A NOTE ON CONCISENESS OF ENGEL WORDS

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Abstract. It is still an open problem to determine whether the n-th Engel word $[x,_n y]$ is concise, that is, if for every group G such that the set of values $e_n(G)$ taken by $[x,_n y]$ on G is finite it follows that the verbal subgroup $E_n(G)$ generated by $e_n(G)$ is also finite. We prove that if $e_n(G)$ is finite then $[E_n(G), G]$ is finite, and either $G/[E_n(G), G]$ is locally nilpotent and $E_n(G)$ is finite, or G has a finitely generated section that is an infinite simple n-Engel group. It follows that $[x,_n y]$ is concise if n is at most four.

1. Introduction

Let x, y be two symbols, to which we refer as *indeterminates*, and let F be the free group having x, y as a free basis. The n-th Engel word $[x,_n y]$ can be identified with the element of F defined inductively by

$$[x,_0 y] = x; \quad [x,_n y] = [[x,_{n-1} y], y],$$

for all positive integers n.

Given a group G, we think of $[x,_n y]$ as a function from G^2 to G, by substituting group elements for the indeterminates. Thus we can consider the *set* $e_n(G)$ of all values taken by this function, that is,

$$e_n(G) = \{[g,_n h] \mid g, h \in G\}.$$

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The subgroup generated by $e_n(G)$ is called the *n-th Engel verbal subgroup* of G, and is denoted by $E_n(G)$.

In this paper we address the problem of determining whether $[x,_n y]$ is a concise word. In general, a word ω in some alphabet $x_1, \ldots, x_t$ is said to be concise if for every group G such that ω takes only a finite number of values in G it follows that the verbal subgroup $\omega(G)$ is also finite.

As mentioned in [9], Philip Hall had conjectured that every word is concise, and he proved this for every non-commutator word (i.e. a word outside the commutator subgroup of the free group), and for lower central words. In [10], Turner-Smith showed that derived words are also concise, and Jeremy Wilson [11] subsequently extended this result to all outer commutator words (which are words obtained by nesting commutators, but using always different indeterminates). On the other hand, Hall's conjecture was eventually refuted in 1989 by Ivanov, see [5]. Note that $[x,_n y]$ is not an outer commutator word if $n > 1$ (because the indeterminate y occurs more than once), and that the problem of determining its conciseness is still open.

In this paper we are able to prove a partial result in this direction, namely, that if G is a group such that $e_n(G)$ is finite, then $[E_n(G), G]$ is finite. Moreover, a strong dichotomy result holds.

Dichotomy Theorem. *Let G be a group, and assume that $e_n(G)$ has order m. Then $[E_n(G), G]$ is finite of (n, m)-bounded order. Furthermore, there exists a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for each $n \in \mathbb{N}$, exactly one of the following holds:*

(1) *$G/[E_n(G), G]$ is locally nilpotent and $E_n(G)$ is finite of order at most $f(n, m)$.*
(2) *G has a finitely generated section that is an infinite simple n-Engel group.*

Note that if G is locally solvable, or locally finite, or more generally locally graded, then case (1) occurs by Corollary 6 of [6]. Furthermore, since it

was proved in [4] that 4-Engel groups are locally nilpotent, we obtain as a consequence that $[x,_n y]$ is concise for every $n \leq 4$.

Corollary. *Let G be a group and let $n \leq 4$. If $e_n(G)$ is finite of order m, then $E_n(G)$ is finite of m-bounded order.*

The existence of infinite simple n-Engel groups is a longstanding problem which is open for $n > 4$. Note however that it might happen that such a group exists for some n, and still the n-th Engel word is concise.

2. Proofs of the results

We first prove that if G is a group such that $e_n(G)$ is finite of order m then $[E_n(G), G]$ is finite of (n, m)-bounded order. The first step of the proof is to show that $E_n(G)'$ is finite of m-bounded order. We obtain this result as a particular case of the following general proposition. In the remainder, if ω is a group word, we use the notation G_ω for the set of all values of ω in a group G.

