,y)^2d^{-6} \\
[y,x,x,y,y] & = & a(y,x) \\
[y,x,y,x,y] & = & a(y,x)d^3 \\
[y,x,y,y,x] & = & c(y,x)^{-1}a(y,x)^{-2}d^{-6}.
\end{array} \tag{18}
$$

Next

$$
\begin{array}{rcl}
1 & = & [y,x,x,x^{-1},y]^x[x,y^{-1},[y,x,x]]^y[y,[y,x,x]^{-1},x]^{[y,x,x]} \\
 & = & [y,x,x,x,y]^{-1}[y,x,x,[y,x]]^{-1}[y,x,x,y,x][x,y,y,[y,x,x]][y,x,x,[y,x],y]^{-1},
\end{array}
$$

that gives

$$
\begin{array}{rcl}
[y,x,x,[y,x]] & = & c(y,x)^{-1}a(x,y)^{-3}d^6 \\
[y,x,y,[y,x]] & = & c(y,x)^{-1}a(y,x)^{-3}d^{-9}.
\end{array} \tag{19}
$$

Then

$$
\begin{array}{rcl}
1 & = & [y,x,x^{-1},y]^x[x,y^{-1},[y,x]]^y[y,[y,x]^{-1},x]^{[y,x]} \\
 & = & [y,x,x,y]^{-1}[y,x,y,x][y,x,x,x,y][y,x,x,y,x]^{-1} \\
 & & [x,y,y,[y,x]][y,[y,x],[y,x],x][y,x,y,x,[y,x]]
\end{array}
$$

and

$$[y,x,x,y] = [y,x,x,y]a(x,y)^3 a(y,x)^3 d^3. \tag{20}$$

From these relations, it is now straightforward to come up with a power commutator presentation for $E(2,4)$. Let $a_1 = x, a_2 = y, a_3 = [y,x], a_4 = [y,x,x], a_5 = [y,x,y], a_6 = [y,x,x,x], a_7 = [y,x,y,x]$, $a_8 = [y,x,y,y]$, $a_9 = a(x,y)^{-1}$, $a_{10} = a(y,x)d^3$, $a_{11} = c(y,x)^{-1}$ and $a_{12} = d$. We see from (17)-(21), that the following relations are satisfied (the trivial commutators are omitted)

$$a_{11}^2 = a_{12}^3$$

$$\begin{array}{llll}
[a_2,a_1] = a_3, & & & \\
[a_3,a_1] = a_4, & [a_3,a_2] = a_5, & & \\
[a_4,a_1] = a_6, & [a_4,a_2] = a_7 a_9^{-3} a_{10}^3 a_{12}^{-6}, & [a_4,a_3] = a_9^3 a_{11} a_{12}^6, & \\
[a_5,a_1] = a_7, & [a_5,a_2] = a_8, & [a_5,a_3] = a_{10}^{-3} a_{11}, & [a_5,a_4] = a_{12}^{-3}, \\
 & [a_6,a_2] = a_9^{-2} a_{11} a_{12}^{-6}, & & \\
[a_7,a_1] = a_9, & [a_7,a_2] = a_{10}, & & \\
[a_8,a_1] = a_{10}^{-2} a_{11}, & & & \\
 & [a_9,a_2] = a_{12}, & & \\
[a_{10},a_1] = a_{12}. & & &
\end{array}$$

One can easily check that conversely, the group F defined by this presentation is a 4-Engel group and thus $F = E(2,4)$. As the group is nilpotent of class at most 6, one just has to expand $[y^r x^s [y,x]^t,_4 y^m x^n [y,x]^p]$ and check that it is trivial. This is however not needed for the proof of the local nilpotence result. For that we only need to know that $E(2,4)$ satisfies the relations above or in other words that it is a quotient of the group F. One can also check that the presentation is consistent, but again this is not needed for the proof.

Appendix B. 4-Engel groups of type $(1,2,3)$

Let $G = \langle a,b,c \rangle$ be a 4-Engel group of type $(1,2,3)$. By Proposition 4.1, this group is nilpotent and in this section, we show that G is nilpotent of class at most 4. Without loss of generality we can then assume that $\gamma_6(G) = \{1\}$. We know that

$$\gamma_2(\langle a,b \rangle) = \gamma_3(\langle a,c \rangle) = \gamma_4(\langle b,c \rangle) = \{1\}. \tag{21}$$

We need to show that any commutator of weight 5 in a, b, c is trivial. By (22) it is clear that we only need to deal with the multiweights $(1,1,3)$, $(1,2,2)$, $(1,3,1)$, $(3,1,1)$, $(2,1,2)$ and $(2,2,1)$ in a, b, c. The only commutators of these multiweights that we need to consider are

$$\begin{array}{ll}
\text{Weight}\,(1,1,3): & [a,c,b,c,c] \\
\text{Weight}\,(1,3,1): & [a,c,b,b,b] \\
\text{Weight}\,(1,2,2): & [a,c,b,b,c],\ [a,c,b,c,b] \\
\text{Weight}\,(3,1,1): & [b,c,a,a,a] \\
\text{Weight}\,(2,1,2): & [b,c,c,a,a],\ [b,c,a,c,a],\ [b,c,a,a,c] \\
\text{Weight}\,(2,2,1): & [c,a,a,b,b]
\end{array}$$

