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REFINED SOLVABLE PRESENTATIONS FOR POLYCYCLIC GROUPS

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Abstract. We describe a new type of presentation that, when consistent, describes a polycyclic group. This presentation is obtained by refining a series of normal subgroups with abelian sections. These presentations can be described effectively in computer-algebra-systems like Gap or Magma. We study these presentations and, in particular, we obtain consistency criteria for them. The consistency implementation demonstrates that there are situations where the new method is faster than the existing methods for polycyclic groups.

1. Introduction

A group G is polycyclic if there exists a finite series of subnormal subgroups

$$G = G_1 \trianglerighteq G_2 \trianglerighteq \ldots \trianglerighteq G_m \trianglerighteq G_{m+1} = \{1\}$$

so that each section G_i/G_{i+1} is cyclic. Polycyclic groups play an important role in group theory as, for instance, each finite group with odd order is polycyclic. Moreover, polycyclic groups form a special class of finitely presented groups for which various algorithmic problems are solvable. For instance, it is well-known that the word problem in a polycyclic group is solvable. More precisely, a polycyclic group G can be described by a polycyclic presentation. This is a finite presentation with generators

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$\{a_1,\dots,a_m\}$ and relations of the form

$$\begin{array}{rcll} a_i^{n_i} &=& a_{i+1}^{\alpha_{i,i+1}}\cdots a_m^{\alpha_{i,m}}, & i\in\mathcal{I}\\ a_i^{-1}a_ja_i &=& a_{i+1}^{\beta_{i,j,i+1}}\cdots a_m^{\beta_{i,j,m}}, & 1\le i<j\le m\\ a_i^{-1}a_j^{-1}a_i &=& a_{i+1}^{\gamma_{i,j,i+1}}\cdots a_m^{\gamma_{i,j,m}}, & 1\le i<j\le m,\ j\notin\mathcal{I}\\ a_ia_ja_i^{-1} &=& a_{i+1}^{\delta_{i,j,i+1}}\cdots a_m^{\delta_{i,j,m}}, & 1\le i<j\le m,\ i\notin\mathcal{I}\\ a_ia_j^{-1}a_i^{-1} &=& a_{i+1}^{\varepsilon_{i,j,i+1}}\cdots a_m^{\varepsilon_{i,j,m}}, & 1\le i<j\le m,\ i,j\notin\mathcal{I} \end{array}$$

for a subset $\mathcal{I}\subseteq\{1,\dots,m\}$ and integers $\alpha_{i,\ell},\beta_{i,j,\ell},\gamma_{i,j,\ell},\delta_{i,j,\ell},\varepsilon_{i,j,\ell}\in\mathbb{Z}$ that satisfy n_ℓ a positive integer and

$$0\le\alpha_{i,\ell},\beta_{i,j,\ell},\gamma_{i,j,\ell},\delta_{i,j,\ell},\varepsilon_{i,j,\ell}<n_\ell$$

whenever $\ell\in\mathcal{I}$ holds. For further details on polycyclic presentations we refer to Section 9.4 of [16].

Given any finite presentation of a polycyclic group, the polycyclic quotient algorithm [13,14] allows one to compute a polycyclic presentation defining the same group. If, additionally, the polycyclic group is nilpotent, then a finite presentation can be transformed into a polycyclic presentation with the nilpotent quotient algorithm [15]. We further note that even certain infinite presentations (so-called finite L-presentations; see [2]) of a nilpotent and polycyclic group can be transformed into a polycyclic presentation [3]. We may therefore always assume that a polycyclic group is given by a polycyclic presentation.

In the group G, every element is represented by a word $a_1^{e_1}a_2^{e_2}\cdots a_m^{e_m}$ with $0\le e_i<n_i$ whenever $i\in\mathcal{I}$ holds. If this representation is unique, then the polycyclic presentation is consistent and it yields a normal form for elements in the group. This is a basis for symbolic computations within polycyclic groups. Various strategies for computing normal forms in a polycyclic group have been studied so far [12,18,8,1]. The current state of the art algorithm is *collection from the left*. But it is known that even 'collection from the left' is exponential in the number of generators [12]; see also [1].

In this paper, we concentrate on a certain type of presentations that we call refined solvable presentations. When such a presentation is consistent it defines a polycyclic group with a natural ascending series of normal subgroups with abelian factors. This ascending series is indicated in the presentation through a partition of the set X of generators. We describe these presentations in Section 2. Each weighted nilpotent presentation, as used extensively in the nilpotent quotient algorithms [15,3] and in [17], is of this type. A solvable presentation can be described effectively by presentation maps which we define in Section 2. Presentation maps can be considered as the basic data structure to define a polycyclic group in computer-algebra-systems like Gap or Magma. We obtain consistency criteria for refined solvable presentations in Section 3. This consistency check has been implemented in the NQL-package [10]. Our implementation shows that there are situations where the consistency checks for refined solvable presentations are faster than the general methods for polycyclic groups. We demonstrate that this is the case when dealing with nilpotent quotients of the Basilica group [9] and the Brunner-Sidki-Vieira-Group BSV [5]. We have focused here on nilpotent groups and our hope is that future investigations involving more general examples may provide further evidence for the new method.

Fast algorithms for working with presentations of polycyclic groups are of special interest as, for instance, the algorithm in [11] attempts to find periodicities in the Dwyer quotients of the Schur multiplier of a group. In order to observe these periodicities, the algorithm needs to compute with polycyclic presentations with some hundreds of generators and therefore fast algorithms for polycyclic groups are needed.