Proposition 1. *Let ω be a group word, and let G be a group such that $|G_\omega| = m$. Then $|\omega(G)'| < ((m-1)(m-2))^{m^2}$.*

Proof. The proof is modelled on the second part of the proof of Theorem A in [3] (and so also, indirectly, on the second part of the proof of Theorem 3.4 in [1]). Suppose that ω depends on the indeterminates $x_1, \ldots, x_k$, and choose a new set $y_1, \ldots, y_k$ of indeterminates. If we define a new word α by

$$\alpha = [\omega(x_1, \ldots, x_k), \omega(y_1, \ldots, y_k)],$$

then $|G_\alpha| \leq m^2$ and $\alpha(G) = \omega(G)'$.

We claim that the order of an element $g \in G_\alpha$ is at most $(m-1)(m-2)$. Of course, we may assume $g \neq 1$. Let us write $g = [a, b]$ with $a, b \in G_\omega$, and consider the subgroup $H = \langle a, b \rangle$. Put $C = C_H(a)$. Since $a \in G_\omega \setminus \{1\}$, it has at most $m-1$ conjugates in G, and consequently $|H : C| \leq m-1$. Now C permutes the $m-1$ non-trivial values of G_ω, and leaves the element a fixed

by definition. Thus $|C : C_C(b)| \leq m-2$, and consequently $|H : Z(H)| = |H : C_H(a) \cap C_H(b)| \leq (m-1)(m-2)$. By applying Schur's Theorem [7, 10.1.4] to H, it follows that the exponent of H' is at most $(m-1)(m-2)$, which proves the claim.

Since G_α is a normal finite set of elements of finite order, we can apply Dietzmann's Lemma [7, 14.5.7] (more precisely, its proof) to conclude that $\omega(G)' = \langle G_\alpha \rangle$ is finite, of order at most $((m-1)(m-2))^{m^2}$. □

We need two more lemmas.

Lemma 2. *Let G be a group, let $g, h \in G$ and assume that $[g,_{i+1}\ h] = 1$ for some positive integer i. Then*

$$[g,_{i-1}\ h, h^s] = [g,_i\ h]^s$$

for every positive integer s.

Proof. The proof is by induction on s. The result is obviously true for $s = 1$, so we assume that $s \geq 2$ and that the result holds for $s-1$. By using the induction hypothesis and the fact that $[g,_{i+1}\ h] = 1$, we have

$$\begin{aligned}[g,_{i-1}\ h, h^s] &= [g,_i\ h][g,_{i-1}\ h, h^{s-1}]^h = [g,_i\ h]([g,_i\ h]^{s-1})^h \\ &= [g,_i\ h]([g,_i\ h]^h)^{s-1} = [g,_i\ h]([g,_i\ h][g,_i\ h, h])^{s-1} = [g,_i\ h]^s,\end{aligned}$$

which proves the lemma. □

Since G acts by conjugation on the normal subgroup $E_n(G)$, if $E_n(G)$ is abelian then it is actually a $\mathbb{Z}[G]$-module. In this case, we will keep the multiplicative notation in $E_n(G)$, and the action of an element $z \in \mathbb{Z}[G]$ on an element $v \in E_n(G)$ will be denoted by v^z.

Lemma 3. *Let G be a group such that $E_n(G)$ is abelian. If $u \in e_{2n}(G)$ then $u^s \in e_n(G)$ for every positive integer s.*

Proof. Write $u = [g,_{2n}\ h] \in e_{2n}(G)$, with $g, h \in G$, and let $a = [g,_n\ h]$. Hence $u = [a,_n\ h]$. If we view $E_n(G)$ as a $\mathbb{Z}[G]$-module, then

$$u^s = (a^{(h-1)^n})^s = (a^s)^{(h-1)^n} = [a^s,_n\ h] \in e_n(G),$$

as desired. □

Proposition 4. *Let G be a group such that $|e_n(G)| = m$. Then $[E_n(G), G]$ is finite of (n, m)-bounded order.*

Proof. By Proposition 1, we may assume that $E_n(G)$ is abelian.

Let $|e_i(G)| = l_i$ for $i = n+1, \dots, 2n$. If $u \in e_{2n}(G)$, then since $|e_n(G)| = m$, it follows from Lemma 3 that there exist positive integers s, t with $1 \leq s < t \leq m+1$ such that $u^s = u^t$. Hence u has finite order, which is at most $t - s \leq m$. It follows that $E_{2n}(G)$ has order at most $m^{l_{2n}}$, since it is an abelian group generated by l_{2n} elements of order at most m.