The following calculations, show that these are all trivial. Firstly it follows form (22) that

$$1 = [a,[b,c,c,c]] = [a,c,b,c,c]^{-3}$$

and thus $[a,c,b,c,c] = 1$ as G is 3-torsion free. Next using (22) again

$$1 = [a,[c,b,b,b]] = [a,c,b,b,b].$$

Having dealt with weights $(1,1,3)$ and $(1,3,1)$, we turn to weight $(1,2,2)$. Firstly

$$1 = [a,[b,c,c,b]] = [a,[b,c,c],b] = [a,c,b,c,b]^{-2},$$

and as G has no elements of order 2, it follows that $[a,c,b,c,b] = 1$. Next using the linearised 4-Engel identity (obtained by expanding $[a,_4 x_1x_2x_3x_4] = 1$)

$$1 = \prod_{\sigma \in S_4} [a, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}],$$

for $x_1 = x_2 = b$ and $x_3 = x_4 = c$, we obtain $1 = [a,c,b,b,c]^4$ that gives us $[a,c,b,b,c] = 1$ as there is no 2-torsion. For weight $(3,1,1)$ simply notice that

$$1 = [b,[c,a,a],a] = [b,c,a,a,a].$$

For weight $(2,1,2)$, we need to deal with three commutators. First notice that

$$1 = [b,[c,a,a],c] = [b,c,a,a,c]$$

and using this we furthermore have

$$1 = [b,c,[c,a,a]] = [b,c,c,a,a][b,c,a,c,a]^{-2}.$$

Using again the linearised 4-Engel identity for $x_1 = x_2 = a$ and $x_3 = x_4 = c$, we get

$$1 = [b, c, c, a, a]^4 [b, c, a, c, a]^4$$

From this and the last identity it follows that $[b, c, a, c, a]^3 = 1$ that gives us $[b, c, a, c, a] = 1$ and then also $[b, c, c, a, a] = 1$. It now only remains to deal with weight $(2, 2, 1)$. We apply once again the linearised 4-Engel identity, this time with $x_1 = x_2 = a$ and $x_3 = x_4 = b$. This gives

$$1 = [c, a, a, b, b]^{4!} = [c, a, a, b, b]^{3 \cdot 8}.$$

Again, as G is $\{2, 3\}$-torsion free, it follows that $[c, a, a, b, b] = 1$. We have thus shown that G is nilpotent of class at most 5.

Now suppose that the $G = \langle a, b, c\rangle$ is of type $(1, 2, 2)$. We show that in this case we get the stronger result that the nilpotency class is at most 3. We only need to deal with commutators of multiweight $(1, 1, 2), (1, 2, 1)$ and $(2, 1, 1)$. As $\langle a, c\rangle$ and $\langle b, c\rangle$ are nilpotent of class at most 2 it is clear that $[c, a, a, b] = [c, b, b, a] = 1$. We are thus only left with the multiweight $(1, 1, 2)$ and as $[a, c, c, b] = 1$ only the commutator $[a, c, b, c]$ remains to be dealt with. However,

$$1 = [a, [b, c, c]] = [a, c, b, c]^{-2}.$$

and as G is 2-torsion free it follows that $[a, c, b, c] = 1$.

Appendix C. Subgroups generated by 3 conjugates

I. The subgroup $\langle a, a^{a^{a^x}}, a^y\rangle$

Let $b = a^{a^{a^x}}$ and $c = a^y$. In Section 5.1 we have already proved that $A = \langle a, b, c\rangle$ is nilpotent. In this section we show that the nilpotency class is at most 4. We will use the following facts that were established in Section 5.1.

(1) $\langle a, b\rangle$ is abelian.
(2) $\langle a, c\rangle$ and $\langle b, c\rangle$ are nilpotent of class at most 3.
(3) $\langle [c, a, a], [a, c, c], b\rangle$ and $\langle [c, b, b], [b, c, c], a\rangle$ are nilpotent of class at most 2.

Notice that it suffices to show that $\gamma_5(G)/\gamma_6(G)$ is a $\{2,3\}$-group. The reason for this is that it will then follow that $\gamma_5(G)$ is a $\{2,3\}$-group and thus trivial as G is $\{2,3\}$-torsion free. We can thus assume that the $\{2,3\}$-isolator of $\gamma_6(G)$ is trivial and we want to show that all commutators of weight 5 are trivial. Notice that quotient of G modulo the isolator of $\gamma_6(G)$ is $\{2,3\}$-torsion free. We can thus assume that G is nilpotent of class at most 5 and $\{2,3\}$-torsion free and we want to show that G is in fact of class at most 4. As (1),(2),(3) are symmetrical in a and b, it suffices to deal with commutators with multiweight in a, b, c of the form (r, s, t) where $r \leq s$ and $r+s+t = 5$. It is not difficult to see that it suffices to deal with the following commutators.

$$\begin{array}{ll}
\text{Weight } (1,1,3): & [a,c,b,c,c],\ [a,c,c,b,c], \\
\text{Weight } (1,2,2): & [a,c,b,b,c],\ [a,c,b,c,b], \\
\text{Weight } (1,3,1): & [a,c,b,b,b], \\
\text{Weight } (2,2,1): & [c,a,a,b,b].
\end{array}$$

The following calculations show that these are all trivial. First for weight $(1,1,3)$ notice that

$$1 = [a,[b,c,c,c]] = [a,c,b,c,c]^{-3}[a,c,c,b,c]^3$$

and, using the linearised 4-Engel identity $1 = \prod_{\sigma \in S_4}[y, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}]$ with $y = a$, $x_1 = b$, $x_2 = x_3 = x_4 = c$,

$$1 = [a,c,b,c,c]^6[a,c,c,b,c]^6.$$

From this it follows that $[a,c,c,b,c]^{12} = [a,c,b,c,c]^{12} = 1$. As G is $\{2,3\}$-torsion free it follows that $[a,c,c,b,c] = [a,c,b,c,c] = 1$.