2. **Refined solvable presentations**

Let G be a polycyclic group with a strictly ascending chain of normal subgroups

$$\{1\} = G_0 < G_1 < \cdots < G_r = G$$

where G_i/G_{i-1} is abelian for $i = 1, \ldots, r$. Since each subgroup of a polycyclic group is finitely generated, we can choose a finite generating set X for G which partitions as $X = X_1 \cup X_2 \cup \cdots \cup X_r$ such that

$$G_i/G_{i-1} = \bigoplus_{x \in X_i} \langle xG_{i-1} \rangle$$

for $i = 1, \ldots, r$ and where all the direct summands are non-trivial. We can furthermore make our choice so that for each $x \in X_i$, the order, $o(xG_{i-1})$, of xG_{i-1} is either infinite or a power of a prime. Let $\mathcal{P}$ denote the set of all primes. For each $p \in \mathcal{P}$, let

$$X_i(p) = \{x \in X_i : o(xG_{i-1}) \text{ is a power of } p\}$$

and let

$$X_i(\infty) = \{x \in X_i : o(xG_{i-1}) = \infty\}.$$

Notice that the Sylow p-subgroup of G_i/G_{i-1} is

$$(G_i/G_{i-1})_p = \bigoplus_{x \in X_i(p)} \langle xG_{i-1} \rangle.$$

We order the generators in X such that the generators in X_i precede those in X_j whenever $i < j$. Suppose that $X = \{g_1, \ldots, g_m\}$ with $g_1 < g_2 < \ldots < g_m$. For each $x \in X_i$ let $n(x) = o(xG_{i-1})$. If $n(x) = \infty$, let $\mathbb{Z}_x = \mathbb{Z}$ and otherwise let $\mathbb{Z}_x = \{0, \ldots, n(x) - 1\}$. Each element $g \in G$ has a unique normal form expression

$$g = g_m^{r_m} g_{m-1}^{r_{m-1}} \cdots g_1^{r_1}$$

where $r_i \in \mathbb{Z}_{g_i}$.

We next describe some relations that hold in the generators $g_1, \ldots, g_m$. If $x \in X_s(p)$ then we get a *power relation* of the form

$$x^{n(x)} = g_m^{\alpha_x(m)} \cdots g_1^{\alpha_x(1)} \tag{2.1}$$

with $\alpha_x(i) \in \mathbb{Z}_{g_i}$ and where $\alpha_x(i) = 0$ if $g_i \notin X_1 \cup \cdots \cup X_{s-1}$.

For each pair of generators $x, y \in X$ with $x < y$ we also get a *conjugacy relation*

$$x^y = g_m^{\beta_{(x,y)}(m)} \cdots g_1^{\beta_{(x,y)}(1)} \tag{2.2}$$

where $\beta_{(x,y)}(i) \in \mathbb{Z}_{g_i}$.

Remark. There are three types of relations of the form (2).

Type 1. If $x, y \in X_s$ then x and y commute modulo G_{s-1} and thus we get that $\beta_{(x,y)}(i) = 0$ if $g_i \notin X_1 \cup \cdots \cup X_{s-1} \cup \{x\}$ and that $\beta_{(x,y)}(i) = 1$ if $g_i = x$.

Now suppose that $s < t$.

Type 2. If $x \in X_s(p)$ and $y \in X_t$ then $x^y G_{s-1} \in (G_s/G_{s-1})_p$ and thus we get a relation of the form (2) where $\beta_{(x,y)}(i) = 0$ if $g_i \notin X_1 \cup \cdots \cup X_{s-1} \cup X_s(p)$.

Type 3. Finally if $x \in X_s(\infty)$ and $y \in X_t$ then $x^y \in G_s$ and we get a relation of the type (2) where $\beta_{(x,y)}(i) = 0$ if $g_i \notin X_1 \cup \cdots \cup X_s$.

Remark. By an easy induction on m, one can see that (1) and (2) also give us, for every pair of generators $x, y \in X$ such that $x < y$, a relation $x^{y^{-1}} = \mu(x, y)$, where $\mu(x, y)$ is a normal form expression in $g_1, g_2, \ldots, g_m$. Thus using only relations (1) and the three types of relations (2), we have full information about G and we can calculate inverses and products of elements in normal form and turn the result into a normal form expression using for example collection from the left.

The claim holds trivially for $m = 1$. Now suppose that $m \geq 2$ and that the claim holds for all smaller values of m. Consider the subgroup $H = \langle g_1, \ldots, g_{m-1} \rangle$. By the inductive hypothesis, every element in H can be turned into a normal form expression using only relations (1) and (2). Now (2) gives us normal form expressions for $g_1^{g_m}, \ldots, g_{m-1}^{g_m}$ and this determines an automorphism $\phi \in \mathrm{Aut}\,(H)$ induced by conjugation by g_m. This then gives us ϕ^{-1} that gives us in turn normal form expressions for $g_1^{g_m^{-1}}, \ldots, g_{m-1}^{g_m^{-1}}$. This finishes the proof of the inductive step.

The point about this is that the relations $x^{y^{-1}} = \mu(x, y)$ are consequences of (1) and (2). So for a polycyclic group G we only need (1) and (2) to define it. For practical reasons we need however to determine the relations $x^{y^{-1}} = \mu(x, y)$ first to be able to perform calculations in G. At the end of Section 3, we describe an efficient method for doing this for the presentations that we are about to introduce next, refined solvable presentations.

Suppose now conversely that we have a finite alphabet $X = \{x_1, x_2, \ldots, x_m\}$ with an ordering $x_1 < x_2 < \ldots < x_m$. If $X = \emptyset$ we associate to this data the presentation for the trivial group that has no generators and no relations. Suppose now that $X \neq \emptyset$. Partition X into some disjoint non-empty subsets $X_1, \ldots, X_r$ such that the elements of X_i precede those in X_j whenever $i < j$. Then partition further each X_i as a union of disjoint subsets (most empty of course)

$$X_i = (\bigcup_{p \in \mathcal{P}} X_i(p)) \cup X_i(\infty).$$

We introduce three maps that we will refer to as *presentation maps*. The first one is

$$n : X \to \mathbb{N} \cup \{\infty\}$$

such that $n(x) = \infty$ if $x \in X_i(\infty)$ and $n(x)$ is a non-trivial power of p if $x \in X_i(p)$. We let $\mathbb{Z}_x = \mathbb{Z}$ if $n(x) = \infty$ and otherwise we let $\mathbb{Z}_x = \{0, 1, \ldots, n(x) - 1\}$. Let $Y = X \setminus \{x \in X : n(x) = \infty\}$ and $Z = \{(x, y) \in X \times X : x < y\}$. Let F be the free group on X. The second presentation map is

