# Symplectic alternating algebras

Gunnar Traustason
Department of Mathematical Sciences, University of Bath,
Bath BA2 7AY, UK

**Abstract**

This paper begins the development of a theory of what we will call symplectic alternating algebras. They have arisen in the study of 2-Engel groups but seem also to be of interest in their own right. The main part of the paper deals with the challenging classification of some algebras of this kind which arise in the context of 2-Engel groups and give some new information about these groups. The main result is that there are 31 such algebras with rank 6 over the field of three elements.

## 1 Introduction

**Definition 1.1** *Let $F$ be a field. A symplectic alternating algebra over $F$ is a triple $(V, (\ ,\ ), \cdot)$ where $V$ is a symplectic vector space over $F$ with respect to a non-degenerate alternating form $(\ ,\ )$ and $\cdot$ is a bilinear and alternating binary operation on $V$ such that*

$$(u \cdot v, w) = (v \cdot w, u)$$

*for all $u, v, w \in V$.*

This paper begins the theory of symplectic alternating algebras. They have arisen in the study of 2-Engel groups [1,2] and before going further we describe briefly how these structures and 2-Engel groups are related. In [1,2] we worked with powerful 2-Engel groups. In particular we classified all powerful 2-Engel groups of class three that are minimal in the sense that every proper powerful section has class at most two. Our initial expectation was that all powerful 2-Engel groups would be of class 2 but to our surprise there turned out to be a rich class of minimal examples. One of the motivations for this study is the following problem raised by Caranti:

**Problem**. Does there exist a finite 2-Engel 3-group of class three such that $\mathrm{Aut}\, G = \mathrm{Aut}_c G \cdot \mathrm{Inn}\, G$ where $\mathrm{Aut}_c G$ is the group of central automorphisms of $G$.

Our hope was to find such an example within our rich class of minimal examples or if not that this study would be a stepping stone towards showing that such examples do not exist. None of our minimal examples turned out to answer the problem above. However our study led us to a special class of 2-Engel groups that seems likely to contain strong candidates for such an example. Let us describe these groups. We start with any symplectic alternating algebra $L$ of rank $2r$ with basis $u_1, u_2, \ldots, u_{2r}$ over the field of three elements such that $(u_{2k}, u_{2k-1}) = 1$ and $(u_j, u_i) = 0$ for all other pairs $(i,j)$, $i<j$. We let $F(L)$ be the largest 2-Engel group on generators $x, x_1, x_2, \ldots, x_{2r}$ satisfying the relations

$$\text{F(L):}\quad \begin{aligned} x^{27} &= 1 \\ x_i^9 &= 1 \\ [x, x_i] &= 1 \\ [x_j, x_i] &= x_1^{3(u_j u_i, -u_2)} x_2^{3(u_j u_i, u_1)} \cdots x_{2r-1}^{3(u_j u_i, -u_{2r})} x_{2r}^{3(u_j u_i, u_{2r-1})} x^{3(u_j, u_i)}. \end{aligned}$$

The group $F(L)$ is then a 2-Engel group of class 3, exponent 27 and rank $2r+1$. In [2] we also proved that $F(L_1)$ is isomorphic to $F(L_2)$ if and only if $L_1$ and $L_2$ are isomorphic. One can see that for $F(L)$ to be an example for the Caranti question, the automorphism group of $L$ would have to be trivial. As there seems to be some scope for asymetry in the structure of symplectic alternating algebras this does not seem impossible.

In the first part of this paper we will begin the development of a general theory for symplectic alternating algebras. These are in general neither associative nor Lie algebras and apart from bilinearity and the alternating property there is no law that holds for the multiplication. Having said that there is a strong interaction between the multiplication and the alternating form that leads to many beautiful properties.

It is easily seen that up to isomorphism there is only one symplectic alternating algebra of rank 2, and two of rank 4. The details will be given later. The classification problem however seems very hard in general and gets soon much harder as the rank grows. The main part of this paper deals with the classification of the symplectic alternating algebras of rank 6 over the field of three elements. This turns out to be quite a challenge. The main result is that there are 31 such algebras of which 15 are simple. None of the 31 algebras has a trivial automorphism group.

There remains much to be explored and our study seems to provide some interesting questions:

**Question 1**. What is the structure like if one assumes that the underlying field is algebraically closed and/or of characteristic zero? Does the theory become simpler?

**Question 2**. Does there exist a (non-trivial) symplectic algebra with trivial automorphism group? (We have seen that this question has a relevance to 2-Engel groups when the underlying field is the field of three elements).

**Question 3**. What can one say about the structure of symplectic nil-algebras, that is symplectic alternating algebras satisfying the extra law $yx^n = 0$ for some positive integer $n$. In particular does a symplectic nil-algebra have to be nilpotent. (Easy to see that this is the case when $n \leq 2$).

We end this introduction with some notation that will be used much later.

**Notation**. If we have a symplectic alternating algebra of rank $2r$ then we will refer to a basis $x_1, y_1, \ldots, x_r, y_r$ with the property that $(x_i, x_j) = (y_i, y_j) = 0$ and

$(x_i, y_j) = \delta_{ij}$ as a standard basis.

# 2 A general theory for symplectic alternating algebras

In this section we develop some theory of symplectic alternating algebras. First we describe some general results and then we move on to algebras of rank 6 as a preparation of the classification of all symplectic alternating algebras over the field of three elements of rank up to 6.

## 2.1 Some general properties

Let $L$ be a symplectic alternating algebra over any field.

**Lemma 2.1** *If $I$ is an ideal of $L$ then $I^{\perp}$ is also and ideal of $L$.*

**Proof**. Let $a \in I$ and $b \in I^{\perp}$. For any $x \in L$ we have that

$$(a, bx) = (ax, b) = 0$$

since $ax \in I$ as $I$ is an ideal. This shows that $bx \in I^{\perp}$. $\Box$

For subspaces $U, V$ of $L$ we define $UV$ in the usual way as the subspace consisting of all linear spans of elements of the form $uv$ where $u \in U$ and $v \in V$. We define the lower central series $(L^i)_{i \geq 1}$ inductively by $L^1 = L$ and $L^{i+1} = L^i L$. Clearly

$$L^1 \geq L^2 \geq \cdots$$

which implies in particular that every $L^i$ is an ideal. We can also define the upper central series $(Z^i(L))_{i \geq 1}$ naturally by $Z^1(L) = Z(L) = \{a \in L : ax = 0 \text{ for all } x \in L\}$ and $Z^{i+1}(L) = \{a \in L : xa \in Z^i(L) \text{ for all } x \in L\}$. The next lemma gives us a beautiful relationship between the lower and the upper central series.

**Lemma 2.2** *Let $L$ be any symplectic alternating algebra. Then*

$$Z^i(L) = (L^{i+1})^{\perp}.$$

**Proof** We have

$$\begin{aligned} a \in Z^i(L) \quad &\Leftrightarrow \quad a x_1 \cdots x_i = 0, \ \forall x_1, \ldots, x_i \in L \\ &\Leftrightarrow \quad (a x_1 \cdots x_i, x_{i+1}) = 0, \ \forall x_1, \ldots, x_{i+1} \in L \\ &\Leftrightarrow \quad (a, x_{i+1} x_i \cdots x_1) = 0, \ \forall x_1, \ldots, x_{i+1} \in L \\ &\Leftrightarrow \quad a \in (L^{i+1})^{\perp}. \ \Box \end{aligned}$$

In particular it follows that $Z^i(L)$ is ideal, but this also follows directly from $Z^{i+1}(L) \cdot L \leq Z^i(L)$. Notice also that the $\dim(Z^i(L)) + \dim(L^{i+1}) = \dim(L)$.

If $a \in L$ then

$$I(a) = aL + (aL)L + ((aL)L)L + \cdots$$

is clearly the smallest ideal of $L$ containing $L$. We say that an ideal is cyclic if it is of the from $I(a)$ for some $a \in L$. We define simplicity for symplectic alternating algebras in the natural way. So $L \neq \{0\}$ is simple if it has no proper non-trivial ideals.

**Lemma 2.3** *Any isotropic ideal $I$ of $L$ is abelian. Furthermore any ideal of dimension 1 is contained in $Z(L)$ and any ideal of dimension 2 is abelian and contains a non-trivial element from the center.*

**Proof** Suppose first that $I$ is an isotropic ideal of $L$. Let $a, b \in I$ and $x \in L$ then

$$(ab, x) = (xa, b) = 0$$

since $ax, b \in I$. As $L^{\perp} = \{0\}$ this shows that $I$ is abelian.

Now let $Fu$ be an ideal of dimension 1. We want to show that $u \in Z(L)$. We argue by contradiction and suppose that $uv = \alpha u \neq 0$. We can then extend $u, v$ to a basis $u, v, v_1, \ldots, v_{2s}$ for $L$ and where $uv_i = 0$, for $1 \leq i \leq 2s$. Consider the basis $v, w = v + u, w_1 = v + v_1, \ldots, w_{2s} = v + v_{2s}$. Then $uw = uw_1 = \cdots = uw_{2s} = \alpha u$ and it follows that $u$ is orthogonal to $u, w, w_1, \ldots, w_{2s}$ and therefore $u \in L^{\perp} = \{0\}$. This contradiction shows that $u$ must be in $Z(L)$.