Next we turn to weight $(1,2,2)$. We have

$$1 = [a,[b,c,c,b]] = [a,[b,c,c],b] = [a,c,b,c,b]^{-2}$$

and thus $[a,c,b,c,b] = 1$ as there is no 2-torsion. From the linearised 4-Engel identity with $y = a, x_1 = x_2 = b, x_3 = x_4 = c$ we also have

$$1 = [a,c,b,c,b]^4[a,c,b,b,c]^4 = [a,c,b,b,c]^4$$

that gives $[a,c,b,b,c] = 1$. For weight $(1,3,1)$, simply notice that

$$1 = [a,[c,b,b,b]] = [a,c,b,b,b].$$

Finally, for weight $(2,2,1)$, we use the linearised 4-Engel identity with $y = c, x_1 = x_2 = a, x_3 = x_4 = b$ and we get

$$1 = [c,a,a,b,b]^{4!} = [c,a,a,b,b]^{3\cdot 8}.$$

This again gives $[c,a,a,b,b] = 1$ as G is $\{2,3\}$-torsion free.

II. The subgroup $\langle a, a^{a^x}, a^{a^y} \rangle$

Let $b = a^{a^x}$ and $c = a^{a^y}$. We have shown in Section 5.2 that $G = \langle a,b,c \rangle$ is nilpotent. Here we show that the nilpotency class is at most 4 using relations (11) and (12). As explained at the beginning of Section I, we can without loss of generality suppose that $\gamma_6(G) = \{1\}$ and that G is $\{2,3\}$-torsion free. Using the fact that (11) and (12) are symmetric in b and c, it suffices to deal with multiweights $(1,1,3),(3,1,1),(1,2,2)$ and $(2,2,1)$. Furthermore it is not difficult to see that it suffices to consider the following commutators

$$\begin{array}{ll}
\text{Weight}\,(1,1,3): & [a,c,b,c,c], \\
\text{Weight}\,(3,1,1): & [c,b,a,a,a],\ [c,a,b,a,a], \\
\text{Weight}\,(1,2,2): & [a,c,b,c,b],\ [a,b,c,b,c], \\
\text{Weight}\,(2,2,1): & [c,a,b,a,b],\ [c,b,b,a,a],\ [c,b,a,a,b],\ [c,b,a,b,a].
\end{array}$$

The following calculations show that these are all trivial. Recall that G is $\{2,3\}$-torsion free. Firstly

$$1 = [a,[b,c,c,c]] = [a,c,b,c,c]^{-3},$$

that deals with weight $(1,1,3)$. Turning next to weight $(3,1,1)$, we have

$$1 = [c,a,[b,a,a]] = [c,a,b,a,a]$$

and then also

$$1 = [c,[b,a,a],a] = [c,b,a,a,a][c,a,b,a,a]^{-2} = [c,b,a,a,a].$$

For weight $(1,2,2)$, we first have

$$1 = [a,[c,b,b,c]] = [a,[c,b,b],c][a,c,[c,b,b]]^{-1} = [a,b,c,b,c]^{-2}[a,c,b,c,b]^2.$$

Thus $[a,b,c,b,c] = [a,c,b,c,b]$ and using the linearised 4-Engel identity $1 = \prod_{\sigma\in S_4}[y,x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)}]$ with $x = a, x_1 = x_2 = b, x_3 = x_4 = c$, we get

$$1 = [a,b,c,b,c]^4[a,c,b,c,b]^4 = [a,b,c,b,c]^8$$

that implies that $[a, c, b, c, b] = [a, b, c, b, c] = 1$. This leaves us with weight $(2, 2, 1)$. We see that

$$1 = [c, a, [a, b, b]] = [c, a, b, a, b]^{-2},$$

$$1 = [c, [b, a, a], b] = [c, b, a, a, b][c, a, b, a, b]^2 = [c, b, a, a, b]$$

and

$$1 = [c, [a, b, b], a] = [c, b, a, b, a]^{-2}[c, b, b, a, a].$$

Furthermore, using the linearised 4-Engel identity, we have

$$1 = [c, b, b, a, a]^4[c, b, a, b, a]^4 = [c, b, a, b, a]^{12}.$$

As G is $\{2, 3\}$-torsion free, it follows that $[c, b, b, a, a] = [c, b, a, b, a] = 1$. This deals with all commutators of weight 5 and we have shown that G is nilpotent of class at most 4.

III. The subgroup $\langle a, a^{a^x}, a^y \rangle$

Let $b = a^{a^x}$ and $c = a^y$. We have seen in Section 5.3 that $G = \langle a, b, c \rangle$ is nilpotent. That section also included a outline of the proof that the class is at most 4. In this section we provide the missing details. As said in Section 5.3, the proof is in two stages. The first is to show that the class is at most 5, using the 4-Engel identity and

(1) $[c, a, a, a^{-1}b, a^{-1}b] = 1$,
(2) $\langle a, b \rangle$ is nilpotent of class at most 2,
(3) $\langle u, v^w \rangle$ is nilpotent of class at most 3 for all $u, v, w \in \{a, b, c\}$.

We proceed as in the previous two sections. We can assume that $\gamma_7(G) = \{1\}$ and that G is $\{2, 3\}$-torsion free. First we deal with commutators of weight $(1, 1, 4), (1, 4, 1)$ and $(4, 1, 1)$. Notice that

$$1 = [a^b, c, c, c, c] = [a, b, c, c, c].$$

Then

$$\begin{aligned} 1 &= [a, [b, c, c, c], c] \\ &= [a, c, b, c, c, c]^{-3}[a, c, c, b, c, c]^3 \\ 1 &= [a, c, [b, c, c, c]] \\ &= [a, c, b, c, c, c][a, c, c, b, c, c]^{-3}. \end{aligned}$$

