On right \( n \)-Engel subgroups II

Peter G. Crosby & Gunnar Traustason
Department of Mathematical Sciences,
University of Bath,
Bath BA2 7AY, UK

In this sequel to “On right \( n \)-Engel subgroups” we add a new general structure result on right \( n \)-Engel subgroups. We also use one of the structure results to prove some results about right \( n \)-Engel subgroups in finite \( p \)-groups.

1 Introduction

Let \( n \) be a non-negative integer. The \( n \)-Engel word \([y, n, x]\) is defined recursively by \([y, 0, x] = y\) and \([y, m+1, x] = [[y, m, x], x]\). Recall that an element \( a \) in a group \( G \) is said to be right \( n \)-Engel if \([a, n, g] = 1\) for all \( g \in G \) and that the group \( G \) is said to be \( n \)-Engel if every element \( a \in G \) is right \( n \)-Engel. Clearly for every element \( a \) in the \((n+1)\)th term of the upper central series, \( Z_n(G) \), we have that all the elements in \( \langle a \rangle^G \) are right \( n \)-Engel. It is conversely not true in general that right \( n \)-Engel elements need to be in the hypercentre. Take for example the standard wreath product \( C_2 \wr C_2^\infty \), of the cyclic group of order 2 with the infinite countable direct product of groups of order 2. This group is a 3-Engel group with a trivial centre. For a number of classes of groups we do however have that the right Engel elements belong to the hypercentre. For example this is true for finite groups \([2]\) and finitely generated solvable groups \([3]\).

This paper is a sequel to \([6]\). Our work in that paper and the present one is motivated by the following results on the structure of \( n \)-Engel groups. For all four results we let \( G \) be a nilpotent \( n \)-Engel group.
Structure Theorem 1 (Wilson [11]) If $G$ is a $d$-generator group, then the nilpotence class of $G$ is $(d, n)$-bounded.

Structure Theorem 2 (Zel’manov [12]) If $G$ is torsion free, then the nilpotence class of $G$ is $n$-bounded class.

Structure Theorem 3 (Crosby and Traustason [5]) There exist integers $f(n)$ and $c(n)$ such that $G^{f(n)} \leq Z_{c(n)}(G)$.

Structure Theorem 4 (Burns and Medvedev [4]) There exist integers $f(n)$ and $c(n)$ such that $\gamma_{c(n)}(G)^{f(n)} = \{1\}$.

Notice that both the third and the fourth theorem imply the 2nd. It should also be mentioned that the third theorem was preceded by a result of Burns and Medvedev [4], who proved under the same assumptions that there exist integers $f(n), c(n)$ such that $G^{f(n)}$ is nilpotent of class $c(n)$.

We now move to the analogous statements for right Engel subgroups. First a definition.

**Definition.** Let $H$ be a subgroup of a group $G$. Then $H$ is said to be a right $n$-Engel subgroup if all the elements of $H$ are right $n$-Engel elements of $G$.

As we said above, for any group $G$, the normal subgroup $Z_{n}(G)$ consists of right $n$-Engel elements. We are interested in the reverse problem. Suppose $H$ is a normal right $n$-Engel subgroup of $G$ and suppose that $H$ belongs to some term of the upper central series. We refer to the smallest integer $c$ such that $H \leq Z_{c}(G)$ as the upper central degree of $H$. Our analogous results for this situation are.

**Theorem 1** ([6]) If $G$ is a $d$-generator group, then the upper central degree of $H$ is $(d, n)$-bounded.

**Theorem 2** ([6]) If $H$ is torsion-free, then the upper central degree of $H$ is $n$-bounded.
Theorem 3 ([6]) There exist integers $c(n), f(n)$, only depending on $n$, such that $H^{f(n)} \leq Z_{c(n)}(G)$.

Theorem 4 There exist integers $c(n), f(n)$, only depending on $n$, such that $[H,c(n)G]^{f(n)} = \{1\}$.

It is not difficult to see that these theorems imply the structure theorems on Engel groups discussed above and that both Theorem 3 and Theorem 4 imply Theorem 2. The way the proof works is however that one uses Theorems 1 and 2 to prove Theorem 3 and Theorem 4.