Finally, let $I = Fu + Fv$ be an ideal of dimension 2. If $I$ is isotropic we know already that $I$ is abelian. Hence we can suppose that $(u, v) \neq 0$. Then $uv$ is orthogonal to both $u$ and $v$ and thus in $I \cap I^{\perp} = \{0\}$. This shows that $I$ is abelian. We next want to show that $I$ contains an element from the center. We first deal with the case when $(u, v) \neq 0$. Let $x \in L$. Then $(ux, v) = (vu, x) = (0, x) = 0$. Hence $ux$ is orthogonal to both $u$ and $v$ and thus in $I \cap I^{\perp} = \{0\}$. Similarly $vx = 0$ and both $u$ and $v$ are in $Z(L)$. We are left with the case when $(u, v) = 0$. Let $u = x_1$, $v = x_2$ and extend this to a standard basis $x_1, y_1, x_2, y_2, \ldots, x_r, y_r$ for $L$. Firstly if $x \in I^{\perp}$ then for any $y \in L$ we have

$$(x_1 x, y) = (-x_1 y, x) = 0$$

since $x_1 y \in I$ and $x \in I^{\perp}$. This shows that $x_1 x = 0$ and similarly $x_2 x = 0$. It remains to find some non-trivial linear combination $u$ of $x_1$ and $x_2$ that is in the center of $L$. Suppose that $(y_1 y_2, x_1) = a$ and $(y_1 y_2, x_2) = b$. As $I$ is an ideal this forces the following:

$$\begin{array}{ll} x_1 y_1 = a x_2 & x_1 y_2 = -a x_1 \\ x_2 y_1 = b x_2 & x_2 y_2 = -b x_1. \end{array}$$

Now if both $a$ and $b$ are zero it follows that both $x_1$ and $x_2$ are in the center. So suppose this is not the case. Then the equations above imply that $bx_1 - ax_2$ is in the center. $\Box$

**Lemma 2.4** *Let $L$ be a symplectic alternating algebra of rank $2r$. If $M$ is an isotropic abelian subalgebra of dimension $r$ then $M$ must be an ideal.*

**Proof** Let $x \in M$ and $y \in L$. For all $u \in M$, we have

$$(xy, u) = (ux, y) = (0, y) = 0.$$

Hence $xy \in M^{\perp} = M$. $\Box$

**Definition 2.5** *Suppose that $L$ is a symplectic alternating algebra with ideals $I, J$ which both are symplectic alternating algebras and where*

$$L = I \oplus J.$$

*We say then that $L$ is the direct sum of the symplectic alternating algebras $I$ and $J$. This can be extended naturally to a definition of a direct sum $L = I_1 \oplus \cdots \oplus I_n$.*

**Remark**. It follows that $IJ \in I \cap J = \{0\}$.

**Definition 2.6** *We say that a symplectic alternating algebra is semi-simple if is a direct sum of simple symplectic alternating algebras.*

**Theorem 2.7** *Let $L$ be a symplectic alternating algebra. Either $L$ contains an abelian ideal or $L$ is semi-simple. In the latter case the direct summands are uniquely determined as the minimal ideals of $L$.*

**Proof** Suppose $L$ does not contain any abelian ideal. If $L$ is simple we are done. Otherwise $L$ contains a proper non-trivial ideal $I$. By Lemma 2.1, $I^{\perp}$ is also and ideal. We claim that $I \cap I^{\perp} = \{0\}$. Otherwise $I \cap I^{\perp}$ would be a non-trivial isotropic ideal and therefore abelian by Lemma 2.3. It follows that $I$ and $I^{\perp}$ are symplectic algebras and $L = I \oplus I^{\perp}$. Either $I$ and $I^{\perp}$ are simple, and we are done, or we can repeat the previous argument for $I$ and $I^{\perp}$. This process eventually gives us a direct sum of simple algebras.

We turn to the uniqueness. Suppose that $L$ is semi-simple and that $L = L_1 \oplus \cdots \oplus L_s$ is a decomposition of $L$ as a direct sum of simple algebras. We of course only have to deal with the case when $s \geq 2$. Suppose that $I(a)$ is a minimal ideal of $L$ where $a = a_1 + \cdots + a_s$ with $a_i \in L_i$. As $I(a)$ is not abelian there is some $v_j \in L_j$, $1 \leq j \leq s$ such that $av_j \neq 0$. Then $I(av_j)$ is also an ideal of $L$ and by minimality $I(av_j) = I(a)$. As $I(av_j) \leq L_j$ and $L_j$ is simple, it follows that $I(a) = L_j$. This shows that the summands are the minimal ideals of $L$. $\Box$

From last result we have that any symplectic alternating algebra that is not semi-simple must contain a cyclic ideal $I(a)$ that is abelian. The next lemma gives a criterion for $I(a)$ to be abelian. First a definition.

**Definition 2.8** *Let $m$ be a positive integer. We say that $a \in L$ is a $m$-sandwich if*

$$ax_1x_2 \cdots x_m a = 0$$

*for all $x_1 \ldots x_m \in L$. We say that $a$ is a $\infty$-sandwich if $a$ is a $m$-sandwich for all $m \geq 1$.*

**Lemma 2.9** *We have that $I(a)$ is an abelian ideal if and only if $a$ is an $\infty$-sandwich.*

**Proof** We have that $I(a)$ is an abelian ideal if and only if for all $r, s \geq 0$

$$\begin{array}{rcl} (ax_1 \cdots x_r)(ay_1 \cdots y_s) & = & 0 \;\; \forall x_1, \ldots, x_r, y_1, \ldots, y_s \in L \\ & \Updownarrow & \\ ((ax_1 \cdots x_r)(ay_1 \cdots y_s), y_{s+1}) & = & 0 \;\; \forall x_1, \ldots, x_r, y_1, \ldots, y_{s+1} \in L \end{array}$$

$$\begin{aligned}
&\Updownarrow\\
(ax_1\cdots x_r, ay_1\cdots y_s y_{s+1}) &= 0 \ \ \forall x_1,\ldots,x_r,y_1,\ldots,y_{s+1}\in L\\
&\Updownarrow\\
(ax_1\cdots x_r y_{s+1}\cdots y_2, ay_1) &= 0 \ \ \forall x_1,\ldots,x_r,y_1,\ldots,y_{s+1}\in L\\
&\Updownarrow\\
(ax_1\cdots x_r y_{s+1}\cdots y_2 a, y_1) &= 0 \ \ \forall x_1,\ldots,x_r,y_1,\ldots,y_{s+1}\in L\\
&\Updownarrow\\
ax_1\cdots x_r y_{s+1}\cdots y_2 a &= 0 \ \ \forall x_1,\ldots,x_r,y_2,\ldots,y_{s+1}\in L
\end{aligned}$$

The last property says that $a$ is an $(r+s)$-sandwich. As this is true for all $r,s\geq 0$ it follows that $I(a)$ is an ideal if and only if $a$ is an $\infty$-sandwich. $\Box$

Next lemma gives us another useful property

**Lemma 2.10** *Let $L$ be any symplectic alternating algebra and $x,y\in L$ then the subspace generated by*

$$y, yx, yxx, \ldots$$

*is isotropic.*

**Proof** We use the property $(Ux,V)=(U,Vx)$ to transfer any $(yx^r,yx^s)$ to a alternating product of the form $(yx^m,yx^m)$ or $(yx^m,yx^{m+1})$. Clearly the first is zero. For the latter notice that

$$(yx^m, yx^m x) = -((yx^m)(yx^m), x) = (0,x) = 0.$$

This finishes the proof. $\Box$

We are interested in the classification of symplectic alternating algebras. We know that every symplectic alternating algebra $L$ must be of even rank. Suppose that $L$ is a symplectic space with basis

$$x_1,\ldots,x_{2r}$$

for $L$ such that $(x_{2i},x_{2i-1})=1$ but $(x_j,x_i)=0$ otherwise for any $1\leq i<j\leq 2r$. We want to turn this into a symplectic alternating algebra by defining a bilinear alternating product on $L$ that interacts with the alternating form such that $(uv,w)=(vw,u)$ for all $u,v,w\in L$. As the product and the alternating form are both bilinear, everything reduces to working with the basis vectors. One constraint is

$$(x_ix_j,x_k) = -(x_jx_i,x_k) \ \text{ and } \ (x_ix_j,x_k)=(x_jx_k,x_i)$$

for all $1\leq i,j,k\leq 2r$. Notice that every product $(x_ix_j,x_k)=0$ if there is a repeated occurrence of a basic vector. So for $(x_ix_j,x_k)$ to be non-trivial we need $i,j,k$ to be pairwise distinct.

**Proposition 2.11** *Let $r\geq 2$ and for each $(i,j,k)$, $1\leq i<j<k\leq 2r$, choose a number $\alpha(i,j,k)$ in the field $F$. There is a unique symplectic alternating algebra of rank $2r$ over $F$ satisfying*

$$(x_ix_j,x_k)=\alpha(i,j,k)$$

*for $1\leq i<j<k\leq 2r$.*

**Proof** We start by extending the function $\alpha$ to all triplets of pairwise disjoint numbers $1 \leq i, j, k \leq 2r$ subject to $\alpha(i,j,k) = \alpha(j,k,i)$ and $\alpha(i,j,k) = -\alpha(j,i,k)$. A bilinear alternating product is determined completely from $x_j x_i,\ i < j$. If

$$x_j x_i = \alpha_1 x_1 + \cdots + \alpha_{2r} x_{2r}$$

then $\alpha_{2j} = (x_j x_i, x_{2j-1})$ and $\alpha_{2j-1} = (x_j x_i, -x_{2j})$. So it is clear that there is at most one symplectic alternating algebra satisfying the property above. The existence is also clear. Simply define for each $i < j$

$$x_j x_i = -\alpha(j,i,2)x_1 + \alpha(j,i,1)x_2 + \cdots - \alpha(j,i,2r)x_{2r-1} + \alpha(j,i,2r-1)x_{2r}. \quad \Box$$

We'll next discuss a different way of describing a symplectic algebra. Let $x_1, y_1, \ldots x_r, y_r$ be a standard basis for $L$. Consider the two isotropic subspaces

$$Fx_1 + \cdots + Fx_r, \quad \text{and} \quad Fy_1 + \cdots + Fy_r.$$

It suffices then to write only down the products of $x_i x_j, y_i y_j,\ 1 \leq i < j \leq r$. The reason for this is that having determined these products we have determined all triples $(uv, w)$ of basis vectors $u, v, w$ since two of those are either some $x_i, x_j$ or some $y_i, y_j$ in which case the triple is determined from $x_i x_j$ or $y_i y_j$. The only restraints on the products $x_i x_j$ and $y_i y_j$ come from

$$(x_i x_j, x_k) = (x_j x_k, x_i) \text{ and } (y_i y_j, y_k) = (y_j y_k, y_i).$$