From which we get $[a,c,b,c,c,c]^2=1$ and $[a,c,c,b,c,c]^6=1$. As we have no $\{2,3\}$-torsion elements, it follows that all commutators of weight $(1,1,4)$ are trivial. Now turn to weight $(4,1,1)$. We have

$$1=[c^b,a,a,a,a]=[c,b,a,a,a,a].$$

Then

$$1=[c,a,a,[b,a,a]]=[c,a,a,b,a,a]$$

and

$$1=[c,a,[b,a,a],a]=[c,a,b,a,a,a][c,a,a,b,a,a]^{-2}=[c,a,b,a,a,a].$$

Thus all commutators of weight $(4,1,1)$ are trivial and by swapping the role of a and b we see that the same is true for weight $(1,4,1)$. We next turn to the weights $(2,3,1)$ and $(3,2,1)$. Firstly

$$1=[c^a,b,b,b,a]=[c,a,b,b,b,a].$$

Then, using this and the linearised 4-Engel identity $\prod_{\sigma\in S_4}[a,x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)},a]$ with $x_1=x_2=x_3=b$ and $x_4=c$, we see that

$$1=[a,c,b,b,b,a]^6[a,b,c,b,b,a]^6=[c,[a,b],b,b,a]^{-6}=[c,b,a,b,b,a]^6.$$

Then

$$1=[c,[a,b,b],b,a]=[c,b,b,a,b,a]$$

and all simple commutators of weight $(2,3,1)$ ending in a are trivial. We next use another consequence of the 4-Engel identity. Namely if we let x_1,x_2,x_3,x_4,y,z be variables and we expand the commutator

$$[yz,_4\, x_1x_2x_3x_4]$$

modulo commutators of weight 7 we see that the component of multiweight $(1,1,1,1,1,1)$ in y,z,x_1,x_2,x_3,x_4 is

$$\prod_{\sigma\in S_4}[y,x_{\sigma(1)},z,x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)}].$$

From this one easily sees that law

$$\prod_{\sigma\in S_4}[y,x_{\sigma(1)},z,x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)}]=1$$

is satisfied in $G/\gamma_7(G)$. We use this identity for $y = z = a$, $x_1 = x_2 = x_3 = b$ and $x_4 = c$. This gives (recall that $\langle a, b\rangle$ is nilpotent of class at most 2)

$$1 = [a, c, a, b, b, b]^6$$

and thus $[c, a, a, b, b, b] = 1$. We had shown that all commutators ending in a are trivial. Thus

$$1 = [c, b, b, [b, a, a]] = [c, b, b, a, a, b].$$

We are now only left with simple commutators ending in b and starting either with $[c, a, b]$ or $[c, b, a]$. These are the commutators $[c, a, b, a, b, b]$, $[c, a, b, b, a, b]$,$[c, b, a, a, b, b]$ and $[c, b, a, b, a, b]$. It is now easy to see that these are all trivial. Firstly we use the new multilinear identity above with $y = a, z = b, x_1 = c, x_2 = a, x_3 = x_4 = b$. This gives

$$1 = [a, c, b, b, a, b]^2[a, c, b, a, b, b]^2.$$

From this and

$$1 = [a, c, b, [a, b, b]] = [a, c, b, a, b, b][a, c, b, b, a, b]^{-2}$$

we get $[a, c, b, b, a, b]^3 = [a, c, b, a, b, b]^3 = 1$ and thus $[a, c, b, b, a, b] = [a, c, b, a, b, b] = 1$. Similarly letting $y = b, z = a, x_1 = c, x_2 = a, x_3 = x_4 = b$ in the former identity and using $1 = [c, b, a, [a, b, b]]$, we see that $[c, b, a, a, b, b] = [c, b, a, b, a, b] = 1$. Thus all commutators of weight $(2, 3, 1)$ are trivial. Swapping the role of a and b in the calculations above we see that the same is true for weight $(3, 2, 1)$.

We now turn to the weights $(2, 1, 3)$ and $(1, 2, 3)$. To start with

$$1 = [a^b, c, c, c, a]^{-1} = [b, a, c, c, c, a].$$

Then

$$1 = [b, [a, c, c, c], a] = [b, c, a, c, c, a]^{-3}[b, c, c, a, c, a]^3.$$

Using the linearised 4-Engel identity we have $1 = \prod_{\sigma\in S_4}[b, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, a]$ for $x_1 = a$ and $x_2 = x_3 = x_4 = c$. This gives

$$1 = [b, c, a, a, c, c, a]^6[b, c, c, a, c, a]^6.$$

From this and the previous displayed identity we see that $[b, c, a, c, c, a] = [b, c, c, a, c, a] = 1$ and thus all commutators of weight $(2, 1, 3)$ that end in a are trivial. Next we use the linearised 4-Engel again. This time we apply it for $\prod_{\sigma \in S_4} [b, a, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}] = 1$ with $x_1 = a$ and $x_2 = x_3 = x_4 = c$. This gives

$$1 = [b, a, c, a, c, c]^6 [b, a, c, c, a, c]^6.$$

This together with

$$1 = [b, a, [a, c, c, c]] = [b, a, c, a, c, c]^{-3} [b, a, c, c, a, c]^3$$

gives that $[b, a, c, a, c, c] = [b, a, c, c, a, c] = 1$. We are thus only left with simple commutators of weight $(2, 1, 3)$ that begin in $[b, c]$ and end in c. These are $[b, c, c, a, a, c]$, $[b, c, a, c, a, c]$ and $[b, c, a, a, c, c]$. The linearised 4-Engel identity and the other linearised multilinear identity give (we have underlined those variables that are permuted)

$$\begin{array}{rcl} 1 & = & [b, \underline{c}, \underline{a}, \underline{c}, \underline{a}, c]^4 [b, \underline{c}, \underline{c}, \underline{a}, \underline{a}, c]^4 [b, \underline{c}, \underline{a}, \underline{a}, \underline{c}, c]^4 \\ 1 & = & [b, \underline{c}, a, \underline{c}, \underline{a}, \underline{c}]^6 [b, \underline{c}, a, \underline{a}, \underline{c}, c]^6. \end{array}$$

This gives $[b, c, c, a, a, c] = 1$. Then

$$1 = [b, c, [c, a, a, c]] =][b, c, [c, a, a], c][b, c, c, [c, a, a]]^{-1} = [b, c, a, c, a, c]^{-2}][b, c, a, a, c, c].$$

From this and the previous identities we have $[b, c, a, c, a, c] = [b, c, a, a, c, c] = 1$. Thus all commutators of weight $(2, 1, 3)$ are trivial and swapping the role of a and b the same argument shows that all commutators of weight $(1, 2, 3)$ are trivial.

