A remark on the structure of n-Engel groups

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We observe that there exists a positive integer $c(n)$ such that for every locally nilpotent n-Engel group G we have that $G/Z_{c(n)}(G)$ is of n-bounded exponent. This strengthens a result of Burns and Medvedev.

By the solution to the restricted Burnside problem we have that, for any given positive integer n, the locally nilpotent groups of exponent n form a variety $\overline{\mathcal{B}}_n$. Building also on Zel'manov's work, Wilson [8] has proved an analogous result for the variety of n-Engel groups. Thus, for any positive integer n, the locally nilpotent n-Engel groups form a subvariety $\overline{\mathcal{E}}_n$. Now clearly $\overline{\mathcal{B}}_n$ is contained in some $\overline{\mathcal{E}}_{m(n)}$ and in fact these two types of varieties are closely related. Let $\mathcal{N}_c$ be the variety of groups that are nilpotent of class at most c. In [2] Burns and Medvedev have shown that there exist positive integers $e(n), c(n)$ such that every locally nilpotent n-Engel group is in the the intersection of the varieties $\mathcal{N}_{c(n)}\overline{\mathcal{B}}_{e(n)}$ and $\overline{\mathcal{B}}_{e(n)}\mathcal{N}_{c(n)}$. In this note we will show that the integers $e(n)$ and $c(n)$ can be chosen such that the former variety can be replaced by the variety of groups that are ($c(n)$th centre)-by-$\overline{\mathcal{B}}_{e(n)}$. Notice that every locally nilpotent variety $\mathcal{V}$ is a subvariety of some $\overline{\mathcal{E}}_n$ so that it follows from our result that there exist integers $c = c_{\mathcal{V}}$ and $e = e_{\mathcal{V}}$ such that the law

$$[x^e, x_1, \ldots, x_c] = 1$$

holds in $\mathcal{V}$.

Let F be the relatively free locally nilpotent n-Engel group on a countably infinite set X and let T be the torsion subgroup of F. Then, as F/T is torsion-free, it is nilpotent [9] of class say $c = c(n)$. It follows that, for $x, x_1, \ldots, x_c \in X$,

$$[x, x_1, \cdots, x_c]^l = 1$$

where l is the order of $[x, x_1, \ldots, x_c]$ in F. Then every locally nilpotent n-Engel group satisfies this law. Now let G be the relatively free nilpotent n-Engel group on $c(n)+1$ generators. Suppose that the nilpotency class of G is $r = r(n)$. Notice that $r(n) \geq c(n)$.

Theorem *Every locally nilpotent n-Engel group satisfies the law*

$$[x^{l^{r-c}}, x_1, x_2, \ldots, x_c] = 1.$$

Proof. We show by reverse induction on m that the law

$$[x^{l^{r-c-m}}, x_1, \ldots, x_c, y_1, \cdots, y_m] = 1$$

holds in G for $m = 0, \ldots, r-c$. As G is nilpotent of class r this is clearly the case for $m = r-c$. Now suppose that $0 \leq m \leq r-c-1$ and that the result holds for larger values of m in $\{0, 1, \ldots, r-c\}$. By the induction hypothesis we have that $x^{l^{r-c-m-1}}$ is in the $(c+m+1)$st centre. Hence

$$\begin{aligned}
[x^{l^{r-c-m}}, x_1, \ldots, x_c, y_1, \ldots, y_m] &= [(x^{l^{r-c-m-1}})^l, x_1, \ldots, x_c, y_1, \ldots, y_m] \\
&= [x^{l^{r-c-m-1}}, x_1, \ldots, x_c, y_1, \ldots, y_m]^l \\
&= 1.
\end{aligned}$$

This finishes the inductive proof. Letting $m = 0$ gives the result. $\square$

Remarks. (1) The integer $c(n)$ above has been chosen to be the class of F/T where F was the free locally nilpotent n-Engel group of countably infinite rank. We then have laws

$$[x, x_1, \ldots, x_{c(n)}]^{l(n)} = 1 \quad (1)$$

$$[x^{e(n)}, x_1, \ldots, x_{c(n)}] = 1 \quad (2)$$

where $l(n)$ is the order of $[x_1, \ldots, x_{c(n)}]$ and $e(n)$ is the exponent of $F/Z_{c(n)}(F)$. The prime divisors of $l(n)$ and $e(n)$ are the same. These are the primes p

where there exists a finite n-Engel p-group with class greater than $c(n)$.