Suppose that $F$ is the field of three elements. It follows from last proposition that for a given choice of basis there are exactly

$$3^{\binom{2r}{3}}$$

ways of defining a product on $L$ making it into a symplectic algebra. Of course we are not talking about isomorphism classes here. The problem of determining all symplectic alternating algebras of rank $2r$ up to isomorphism seems to be very difficult. As we will see later even the case $r = 3$ is a non-trivial exercise. However it is easy to find all symplectic alternating algebras of rank 2 or 4. Let us deal with these first. As $(uv, u) = (uv, v) = 0$ it is clear that the only symplectic alternating algebra of rank 2 is the abelian one. We will now see that there are up to isomorphism, two symplectic alternating algebras of rank 4. In order to see this suppose that $L$ is a non-abelian symplectic algebra of rank 4. We can then choose two elements $x_1, y_1$ whose product is non-zero. Clearly we can assume that these are not orthogonal and furthermore that $(x_1, y_1) = 1$. Then $x_2 = x_1 y_1$ is orthogonal to both $x_1$ and $y_1$ and therefore $x_1, y_1, x_2$ are linearly independent. Extend this to a standard basis $x_1, y_1, x_2, y_2$ where $(x_i, y_i) = 1$ and $(x_1, x_2) = (y_1, y_2) = (x_i, y_j) = 0$. By lemma 2.9 we have that the elements

$$x_2 x_1 = -y_1 x_1^2, \quad x_2 y_1 = x_1 y_1^2$$

are orthogonal to $x_1, y_1$ and $x_2$ and therefore multiples of $x_2$. Say $x_2 x_1 = a x_2$ and $x_2 y_1 = b x_2$. It follows from these equations that

$$u = -bx_1 + ay_1 + x_2$$

satisfies $ux_1 = ux_2 = uy_1 = 0$. Now if $v$ is any of $x_1, x_2, y_1$ we have $(uy_2, v) = (uv, y_2) = (0, y_2) = 0$ and as $(uy_2, y_2) = 0$, $uy_2$ is in $L^\perp = \{0\}$. Therefore $u \in Z(L)$. This shows that $Z(L) \neq \{0\}$. But as $L$ is non-abelian it follows that $Z(L)$ cannot have dimension larger than 1. (Since otherwise we would have a basis with at least two elements from the centre and we could not have a non-trival triple $(uv, w)$). So $Z(L)$ has dimension 1. We can then choose a new basis for $L$ so that one of the basis vectors spans $Z(L)$. We can therefore assume that within our standard basis $x_1, y_1, x_2, y_2$, the vector $x_2$ is in the center. It follows that the only candidate for a non-trival triple $(uv, w)$ in the basis vectors must be for $\{u, v, w\} = \{x_1, y_1, y_2\}$. In particular $x_1y_1 \neq 0$. By Lemma 2.2, $L^2 = Z(L)^\perp = Fx_1 + Fy_1 + Fx_2$. As $x_1y_1$ is orthogonal to both $x_1$ and $y_1$, it follows that $x_1y_1$ is a multiple of $x_2$ and by replacing $x_2$ by this multiple we can assume that $x_1y_1 = x_2$. Hence $(x_1y_1, y_2) = 1$ and the structure of $L$ is determined. One can check that $L$ has the following presentation

$$L: \quad \begin{array}{rcl} x_1x_2 & = & 0 \\ y_1y_2 & = & -y_1 \\ x_1y_1 & = & x_2 \\ x_1y_2 & = & -x_1 \\ x_2y_1 & = & 0 \\ x_2y_2 & = & 0 \end{array}$$

## 2.2 Symplectic alternating algebras of dimension six

Before embarking on the classification of the symplectic alternating algebras of rank 6 over the field of three elements, we collect together some properties that we will be using and that are special for rank 6. Throughout this subsection, $L$ will be a given symplectic alternating algebra of rank 6. We will not make any assumptions about the underlying field.

**Lemma 2.12** *Either $L$ is simple or it has an isotropic ideal of dimension 3. Furthermore if $L$ is not simple and if $Z(L) = \{0\}$, then there is a unique non-trivial proper ideal and this ideal is isotropic and of dimension 3.*

**Proof** Suppose that $L$ is not simple. If $I$ is an ideal then $I^\perp$ is also an ideal. This means that there is an ideal of dimension 1, 2 or 3. Let us first deal with the case when there is an ideal $I$ of dimension 3. If $I$ is isotropic, we are of course done. Suppose then that this is not the case. It follows that $I \cap I^\perp$ is an ideal of dimension 1. Hence we have reduced to the case when there is an ideal of dimension 1 or 2. By Lemma 2.3 we know that in both cases $L$ contains then a 1-dimensional ideal $I = Fx_1$ which must be contained in the center. Then $I^\perp/I$ is a symplectic alternating algebra of rank 4 and we have seen that there are up to isomorphism only two such each with a non-trivial center. Pick $x_2$ from $I^\perp$ such that $x_2 + I$ is in the center of $I^\perp/I$. We can then extend $x_1, x_2$ to a standard basis $x_1, y_1, x_2, y_2, x_3, y_3$ such that $x_1, x_2, y_2, x_3, y_3$ form a basis for $I^\perp$ and furthermore we can assume that $x_2x_3 = 0$. But then

$$J = Fx_1 + Fx_2 + Fx_3$$

is an abelian isotropic subalgebra or dimension 3 and by Lemma 2.4 this is an ideal.

Now suppose that $L$ is not simple with trivial center. If there was an ideal $I$ of dimension $1, 2, 4$ or $5$ then by considering also the ideal $I^{\perp}$, we would have an ideal of dimension 1 or 2. But in this case Lemma 2.3 implies that the center is non-trivial. Hence all proper non-trivial ideals must be of dimension three. Furthermore, every such ideal $I$ must be isotropic. Otherwise $I \cap I^{\perp}$ would be an ideal of dimension 1 and thus contained in the center. By Lemma 2.3 every isotropic ideal of dimension three is abelian. It remains to show that there is a unique such ideal. We argue by contradiction and suppose that $J$ is another such ideal. Then $I \cap J$ is of smaller dimension and therefore trivial. It follows that $I \cdot J \leq I \cap J = \{0\}$ and that $L = I \oplus J$. But then $L^2 = I^2 + IJ + J^2 = \{0\}$ and $L$ is abelian. This of course contradicts the assumption that $L$ has a non-trivial center. $\Box$

Next result will be important for the classification. First a notation. Suppose that

$$L = L_1 \oplus L_2 \oplus L_3$$

where each $L_i$ is a symplectic alternating algebra of dimension two and where the subspaces are pairwise orthogonal. We will refer to such a decomposition as a canonical . Choosing basis vectors $x_i, y_i$ for $L_i$ such that $(x_i, y_i) = 1$ will give us a standard basis for this canonical decomposition. Notice that when we have a standard basis then we have a presentation of the form

$$\begin{array}{ll}
x_2x_3 = a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + ay_1 & \quad y_2y_3 = b_{11}y_1 + b_{21}y_2 + b_{31}y_3 + bx_1 \\
x_3x_1 = a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + ay_2 & \quad y_3y_1 = b_{12}y_1 + b_{22}y_2 + b_{32}y_3 + bx_2 \\
x_1x_2 = a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + ay_3 & \quad y_1y_2 = b_{13}y_1 + b_{23}y_2 + b_{33}y_3 + bx_3
\end{array}$$

where $a = (x_1x_2, x_3) = (x_2x_3, x_1) = (x_3x_1, x_2)$ and $-b = (y_1y_2, y_3) = (y_2y_3, y_1) = (y_3y_1, y_2)$.

**Lemma 2.13** *For each canonical decomposition of $L$ there exists a corresponding standard basis $x_1, y_1, x_2, y_2, x_3, y_3$ such that*

$$Fx_1 + Fx_2 + Fx_3, \quad \textit{and} \quad Fy_1 + Fy_2 + Fy_3$$

*are subalgebras of $L$. Furthermore if $Fx_1 + Fx_2 + Fx_3$ is abelian then it is an ideal of $L$.*

**Proof**. This is equivalent to finding a standard basis where $(x_1x_2, x_3) = (y_1y_2, y_3) = 0$. Take any standard basis $x_1, y_1, x_2, y_2, x_3, y_3$ for $L$ with respect to the given canonical decomposition. By replacing $x_3$ by some $ax_3 + by_3$ and $y_3$ accordingly if necessary we can assume that $(x_1x_2, x_3) = 0$ and thus that $Fx_1 + Fx_2 + Fx_3$ is a subalgebra. Now for any $z_i = y_i + c_ix_i$ we get another standard basis $x_1, z_1, x_2, z_2, x_3, z_3$ with respect to the same canonical decomposition. If we can choose $c_1, c_2, c_3$ such that $(z_1z_2, z_3) = 0$ then we are done. So suppose $(z_1z_2, z_3) \neq 0$ for all choices of $c_1, c_2, c_3$. Then $(y_1y_2, y_3) \neq 0$ and letting $c_2 = c_3 = 0$ we see that we must have $(x_1y_2, y_3) = 0$. Similarly we see that $(y_1x_2, y_3) = (y_1y_2, x_3) = 0$ and then taking only $c_3 = 0$ we see that $(x_1x_2, y_3) = 0$ and similarly we see that $(x_1y_2, x_3) = (x_1x_2, y_3) = 0$. In particular it follows from this that

$$(x_1x_2, y_3) = (y_1y_2, -x_3) = 0.$$

This implies that

$$Fx_1 + Fx_2 + Fy_3 \;\text{ and }\; Fy_1 + Fy_2 + F(-x_3)$$

are abelian. Clearly $x_1, y_1, x_2, y_2, y_3, -x_3$ is a standard basis for the given canonical decomposition.