We next turn to weights $(3, 1, 2)$ and $(1, 3, 2)$. We have

$$1 = [b^c, a, a, a, c] = [b, c, a, a, a, c],$$

and using this we also have

$$1 = [b, [c, a, a, a], c] = [b, a, c, a, a, c]^{-3}.$$

Thus all commutators of weight $(3, 1, 2)$ that end in c are trivial. The next step is to deal with all those commutators beginning in $[b, a]$. We use one of the multilinear identities (again we have underlined those that elements that are permuted). This gives us

$$1 = [b, \underline{c}, a, \underline{c}, \underline{a}, \underline{a}]^4 [b, \underline{c}, a, \underline{a}, \underline{c}, \underline{a}]^4$$

and

$$1 = [a, \underline{c}, b, \underline{c}, \underline{a}, \underline{a}]^4 [a, \underline{c}, b, \underline{a}, \underline{c}, \underline{a}]^4.$$

From these two identities, it follows that

$$1 = [b, a, c, c, a, a]^4 [b, a, c, a, c, a]^4.$$

We also have

$$1 = [b, a, [a, c, c, a]] = [b, a, [a, c, c], a] = [b, a, c, c, a, a][b, a, c, a, c, a]^{-2}.$$

From this it follows that $[b, a, c, c, a, a] = [b, a, c, a, c, a] = 1$. We are thus only left with commutators that begin in $[b, c]$ and end in a. First using one of the multilinear identities we see that

$$1 = [b, \underline{c}, c, \underline{a}, \underline{a}, \underline{a}]^6.$$

Then

$$1 = [b, [a, c, c, a], a] = [b, [a, c, c], a, a] = [b, c, a, c, a, a]^{-2}$$

and we are only left with the commutator $[b, c, a, a, c, a]$. However

$$1 = [b, c, [c, a, a, a]] = [b, c, a, a, c, a]^{-3}$$

and all commutators of weight $(3, 1, 2)$ are trivial. Swapping the roles of a and b we see that the same is true for commutators of weight $((1, 3, 2)$.

We are now only left with multiweight $(2, 2, 2)$. We now use for the first time identity (1) above. Using the work above we have

$$1 = [c, a, a, a^{-1}b, a^{-1}b, c] = [c, a, a, b, b, c],$$

$$1 = [a^c, b, b, a^c, c] = [c, a, b, b, a, c]^{-1},$$

$$1 = [b^c, a, a, b^c, c] = [c, b, a, a, b, c]^{-1},$$

$$1 = [c, a, [a, b, b], c] = [c, a, b, a, b, c]^{-2}.$$

Then, using $[c, b, [b, a, a], c] = 1$ and the linearised 4-Engel identity,

$$\begin{aligned} 1 &= [c, b, b, a, a, c][[c, b, a, b, a, c]^{-2} \\ 1 &= [c, \underline{b}, \underline{b}, \underline{a}, \underline{a}, c][c, \underline{b}, \underline{a}, \underline{b}, \underline{a}, c] \end{aligned}$$

gives $[c, b, b, a, a, c] = [c, b, a, b, a, c] = 1$ and all simple commutators of weight $(2, 2, 2)$ ending in c are trivial. Using one of the multilinear identities we see that

$$\begin{aligned} 1 &= [a, \underline{c}, a, \underline{b}, \underline{c}, \underline{b}]^4 \\ 1 &= [b, \underline{c}, b, \underline{a}, \underline{c}, \underline{a}]^4. \end{aligned}$$

Thus $[c, a, a, b, c, b] = [c, b, b, a, c, a] = 1$. Next

$$1 = [c^a, b, b, c^a, a] = [c, a, b, b, c, a][c, b, b, [c, a], a] = [c, a, b, b, c, a].$$

Also

$$1 = [c, [a, b, b], c, a] = [c, b, a, b, c, a]^{-2}$$

and thus all commutators with the last two entries c, a are trivial. Swapping the role of a and b we see that the same is true for the commutators with last two entries c, b. Next we consider the case when c occurs as the 4th entry. Using one of the multilinear identities, we see that

$$1 = [a, \underline{c}, b, \underline{c}, \underline{a}, \underline{b}]^2[a, \underline{c}, b, \underline{c}, \underline{b}, \underline{a}]^2.$$

But also

$$1 = [c, a, b, [c, a, b]] = [c, a, b, c, a, b]$$

and thus $[c, a, b, c, a, b] = [c, a, b, c, b, a] = 1$. Swapping the role of a and b, we also have $[c, b, a, c, a, b] = [c, b, a, c, b, a] = 1$. We are now only left with commutators starting with either $[c, a, c]$ or $[c, b, c]$. Using, $[c, a, c, [a, b, b]] = 1$ and the linearised 4-Engel identity, we see that

$$\begin{aligned} 1 &= [c, a, c, b, a, b]^{-2}[c, a, c, b, b, a] \\ 1 &= [c, a, \underline{c}, \underline{b}, \underline{a}, \underline{b}]^2[c, a, \underline{c}, \underline{b}, \underline{b}, \underline{a}]^2 \end{aligned}$$

that give $[c, a, c, b, a, b] = [c, a, c, b, b, a] = 1$. Swapping the role of a and b gives also $[c, b, c, a, b, a] = [c, b, c, b, b, a] = 1$. We have thus shown that all commutators of weight $(2, 2, 2)$ are trivial and thus G is nilpotent of class at most 5.