Theorems 1, 2 and 3 were proved in [6]. In Section 2 we will prove Theorem 4 and we will also use Theorem 3 to obtain some results concerning right $n$-Engel subgroups of finite $p$-groups. These results are analogues to the following results on the structure of $n$-Engel $p$-groups [1]. Let $p$ be a prime and let $r = r(n,p)$ be the integer satisfying $p^{r-1} < n \leq p^r$.

Structure Theorem 5 ([1]) There exists a positive integer $s = s(n)$ such that any finite powerful $n$-Engel $p$-group is nilpotent of class at most $s$.

Structure Theorem 6 ([1]) Let $G$ be a finite $n$-Engel $p$-group.

(a) If $p$ is odd, then $G^{p^r}$ is powerful.
(b) If $p = 2$, then $(G^{2^r})^2$ is powerful.

As corollary of Theorems 5 and 6 we then have

Structure Theorem 7 ([1]) Let $G$ be a locally finite $n$-Engel $p$-group.

(a) If $p$ is odd, then $G^{p^r}$ is nilpotent of $n$-bounded class.
(b) If $p = 2$ then $(G^{2^r})^2$ is nilpotent of $n$-bounded class.

The analogous results for right $n$-Engel subgroups are

Theorem 5 There exists a positive integer $s(n)$ such that, for any finite $p$-group $G$ and right $n$-Engel subgroup $H$ which is powerfully embedded in $G$, $[H,s(n)G] = 1$. 
Theorem 6 Let $G$ be a finite $p$-group and $H$ be a normal right $n$-Engel subgroup of $G$.

(a) If $p$ is odd, $H^{p^r}$ is powerfully embedded in $G^{p^r}$.
(b) If $p = 2$, $(H^{2r})^2$ is powerfully embedded in $(G^{2r})^2$.

Theorem 7 Let $G$ be a locally finite $p$-group and $H$ be a normal right $n$-Engel subgroup of $G$. There exists an integer $s = s(n)$ such that the following hold.

(a) If $p$ is odd, $[H^{p^r}, sG^{p^r}] = 1$.
(b) If $p = 2$, $[(H^{2r})^2, s(G^{2r})^2] = 1$.

Remark. The $r$ given in Theorem 7 is close to being the best bound. Let $t$ be the smallest positive integer such that $H^{p^t}$ is upper central of $n$-bounded degree. Then $r \in \{r - 1, r\}$ if $p$ is odd and $t \in \{r - 1, r, r + 1\}$ if $p = 2$ [1].

2 Proofs

In this section we prove Theorems 4, 5, 6 and 7. We start with Theorem 4.

Proof of Theorem 4. By Lemma 3 [6], we know that there exist positive integers $m = m(n)$ and $l = l(n)$ such that, for any $h \in H$ and $g_1, \ldots, g_m \in G$, $[h, g_1, \ldots, g_m] = 1$. Fix $h \in H$ and $g_1, \ldots, g_{m+1} \in G$ and let $K = \langle [h, g_1, \ldots, g_m], g_{m+1} \rangle$. Then $K' / [K', K']$ is abelian of exponent dividing $l$. Let $k = [h, g_1, \ldots, g_m]$. By Theorem 1, there exists a positive integer $s = s(n)$ such that $<[k]>$ in $K', K'[sK']$ is abelian of exponent dividing $l^s$. Let $e$ be an integer such that $(l^e)_{K'}$ is divisible by $l^e$ for $k = 1, \ldots, s$ and set $f = l^e$. Let $g = a_1 \cdots a_t$ be any product of commutators of the form $[y, x_1, \ldots, x_m]$, with $y \in H$ and $x_1, \ldots, x_m \in G$. We prove, by induction on $t$, that $g^f = 1$. If $t = 1$ this is trivial. Now suppose $t \geq 2$ and that the inductive hypothesis holds for smaller values of $t$. Let $z = a_2 \cdots a_t$ and $a = a_1$. Applying the well known Hall-Petrescu identity, we have that

$$a^f z^f = (az)^f w_2^{(f)} w_3^{(f)} \cdots w_s^{(f)}$$

with $w_i \in \gamma_i(\langle a, z \rangle)$. By inductive hypothesis the left hand side is trivial and by definition of $f$ every $w_i^{(f)}$ is also trivial, since each $w_i$ is in $\langle a_1, z \rangle'$. Hence
\((a_1 \cdots a_t)^f = (az)^f = 1\). This finishes the inductive proof and we conclude that \(g^f = 1\) for all \(g \in [H, m G]\). \(\square\)