$$\pi : Y \to F$$

where, if $x \in X_s(p)$, $\pi(x) = x_m^{\alpha_x(m)} \cdots x_1^{\alpha_x(1)}$ with $\alpha_x(i) \in \mathbb{Z}_{x_i}$ and $\alpha_x(i) = 0$ whenever $x_i \notin X_1 \cup \cdots \cup X_{s-1}$. Notice that these are the conditions for the right hand side of the power relation (1) with $g_1, \ldots, g_m$ replaced by $x_1, \ldots, x_m$. The final presentation map is

$$\delta : Z \to F$$

where $\delta(x, y) = x_m^{\beta_{(x,y)}(m)} \cdots x_1^{\beta_{(x,y)}(1)}$ and the conditions for the right hand side of (2) above hold as indicated in the remark that follows it with $g_1, \ldots, g_m$ replaced by $x_1, \ldots, x_m$. So we have a data structure that consists of an ordered alphabet X with a partition and three presentation maps. To this data we associate a presentation with generators $x_1, \ldots, x_m$, power relations

$$x^{n(x)} = \pi(x)$$

for any $x \in X$ such that $n(x) \neq \infty$, and conjugacy relations

$$x^y = \delta(x, y)$$

for each pair $(x, y) \in X \times X$ such that $x < y$. We call such a presentation a *refined solvable presentation.*

Let N be the normal subgroup of the free group F that is generated as a normal subgroup by the relators of the presentation and let $G = F/N$. Let $g_i = x_i N$ for $i = 1, \ldots, m$ and let $G_j = \langle X_1 \cup \cdots \cup X_j \rangle N$

for $j = 1, \ldots, r$.

Definition. We say that the refined solvable presentation is *consistent* if the following two conditions hold:

(1) $G_1, \ldots, G_r \trianglelefteq G$.
(2) Every element $g \in G$ has a unique normal form expression

$$g = g_m^{r_m} \cdots g_1^{r_1}$$

with $r_i \in \mathbb{Z}_{x_i}$.

Remark. Suppose we have a consistent refined solvable presentation defining a group G. It follows from the relations that G_i/G_{i-1} is abelian. As $G_1, \ldots, G_r$ are finitely generated normal subgroups of G it follows that G is polycyclic. Thus every consistent refined solvable presentation defines a polycyclic group. Conversely we have seen above that every polycyclic group can be defined by a consistent refined solvable presentation.

Remark. Notice that there are groups with a refined solvable presentation that are not polycyclic. Take for example two variables $x_1 < x_2$ and let $X_1 = X_1(\infty) = \{x_1\}$, $X_2 = X_2(\infty) = \{x_2\}$. Here $Y = \emptyset$ and $Z = \{(x_1, x_2)\}$. For the presentation maps $n : X \to \mathbb{N} \cup \{\infty\}$ and $\pi : Y \to F$, we must have $n(x_1) = n(x_2) = \infty$ and π must be empty. Suppose we choose $\delta : Z \to F$ such that $\delta(x_1, x_2) = x_1^2$. Then we get a presentation with two generators x_1, x_2 and one relation

$$x_1^{x_2} = x_1^2.$$

The group defined by this presentation is a Baumslag-Solitar group [4] that is not polycyclic. Thus the presentation is not consistent as can be seen directly from the fact that $G_1 = \langle x_1 N \rangle$ is not normal in $G = F/N$.

In Section 3 we will describe consistency criteria for refined solvable presentations.

3. The consistency criteria

Before establishing our consistency criteria, we first describe constructions that are central to what follows. Suppose we have a polycyclic group G defined by a consistent refined solvable presentation as described in Section 2. Thus $G_1 < \ldots < G_r$ is the corresponding ascending chain of normal subgroups with abelian factors. Let $\phi \in \mathrm{Aut}\,(G)$ such that $G_1^\phi = G_1, \ldots, G_r^\phi = G_r$. Let $\mathcal{G} \subseteq F$ be the set of all normal form expressions in the variables $x_1, \ldots, x_m$. That is, we take all the normal form expressions in $g_1, \ldots, g_m$ and in each of these we replace $g_1, \ldots, g_m$ by $x_1, \ldots, x_m$. As the presentation is consistent, this gives us an identification of G with $\mathcal{G}$ that induces a group structure on $\mathcal{G}$ such that $\mathcal{G} \cong G$. We can thus think of ϕ as acting on $\mathcal{G}$.

We will consider two situations where we can use this data to get a consistent refined solvable presentation for a larger polycyclic group $\tilde{G}$. Add a new variable x_{m+1} and extend our order on $\tilde{X} = X \cup \{x_{m+1}\}$ such that x_{m+1} is larger than the elements in X. Let $\tilde{F}$ be the free group on $\tilde{X}$. Let H be the semidirect product of $\mathcal{G} \cong G$ with a infinite cyclic group $C_\infty = \langle u \rangle$ where the action of C_∞ on G is given by $g^u = g^\phi$.

For the first situation let $\tilde{G} = H$. We extend the presentation maps n, π, δ to $\tilde{n}, \tilde{\pi}, \tilde{\delta}$ so they involve $\tilde{X}$. We do this by letting $\tilde{n}(x_{m+1}) = \infty$ and

$$\tilde{\delta}(x_i, x_{m+1}) = x_i^\phi \text{ (a normal form expression in } x_1, \ldots, x_m)$$

for $i = 1, \ldots, x_m$. Notice that, since $\tilde{n}(x_{m+1}) = \infty$, $\tilde{\pi} = \pi$. The refined solvable presentation that we get using the extended presentation maps has all the relations for G together with m extra relations

$$x_i^{x_{m+1}} = \tilde{\delta}(x_i, x_{m+1}) = x_i^\phi$$

for $i = 1, \ldots, m$. Let $\tilde{N}$ be the normal subgroup of $\tilde{F}$ generated as normal subgroup by all the relators for the new refined solvable presentation.