We next want to show that one can strengthen this and show that the class is at most 4. How this is achieved was explained in Section 5.3. We will in the missing details. First we proof the following result that was used there.

Claim *Let* $H = \langle \bar{a}, \bar{b}, \bar{c} \rangle$ *be a nilpotent* 4*-Engel group where* $\langle \bar{a}, \bar{b} \rangle$, $\langle \bar{a}, \bar{c} \rangle$ *are nilpotent of class at most* 2 *and where* $\langle u, v^w \rangle$ *is nilpotent of class at most* 3 *for all* $u, v, w \in \{\bar{a}, \bar{b}, \bar{c}\}$. *Then all commutators of multiweight* $(2, 1, 2)$ *in* $\bar{a}, \bar{b}, \bar{c}$ *are trivial.*

Proof of Claim. We have

$$1 = [\bar{a}^{\bar{b}}, \bar{c}, \bar{c}, \bar{a}^{\bar{b}}] = [\bar{b}, \bar{a}, \bar{c}, \bar{c}, \bar{a}]^{-1}.$$

Then

$$1 = [\bar{b}, \bar{a}, [\bar{a}, \bar{c}, \bar{c}]] = [\bar{b}, \bar{a}, \bar{c}, \bar{a}, \bar{c}]^{-2}$$

that gives $[\bar{b}, \bar{a}, \bar{c}, \bar{a}, \bar{c}] = 1$. Then

$$1 = [\bar{c}^{\bar{b}}, \bar{a}, \bar{a}, \bar{c}^{\bar{b}}] = [\bar{c}, \bar{b}, \bar{a}, \bar{a}, \bar{c}] = [\bar{b}, \bar{c}, \bar{a}, \bar{a}, \bar{c}]^{-1}.$$

We are now only left with the simple commutators $[\bar{b}, \bar{c}, \bar{c}, \bar{a}, \bar{a}]$ and $[\bar{b}, \bar{c}, \bar{a}, \bar{c}, \bar{a}]$. Notice that

$$1 = [\bar{b}, \bar{c}, [\bar{c}, \bar{a}, \bar{a}]] = [\bar{b}, \bar{c}, \bar{c}, \bar{a}, \bar{a}][\bar{b}, \bar{c}, \bar{a}, \bar{c}, \bar{a}]^{-2}$$

that gives $[\bar{b}, \bar{c}, \bar{c}, \bar{a}, \bar{a}] = [\bar{b}, \bar{c}, \bar{a}, \bar{c}, \bar{a}]^2$. Finally the 4-Engel identity gives

$$1 = [\bar{b}, \underline{\bar{c}}, \underline{\bar{c}}, \underline{\bar{a}}, \underline{\bar{a}}]^4 [\bar{b}, \underline{\bar{c}}, \underline{\bar{a}}, \underline{\bar{c}}, \underline{\bar{a}}]^4 = [\bar{b}, \bar{c}, \bar{a}, \bar{c}, \bar{a}]^{12}$$

and $[\bar{b}, \bar{c}, \bar{c}, \bar{a}, \bar{a}] = [\bar{b}, \bar{c}, \bar{a}, \bar{c}, \bar{a}] = 1$. This finishes the proof of the Claim.

We are now ready to show that the class of G is at most 4. We start with multiweights $(1, 3, 1)$ and $(3, 1, 1)$. We have

$$1 = [a^c, b, b, b] = [c, a, b, b, b]^{-1}.$$

Using this, $[c, [a, b, b], b] = 1$ and the linearised 4-Engel identity, we see that

$$\begin{aligned} 1 &= [c, b, a, b, b]^{-2}[c, b, b, a, b] \\ 1 &= [c, b, a, b, b]^6[c, b, b, a, b]^6. \end{aligned}$$

This gives $[c,b,a,b,b] = [c,b,b,a,b] = 1$ and all commutators of weight $(1,3,1)$ are trivial. Swapping the role of a and b similarly shows that all commutators of weight $(3,1,1)$ are trivial. Next we turn to weight $(1,1,3)$. Here

$$1 = [a^b, c, c, c] = [a, b, c, c, c]$$

and then, using $[a, [b, c, c, c]] = 1$ and the linearised 4-Engel identity gives

$$\begin{aligned} 1 &= [a,c,b,c,c]^{-3}[a,c,c,b,c]^3 \\ 1 &= [a,c,b,c,c]^6[a,c,c,b,c]^6 \end{aligned}$$

that gives $[a,c,b,c,c] = [a,c,c,b,c] = 1$. Next we move to multiweight $(2,2,1)$. Building on the previous work we have

$$1 = [c, a, a, a^{-1}b, a^{-1}b] = [c, a, a, b, b],$$

$$1 = [a^c, b, b, a^c] = [c, a, b, b, a]^{-1},$$

$$1 = [c, a, [a, b, b]] = [c, a, b, a, b]^{-2},$$

and

$$1 = [b^c, a, a, b^c] = [c, b, a, a, b]^{-1}.$$

This leaves us with the commutators $[c,b,b,a,a]$ and $[c,b,a,b,a]$. Then, using $[c, b, [b, a, a]] = 1$ and the 4-Engel identity we have

$$\begin{aligned} 1 &= [c,b,b,a,a][c,b,a,b,a]^{-2} \\ 1 &= [c,b,b,a,a]^4[c,b,a,b,a]^4 \end{aligned}$$

that gives $[c,b,b,a,a] = [c,b,a,b,a] = 1$.