We now move to Theorem 5. Let \(G\) be a finite \(p\)-group. Recall that a group \(H\) is powerfully embedded in \(G\) if \([H, G] \leq H^p\) provided that \(p\) is odd. If \(p = 2\), we require that \([H, G] \leq H^4\). For the proof we need to apply few well known properties of powerfully embedded subgroups. Firstly if \(H\) is powerfully embedded in \(G\), then \(H^p\) is also powerfully embedded in \(G\). Secondly, if \(H\) is powerfully embedded in \(G\), then for each positive integer \(m\) we have that \(H^m = \{h^m : h \in H\}\). The details can be found in [7] for example.

**Proof of Theorem 5.** As \(H\) is powerfully embedded in \(G\) we have that \(H^p^k\) is powerfully embedded for any positive integer \(k\). Furthermore \(H^p^k = \{h^p^k : h \in H\}\). We use these properties to show by induction on \(k \geq 1\) that \([H, k G] \leq H^p^k\). This finishes the inductive proof. Let \(c\) and \(f\) be as in Theorem 3, and let \(v = v(n)\) be the largest power of any prime that occurs in \(f(n)\). Then \([H^p^v, c G] = \{1\}\) and thus \([H^p^v, G] \leq [H^p^v, G] = \{1\}\).

We finally turn to Theorems 6 and 7. Let \(p\) be a fixed prime and \(n\) be a fixed positive integer. Let \(r\) be the integer satisfying \(p^r - 1 < n \leq p^r\).

**Proof of Theorem 6.** (a) We can assume that \((H^r)^p = \{1\}\) and then the aim is to show that \([H^p^r, G^p^r] = \{1\}\). Let \(g \in G\) be arbitrary and set \(V = H^p^r\). Since \(H\) is a finite \(n\)-Engel \(p\)-group, we have by Structure Theorem 6 that \(V\) is powerful and hence elementary abelian. Since \(H\) is a right \(n\)-Engel subgroup, for each \(v \in V\), \([v, g] = 1\) and thus \([v, g^r] = 1\). Hence, in \(\text{End}(V)\), \(0 = (-1 + g)^p = g^p - 1\). So \([v, g^p] = 1\) as required.

(b) Let \(K = (H^r)^2\). We may assume that \(K^4 = 1\) and the aim is then to show that \([K, (G^r)^2] = \{1\}\). Let \(g \in G\) be arbitrary and set \(V = K/K^2\). As \(H\) is a right \(n\)-Engel subgroup, we have by Structure Theorem 6 that \(K\) is powerful and hence abelian. It follows that \(V\) is an elementary abelian 2-group and
\[ v, 2, g \] = 1. We can conclude that in End \((V)\), \(0 = (-1 + g)^2 = t^2 - 1\). This shows that \([K, G^{2^r}] \leq K^2\). Let \(k \in K\), then

\[ [k, (g^{2^r})^2] = [k, g^{2^r}][k, g^{2^r}, g^{2^r}] \]

and since \([k, g^{2^r}] \in K^2\) we have that \([k, g^{2^r}]^2 = 1\). It remains to see that \([k, g^{2^r}, g^{2^r}] = 1\) and for this it suffices to show that \([K^2, G^{2^r}] = \{1\}\). But as \(K^2\) is right \(n\)-Engel we have as before that in End \((K^2)\), \(0 = (-1 + g)^2 = g^2 - 1\), and so \([K^2, G^{2^r}] = \{1\}\). This finishes the proof. \(\Box\)

**Proof of Theorem 7.** (a) Let \(s = s(n)\) be as in Theorem 5. Let \(h \in H^{p^r}\) and \(g_1, \ldots, g_s \in G^{p^r}\). Then \(h, g_1, \ldots, g_s \in K^{p^r}\) for some finitely generated, and hence finite subgroup \(K\) of \(G\). By Theorem 6, \((\langle h \rangle K)^{p^r}\) is powerfully embedded in \(K^{p^r}\). Hence, by Theorem 5, \([h, g_1, \ldots, g_s] = 1\). This finishes the proof of part (a). Part (b) is proved similarly. \(\Box\)