We next turn to the partition of $\tilde{X} = \{x_1, \ldots, x_{m+1}\}$. The partition could be into $\tilde{X}_1 = X_1, \ldots, \tilde{X}_r = X_r, \tilde{X}_{r+1} = \{x_{m+1}\}$. If furthermore $x^{-1}x^\phi \in \langle X_1 \cup \cdots \cup X_{r-1} \rangle$ for all $x \in X_r$ we could instead choose a partition with $\tilde{X}_1 = X_1, \ldots, \tilde{X}_{r-1} = X_{r-1}, \tilde{X}_r = X_r \cup \{x_{m+1}\}$.

Let $\tilde{G}_i = \langle \tilde{X}_1 \cup \cdots \cup \tilde{X}_i \rangle \tilde{N}$ for $i = 1, \ldots, r+1$. As $G_1^\phi = G_1, \ldots, G_r^\phi = G_r$, we still have that $\tilde{G}_1, \ldots, \tilde{G}_r$ are normal in $\tilde{G}_{r+1} = \tilde{F}/\tilde{N} \cong \tilde{G}$. For both choices of partition for $\tilde{X}$ we clearly have that the factors of the ascending series

$$\{1\} < \tilde{G}_1 < \ldots < \tilde{G}_{r+1} = \tilde{F}/\tilde{N}$$

are abelian. Also every element $g \in \tilde{G}$ has a unique expression of the form $g = u^m a$ with $m \in \mathbb{Z}$ and $a \in \mathcal{G}$. Thus the new refined solvable presentation is consistent as well.

The second situation is a variant of the first. Now suppose furthermore that for some integer $e \geq 2$, that is a power of a prime p, and $g \in \mathcal{G}$ we have that

$$a^g = a^{\phi^e} \quad \text{(for all } a \in \mathcal{G}) \tag{3.1}$$

$$g^\phi = g. \tag{3.2}$$

In this case $M = \langle g^{-1}x^e \rangle$ is a subgroup of the centre of H. Let $\tilde{G} = H/M$. $\mathcal{G}$ embeds naturally into $\tilde{G}$ and we identify it with its image. We now extend the presentation maps n, π, δ to $\tilde{n}, \tilde{\pi}, \tilde{\delta}$ as follows. First we let $\tilde{n}(x_{m+1}) = e$ and $\tilde{\pi}(x_{m+1}) = g$. Notice that, as $g \in \mathcal{G}$, g is a normal form expression in $x_1, \ldots, x_m$. Finally, as before, let $\tilde{\delta}(x_i, x_{m+1}) = x_i^\phi$ which is again in $\mathcal{G}$ and thus a normal form expression in $x_1, \ldots, x_m$. The refined solvable presentation with respect to the presentation maps $\tilde{n}, \tilde{\pi}$ and $\tilde{\delta}$ is then a presentation with all the relations for G and the extra relations

$$x_{m+1}^{n(x_{m+1})} = \tilde{\pi}(x_{m+1}) = g$$

together with

$$x_i^{x_{m+1}} = \tilde{\delta}(x_i, x_{m+1}) = x_i^{\phi}$$

for $1 \leq i \leq m$. Similar considerations hold for the partition for $\tilde{X}$ as in the previous situation and one sees similarly that the new solvable presentation is a consistent refined solvable presentation for the polycyclic group $\tilde{G} = H/M$.

We now turn back to our task of finding consistency criteria for refined solvable presentations. Suppose we have a group G defined by a refined solvable presentation as described above. So we have some partition of $X = \{x_1, \ldots, x_m\}$ and presentation maps n, π, δ giving us relations

$$x^{n(x)} = \underbrace{x_m^{\alpha_x(m)} \cdots x_1^{\alpha_x(1)}}_{\pi(x)}$$

for $x \in Y$ and

$$x^y = \underbrace{x_m^{\beta_{(x,y)}(m)} \cdots x_1^{\beta_{(x,y)}(1)}}_{\delta(x,y)}$$

for $x_1 \leq x < y \leq x_m$. For $k = 0, 1, \ldots, m$, let H_k be the group defined by the sub-presentation with generators $x_1, \ldots, x_k$ and those of the relations involving only $x_1 \leq x < y \leq x_k$. Notice that H_0 is the trivial group. The idea is to establish inductively criteria for the refined solvable presentation for H_k to be consistent. The induction basis $k = 0$ doesn't need any work. Now suppose $0 \leq k \leq m-1$ and that the sub-presentation for H_k is consistent. Let $\mathcal{H}_k$ be the set of all normal form expressions in $x_1, \ldots, x_k$. As the sub-presentation for H_k is consistent, we can identify H_k with $\mathcal{H}_k$ as above. Using the presentation map δ we define a function $\delta(x_{k+1}) : \mathcal{H}_k \to \mathcal{H}_k$ by first defining the values of the generators as $x_i^{\delta(x_{k+1})} = \delta(x_i, x_{k+1})$ for $i = 1, \ldots, k$. We then extend this to the whole of $\mathcal{H}_k$ by letting $\delta(x_{k+1})$ act on normal form expressions as follows

$$(x_k^{r_k} \cdots x_1^{r_1})^{\delta(x_{k+1})} = (x_k^{\delta(x_{k+1})})^{r_k} \cdots (x_1^{\delta(x_{k+1})})^{r_1}.$$

Notice that $\delta(x_1) = \mathrm{id}$. Suppose the resulting map $\delta(x_{k+1})$ is an automorphism. If $n(x_{k+1}) = \infty$, we have that the sub-presentation defining H_{k+1} is a consistent refined solvable presentation for the semidirect product of $\mathcal{H}_k$ with the infinite cyclic group $C_\infty = \langle u \rangle$ where $g^u = g^{\delta(x_{k+1})}$. Now suppose that $n(x_{k+1}) \neq \infty$. Using the second construction above and taking into account conditions (3) and (4), the sub-presentation for H_{k+1} is consistent, provided that

$$\begin{aligned} \pi(x_{k+1})^{\delta(x_{k+1})} &= \pi(x_{k+1}) \\ x_i^{\delta(x_{k+1})^{n(x_{k+1})}} &= x_i^{\pi(x_{k+1})} \end{aligned}$$

for $i = 1, \ldots, k$. It remains to find criteria for $\delta(x_{k+1})$ to be an automorphism. We turn next to this problem.