We are now left with multiweights $(1,2,2)$ and $(2,1,2)$. As we saw in 5.3 we get the following extra relations

$$\begin{aligned} &(4)\ [c,a,c,b,b] = [c,b,c,a,b] = 1 \\ &(5)\ [c,b,c,a,a] = [c,a,c,b,a] = 1. \end{aligned}$$

First we turn to weight $(2,1,2)$. We have

$$\begin{aligned} 1 &= [c^a, b, b, c^a] \\ &= [c,a,b,b,c][c,b,b,[c,a]] \\ &= [c,a,b,b,c][c,b,b,a,c]^{-1} \\ &= [a,c,b,b,c]^{-1}[a,[c,b,b],c] \\ &= [a,b,c,b,c]^{-2}. \end{aligned}$$

From (4) we also have

$$1 = [a, [b, c, c], b] = [a, b, c, c, b][a, c, b, c, b]^{-2}$$

and thus (using the other relation in (4)) we have $[a, c, c, b, b] = 1$, $[a, b, c, b, c] = 1$ and $[a, b, c, c, b] = [a, c, b, c, b]^2$. We are left with the commutators $[a, b, c, c, b]$, $[a, c, b, c, b]$ and $[a, c, b, b, c]$. The 4-Engel identity gives us

$$1 = [a, c, b, b, c]^4[a, b, c, c, b]^4[a, c, b, c, b]^4$$

that implies that $[a, c, b, b, c] = [a, c, b, c, b]^{-3}$. Finally

$$1 = [a, [c, b, b, c]] = [a, [c, b, b], c][a, c, [c, b, b]]^{-1} = [a, c, b, b, c][a, c, b, c, b]^2[a, c, b, b, c]^{-1}$$

that gives $[a, c, b, c, b] = 1$ and thus also $[a, b, c, c, b] = [a, c, b, b, c] = 1$. This finishes the proof that all commutators of weight $(2, 1, 2) = 1$. Similarly, using (5), we see that all commutators of weight $(1, 2, 2)$ are trivial. Hence the class of G is at most 4.

IV. The subgroup $\langle a, a^x, a^y\rangle$

Let $b = a^x$ and $c = a^y$. We fill in the missing details of Section 5.4. We start with Step 1. We wish to show first that the subgroup $\langle a, [a, b, b], c\rangle$ is nilpotent of class at most 4. The treatment in Section 5.4 has already given us that this subgroup is nilpotent of class at most 5. Let $x = [a, b, b]$. We will use the following relations that we know that hold from Section 5.4

(1) $[x, a] = 1$,
(2) $[x, [c, a, a]] = 1$,
(3) $[x, [a, c, c]] = 1$,
(4) $[c, x, x] = 1$,
(5) $[x, c, c, c] = 1$.

We will also use the fact established in Section 5.4 that the class of $\langle x, a, c\rangle$ is at most 5. We first see that it follows from this that

(6) $[x, [c, a], c] = 1$.

We saw in Section 5.4 that $[x^8, [c, a], c] = 1$. As the class of $\langle x, a, c\rangle$ is at most 5, we get

$$1 = [x^8, [c, a], c] = [x, [c, a], c]^8 [x, [c, a], x, c]^{28}$$

But $[x, [c, a], x, c] = [c, a, x, x, c]^{-1} = [c, x, x, a, c]^{-1}$, as x and a commute. Hence $[x, [c, a], c]^8 = 1$ and as $G = \langle a, b, c\rangle$ is 2-torsion free it follows that (6) holds. We will now show that (1)-(6) imply that the class of $H = \langle x, a, c\rangle$ is in fact 3. We can assume that $\gamma_5(H) = \{1\}$. We need to consider the multiweights $(2, 1, 1), (1, 2, 1)$ and $(1, 1, 2)$ in x, a, c. Notice first that as a and x commute the only simple commutator of weight $(2, 1, 1)$ that needs to be considered is $[c, x, x, a]$ that is trivial by (4). Similarly the only commutator of weight $(1, 2, 1)$ that needs to be considered is $[x, c, a, a]$ but this is trivial since by (2)

$$1 = [x, [c, a, a]] = [x, c, a, a].$$

To deal with weight $(1, 1, 2)$ we need to use both (3) and (6). From (6) we have $[x, c, a, c] = 1$ and thus the only commutator that needs to be considered is $[x, c, c, a]$. However by (3)

$$1 = [x, [a, c, c]] = [x, c, a, c]^{-2}[x, c, c, a] = [x, c, c, a].$$

Hence $\langle a, [a, b, b], c\rangle$ is nilpotent of class at most 3.

In Section 5.4, we then used this result to show that $G = \langle a, b, c\rangle$ is nilpotent. We next show that G is nilpotent of class at most 5. We will use some facts established in Section 5.4. Namely that any two of $a, b, c, a^b, b^a, a^c, c^a, b^c, c^b$ generate a subgroup that is nilpotent of class at most 3 and that the elements $[a, b, b]$, $[b, a, a]$,$[a, c, c]$, $[c, a, a]$,$[b, c, c]$,$[c, b, b]$ are all in $Z_3(G)$. By symmetry in a,b and c, we only need to consider multiweights $(4, 1, 1)$, $(3, 2, 1)$ and $(2, 2, 2)$ in a, b, c.

We start with multiweight $(4, 1, 1)$. As $[c, a, a] \in Z_3(G)$ we only need to consider the commutators $[c, b, a, a, a, a]$ and $[c, a, b, a, a, a]$. The first is trivial as

$$1 = [c^b, a, a, a, a] = [c, b, a, a, a, a]$$

and then

$$1 = [c, [b, a, a, a], a] = [c, a, b, a, a, a]^{-3}$$

gives that $[c, a, b, a, a, a] = 1$ as well.