### 3 Right 2-Engel subgroups

In this section we consider the simplest non-trivial case of right 2-Engel subgroups. First we determine the integers \(f(2)\) and \(c(2)\) of Theorem 4. Let \(G\) be any group with a normal subgroup right 2-Engel subgroup \(H\). In \([8, 9]\) (see also \([10]\) Theorem 7.13), it is shown that \([h, x, y, z] = 1\) for all right 2-Engel elements \(h\) in \(G\) and all \(x, y, z \in G\) and that \(\langle h \rangle^G\) is an abelian right 2-Engel subgroup. We also have

\[ 1 = [h, x, xy, xy] = [h, x, y, xy] = [h, x, y, x]^y \]

and \([h, x^{-1}] = [h, x]^{-1}\). From this it is clear that any commutator \([h, u_1, \ldots, u_m]\) with \(u_1, \ldots, u_m \in \{x, y\}\) and with a repeated entry of either \(x\) or \(y\) is trivial. In particular, such a commutator is trivial if \(m \geq 3\). It follows that

\[ 1 = [h, xy, xy] = [h, x, y][h, y, x] \]

and \([h, y, x] = [h, y, x]^{-1}\). It follows that if \(h \in H\) and \(x_1, \ldots, x_m \in G\), then any commutator \([h, x_1, \ldots, x_m]\), with some \(x_i\) repeated, is trivial. Thus for \(h \in H\) and \(x, y, z \in G\), we have

\[ [h, x, [y, z]] = [h, x, y, z][h, x, z, y]^{-1} = [h, x, y, z]^2 = 1. \]
It follows that $[H, G, G, G] = \{[h, x, y, z] : h \in H, x, y, z \in G\}$, is abelian and so $[H, G, G, G]^2 = 1$. Examples 1 and 2 in [6] then show that this is the best possible. Thus $c(2) = 3$ and $f(2) = 2$.

We next move to Theorems 5 and 7. Let $s(2, p)$ be the smallest positive integer such that $[H, s(2, p) G] = \{1\}$ for all pairs $(H, G)$, where $G$ is a finite $p$-group and $H$ is a right $n$-Engel subgroup that is powerfully embedded in $G$. Also, let $e(2, p)$ and $f(2, p)$ be integers such that for any pair $H, G$,

$$[H, p^{e(2, p)}] G = \{1\}$$

where $G$ is a locally finite $p$-group and $H$ is a normal right 2-Engel subgroup of $G$. We want to find the value of $s(2, p)$ and the best possible values for $e(2, p)$ and $f(2, p)$.

First we deal with the case when $p$ is odd. From [6], we know that $[a, x, y, z] = 1$ when $a$ is a right 2-Engel element and $x, y, z \in G$. Hence $[a, x, y, z] = 1$ when $p$ is odd. Hence $s(2, p) \leq 3$ and the best possible value for $f(2, p) = 0$. Next example shows that $s(2, p) = 3$ and that the best possible value for $e(2, p)$ is 3.

**Example 1.** For any given positive integer $s$ we let $\mathbb{Z}_{p^s}$ be the congruence class of the integers modulo $p^s$. We let $N(s) = \mathbb{Z}_{p^s}^4$ and we let $M(s)$ be the subgroup of $\text{GL}(4, \mathbb{Z}_{p^s})$, generated by

$$X(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -p & 1 \end{pmatrix}, \quad Y(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $L(s) = N(s) \rtimes M(s)$ where $M(s)$ acts on $N(s)$ by multiplication on the left. Notice that if $v_1, v_2, v_3, v_4$ is the standard basis for the $\mathbb{Z}_{p^s}$-module $N(s)$ then

$$[v_1, X] = pv_2 \quad [v_2, X] = 0 \quad [v_3, X] = -pv_4 \quad [v_4, X] = 0$$

$$[v_1, Y] = pv_3 \quad [v_2, Y] = pv_4 \quad [v_3, Y] = 0 \quad [v_4, Y] = 0.$$

Notice that $L(s)$ is a finite $p$-group and that $N(s)$ is powerfully embedded right 2-Engel subgroup in $L(s)$. Then

$$[p^t v_1, X^{p^t}, Y^{p^t}] = p^{3t+2} v_4,$$

which is non trivial in $L(3t + 3)$. This shows that the best possible value for $e(2, p)$ is 3. Since $[v_1, X, Y] = p^2 v_4$, that is non-trivial in $L(3)$, we also see
that \( c(2, p) = 3 \).