Let G be a polycyclic group defined by a consistent solvable presentation as described above. Let $\mathcal{G}$ be the set of all normal form expressions in $x_1, \ldots, x_m$. As the presentation is consistent, we can identify G with $\mathcal{G}$. For $s = 1, \ldots, r$ let $\mathcal{G}_s$ be the subgroup of $\mathcal{G}$ generated by $X_1 \cup \cdots \cup X_s$, $\mathcal{G}_s(p)$ be the subgroup generated by $X_1 \cup \cdots \cup X_{s-1} \cup X_s(p)$ and let $\tau(\mathcal{G}_s)$ be the subgroup generated by $X_1 \cup \cdots X_{s-1} \cup (\bigcup_{p \in \mathcal{P}} X_s(p))$. For each $x \in X$ choose an element x^ϕ subject to the following conditions:

$$x^\phi \in \mathcal{G}_i \quad \text{if} \quad x \in X_i \tag{3.3}$$
$$x^\phi \in \mathcal{G}_i(p) \quad \text{if} \quad x \in X_i(p).$$

We extend this to a map $\phi : \mathcal{G} \to \mathcal{G}$ by letting ϕ act on normal form expressions as:

$$(x_m^{r_m} \cdots x_1^{r_1})^\phi = (x_m^\phi)^{r_m} \cdots (x_1^\phi)^{r_1}.$$

Notice that the condition (5) implies that ϕ induces maps $\phi_s : \mathcal{G}_s \to \mathcal{G}_s$, $s = 1, \ldots, r$, where $\phi_s = \phi|_{\mathcal{G}_s}$. It also induces maps $\phi_{(s,p)} : \mathcal{G}_s(p)/\mathcal{G}_{s-1} \to \mathcal{G}_s(p)/\mathcal{G}_{s-1}$ and maps $\phi_{(s,\infty)} : \mathcal{G}_s/\tau(\mathcal{G}_s) \to \mathcal{G}_s/\tau(\mathcal{G}_s)$.

Lemma 1. *The map $\phi : \mathcal{G} \to \mathcal{G}$ is an endomorphism if and only if*

(a) $$\pi(x)^\phi = (x^\phi)^{n(x)} \quad (x_1 \leq x \leq x_m)$$

and

(b) $$x^{y\phi} = x^{\phi y^\phi} \qquad (x_1 \leq x < y \leq x_m).$$

We furthermore have that ϕ is an automorphism if for $s = 1, \ldots, r$ we have

(c) $$\begin{aligned} det(\phi_{(s,p)}) &\neq 0 \ (mod\ p) \\ det(\phi_{(s,\infty)}) &= \pm 1. \end{aligned}$$

Proof. Consider the free group $F = \langle x_1, \ldots, x_m \rangle$ and let N be the normal subgroup defined by the relators of the consistent refined solvable presentation defining G. Thus $G = F/N$. Let $\tilde{\phi} : G \to G$ be the map corresponding to the map $\phi : \mathcal{G} \to \mathcal{G}$. Consider the homomorphism $\psi : F \to F$ induced by the values $x^\psi = x^\phi$ for $x_1 \leq x \leq x_m$. Then conditions (a) and (b) imply that

$$(x^{n(x)})^\psi = (x^\psi)^{n(x)} = (x^\phi)^{n(x)} = \pi(x)^\phi = \pi(x)^\psi$$

and

$$(x^y)^\psi = (x^\psi)^{y^\psi} = x^{\phi y^\phi} = (x^y)^\phi = \delta(x,y)^\phi = \delta(x,y)^\psi.$$

Thus $N^\psi \leq N$ and ψ induces a endomorphism on $G = F/N$. This endomorphism is clearly the map $\tilde{\phi}$. Hence ϕ is an endomorphism.

The endomorphism ϕ is bijective if and only if the induced linear maps $\phi_{(s,p)}$ and $\phi_{(s,\infty)}$ are bijective and this happens if and only if condition (c) holds. $\square$

Remark. The condition (a) in the lemma above is of course only relevant when $n(x) < \infty$. To avoid making the statement more complicated we can decide that $\pi(x) = 1$ and $u^{n(x)} = 1$ for all $u \in G$ in the case when $n(x) = \infty$.

We now turn back again to the problem of establishing criteria for a refined solvable presentation to be consistent. Let G be a group defined by a refined solvable presentation as described above with relations

$$\begin{aligned} x^{n(x)} &= \underbrace{x_m^{\alpha_x(m)} \cdots x_1^{\alpha_x(1)}}_{\pi(x)} \quad (x_1 \leq x \leq x_m) \\ x^y &= \underbrace{x_m^{\beta_{(x,y)}(m)} \cdots x_1^{\beta_{(x,y)}(1)}}_{\delta(x,y)} \quad (x_1 \leq x < y \leq x_m). \end{aligned}$$

We let H_k be the group defined by the sub-presentation with generators $x_1, \ldots, x_k$ and those of the relations where $x_1 \leq x < y \leq x_k$. We establish inductively criteria for the sub-presentation for H_k to be consistent. Suppose this has been achieved for some k where $0 \leq k \leq m-1$. We want to add criteria so that the sub-presentation for H_{k+1} is consistent. We let $\delta(x_{k+1}) : \mathcal{H}_k \to \mathcal{H}_k$ be the map induced by the values $x^{\delta(x_{k+1})}$ in $\mathcal{H}_k$ as described above. As we pointed out, the presentation for H_{k+1} is consistent if and only if the map $\delta(x_{k+1})$ is an automorphism and that we have the extra criteria that

$$\begin{aligned} \pi(x_{k+1})^{\delta(x_{k+1})} &= \pi(x_{k+1}) \\ x_i^{\delta(x_{k+1})^{n(x_{k+1})}} &= x_i^{\pi(x_{k+1})}. \end{aligned}$$

From Lemma 1 we have criteria for $\delta(x_{k+1})$ to be an automorphism. Suppose that $x_{k+1} \in X_s$. Then $\delta(x_{k+1})$ acts trivially on $\mathcal{G}_s/\mathcal{G}_{s-1}$ and so to establish that $\delta(x_{k+1})$ is bijective we only need to show that $\delta(x_{k+1})_{(t,p)}$ and $\delta(x_{k+1})_{(t,\infty)}$ are bijective for $1 \leq t < s$.