Now suppose that $Fx_1 + Fx_2 + Fx_3$ is abelian. It follows then from Lemma 2.4 that this is an ideal. $\square$

**Lemma 2.14** *If $L$ is not simple then there exists a canonical decomposition with standard basis $x_1, y_1, x_2, y_2, x_3, y_3$ such that*

$$Fx_1 + Fx_2 + Fx_3 \quad \text{and} \quad Fy_1 + Fy_2 + Fy_3$$

*are subalgebras and $Fx_1 + Fx_2 + Fx_3$ is abelian.*

**Proof** By Lemma 2.12, $L$ has an isotropic ideal $I$ of dimension 3 which by Lemma 2.3 is abelian. Let $x_1, x_2, x_3$ be a basis for $I$ and extend to a standard basis $x_1, y_1, x_2, y_2, x_2, y_3$. One can check that if $x_1, z_1, x_2, z_2, x_3, z_3$ is any other standard basis then we must have

$$\begin{aligned} z_1 &= y_1 + ax_1 + bx_2 + cx_3 \\ z_2 &= y_2 + bx_1 + dx_2 + ex_3 \\ z_3 &= y_3 + cx_1 + ex_2 + fx_3 \end{aligned}$$

for some $a, b, c, d, e, f \in F$. Letting $b = c = e = 0$ and following the same arguement as in the proof of Lemma 2.13 we have

$$(x_1x_2, y_3) = (x_2x_3, y_1) = (x_3x_1, y_2) = (y_1y_2, x_3) = (y_2y_3, x_1) = (y_3y_1, x_2) = 0.$$

Then letting $a = c = d = e = f = 0$ we see that

$$(y_2y_3, x_2) = -(y_3y_1, x_1)$$

and considering similarly the cases $a = b = d = e = f = 0$ and $a = b = c = d = f = 0$ we have

$$\begin{aligned} (y_2y_3, x_3) &= -(y_1y_2, x_1) \\ (y_3y_1, x_3) &= -(y_1y_2, x_2). \end{aligned}$$

It follows from these constraints that we get equations of the form

$$\begin{array}{ll} x_2x_3 = 0 & y_2y_3 = ry_2 - sy_3 + ax_1 \\ x_3x_1 = 0 & y_3y_1 = ty_3 - ry_1 + ax_2 \\ x_1x_2 = 0 & y_1y_2 = sy_1 - ty_2 + ax_3. \end{array}$$

We claim that we can choose a basis $z_1, z_2, z_3$ for $Fy_1 + Fy_2 + Fy_3$ such that $z_1z_2 = 0$ modulo $I = Fx_1 + Fx_2 + Fx_3$. If any two of $r, s, t$ are zero this is clearly possible. So suppose at least two of $r, s, t$ are non-zero. Let $v_1 = y_2y_3$, $v_2 = y_3y_1$ and $v_3 = y_1y_2$. Direct calculations show that modulo the ideal $I$ we have $v_1v_2 = v_2v_3 = v_3v_1 = 0$. But since at most one of $r, s, t$ is zero, some two of $v_1, v_2, v_3$ are linearly independent. By letting these be $z_1, z_2$ and extending to a basis $z_1, z_2, z_3$ we have proved our claim. Without loss of generality we can thus assume that in the equations above $s = t = 0$. We thus have the following equations

$$\begin{array}{ll} x_2x_3 = 0 & y_2y_3 = ry_2 + ax_1 \\ x_3x_1 = 0 & y_3y_1 = -ry_1 + ax_2 \\ x_1x_2 = 0 & y_1y_2 = ax_3. \end{array}$$

Now $(x_3x_1, y_2) = 0$ and $(y_3y_1, x_2) = 0$. Hence

$$Fx_1 + Fy_2 + Fx_3 \quad \text{and} \quad Fy_1 + F(-x_2) + Fy_3$$

are subalgebras. We finish the proof by showing that the former is abelian. Firstly $x_3x_1 = 0$ as these are in $I$. As $I$ is abelian we also have that $y_2x_3$ is orthogonal to $x_1, x_2, x_3$ and as $y_2y_1 \in I$ we also have that $y_2x_3$ is orthogonal to $y_1$ and $y_2$. As

$$(y_2x_3, y_3) = (y_3y_2, x_3) = 0.$$

It follows that $y_2x_3 = 0$. It remains to show that $x_1y_2 = 0$. Firstly as $I$ is abelian and $y_2y_1 \in I$ we have that $x_1y_2$ is orthogonal to $x_1, x_2, x_3, y_1, y_2$. As

$$(x_1y_2, y_3) = (y_2y_3, x_1) = 0$$

it follows that $x_1y_2 = 0$. $\Box$

Let $L_1 = Fx_1 + Fx_2 + Fx_3$ and $L_2 = Fy_1 + Fy_2 + Fy_3$ be as above with $L_1$, $L_2$ subalgebras. We will then refer to the decomposition $L = L_1 \oplus L_2$ as a standard decomposition of $L$. As we will be dealing much with these later on it is useful to be able to locate the center from such a decomposition.

**Lemma 2.15** *Let $L = L_1 \oplus L_2$ be a standard decomposition of $L$. If $x \in L_1$ and $y \in L_2$ such that $x + y \in Z(L)$ then*

$$x \in L_1 \cap (L_2)^2)^{\perp} \quad \textit{and} \quad y \in L_2 \cap (L_1^2)^{\perp}$$

**Proof** By Lemma 2.2, we have for all $y_1, y_2 \in L_2$ and all $x_1, x_2 \in L_1$ that

$$\begin{aligned} 0 &= (x + y, y_1y_2) = (x, y_1y_2) \\ 0 &= (x + y, x_1x_2) = (y, x_1x_2) \quad \Box \end{aligned}$$

**Proposition 2.16** *Let $L = L_1 \oplus L_2$ be a standard decomposition of $L$ where $L_1$ is abelian. Then*

$$Z(L) = L_1 \cap (L_2^2)^{\perp} + Z(L_2).$$

*In particular*

*(a) If $L_2^2 = L_2$ then $Z(L) = \{0\}$.*
*(b) If $L_2^2$ is one dimensional then $Z(L)$ is three dimensional.*
*(c) If $L_2^2$ is two dimensional then $Z(L)$ is one dimensional.*
*(d) If $L_2^2 = \{0\}$ then $L$ is abelian.*

**Proof** Let $x \in L_1 \cap (L_2^2)^{\perp}$. Then $x$ commutes with all elements in $L_1$ and is orthogonal to everything in $L_2^2$. It follows that for all $x_1, x_2 \in L_1$ and $y_1, y_2 \in L_2$ we have

$$\begin{aligned} &(x, x_1x_2) = (x, 0) = 0 \\ &(x, y_1x_1) = (xx_1, y) = (0, y) = 0 \\ &(x, y_1y_2) = 0. \end{aligned}$$

and $x \in (L^2)^{\perp}$ which by Lemma 2.2 implies that $x \in Z(L)$.

If $y \in Z(L_2)$ then

$$\begin{aligned} &(y, x_1x_2) = (y, 0) = 0 \\ &(y, x_1y_1) = (yy_1, x_1) = (0, x_1) = 0 \\ &(y, y_1y_2) = 0 \end{aligned}$$

and $y \in (L^2)^\perp = Z(L)$.

This proves that $L_1 \cap (L_2^2)^\perp + Z(L_2) \subseteq Z(L)$. For the converse suppose that $x \in L_1$ and $y \in L_2$ such that $x + y \in Z(L)$. By Lemma 2.15 $x \in L_1 \cap (L_2^2)^\perp$ that we have seen that is contained in $Z(L)$. Hence $y$ is also in $Z(L)$ and thus in particular in $Z(L_2)$.

Now for parts (a)-(d). First (d) is obvious from the first part. If $L_2^2 = L_2$ then $Z(L_2) = \{0\}$ and $L_1 \cap (L_2^2)^\perp = L_1 \cap L_2^\perp = L_1 \cap L_2 = \{0\}$. This proves (a).

Next we turn to (c). If $\dim(L_2^2) = 2$ then $Z(L_2) = \{0\}$ (otherwise $\dim(L_2^2) \leq 1$) and $\dim(L_1 \cap (L_2^2)^\perp) = 1$. Hence, by the first part, $\dim(Z(L)) = 1$.

It remains to prove (b). As $\dim(L_1 \cap (L_2^2)^\perp) = 2$ it suffices to show that $\dim(Z(L_2)) = 1$. But if $L_2^2 = Fy_1$ and $y_2, y_3$ complete the basis then

$$
\begin{aligned}
y_2 y_3 &= \alpha y_1 \\
y_3 y_1 &= \beta y_1 \\
y_1 y_2 &= \gamma y_1
\end{aligned}
$$

where not all of $\alpha, \beta, \gamma$ are zero. One can check in this case that $\alpha y_1 + \beta y_2 + \gamma y_3$ is in $Z(L_2)$. $\square$

Our next aim will be to classify all the non-simple symplectic alternating algebras of dimension three over the field of three elements. Because of Lemma 2.14 we first classify all the alternating algebras of dimension three.

# 3 The alternating algebras of dimension three

In this section $F$ will be the field of three elements. Let $L = Fx_1 + Fx_2 + Fx_3$ be any alternating algebra over $F$. If

$$
\begin{aligned}
x_2 x_3 &= a_{11} x_1 + a_{21} x_2 + a_{31} x_3 \\
x_3 x_1 &= a_{12} x_1 + a_{22} x_2 + a_{32} x_3 \\
x_1 x_2 &= a_{13} x_1 + a_{23} x_2 + a_{33} x_3
\end{aligned}
$$

then we represent the presentation with the matrix

$$
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
$$

Now choose a different basis for $L$

$$
\begin{aligned}
y_1 &= g_{11} x_1 + g_{21} x_2 + g_{31} x_3 \\
y_2 &= g_{12} x_1 + g_{22} x_2 + g_{32} x_3 \\
y_3 &= g_{13} x_1 + g_{23} x_2 + g_{33} x_3
\end{aligned}
$$

and let $g$ be the linear map that corresponds to the matrix $(g_{ij})$. We want to determine the matrix for $L$ with respect to $y_1, y_2, y_3$. Calculations give

$$
y_2 y_3 = \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix} x_2 x_3 - \begin{vmatrix} g_{12} & g_{13} \\ g_{32} & g_{33} \end{vmatrix} x_3 x_1 + \begin{vmatrix} g_{12} & g_{13} \\ g_{22} & g_{23} \end{vmatrix} x_1 x_2
$$

$$\begin{aligned} y_3y_1 &= -\begin{vmatrix} g_{21} & g_{23} \\ g_{31} & g_{33} \end{vmatrix} x_2x_3 + \begin{vmatrix} g_{11} & g_{13} \\ g_{31} & g_{33} \end{vmatrix} x_3x_1 - \begin{vmatrix} g_{11} & g_{13} \\ g_{21} & g_{23} \end{vmatrix} x_1x_2 \\ y_1y_2 &= \begin{vmatrix} g_{21} & g_{22} \\ g_{31} & g_{32} \end{vmatrix} x_2x_3 - \begin{vmatrix} g_{12} & g_{12} \\ g_{31} & g_{32} \end{vmatrix} x_3x_1 + \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} x_1x_2. \end{aligned}$$

Thus the matrix for $L$ with respect to $y_1, y_2, y_3$ is

$$g^{-1} \cdot A \cdot \operatorname{adj}(g)^t = \det(g)\, g^{-1} \cdot A \cdot (g^{-1})^t.$$

Define an action of $\mathrm{GL}(3, F)$ on $M(3, F)$ by

$$A^g := \det(g) g^t \cdot A \cdot g.$$

We have seen that two matrices $A, B$ represent the same alternating algebra if and only if $B = A^g$ for some $g \in GL(3, F)$.