It remains to deal with \( p = 2 \). Notice that Example 1 in fact shows also that the best value for \( e(2, 2) \) is 3 as we know that \( [a^2, x^2, y^2, z^2] = 1 \) for \( a \in H \) and \( x, y, z \in G \). This also shows that the best value for \( f(2, 2) \) is at most 1. The group \( N(2) \) is however not powerfully embedded in \( L(2) \) so it remains to deal with \( s(2, 2) \) and \( f(2, 2) \). The first example shows that the best possible value of \( f(2, 2) \) is 1.

**Example 2.** For each \( t \in \mathbb{N} \), we let

\[
R(t) = C_2 \wr C_2^t = \prod_{g \in C_2^t} \langle a^g \rangle \times C_2^t.
\]

The base group \( B(t) = \prod_{g \in C_2^t} \langle a^g \rangle \) is then a normal right 2-Engel subgroup. However if \( g_1, \ldots, g_t \) is the standard basis for \( C_2^t \) then

\[
[a, g_1, \ldots, g_t] = a^{(-1+g_1)\cdots(-1+g_t)} \neq 1.
\]

This shows that the best possible value of \( f(2, 2) \) is 1.

It now only remains to deal with \( s(2, 2) \). As we have remarked before we know that \( [H^2, 3 G] = \{1\} \) for any pair \((H, G)\) where \( H \) is a normal right 2-Engel subgroup of \( G \). Thus, if \( H \) is a powerfully embedded subgroup of a finite \( p \)-group \( G \), then

\[
[H, 4 G] \leq [H^4, 3 G] = \{1\}.
\]

We now show that \( s(2, 2) = 4 \), by giving an example that shows that \( s(2, 2) > 3 \).

**Example 3.** The construction is similar to the one in Example 1. This time we let \( N(s) = \mathbb{Z}_2^8 \), and we consider the subgroup of \( \text{GL}(8, \mathbb{Z}_2) \) generated by the following three matrices.
\[
X(s) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -4 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 \\
\end{pmatrix}, \quad
Y(s) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]
and
\[
Z(s) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

We then let \(L(s) = N(s) \rtimes M(s)\), where as before \(M(s)\) acts on \(N(s)\) by matrix multiplication on the left. The group \(L(s)\) is then a finite 2-group where \(N(s)\) is powerfully embedded in \(L(s)\). If we let \(v_1, \ldots, v_8\) be the standard basis for \(N(s)\) then

\[
\begin{align*}
[v_1, X] &= 4v_2, & [v_2, X] &= 0, & [v_3, X] &= -4v_5, & [v_4, X] &= -4v_6, \\
[v_5, X] &= 0, & [v_6, X] &= 0, & [v_7, X] &= 4v_8, & [v_8, X] &= 0, \\
[v_1, Y] &= 4v_3, & [v_2, Y] &= 4v_5, & [v_3, Y] &= 0, & [v_4, Y] &= -4v_7, \\
[v_5, Y] &= 0, & [v_6, Y] &= -4v_8, & [v_7, Y] &= 0, & [v_8, Y] &= 0, \\
[v_1, Z] &= 4v_4, & [v_2, Z] &= 4v_6, & [v_3, Z] &= 4v_7, & [v_4, Z] &= 0, \\
[v_5, Z] &= 4v_8, & [v_6, Z] &= 0, & [v_7, Z] &= 0, & [v_8, Z] &= 0.
\end{align*}
\]

Notice that for all \(v \in \{v_1, \ldots, v_8\}\) that

\[
[v, X, X] = [v, Y, Y] = [v, Z, Z] = 0
\]

and

\[
[v, X, Y] = -[v, Y, X], \quad [v, X, Z] = -[v, Z, X], \quad [v, Y, Z] = -[v, Z, Y].
\]
Notice also that

\[ [v, X, [Y, Z]] = [v, Y, [Z, X]] = [v, Z, [X, Y]] = 2[v, X, Y, Z]. \]

It follows that in \( L(7) \), we have \([v, X, [Y, Z]] = [v, Y, [Z, X]] = [v, Z, [X, Y]] = 2[v, X, Y, Z] \in 2^7N(s) = \{0\} \) which implies that \( N(7) \) is a right 2-Engel subgroup of \( L(7) \). However

\[ [v_1, X, Y, Z] = 2^6v_4 \neq 0. \]

This shows that \( s(2, 2) = 4 \).

References