Now consider the normal series of subgroups

$$1 \leq E_{2n}(G) \leq E_{2n-1}(G) \leq \cdots \leq E_{n+1}(G) \leq E_n(G).$$

We are going to show that

$$|E_i(G)/E_{i+1}(G)| \leq (l_{i-1} - l_{i+1})^{l_i - l_{i+1}},$$

for each $i = n+1, \dots, 2n-1$, and as a consequence that $E_{n+1}(G)$ is finite of (n, m)-bounded order.

Let $\bar{G} = G/E_{i+1}(G)$ and let $\bar{z}$ denote the image of z in $\bar{G}$. We have $|e_i(\bar{G})| \leq l_i - l_{i+1} + 1$ and $|e_{i-1}(\bar{G})| \leq l_{i-1} - l_{i+1} + 1$ (here the $+1$ takes into account the identity).

Now consider an element $[g,_i\ h] \in e_i(G)$ such that $[\bar{g},_i\ \bar{h}] \neq \bar{1}$. Since $[\bar{g},_{i-1}\ \bar{h}]$ has at most $l_{i-1} - l_{i+1}$ conjugates, there exists $s \leq l_{i-1} - l_{i+1}$ such that $\bar{h}^s$ centralizes $[\bar{g},_{i-1}\ \bar{h}]$. It follows from Lemma 2 that $[\bar{g},_i\ \bar{h}]$ has order at most $l_{i-1} - l_{i+1}$, and so

$$|E_i(G)/E_{i+1}(G)| \leq (l_{i-1} - l_{i+1})^{l_i - l_{i+1}},$$

as we wanted.

Now we prove that $[E_n(G), G]$ is finite of (n, m)-bounded order. By the previous paragraphs, we may assume that $E_{n+1}(G) = 1$. Let $\alpha_i = [x,_n\ y,_i\ z]$, where $i \geq 0$. Then α_i takes at most $(2m)^{2^i}$ values in G, since each value of α_i is a product of 2^i elements, each lying in $e_n(G) \cup e_n(G)^{-1}$ (note that $\alpha_i = \alpha_{i-1}^{-1}\alpha_{i-1}^z$). Also the verbal subgroup $\alpha_{n+1}(G)$ is contained in $E_{n+1}(G) = 1$ and $\alpha_1(G) = [E_n(G), G]$.

Consider the normal series of subgroups

$$1 = \alpha_{n+1}(G) \leq \cdots \leq \alpha_i(G) \leq \cdots \leq \alpha_1(G).$$

The same argument as before shows that every value of α_i in $G/\alpha_{i+1}(G)$ has order at most $(2m)^{2^{i-1}}$, and consequently

$$|\alpha_i(G)/\alpha_{i+1}(G)| \leq (2m)^{2^{i-1}(2m)^{2^i}}$$

for each $i = 1, \ldots, n$. Thus $\alpha_1(G)$ is finite of (n, m)-bounded order, which concludes the proof. □

Before embarking on the proof of the Dichotomy Theorem, we consider locally nilpotent groups G in which $e_n(G)$ is finite.

Lemma 5. *Let G be a locally nilpotent group such that $e_n(G)$ is finite of order m. Then G is an $(n + m - 1)$-Engel group.*

Proof. Let $a, b \in G$, and consider the set

$$A = \{[a,_n\ b], [a,_{n+1}\ b], \ldots, [a,_{n+m}\ b]\}.$$

Since A has at most m elements, there exist $i, j \in \{0, \ldots, m\}$ with $j > i$ such that $[a,_{n+i}\ b] = [a,_{n+j}\ b]$. Then $[a,_{n+i}\ b] = [a,_{n+i+\lambda(j-i)}\ b]$ for each positive integer λ and, by the nilpotency of $\langle a, b\rangle$, it follows that $[a,_{n+i}\ b] = 1$. So $[a,_{n+m-1}\ b] = 1$ for each $a, b \in G$, and G is an $(n + m - 1)$-Engel group. □

Proposition 6. *Let G be a locally nilpotent group such that $e_n(G)$ is finite of order m. Then $E_n(G)$ is finite of (n,m)-bounded order.*

Proof. By the previous lemma, G is a locally nilpotent $(n+m-1)$-Engel group, and so by the Main Theorem of [2] there exist (n,m)-bounded constants c and d such that $\gamma_c(G)^d = 1$. Moreover, by Proposition 1 we may assume that $E_n(G)$ is abelian, so it is enough to show that $[a,_n b]$ is of finite (n,m)-bounded order for every $a, b \in G$. Thus we may assume that $G = \langle a, b\rangle$. Also, since $\gamma_c(G)$ is of (n,m)-bounded exponent, we may assume without loss of generality that $\gamma_c(G) = 1$.