Notice that $A^{id} = A$ and that

$$\begin{aligned} (A^g)^h &= (\det(g)g^t \cdot A \cdot g)^h \\ &= \det(h)h^t(\det(g)g^t \cdot A \cdot g)h \\ &= \det(gh)(gh)^t \cdot A \cdot gh \\ &= A^{gh}. \end{aligned}$$

So $M(3, F)$ becomes a $\mathrm{GL}(3, F)$-set under this action.

**Remark**. $A^{-\mathrm{id}} = \det(-\mathrm{id})A = -A$ and therefore $-A$ is equivalent to $A$. (Can also be verified easily without using the action).

Now we can write every matrix $A$ as a sum of a symmetric matrix $\frac{A+A^t}{2}$ and a anti-symmetric matrix $\frac{A-A^t}{2}$. Also

$$(A^t)^g = \det(g)g^t \cdot A^t \cdot g = (\det(g)g^t A g)^t = (A^g)^t.$$

It follows that $A^g$ is symmetric(anti-symmetric) if $A$ is symmetric(skew-symmetric). Suppose $A = A_1 + A_2$ where $A_1$ is the symmetric part and $A_2$ is the anti-symmetric part. Then

$$A^g = A_1^g + A_2^g$$

where, as we have seen, $A_1^g$ is the symmetric part and $A_2^g$ is the anti-symmetric part. Let $G = \mathrm{GL}(3, F)$.

**Lemma 3.1** *Let $A = A_1 + A_2$ where $A_1$ is the symmetric part and $A_2$ is the anti-symmetric part. Then*

$$C_G(A) = C_G(A_1) \cap C_G(A_2)$$

**Proof** We have that $g \in C_G(A)$ if and only if $A^g = A$ if and only if $A_1^g = A_1$ and $A_2^g = A_2$. □

Because of this last lemma we first determine the orbits of the symmetric and the anti-symmetric matrices. We start by dividing the $3^6$ symmetric matrices into $G$-orbits.

**Lemma 3.2** *Every symmetric matrix is conjugate to a diagonal matrix.*

**Proof** As $B$ is conjugate to $-B$ we can always correct modulo $\pm 1$ and we can also ignore the $\det(g)$ part in $A^g$. Let $A = (a_{ij})$ be a symmetric matrix. Let

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^3 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz$$

be the symmetric form related to $A$. We want to diagonalise this form. Suppose first that one of the diagonal entries are non-trivial. Without loss of generality we can suppose that this is $a_{11}$ and that $a_{11} = 1$. Then the form can be rewritten as

$$(x + a_{12}y + a_{13}z)^2 + (a_{22} - a_{12}^2)y^2 + (a_{33} - a_{13}^2)z^2 + 2(a_{23} - a_{12}a_{13})yz.$$

The rest is now easy. If one of the coefficients of $y^2$ and $z^2$ is non-trivial this is about completing the square. If both the coefficients are zero, use $yz = (y+z)^2 - (y-z)^2$.

We are left with the case when all the diagonal entries are zero. Then use one of

$$\begin{aligned} (x+y-z)^2 + (x-y+z)^2 + (-x+y+z)^2 &= xy + xz + yz \\ (x+y+z)^2 - (x-y)^2 - z^2 &= xy - xz - yz. \end{aligned} \qquad \square$$

Notice that the rank of $A$ is always preserved by the action of $G$.

**Lemma 3.3** *The invertible matrices form a single orbit. Furthermore*

$$C_G(I) = \{g \in G : g^t g = id \text{ and } \det(g) = 1\},$$

$|C_G(I)| = 2^3 \cdot 3$ *and the size of the orbit is* $[G : C_G(I)] = 2^2 \cdot 3^2 \cdot 13$.

**Proof** Let

$$g_1 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Calculations show that

$$\begin{aligned} I^{g_1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ I^{g_2} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ I^{g_3} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

As $B$ is conjugate to $-B$ this shows that all the invertible diagonal matrices are conjugate. By Lemma 3.2 all the invertible symmetric matrices are therefore conjugate. We next determine $C_G(I)$. We have

$$I^g = I \Leftrightarrow \det(g) g^t \cdot g = I.$$

As we are working with the field of three elements, we have $\det(aA) = a^3 \det(A) = a\det(A)$ for all matrices $A$ and all $a \in F$. The equation above therefore gives that $\det(g) = 1$. A vector $(a, b, c)$ in $F^3$ has norm $a^2 + b^2 + c^2 = 1$ if and only if exactly one of $a, b, c$ is non-zero. Hence there are 6 such vectors. Clearly there are then $6 \cdot 4 \cdot 2$ triples of pairwise orthogonal vectors from this set. Half of the corresponding matrices have determinant 1. Hence $|C_G(I)| = 24 = 2^3 \cdot 3$. As $|G| = (3^3 - 1) \cdot (3^3 - 3) \cdot (3^3 - 3^2) = 2^5 \cdot 3^3 \cdot 13$ we get the size of the orbit as well. $\square$

**Lemma 3.4** *The symmetric matrices of rank* 1 *form a single orbit* $J_1^G$ *where*

$$J_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

*Furthermore*

$$C_G(J_1) = \{ \begin{bmatrix} det(D) & 0 & 0 \\ a & & \\ b & \multicolumn{2}{c}{D} \end{bmatrix} : D \text{ is an invertible } 2 \times 2 \text{ matrix and } a, b \in F\}$$

$|C_G(J_1)| = 2^4 \cdot 3^3$ *and* $|J_1^G| = 2 \cdot 13$.

**Proof** Let

$$g_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Calculations show that

$$I^{g_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$I^{g_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Again as $-B$ is conjugate to $B$ it follows from Lemma 3.2 that all symmetric matrices of rank 1 are conjugate to $J_1$. We leave the remaining straightforward calculations to the reader. □

**Lemma 3.5** *There are two orbits of matrices of rank* 2. *These are* $J_2^G$ *and* $J_3^G$ *where*

$$J_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad J_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

*Furthermore*

$$C_G(J_2) = \{ \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ a & b & \epsilon_1\epsilon_2 \end{bmatrix}, \begin{bmatrix} 0 & \epsilon_1 & 0 \\ \epsilon_2 & 0 & 0 \\ a & b & -\epsilon_1\epsilon_2 \end{bmatrix}, \begin{bmatrix} \epsilon_1 & -\epsilon_1 & 0 \\ \epsilon_2 & \epsilon_2 & 0 \\ a & b & \epsilon_1\epsilon_2 \end{bmatrix}, \begin{bmatrix} \epsilon_1 & -\epsilon_1 & 0 \\ \epsilon_2 & -\epsilon_2 & 0 \\ a & b & -\epsilon_1\epsilon_2 \end{bmatrix} : \epsilon_1, \epsilon_2 = \pm 1, \ a, b \in F\}$$

*and*

$$C_G(J_3) = \{ \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ a & b & \epsilon_1\epsilon_2 \end{bmatrix}, \begin{bmatrix} 0 & \epsilon_1 & 0 \\ \epsilon_2 & 0 & 0 \\ a & b & \epsilon_1\epsilon_2 \end{bmatrix} : \epsilon_1, \epsilon_2 = \pm 1, \ a, b \in F\}.$$

*In particular* $|C_G(J_2)| = 2^4 \cdot 3^2$ *and* $|C_G(J_3)| = 2^3 \cdot 3^2$ *and thus* $|J_2^G| = 2 \cdot 3 \cdot 13$ *and* $|J_3^G| = 2^2 \cdot 3 \cdot 13$.

**Proof**. Again we leave it to the reader to determine $C_G(J_1)$ and $C_G(J_2)$. As these have different orders it follows that $J_2$ and $J_1$ are not conjugate. It remains to see that all diagonal matrices of rank 2 are conjugate to one of these. Let

$$g_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Calculations show that

$$\begin{aligned}
J_2^{g_1} &= -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
J_2^{g_2} &= -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
J_3^{g_1} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
J_3^{g_2} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

As $B$ is conjugate to $-B$ we see that all diagonal matrices of rank 2 are included. $\square$

This finishes the study of the symmetric matrices. Let us do some counting to see that everything fits. We have

$$|I^G| + |J_1^G| + |J_2^G| + |J_3^G| + |0^G| = 2^2 \cdot 3^2 \cdot 13 + 2 \cdot 13 + 2 \cdot 3 \cdot 13 + 2^2 \cdot 3 \cdot 13 + 1 = 3^6.$$

So up to isomorphism there are exactly 5 alternating algebras with symmetric presentation. We then turn to the anti-symmetric matrices.

**Lemma 3.6** *Apart from $0^G$ there is only one orbit $J^G$ of anti-symmetric algebras where*

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

*Furthermore*

$$C_G(J) = \{ \begin{bmatrix} \multicolumn{2}{c}{D} & 0 \\ & & 0 \\ a & b & 1 \end{bmatrix}, \ D \text{ is an invertible } 2 \times 2 \text{ matrix and } a, b \in F \},$$

*$|C_G(J)| = 2^4 3^3$ and $|J^G| = 2 \cdot 13 = 26$.*

**Proof**. The reader can verify that $C_G(J)$ is as stated. As $|J^G| = 26$ all the non-zero anti-symmetric matrices are conjugate to $J$. $\square$

We now consider the general situation. Let $C = A + B$ be any $3 \times 3$ matrix over $F$ with symmetric part $A$ and anti-symmetric part $B$. We want to determine the orbits. By taking an appropriate conjugate of $C$ we can assume that $A$ is one of $0, I, J_1, J_2, J_3$. The question is to determine when $A + B_1$ and $A + B_2$ are conjugate. So we consider the action of $C_G(A)$ on the set of anti-symmetric matrices. We have already dealt with the case when $A = 0$. So we have 4 remaining cases to study. We will skip over the technical details and leave to the reader to verify that the orbits are as stated. Notice that in this case we are working with a set with only 27 matrices so this is not a very difficult task.