(2) Let $\mathcal{P} = \mathcal{P}_n$ be the set of primes p where there is an upper bound for the nilpotency classes of finite n-Engel p-groups. For each $p \in \mathcal{P}$, let $c_p(n)$ be the least upper bound for these nilpotency classes. As every finitely generated torsion-free nilpotent group is residually a finite p-group for any prime p it is clear that $c(n) = \min\{c_p(n) : p \in \mathcal{P}\}$. In fact $c(n) = c_p(n)$ for all primes $p \in \mathcal{P}$ except those that are the divisors of $l(n)$ and $e(n)$. Now let $\bar{c}(n) = \max\{c_p(n) : p \in \mathcal{P}\}$ and let $f(n)$ be the exponent of $F/Z_\infty(F)$. It is a routine to show that there is an integer $d(n) \geq \bar{c}(n)$ that is minimal subject to the conditions that $F/Z_{d(n)}(F)$ is of exponent $f(n)$ and such that (for $x, x_1, \ldots, x_{d(n)} \in X$) $[x, x_1, x_2, \ldots, x_{d(n)}]$ is of least possible order $s(n)$. Then we get identities

$$[x, x_1, \ldots, x_{d(n)}]^{s(n)} = 1 \tag{3}$$
$$[x^{f(n)}, x_1, \ldots, x_{d(n)}] = 1 \tag{4}$$

Those primes that divide $s(n)$ are the same as those that divide $f(n)$ and are those primes p where there is no upper bound for the nilpotency class of n-Engel p-groups.

Let us consider the situation for small values of n. For $n = 1$ we have $c(1) = d(1) = l(1) = s(1) = e(1) = f(1) = 1$ and all the four identities above become $[x, x_1] = 1$. For $n = 2$ it is well known that we get the identities

$$\begin{aligned}
[x, x_1, x_2]^3 &= 1 \\
[x^3, x_1, x_2] &= 1 \\
[x, x_1, x_2, x_3] &= 1
\end{aligned}$$

where $c(2) = 2, d(2) = 3$, $e(2) = l(2) = 3$ and $f(2) = s(2) = 1$.

We end this article by describing the picture for 3-Engel groups. From the work of Heineken [5] and Bachmuth and Mochizuki [1] we know that $c(3) = 4$ and that the exceptional primes are 2 and 5. Gupta and Newman [4] have also shown that $l(3) = 20$ and that $s(3) = 10$. In particular we have the identities

$$\begin{aligned}
[x, x_1, x_2, x_3, x_4]^{20} &= 1 \\
[x, x_1, x_2, x_3, x_4, x_5]^{10} &= 1
\end{aligned}$$

(in fact they prove that $\gamma_5(G)^{20} = \gamma_6(G)^{10} = 1$). We will see that $d(3) = 5$ and that $e(3) = f(3) = 20$. First we show that every 3-Engel group satisfies the law

$$[x^{20}, x_1, x_2, x_3, x_4] = 1.$$

Using the fact [6] that all commutators with a triple entry are trivial we have that $[y^{20}, z] = [y, z]^{20}[y, z, y]^{\binom{20}{2}}$ for all y, z. Using this fact and the two Gupta-Newman identities above, we have

$$\begin{aligned}
[x^{20}, x_1, x_2, x_3, x_4] =& [[x, x_1]^{20}[x, x_1, x]^{190}, x_2, x_3, x_4] \\
=& [[x, x_1]^{20}, x_2, x_3, x_4][[x, x_1, x]^{190}], x_2, x_3, x_4] \\
=& [[x, x_1, x_2]^{20}[x, x_1, x_2, [x, x_1]]^{190}, x_3, x_4][x, x_1, x, x_2, x_3, x_4]^{190} \\
=& [[x, x_1, x_2]^{20}, x_3, x_4][x, x_1, x_2, [x, x_1], x_3, x_4]^{190} \\
=& [[x, x_1, x_2, x_3]^{20}, x_4][x, x_1, x_2, x_3, [x, x_1, x_2], x_4]^{190} \\
=& [x, x_1, x_2, x_3, x_4]^{20}[x, x_1, x_2, x_3, x_4, [x, x_1, x_2, x_3]]^{190} \\
=& fl1
\end{aligned}$$

If G is an n-Engel group, it is well known that for all the primes $p < n$ there are finite n-Engel p-groups of arbitrary large class. If every finite 3-Engel 2-group were to satisfy the identity

$$[x^{10}, x_1, \cdots, x_t] = 1$$

then, for any finite 3-Engel 2-group G, we would have $G/Z_t(G)$ of exponent 2 and thus abelian. Hence we would get the contradiction that the class of finite 3-Engel 2-groups would be bounded above by $t + 1$. This shows that $e(3) = f(3) = 20$ and that $d(3) = 5$.

For 4-Engel groups it is known that $c(4) = 7$ [3,7] and that the exceptional primes are $2, 3$ and 5. Thus $e(4), f(4), l(4), s(4)$ have $2, 3$ and 5 as the only prime divisors. The exact values are unknown and as far as we are aware the value of $d(4)$ is also unknown.

References

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