For $z \in X$ let $r(z)$ be the integer such that $z \in X_{r(z)}$. Adding up for $k = 0, \ldots, m-1$, we obtain the 'if' part of the following theorem.

Theorem 2. *Let G be a group defined by a refined solvable presentation as described above. The refined solvable presentation is consistent if and only if the following criteria hold. Firstly we must* *have for all $x_2 \leq z \leq x_m$ that*

$$\begin{array}{llcll}
\text{(i)} & \pi(z)^{\delta(z)} & = & \pi(z) & \\
\text{(ii)} & \pi(x)^{\delta(z)} & = & (x^{\delta(z)})^{n(x)} & (x_1 \leq x < z) \\
\text{(iii)} & x^{\delta(z)^{n(z)}} & = & x^{\pi(z)} & (x_1 \leq x < z) \\
\text{(iv)} & x^{y\delta(z)} & = & x^{\delta(z)y^{\delta(z)}} & (x_1 \leq x < y < z).
\end{array}$$

We also need for $1 \leq s < r(z)$ that

$$\begin{array}{llcl}
\text{(v)} & det(\delta(z)_{(s,p)}) & \neq & 0 \ (mod\ p) \\
 & det(\delta(z)_{(s,\infty)}) & = & \pm 1.
\end{array}$$

Proof It only remains to deal with the 'only if' part. Suppose that the presentation is consistent and let as before $\mathcal{G}$ be the set of all normal form expressions in $x_1, \ldots, x_m$. We know that we can identify G with $\mathcal{G}$. Furthermore the ascending chain of normal subgroups $G_1 < \ldots < G_r$ of G induces a corresponding ascending chain of normal subgroups $\mathcal{G}_1 < \ldots < \mathcal{G}_r$ for $\mathcal{G}$. As the factors are abelian this implies that $\langle x_1, \ldots, x_i \rangle \trianglelefteq \langle x_1, \ldots, x_{i+1} \rangle$ and thus $\delta(x_{i+1})$ is the same as the conjugation by x_{i+1} on the $\langle x_1, \ldots, x_i \rangle$. This map is clearly bijective where $\delta(x_{i+1})^{-1}$ is induced by the conjugation by x_{i+1}^{-1}. As the groups $\mathcal{G}_s(p)$ and $\tau(\mathcal{G}_s)$ are normal in $\mathcal{G}$, we have that it is the conjugation by z that induces the linear operators $\delta(z)_{(s,p)}$ on $\mathcal{G}_s(p)/\mathcal{G}_{s-1}$ and $\delta(z)_{(s,\infty)}$ on $\mathcal{G}_s/\tau(\mathcal{G}_s)$. The linear operators are bijective with inverses that are induced by the conjugation by z^{-1}. Thus (v) holds. As $\mathcal{G}$ satisfies the relations of the consistent refined solvable presentation we also have $\pi(z)^z = (z^{n(z)})^z = z^{n(z)} = \pi(z)$ and $\pi(x)^z = (x^{n(x)})^z = (x^z)^{n(x)}$. Thus (i) and (ii) hold. Then taking the iterated conjugation action $n(z)$ times by z on x is the same as $x^{z^n} = x^{\pi(z)}$ that establishes (iii). Finally (iv) follows from the fact that $\delta(z)$ is a homomorphism. $\Box$

Remarks. (I) Recall that we established the consistency of the sub-presentation defining H_k recursively for $k = 0, 1, \ldots, m$. So according to the proof we should check (i)-(v) for $z = x_2, \ldots, x_m$ in ascending order. If $z = x_{k+1}$ then the consistency of the presentation for H_{k+1} follows from the consistency of the presentation for H_k together with relations (i)-(v) of Theorem 2 where $z = x_{k+1}$. So when doing the check for $z = x_{k+1}$ we can assume that the presentation for H_k is consistent. Using the definition of $\delta(z)$ we first transform all the expressions in (i)-(iv) into expressions in H_k. Then we turn each side of the equations into normal form in H_k and compare. It is interesting to note that (provided the check has been positive so far) H_k has a consistent presentation and so the normal form in each case is independent of how we calculate. We can however do the check in any order we like (and still sticking to the assumption that H_k has a consistent presentation). The reason for this is that we will at some point reach the smallest z where the check fails (provided that we haven't got a negative result in the mean time). Hence if the presentation is not a consistent this will be recognised.

(II) How does this approach compare to the existing ones? Our approach is to consider the function $\delta(x_{k+1})$ defined on H_k. Modulo consistency of the sub-presentation for H_k the conditions (i)-(v) in Theorem 2 are conditions for the map $\delta(x_{k+1})$ to be an automorphism ((ii), (iv) and (v)) and for the resulting cyclic extension to have a consistent presentation ((i) and (iii)). The emphasis is thus on the function $\delta(x_{k+1})$ rather than the group operation (as in [16]). It is our belief that this viewpoint makes things look a bit clearer.

(III) It should be noted however that our conditions (i)-(iv) have equivalent criteria in the standard approach. See the list (*) in [16], page 424. The 'overlaps' (1),(2),(3) and (5) in that list correspond to (iv),(ii),(iii) and (i) in Theorem 2. The condition (v) is however new and is a by-product of working with an ascending normal solvable series. In the standard approach one works with an ascending subnormal series with cyclic factors. It should also be noted that the idea of obtaining consistency recursively for H_k, $k = 0, \ldots, m$, through working with $\delta(z)$, is also implicit in [16] but is kept in the background within the proof. Our conditions (i)-(v) bring this to the surface.