**Lemma 3.7**

*(a) For $H = C_G(I)$, there are three non-trivial $H$-orbits*

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^H, \quad \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}^H, \quad \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^H$$

*of order* 6*,* 12 *and* 8 *respectively.*

*(b) For* $H = C_G(J_1)$*, there are two non-trivial* $H$*-orbits*

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^H, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}^H$$

*of order* 8 *and* 18*.*

*(c) For* $H = C_G(J_2)$*, there are two non-trivial* $H$*-orbits*

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^H, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}^H$$

*of order* 2 *and* 24*.*

*(d) For* $H = C_G(J_3)$*, there are three non-trivial* $H$*-orbits*

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^H, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}^H, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}^H$$

*of order* 2*,* 12 *and* 12 *respectively.*

It follows that apart from 6 orbits that are either symmetric or alternating, there are 10 others. This gives us 16 non-simple algebras. We will see in next section that they are pairwise non-isomorphic. Notice that for standard presentations $R_1 \oplus S_1$ isomorphic to $R_2 \oplus S_2$ does not in general imply that $R_1$ is isomorphic to $R_2$ and $S_1$ isomorphic to $S_2$. We will see however that this is the case when $S_1, S_2$ are abelian.

# 4 Classification of the non-simple algebras

According to previous section there are up to isomorphism 16 alternating algebras of dimension 3 over $F$. These are (the order is not the same as before, for reasons that will become clear soon)

$$A_1: \begin{array}{l} y_2y_3 = y_1 \\ y_3y_1 = y_2 \\ y_1y_2 = y_3 \end{array} \qquad A_2: \begin{array}{l} y_2y_3 = y_1 - y_2 \\ y_3y_1 = y_1 + y_2 \\ y_1y_2 = y_3 \end{array} \qquad A_3: \begin{array}{l} y_2y_3 = y_1 - y_2 - y_3 \\ y_3y_1 = y_1 + y_2 - y_3 \\ y_1y_2 = y_1 + y_2 + y_3 \end{array}$$

$$A_4: \begin{array}{l} y_2y_3 = y_1 \\ y_3y_1 = -y_3 \\ y_1y_2 = y_2 \end{array} \qquad A_5: \begin{array}{l} y_2y_3 = y_1 - y_3 \\ y_3y_1 = y_2 \\ y_1y_2 = y_1 \end{array} \qquad A_6: \begin{array}{l} y_2y_3 = y_1 - y_3 \\ y_3y_1 = -y_2 \\ y_1y_2 = y_1 \end{array}$$

$$A_7: \begin{array}{l} y_2y_3 = y_1 \\ y_3y_1 = -y_2 - y_3 \\ y_1y_2 = y_2 \end{array}$$

$$B_1: \begin{array}{l} y_2y_3 = -y_2 \\ y_3y_1 = y_1 \\ y_1y_2 = 0 \end{array} \qquad B_2: \begin{array}{l} y_2y_3 = y_1 - y_2 - y_3 \\ y_3y_1 = y_1 + y_2 \\ y_1y_2 = y_1 + y_3 \end{array} \qquad B_3: \begin{array}{l} y_2y_3 = y_1 - y_2 \\ y_3y_1 = y_1 \\ y_1y_2 = 0 \end{array}$$

$$B_4: \begin{array}{l} y_2y_3 = y_1 - y_2 \\ y_3y_1 = y_1 + y_2 \\ y_1y_2 = 0 \end{array} \qquad B_5: \begin{array}{l} y_2y_3 = y_1 \\ y_3y_1 = y_2 \\ y_1y_2 = 0 \end{array} \qquad B_6: \begin{array}{l} y_2y_3 = y_1 \\ y_3y_1 = -y_2 \\ y_1y_2 = 0 \end{array}$$

$$C_1: \begin{array}{l} y_2y_3 = y_1 \\ y_3y_1 = 0 \\ y_1y_2 = 0 \end{array} \qquad C_2: \begin{array}{l} y_2y_3 = y_1 - y_2 \\ y_3y_1 = y_1 - y_2 \\ y_1y_2 = 0 \end{array} \qquad O: \begin{array}{l} y_2y_3 = 0 \\ y_3y_1 = 0 \\ y_1y_2 = 0 \end{array}$$

By the classification above, we know that any non-simple symplectic alternating algebra of dimension 6 over $F$ is isomorphic to one of

$$O \oplus A_1, \ldots, O \oplus A_7,\ O \oplus B_1, \ldots, O \oplus B_6,\ O \oplus C_1,\ O \oplus C_2,\ O \oplus O.$$

Inspection shows that $A_i^2 = A_i$, $\dim(B_i^2) = 2$ and $\dim(C_i^2) = 1$. By Proposition 2.16, $O \oplus A_i$ has trivial center, $O \oplus B_i$ has one dimensional center and $O \oplus C_1, O \oplus C_2$ have three dimensional center.

**Proposition 4.1** *The algebras in the list above are pairwise non-isomorphic.*

**Proof** By previous remarks it suffices to show that the $O \oplus A_1, \ldots, O \oplus A_7$ are pairwise non-isomorphic, that $O \oplus B_1, \ldots, O \oplus B_6$ are pairwise non-isomorphic and that $O \oplus C_1$ and $O \oplus C_2$ are not isomorphic. We deal first with $O \oplus A_1, \ldots, O \oplus A_7$. As the center is trivial, we have by Lemma 2.12 that the subalgebra $O$ is the unique proper non-trivial ideal of $O \oplus A_i$. It follows that $A_i$ is determined as the quotient $(O \oplus A_i)/O$. It follows that if $O \oplus A_i$ and $O \oplus A_j$ were isomorphic it would follow that $A_i$ and $A_j$ were isomorphic.

Next we show that $O \oplus C_1$ and $O \oplus C_2$ are not isomorphic. We do this by showing that their centres differ. Firstly, according to Proposition 2.16,

$$Z(O \oplus C_1) = O \cap (C_1^2)^{\perp} + Z(C_1) = Fx_2 + Fx_3 + Fy_1$$

which is isotropic. On the other hand $Z(C_2) = K(y_1 + y_2)$ and thus

$$Z(O \oplus C_2) = O \cap (C_2^2)^{\perp} + Z(C_2) = F(x_1 + x_2) + Fx_3 + F(y_1 + y_2)$$

which is not isotropic.

It remains to deal with $L_1 = O \oplus B_1, \ldots, L_6 = O \oplus B_6$. We start by determining $Z(L_i)$ and $L_i^2$. In this case the center of $B_i$ is trivial and thus, by Proposition 2.16, $Z(L_i) = O \cap (B_i^2)^\perp$. Then $L_i^2 = Z(L_i)^\perp$. Inspection gives that these are

$$\begin{array}{ll} Z(L_2) = Fx_1 - Fx_2 - Fx_3 & L_2^2 = Fx_1 + Fx_2 + Fx_3 + F(y_1 + y_2) + F(y_1 + y_3) \\ Z(L_i) = Fx_3 & L_i^2 = Fx_1 + Fx_2 + Fx_3 + Fy_1 + Fy_2, \quad i \neq 2. \end{array}$$

Now for $L_1, L_3, L_4, L_5, L_6$ we have that $y_1y_2 = 0$. It follows then that $x_1y_1$, $x_1y_2$, $x_2y_1$ and $x_2y_2$ are orthogonal to $x_1, x_2, x_3, y_1, y_2$ and hence all these products are in $Fx_3 = Z(L_i)$. Hence

$$L_i^2/Z(L_i) \text{ is abelian if } i \neq 2.$$

On the other hand in $L_2$, we have

$$(y_1 + y_2)(y_1 + y_3) = -y_1 + y_2 + y_3 \notin Z(L_2).$$

This implies that $L_2$ is distinct from the rest. Now we deal with the remaining 5 algebras. Note that for any $i \neq 2$

$$L_i/L_i^2 = F(\bar{y_3})$$

is one dimensional and generated by $\bar{a} = \pm\bar{y_3}$. As $L_i^2/Z(L_i)$ is abelian, there is a well define action of $L_i/L_i^2$ on $L_i^2/Z(L_i)$ by setting

$$\bar{a} \cdot \bar{u} = \bar{au}.$$

Let us determine the minimal polynomial of $\bar{a}$ as a linear map on $L_i^2/Z(L_i)$.

Inspection shows that for $i = 1$ the minimal polynomial is $t + 1$ or $t - 1$ depending on whether $\bar{a} = \bar{y_3}$ or $\bar{a} = -\bar{y_3}$. That the minimal polynomial is either $(t+1)^2$ or $(t-1)^2$ if $i = 3$. That it is $t^2 - t - 1$ or $t^2 + t - 1$ if $i = 4$, $t^2 + 1$ if $i = 5$ and finally that it is $t^2 - 1$ if $i = 6$. Hence these algebras are pairwise non-isomorphic. □

We conclude that there are exactly 16 non-simple symplectic algebras over $F$.