Now let $F = \langle x, y\rangle$ be the free nilpotent group of class $c-1$ and rank 2. Then F is torsion-free and

$$F/\gamma_2(F),\ \gamma_2(F)/\gamma_3(F),\ \dots\ ,\gamma_{c-1}(F)/\gamma_c(F)$$

are free abelian groups of finite (n,m)-bounded rank. Consider the verbal subgroup $E = E_n(F)$, and let $E_i = E \cap \gamma_i(F)$. Notice that $E \leq \gamma_{n+1}(F)$. Then

$$E_i/E_{i+1} \cong E_i\gamma_{i+1}(F)/\gamma_{i+1}(F)$$

is a free abelian group of rank at most the rank of $\gamma_i(F)/\gamma_{i+1}(F)$. We can thus choose a set $T = T_{n+1} \cup T_{n+2} \cup \cdots \cup T_{c-1}$ of generators for E such that the elements

$$\{\omega\gamma_{i+1}(F) : \omega \in T_i\}$$

freely span $E_i\gamma_{i+1}(F)/\gamma_{i+1}(F)$ as a free abelian group. Notice that $|T|$ is (n,m)-bounded (at most the sum of the ranks of the groups $\gamma_{n+1}(F)/\gamma_{n+2}(F)$, $\cdots, \gamma_{c-1}(F)/\gamma_c(F)$). As a consequence, every element $\omega \in T$ is a product in $e_n(F) \cup e_n(F)^{-1}$ of (n,m)-bounded length.

We now move back to the original setting with the group $G = \langle a, b\rangle$ such that $\gamma_c(G) = 1$, and let $A_i = E_n(G) \cap \gamma_i(G)$. Consider some word $\omega \in T_i$, which is a product of left-normed commutators of weight i (in x, y) modulo

$\gamma_{i+1}(F)$. Now ω takes only a finite number of values l in $G/\gamma_{i+1}(G)$, where l is (n,m)-bounded.

We want to prove that $A_i/A_{i+1} \cong A_i\gamma_{i+1}(G)/\gamma_{i+1}(G)$ is of (n,m)-bounded exponent, so we assume that $\gamma_{i+1}(G) = 1$. Now

$$\omega(a^r, b^r) = \omega(a,b)^{r^i},$$

and as ω only takes l values, we have $\omega(a,b)^{r^i} = \omega(a,b)^{r^j}$ for some $0 \leq i < j \leq l$. Then $\omega(a,b)^{r^j - r^i} = 1$. This proves that A_i/A_{i+1} is of (n,m)-bounded exponent for each $i = n+1, \ldots, c-1$.

Since c is (n,m)-bounded, it follows that $A_{n+1} = E_n(G)$ is of (n,m)-bounded exponent. In particular $[a,_n b]$ is of (n,m)-bounded order, as we wanted to prove. □

Now we can prove our main theorem and its corollary.

Proof of the Dichotomy Theorem. Let G be a group such that $e_n(G)$ has order at most m. It follows from Proposition 4 that $[E_n(G), G]$ is finite of (n,m)-bounded order.

First suppose that $G/[E_n(G), G]$ is not locally nilpotent. This group is an $(n+1)$-Engel group, and by a folklore result on Engel groups, $G/[E_n(G), G]$ has a finitely generated infinite simple section H (see Theorem 4.1 of [8] for a proof). Now H is centre-by-(n-Engel), and since it is simple non-abelian, H is necessarily an n-Engel group. Hence G is of type (2). We are left with the situation when $G/[E_n(G), G]$ is locally nilpotent. Then Proposition 6 implies that G is of type (1). □

Proof the Corollary. Let $n \leq 4$ and let G be a group such that $e_n(G)$ has order m. By the Main Theorem in [4], it follows that $G/E_n(G)$ is locally nilpotent. So $G/[E_n(G), G]$ is also locally nilpotent, and by the Dichotomy Theorem, $E_n(G)$ is finite of bounded order. □

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