A method for obtaining inverse conjugation relations. For practical checks using these consistency criteria one needs to determine first normal form expressions $x^{z^{-1}}$ for $x < z < x_m$ (in order to be able to transform any expression in H_k to an normal form expression). Note however that this is of course only needed when z is of infinite order. Another advantage of our approach is that it becomes quite simple and effective to determine these after having produced all the linear maps $\delta(z)_{(s,p)}$ and $\delta(z)_{(s,\infty)}$, $2 \leq s < r$. Suppose that $z \in X_s$ for some $2 \leq s < r$. We now describe how to obtain normal form expressions for $x^{z^{-1}}$ recursively for $x < z$.

We can suppose that we already know that the sub-presentation for the group G^* generated by the generators $\{x \in X : x < z\}$ (using only the relations involving these generators) is consistent. The presentation for G^* is built around an ascending normal z-invariant series with each factor either a finite abelian p-group or a finitely generated torsion-free abelian group.

Now suppose that we are looking at one such factor K/H and that the extra generators needed to generate K are $y_1, \ldots, y_e$. We can suppose inductively that we have obtained normal form expressions for all $x^{z^{-1}}$ when x is a generator of H. We want to extend this to $y_i^{z^{-1}}$ for $i = 1, \ldots, e$.

Let $v_1 = y_1H, \ldots, v_e = y_eH$ be the generators of K/H. Let ϕ be the automorphism on K/H induced by the conjugation action by z and let ψ be the inverse of ϕ. Suppose ψ is represented by the matrix $B = (b_{ij})$. Since $\phi(\psi(v_i)) = v_i$, we have

$$b_{ei}\phi(v_e) + \cdots + b_{2i}\phi(v_2) + b_{1i}\phi(v_1) = v_i.$$

It follows that (using the presentation and calculating in K) we get

$$(y_e^z)^{b_{ei}} \cdots (y_2^z)^{b_{2i}} (y_1^z)^{b_{1i}} = y_i u$$

where u is a normal form expression in the generators of H (and we already know how z^{-1} acts on u. It follows that

$$y_i^{z^{-1}} = y_e^{b_{ei}} \cdots y_2^{b_{2i}} y_1^{b_{1i}} u^{-z^{-1}}.$$

Example. To illustrate the method, we will now consider a simple example of a polycyclic group, E_∞, that is non-nilpotent and non-torsion. This is an example from [6] and is the standard wreath product of the infinite dihedral group by a cyclic group of order 2. That is

$$E_\infty = D_\infty \wr \mathbb{Z}_2.$$

Suppose $D_\infty = \langle a, b : \ a^b = a^{-1},\ b^2 = 1 \rangle$ and $\mathbb{Z}_2 = \langle c \rangle$. Let $x_1 = a$, $x_2 = a^c$, $x_3 = b$, $x_4 = b^c$ and $x_5 = c$. The group E_∞ then has the following refined solvable presentation:

<u>Generators</u>

$X_1 :\ x_1, x_2$
$X_2 :\ x_3, x_4$
$X_3 :\ x_5$

<u>Relations</u> (We omit relations of the form $x_i^{x_j} = x_i$ with $i < j$)

$x_3^2 = 1,\ x_4^2 = 1,\ x_5^2 = 1,$
$x_1^{x_3} = x_1^{-1},\ x_1^{x_5} = x_2,\ x_2^{x_4} = x_2^{-1},\ x_2^{x_5} = x_1,\ x_3^{x_5} = x_4,\ x_4^{x_5} = x_3.$

We now apply the consistency check described in Theorem 2. We start with condition (v). First notice that $X_1 = X_1(\infty)$, $X_2 = X_2(2)$ and $X_3 = X_3(2)$. We need to calculate the determinants of 4 linear maps. These are $\delta(x_3)_{(1,\infty)}$, $\delta(x_4)_{(1,\infty)}$, $\delta(x_5)_{(1,\infty)}$ and $\delta(x_5)_{(2,2)}$. These are all easy to calculate by hand. For example we have that the matrix for $\delta(x_5)_{(2,2)}$ with respect to the basis $x_3\langle X_1\rangle, x_4\langle X_1\rangle$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

that has determinant -1. Similarly all the other determinants satisfy condition (v).

Before moving on to the other four conditions we should determine the normal form for $x^{y^{-1}}$ for all $x_1 \le x < y \le x_5$ using the method described earlier. For this example this is however trivial as we only need to consider $y \in \{x_3, x_4, x_5\}$ in which case y is an involution.

Condition (i). We only need to check $\pi(z)^{\delta(z)} = \pi(z)$ for $z \in \{x_3, x_4, x_5\}$ and this is straightforward. For example $\pi(x_3)^{\delta(x_3)} = 1^{\delta(x_3)} = 1 = \pi(x_3)$.

Condition (ii). We need to check $\pi(x)^{\delta(z)} = (x^{\delta(z)})^{n(x)}$ for $x_1 \leq x < z$ where $z \in \{x_3, x_4, x_5\}$. Again there are very few checks. For example

$$\pi(x_3)^{\delta(x_5)} = 1^{\delta(x_5)} = 1,$$

and

$$(x_3^{\delta(x_5)})^{n(x_3)} = x_4^2 = 1.$$

Condition (iii). We need to check $x^{\delta(z)^{n(z)}} = x^{\pi(z)}$ for $x_1 \leq x < z$ where $z \in \{x_3, x_4, x_5\}$. For example

$$x_3^{\pi(x_5)} = x_3^1 = x_3,$$

and

$$x_3^{\delta(x_5)^{n(x_5)}} = x_3^{\delta(x_5)^2} = x_4^{\delta(x_5)} = x_3.$$

Condition (iv). Here we will need to check $x^{y\delta(z)} = x^{\delta(z)y^{\delta(z)}}$ for $x_1 \leq x < y < z$. In this example this can be done quite quickly as one only needs to consider 10 triples (x, y, z) of group elements. For example