Now we turn to multiweight $(3, 2, 1)$. Firstly

$$1 = [c^b, a, a, a, b] = [c, b, a, a, a, b]$$

and (using the fact that $[b, a, a[\in Z_3(G))$

$$1 = [b^c, a, a, b^c, a]^{-1} = [c, b, a, a, b, a].$$

As $[c, b, b], [b, a, a] \in Z_3(G)$, we then have

$$1 = [c, b, [b, a, a], a] = [c, b, a, a, b, a][c, b, a, b, a, a]^{-2} = [c, b, a, b, a, a]^{-2}$$

and $[c, b, a, b, a, a] = 1$. We have thus seen that all simple commutators of weight $(3, 2, 1)$ starting in $[c, b]$ are trivial. As $[c, a, a], [c, b, b] \in Z_3(G)$ we now only need to consider commutators starting in $[c, a, b]$. Firstly

$$1 = [a^c, b, b, a^c, a] = [c, a, b, b, a, a]^{-1}$$

and

$$1 = [c, a, [b, a, a], b] = [c, a, b, a, a, b]$$

that only leaves the commutator $[c, a, b, a, b, a]$. However

$$1 = [c, a, b, [b, a, a]] = [c, a, b, a, b, a]^{-1}$$

and thus $[c, a, b, a, b, a] = 1$ as well.

We are now only left with weight $(2, 2, 2)$. The commutators of weight $(2, 2, 2)$ are generated by commutators where the first and last entry are the same. By symmetry we thus only need to show that all commutators $[a, x_1, x_2, x_3, x_4, a]$ of weight $(2, 2, 2)$ are trivial. Now as $[b, c, c] \in Z_3(G)$, we have $1 = [a, [b, c, c], b, a] = [a, b, [b, c, c], a]$. As $[a, c, c] \in Z_3(G)$ it follows from this that

$$\begin{array}{rcl} 1 & = & [a, b, c, c, b, a][a, c, b, c, b, a]^{-2} \\ 1 & = & [a, b, c, c, b, a][a, b, c, b, c, a]^{-2}. \end{array}$$

Thus $[a, b, c, c, b, a] = [a, c, b, c, b, a]^2 = [a, b, c, b, c, a]^2$. By symmetry in b and c we then have

$$[a, b, c, c, b, a] = [a, c, b, b, c, a] = [a, b, c, b, c, a]^2 = [a, b, c, b, c, a]^2.$$

It thus suffices to show that $[a,b,c,b,c,a]=1$. Using the linearised 4-Engel identity we finally have

$$1=[a,\underline{b},\underline{c},\underline{c},\underline{b},a]^4[a,\underline{b},\underline{c},\underline{b},\underline{c},a]^4[a,\underline{c},\underline{b},\underline{b},\underline{c},a]^4[a,\underline{c},\underline{b},\underline{c},\underline{b},a]^4=[a,c,b,c,b,c,a]^{3\cdot 8}.$$

Hence $[a,c,b,c,b,c,a]=1$ and all commutators of weight $(2,2,2)$ are trivial. This finishes the proof that the class of G is at most 5.

Let us next see that every commutator of multiweight $(3,1,1)$ is trivial. Firstly

$$1=[b^c,a,a,a]=[b,c,a,a,a]$$

and as $[b,a,a,a]=1$ we only need to consider the commutators $[b,a,c,a,a]$ and $[b,a,a,c,a]$. Now

$$1=[b,[c,a,a,a]]=[b,a,c,a,a]^{-3}[b,a,a,c,a]^3$$

and $[b,a,a,c,a]=[b,a,c,a,a]$. The the linearised 4-Engel identity gives

$$1=[b,a,a,c,a]^6[b,a,c,a,a]^6=[b,a,c,a,a]^{12}$$

that gives $[b,a,c,a,a]=1$ and thus every commutator of multiweight $(3,1,1)$ is trivial. From this, the fact that $\langle a,b\rangle,\langle a,c\rangle$ are nilpotent of class at most 3 and the fact that G is nilpotent of class at most 5 it follows that $\langle a\rangle^G=\langle a,[a,b],[a,c]\rangle$ is nilpotent of class at most 2.

References

[1] K. W. Gruenberg, Two theorems on Engel groups, *Proc. Cam. Phil. Soc.* **49** (1953), 377-380.

[2] G. Havas and M. R. Vaughan-Lee, 4-Engel groups are locally nilpotent, *Int.J. Algebra Comput.* **15** (2005), 649-682.

[3] H. Heineken, Engelsche Elemente der Länge drei, *Illinois J. Math.* **5** (1961), 681-707.

[4] Y. K. Kim and A. H. Rhemtulla, Weak maximality condition and polycyclic groups, *Proc. Am. Math. Soc* **123**, (1995), 711-714.

[5] P. Longobardi and M. Maj, Semigroup identities and Engel groups, *Groups St. Andrews 1997 in Bath II* 527-531, London Math. Soc. Lecture Note Ser. **261**, Cambridge Univ. Pess, Cambridge 1999.

[6] M. Newell, On right-Engel elements of length three, *Proc. of the Royal Irish Academy* **96A** (1996), 17-24.

[7] W. Nickel, Computation of nilpotent Engel groups, *J. Austral. Math. Soc.* **67** (1999), 214-222.

[8] G. Traustason, On 4-Engel groups, *J. Algebra* **178** (1995), 414-429.

[9] G. Traustason, Semigroup identities in 4-Engel groups, *J. Group Theory* **2**, (1999), 39-46.

[10] G. Traustason, Two generator 4-Engel groups, *Int. J. Alg. Comp.* **15** (2005), 309-316.

[11] G. Traustason, A note on the local nilpotence of 4-Engel groups, *Int. J. Alg. Comp.* **15** (2005), 757-764.

[12] G. Traustason, Engel groups, in *Groups St Andrews 2009 in Bath, Volume II*, 520-550. London Math. Soc. Lecture Note ser. 388. Cambridge University Press, Cambridge 2011.

[13] M. R. Vaughan-Lee, Engel-4 groups of exponent 5, it Proc. London Math. Soc. (3) **74** (1997), 306-334.