# 5 Classification of the simple algebras

Our approach here will be similar in outline. However the situation is more complicated. Here we don't have standard decomposition with one factor abelian and so we need a $6 \times 6$ matrix to describe the presentation. It will be useful to deal here with a general presentation and not only standard decompositions. Let $x_1, y_1, x_2, y_2, x_3, y_3$ be a standard basis for $L$. If

$$\begin{array}{ll} x_2x_3 = a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + ay_1 & y_2y_3 = b_{11}y_1 + b_{21}y_2 + b_{31}y_3 + bx_1 \\ x_3x_1 = a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + ay_2 & y_3y_1 = b_{12}y_1 + b_{22}y_2 + b_{32}y_3 + bx_2 \\ x_1x_2 = a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + ay_3 & y_1y_2 = b_{13}y_1 + b_{23}y_2 + b_{33}y_3 + bx_3 \end{array}$$

then we represent the presentation with the matrix

$$P = \begin{bmatrix} A & bI \\ aI & B \end{bmatrix}$$

where $A = (a_{ij})$ and $B = (b_{ij})$. Any other standard basis is the image of this basis under a symplectic automorphism. For an element $g$ in the symplectic group we define $P^g$ to be the presentation matrix with respect to $x_1^g, y_1^g, x_2^g, y_2^g, x_3^g, y_3^g$. We will see that we can also talk about symmetric and anti-symmetric presentations. The definition will however be different from the usual one. We start by defining the transpose of a presentation matrix by setting

$$\begin{bmatrix} A & bI \\ aI & B \end{bmatrix}^t = \begin{bmatrix} A^t & bI \\ aI & B^t \end{bmatrix}.$$

We say that a presentation matrix $P$ is symmetric if $P^t = P$, that is if both $A$ and $B$ are symmetric in the usual sense. We say that $P$ is anti-symmetric if $P^t = -P$, that is if $A$ and $B$ are anti-symmetric in the usual sense and $a = b = 0$. Notice that $(P^t)^t = P, (rP + sQ)^t = rP^t + sQ^t$ and that

$$P = (\frac{P + P^t}{2}) + (\frac{P - P^t}{2})$$

is the unique decomposition of $P$ into a sum of a symmetric and antisymmetric presentation matrix.

Recall that a linear map is the transvection with respect to a vector $u \in L$ if it maps $x$ to $x + (x, u)u$. Any transvection preserves the alternating form and we know that they generate the symplectic group.

**Lemma 5.1** *Let $P$ be a presentation matrix and $g$ from the symplectic group. Then $(P^t)^g = (P^g)^t$.*

**Proof**. As the symplectic group is generated by transvections, it suffices to prove this when $g$ is a transvection. So let $g$ be the transvection with respect to $u = r_1x_1 + r_2x_2 + r_3x_3 + s_1y_1 + s_2y_2 + s_3y_3$. Suppose that

$$P^g = \begin{bmatrix} C & dI \\ cI & D \end{bmatrix} \text{ and } (P^t)^g = \begin{bmatrix} E & fI \\ eI & F \end{bmatrix}.$$

We will use the $\cdot_t$ for the product according to the presentation $P^t$. Firstly,

$$\begin{aligned}
e &= -(x_1^g \cdot_t x_2^g, x_3^g) \\
&= -((x_1 + s_1u) \cdot_t (x_2 + s_2u), x_3 + s_3u) \\
&= -(x_1 \cdot_t x_2, x_3) - s_3(x_1 \cdot_t x_2, u) - s_2(x_3 \cdot_t x_1, u) - s_1(x_2 \cdot_t x_3, u) \\
&= a + s_3r_3a + s_2r_2a + s_1r_1a \\
&\quad -s_1s_3(x_1 \cdot_t x_2, y_1) - s_2s_3(x_1 \cdot_t x_2, y_2) - s_3s_3(x_1 \cdot_t x_2, y_3) \\
&\quad -s_1s_2(x_3 \cdot_t x_1, y_1) - s_2s_2(x_3 \cdot_t x_1, y_2) - s_3s_2(x_3 \cdot_t x_1, y_3) \\
&\quad -s_1s_1(x_2 \cdot_t x_3, y_1) - s_2s_1(x_2 \cdot_t x_3, y_2) - s_3s_1(x_2 \cdot_t x_3, y_3) \\
&= a(1 + r_1s_1 + r_2s_2 + r_3s_3) \\
&\quad -s_1s_3a_{31} - s_2s_3a_{32} - s_3s_3a_{33} \\
&\quad -s_1s_2a_{21} - s_2s_2a_{22} - s_3s_2a_{23} \\
&\quad -s_1s_1a_{11} - s_2s_1a_{12} - s_3s_1a_{13} \\
&= a + s_3r_3a + s_2r_2a + s_1r_1a \\
&\quad -s_1s_3a_{13} - s_2s_3a_{23} - s_3s_3a_{33} \\
&\quad -s_1s_2a_{12} - s_2s_2a_{22} - s_3s_2a_{32} \\
&\quad -s_1s_1a_{11} - s_2s_1a_{21} - s_3s_1a_{31} \\
&= a + s_3r_3a + s_2r_2a + s_1r_1a \\
&\quad -s_1s_3(x_1 \cdot x_2, y_1) - s_2s_3(x_1 \cdot x_2, y_2) - s_3s_3(x_1 \cdot x_2, y_3)
\end{aligned}$$

$$
\begin{aligned}
& -s_1s_2(x_3\cdot x_1,y_1)-s_2s_2(x_3\cdot x_1,y_2)-s_3s_2(x_3\cdot x_1,y_3)\\
& -s_1s_1(x_2\cdot x_3,y_1)-s_2s_1(x_2\cdot x_3,y_2)-s_3s_1(x_2\cdot x_3,y_3)\\
= & -(x_1\cdot x_2,x_3)-s_3(x_1\cdot x_2,u)-s_2(x_3\cdot x_1,u)-s_1(x_2\cdot x_3,u)\\
= & -((x_1+s_1u)\cdot(x_2+s_2u),x_3+s_3u)\\
= & -(x_1^g\cdot x_2^g,x_3^g)\\
= & \; c.
\end{aligned}
$$

Similarly $f=(y_1^g\cdot_t y_2^g,y_3^g)=\cdots=(y_1\cdot y_2,y_3)=d$. Then (calculating the indices modulo 3)

$$
\begin{aligned}
e_{ij} &= (x_{j+1}^g\cdot_t x_{j-1}^g,y_i^g)\\
&= ((x_{j+1}+s_{j+1}u)\cdot_t(x_{j-1}+s_{j-1}u),y_i-r_iu)\\
&= (x_{j+1}\cdot_t x_{j-1},y_i)-r_i(x_{j+1}\cdot_t x_{j-1},u)+s_{j+1}(x_{j-1}\cdot_t y_i,u)-s_{j-1}(x_{j+1}\cdot_t y_i,u)\\
&= a_{ji}-r_is_1a_{j1}-r_is_2a_{j2}-r_is_3a_{j3}+r_ir_ja\\
&\quad -r_js_{j+1}(x_{j-1}\cdot_t x_j,y_i)+r_{j+1}s_{j+1}(x_{j+1}\cdot_t x_{j-1},y_i)\\
&\quad -r_js_{j-1}(x_j\cdot_t x_{j+1},y_i)+r_{j-1}s_{j-1}(x_{j+1}\cdot_t x_{j-1},y_i)\\
&\quad +s_{i+1}s_{j+1}(y_i\cdot_t y_{i+1},x_{j-1})-s_{i-1}s_{j+1}(y_{i-1}\cdot_t y_i,x_{j-1})\\
&\quad -s_{j-1}s_{i+1}(y_i\cdot_t y_{i+1},x_{j+1})+s_{i-1}s_{j-1}(y_{i-1}\cdot_t y_i,x_{j+1})\\
&= a_{ji}+r_ir_ja-r_is_{i+1}a_{j,i+1}-r_js_{j+1}a_{j+1,i}\\
&\quad -r_is_{i-1}a_{j,i-1}-r_js_{j-1}a_{j-1,i}\\
&\quad r_{j+1}s_{j+1}a_{ji}+r_js_ja_{ji}+r_{j-1}s_{j-1}a_{ji}\\
&\quad -r_is_ia_{ji}-r_js_ja_{ji}-s_{i+1}s_{j+1}b_{i-1,j-1}-s_{j-1}s_{i-1}b_{i+1,j+1}\\
&\quad +s_{i-1}s_{j+1}b_{i+1,j-1}+s_{j-1}s_{i+1}b_{i-1,j+1}
\end{aligned}
$$

and

$$
\begin{aligned}
c_{ji} &= (x_{i+1}^g\cdot x_{i-1}^g,y_j^g)\\
&= ((x_{i+1}+s_{i+1}u)\cdot(x_{i-1}+s_{i-1}u),y_j-r_ju)\\
&= (x_{i+1}\cdot x_{i-1},y_j)-r_j(x_{i+1}\cdot x_{i-1},u)+s_{i+1}(x_{i-1}\cdot y_j,u)-s_{i-1}(x_{i+1}\cdot y_j,u)\\
&= a_{ji}-r_js_1a_{1i}-r_js_2a_{2i}-r_js_3a_{3i}+r_jr_ia\\
&\quad -r_is_{i+1}(x_{i-1}\cdot x_i,y_j)+r_{i+1}s_{i+1}(x_{i+1}\cdot x_{i-1},y_j)\\
&\quad -r_is_{i-1}(x_i\cdot x_{i+1},y_j)+r_{i-1}s_{i-1}(x_{i+1}\cdot x_{i-1},y_j)\\
&\quad +s_{j+1}s_{i+1}(y_j\cdot y_{j+1},x_{i-1})-s_{j-1}s_{i+1}(y_{j-1}\cdot y_j,x_{i-1})\\
&\quad -s_{i-1}s_{j+1}(y_j\cdot y_{j+1},x_{i+1})+s_{j-1}s_{i-1}(y_{j-1}\cdot y_j,x_{i+1})\\
&= a_{ji}+r_jr_ia-r_js_{j+1}a_{j+1,i}-r_is_{i+1}a_{j,i+1}\\
&\quad -r_js_{j-1}a_{j-1,i}-r_is_{i-1}a_{j,i-1}\\
&\quad r_{i+1}s_{i+1}a_{ji}+r_is_ia_{ji}+r_{i-1}s_{i-1}a_{ji}\\
&\quad -r_js_ja_{ji}-r_is_ia_{ji}-s_{j+1}s_{i+1}b_{i-1,j-1}-s_{i-1}s_{j-1}b_{i+1,j+1}\\
&\quad +s_{j-1}s_{i+1}b_{i-1,j+1}+s_{i-1}s_{j+1}b_{i+1,j-1}.
\end{aligned}
$$

Hence $e_{ij}=c_{ji}$ and $E=C^t$. Similarly $F=D^t$ and $f=d$. □

It follows that if $A=A_1+A_2$ where $A_1$ is the symmetric part and $A_2$ is the anti-symmetric part then $A_1^g\,(A_2^g)$ is the symmetric(skew-symmetric) part of $A^g$. Also

**Corollary 5.2** *Let $G$ be the symplectic group $Sp(6,F)$ and let $A=A_1+A_2$ be a presentation matrix where $A_1$ is the symmetric part and $A_2$ is the anti-symmetric part. Then*

$$C_G(A)=C_G(A_1)\cap C_G(A_2).$$

**Proof** We have that $A^g = A$ if and only if $A_i^g = A_i,\ i = 1, 2.$ $\square$

We now follow the same strategy as for the non-simple algebras. First we determine all the symmetric algebras. We will see that apart from the 5 non-simple ones there are two simple algebras.