$$x_1^{x_3\delta(x_5)} = (x_1^{-1})^{\delta(x_5)} = x_2^{-1},$$

and

$$x_1^{\delta(x_5)x_3^{\delta(x_5)}} = x_2^{x_4} = x_2^{-1}.$$

4. Implementation and some applications of our consistency checks

We have implemented our consistency check for nilpotent presentations using the NQL package [10] of the computer-algebra-system GAP; see [7]. In this section, we demonstrate that there are situations where the method yields a considerable speed-up in checking consistency of large polycyclic presentations (with some hundreds of generators). For this purpose, we consider nilpotent quotients of the Basilica group Δ from [9] and the Brunner-Sidki-Vieira-Group BSV from [5]. Both groups are two-generated but infinitely presented. The Basilica group admits the following infinite presentation

$$\Delta \cong \langle \{a, b\} \mid [a, a^b]^{\sigma^i}, i \in \mathbb{N}_0 \rangle$$

where σ is the endomorphism of the free group over a and b induced by the mapping $a \mapsto b^2$ and $b \mapsto a$; see [9]. The BSV group admits the infinite presentation

$$\mathrm{BSV} \cong \langle \{a, b\} \mid [b, b^a]^{\varepsilon^i}, [b, b^{a^3}]^{\varepsilon^i}, i \in \mathbb{N}_0 \rangle,$$

where ε is the endomorphism of the free group over a and b induced by the mapping $a \mapsto a^2$ and $b \mapsto a^2b^{-1}a^2$. The nilpotent quotient algorithm in [3] computes a weighted nilpotent presentation for the lower central series quotient $G/\gamma_c(G)$ for a group G given by an infinite presentation as above (a so-called finite L-presentation; see [2]). A weighted nilpotent presentation is a polycyclic presentation which refines the lower central series of the group. We note that the weighted nilpotent presentations for the quotients $\Delta/\gamma_c\Delta$ and $\mathrm{BSV}/\gamma_c\mathrm{BSV}$ are refined solvable presentations.

In order to verify consistency of a given polycyclic presentation, the algorithm in [16, p. 424] rewrites the overlaps of the rewriting rules and compares the result; that is, the algorithm checks the underlying rewriting system for local confluence. As even the state of art algorithm 'collection from the left' is exponential [12], the number of overlaps is a central bottleneck here. There are improvements known which make use of the structure of a polycyclic presentation in order to reduce the number of overlaps. For instance, for weighted nilpotent presentations, a weight function allows one to reduce the number of overlaps significantly; see [16, p. 431].

Our method replaces some overlaps by the computation of determinants of integer matrices and it can easily be combined with the method for weighted nilpotent presentations. This promising approach yields a considerable speed-up as the following table shows. In the case of BSV the timings were obtained on an AMD Quad Core processor whereas for Δ we used an Intel Pentium 4 processor. In both cases the clock-speed was 2.4 GHz. The method `Usual` denotes our implementation of the algorithm

Quotient	`#gens`	`Usual`	`Solv`	`Weight`	`Solv+Weight`
BSV, class 30	141	0 : 00 : 24	0 : 00 : 21	0 : 00 : 08	0 : 00 : 06
BSV, class 32	155	0 : 00 : 36	0 : 01 : 31	0 : 00 : 13	0 : 00 : 10
BSV, class 34	171	0 : 01 : 32	0 : 01 : 11	0 : 00 : 59	0 : 00 : 43
BSV, class 36	187	0 : 02 : 27	0 : 01 : 46	0 : 01 : 39	0 : 01 : 11
BSV, class 38	203	0 : 03 : 49	0 : 02 : 41	0 : 02 : 39	0 : 01 : 54
BSV, class 40	219	0 : 05 : 33	0 : 03 : 52	0 : 04 : 00	0 : 02 : 53
BSV, class 42	235	0 : 07 : 53	0 : 05 : 30	0 : 05 : 58	0 : 04 : 18
BSV, class 44	251	0 : 11 : 10	0 : 07 : 37	0 : 08 : 35	0 : 06 : 11
BSV, class 46	267	0 : 15 : 21	0 : 10 : 30	0 : 12 : 07	0 : 08 : 44
BSV, class 48	283	0 : 20 : 53	0 : 13 : 56	0 : 16 : 33	0 : 11 : 57
Δ, class 35	185	0 : 00 : 31	0 : 00 : 31	0 : 00 : 02	0 : 00 : 02
Δ, class 80	609	1 : 19 : 22	1 : 15 : 03	0 : 29 : 48	0 : 27 : 36
Δ, class 100	821	8 : 25 : 37	7 : 39 : 54	5 : 45 : 40	5 : 18 : 08

in [16, p. 424] for polycyclic presentations, the method `Solv` denotes our new method, the method `Weight` denotes the method for weighted nilpotent presentation as in [16, p. 431], and the method `Solv+Weight` denotes a combination of both of the latter methods. The number `#gens` denotes the number of generators of the considered polycyclic presentation. More information about the lower central quotients $\gamma_i(\mathrm{BSV})/\gamma_{i+1}(\mathrm{BSV})$ and $\gamma_i(\Delta)/\gamma_{i+1}(\Delta)$ can be found in [2]. In both cases the first two quotients are free abelian of rank 2 and 1 respectively whereas the remaining quotients are abelian 2-groups. Thus if G is one of the two groups then the refined solvable presentation for $G/\gamma_{n+1}(G)$ has a generating set $X_1 \cup X_2 \cup \cdots \cup X_n$ where $X_n = X_n(\infty)$, $X_{n-1} = X_{n-1}(\infty)$ and otherwise $X_i = X_i(2)$.

In summary, our method always yields a noticeable speed-up compared with the standard method for polycyclic groups. These examples were nilpotent and it is our hope that future investigations involving more general examples may provide further evidence for the new method.

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