## 5.1 The simple symmetric algebra $S_1$

Consider the algebra

$$S_1 : \quad \begin{array}{ll} x_2x_3 = y_1 & y_2y_3 = x_1 \\ x_3x_1 = y_2 & y_3y_1 = x_2 \\ x_1x_2 = y_3 & y_1y_2 = x_3 \end{array}$$

**Proposition 5.3** *$S_1$ is simple.*

**Proof** Let

$$L_{-1} = Fx_1 + Fx_2 + Fx_3 \quad \text{and} \quad L_1 = Fy_1 + Fy_2 + Fy_3.$$

Notice that if $u \in L_{-1}$ and $v \in L_1$ then $(uv, x_i) = (x_iu, v) = 0$ and $(uv, y_i) = (vy_i, u) = 0$ since $x_iu \in L_1$ and $vy_i \in L_{-1}$. We therefore have for $L = S_1$ that

$$L = L_{-1} \oplus L_1$$

where $L_{-1}^2 = L_1,\ L_1^2 = L_{-1}$ and $L_{-1}L_1 = \{0\}$. Let us deduce from this that $L$ is simple. Let $I$ be any non-trivial ideal of $L$. Take $0 \neq a + b \in I$ where $a \in L_{-1}$ and $b \in L_1$. We consider the case $a \neq 0$ (the other case $b \neq 0$ being dealt with similarly). Then (as $L_{-1}^2 = L_1$ is three dimensional) $aL_{-1}$ is a two dimensional subspace of $L_1$. As $L_1^2 = L_{-1}$ is three dimensional it follows that $aL_{-1}L_1$ is a three dimensional subspace of $L_{-1}$. Hence

$$aL_{-1}L_1 = L_{-1} \ \text{ and } \ aL_{-1}L_1L_{-1} = L_1$$

and $L = aL_{-1}L_1 + aL_{-1}L_1L_{-1} \subseteq I$. Hence there is no proper non-trivial ideal. $\square$

Notice that for $g \in G$, the presentation matrix $P^g$ with respect to $x_1^g, x_2^g, x_3^g,\ y_1^g, y_2^g, y_3^g$ is the same as $P$ for $x_1, x_2, x_3,\ y_1, y_2, y_3$ if and only if

$$(u^gv^g, w^g) = (uv, w) = ((uv)^g, w^g)$$

for all $u, v, w \in L$. This is true if and only if $(uv)^g = u^g \cdot v^g$. for all $u, v \in L$. Hence

$$C_G(P) = \text{Aut}\,(L).$$

In order to calculate the size of $S_1^G$ we need thus calculate the automorphism group of $L = S_1$. It will be useful to determine first the 1-sandwiches of $L$. Let $u = ax_1 + bx_2 + cx_3 + dy_1 + ey_2 + fy_3$ be a 1-sandwich. Let $U$ be the linear map induced by the multiplication from the right by $u$. Calculations show that the matrix for $U$ with respect to $x_1, x_2, x_3, y_1, y_2, y_3$ is

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & f & -e \\ 0 & 0 & 0 & -f & 0 & d \\ 0 & 0 & 0 & e & -d & 0 \\ 0 & c & -b & 0 & 0 & 0 \\ -c & 0 & a & 0 & 0 & 0 \\ b & -a & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now $u$ is a 1-sandwich if $Luu = 0$. That is if $A^2 = 0$. But

$$A^2 = \begin{bmatrix} -cf-be & ae & af & 0 & 0 & 0 \\ bd & -cf-ad & bf & 0 & 0 & 0 \\ cd & ce & -ad-be & 0 & 0 & 0 \\ 0 & 0 & 0 & -cf-be & bd & cd \\ 0 & 0 & 0 & ae & -cf-ad & ce \\ 0 & 0 & 0 & af & bf & -ad-be \end{bmatrix}.$$

One sees from this that $A^2 = 0$ if and only if $\{a,b,c\} \cdot \{d,e,f\} = \{0\}$. Hence the set of 1-sandwiches is $L_{-1} \cup L_1$. As any automorphism $\phi$ maps a 1-sandwich to a 1-sandwich, we have two possibilities. Either $\phi$ maps the subspaces $L_{-1}$ and $L_1$ back to themselves or $\phi$ swaps these subspaces. As $\phi$ is in the symplectic group this means that it is of one of the following forms

$$\begin{bmatrix} A & 0 \\ 0 & (A^{-1})^t \end{bmatrix} \text{ or } \begin{bmatrix} 0 & A \\ -(A^{-1})^t & 0 \end{bmatrix}.$$

Let us see what extra is required for these to be automorphisms of $S_1$. For both cases we have $(\phi(x_i)\cdot\phi(x_j), \phi(y_k)) = 0 = (x_i \cdot x_j, y_k)$ and $(\phi(y_i)\cdot\phi(y_j), \phi(x_k)) = 0 = (y_i \cdot y_j, x_k)$. Thus we have a automorphism if and only if

$$(\phi(x_1)\cdot\phi(x_2), \phi(x_3)) = (x_1x_2, x_3) = -1 \text{ and } (\phi(y_1)\cdot\phi(y_2), \phi(y_3)) = (y_1y_2, y_3) = 1.$$

Let us look at the latter candidate first. Here calculations show that $(\phi(y_1)\cdot\phi(y_2), \phi(y_3)) = -\det(A)$ and $(\phi(x_1)\cdot\phi(x_2), \phi(x_3)) = \det(-(A^{-1})^t) = -\det(A)$. So the condition here is that $\det(A) = -1$ and $\det(A) = 1$. As this is impossible we don't have any automorphism of this form. This leave us with the first candidate. In this case $(\phi(x_1), \phi(x_2), \phi(x_3)) = -\det(A)$ and $(\phi(y_1)\cdot\phi(y_2), \phi(y_3)) = \det((A^{-1})^t) = \det(A)$. So here both conditions translate into $\det(A) = 1$. Hence

$$\mathrm{Aut}(S_1) = \{\begin{bmatrix} A & 0 \\ 0 & (A^{-1})^t \end{bmatrix} : \det(A) = 1\}.$$

Now as it is well known that $|G| = (3^6-1)(3^5-3)(3^4-3^2)3^6$ and as $|\mathrm{Aut}(S_1)| = \frac{(3^3-1)(3^3-3)(3^3-3^2)}{2}$, it follows that

$$|S_1^G| = \frac{(3^6-1)(3^4-1)(3^2-1)3^9\cdot 2}{(3^3-1)(3^2-1)(3-1)3^3} = (3^3+1)(3^4-1)3^6$$

## 5.2 The simple symmetric algebra $S_2$

Consider the algebra

$$S_2: \begin{array}{ll} x_2x_3 = y_1 & y_2y_3 = -y_1 \\ x_3x_1 = y_2 & y_3y_1 = -y_2 \\ x_1x_2 = y_3 & y_1y_2 = -y_3 \end{array}$$

It follows from these that

$$\begin{array}{lll} x_1y_1 = 0 & x_2y_1 = -x_3 & x_3y_1 = x_2 \\ x_1y_2 = x_3 & x_2y_2 = 0 & x_3y_2 = -x_1 \\ x_1y_3 = -x_2 & x_2y_3 = x_1 & x_3y_3 = 0 \end{array}$$

For $x = ax_1 + bx_2 + cx_3 + dy_1 + ey_2 + fy_3$ we let $X$ be the multiplication by $x$ from the right. Calculations show that

$$X = \begin{bmatrix} 0 & f & -e & 0 & c & -b \\ -f & 0 & d & -c & 0 & a \\ e & -d & 0 & b & -a & 0 \\ 0 & c & -b & 0 & -f & e \\ -c & 0 & a & f & 0 & -d \\ b & -a & 0 & -e & d & 0 \end{bmatrix}$$

and $X^2$ is equal to

$$\begin{bmatrix} -b^2-e^2-c^2-f^2 & ab+de & ac+df & 0 & -bd+ea & -cd+fa \\ ab+de & -a^2-d^2-c^2-f^2 & bc+ef & bd-ea & 0 & -ce+fb \\ ac+df & bc+ef & -b^2-e^2-a^2-d^2 & cd-fa & ce-fb & 0 \\ 0 & bd-ea & cd-fa & -b^2-e^2-c^2-f^2 & ab+de & ac+df \\ -bd+ea & 0 & ce-fb & ab+de & -a^2-d^2-c^2-f^2 & bc+ef \\ -cd+fa & -ce+fb & 0 & ac+df & bc+ef & -b^2-e^2-a^2-d^2 \end{bmatrix}.$$

**Proposition 5.4** *The algebra $S_2$ is simple.*

**Proof**. Otherwise, by Lemmas 2.4 and 2.12, there must be a non-trivial abelian ideal. By Lemma 2.9 this means that there is a non-trivial $\infty$-sandwich. In particular there must be a non-trivial 1-sandwich $x = ax_1 + bx_2 + cx_3 + dy_1 + ey_2 + fy_3$. This is the same as saying that $X^2 = 0$. Let $u = a^2 + d^2$, $v = b^2 + e^2$ and $w = c^2 + f^2$. By the formula above for $X^2$. We see that $u = -v$, $v = -w$ and $w = -u$. It follows that $u = v = w$ and thus $a = b = c = d = e = f = 0$. This shows that there is no non-trivial 1-sandwich and $S_2$ must be simple. $\square$

Notice that as $S_1$ contains 1-sandwiches but $S_2$ don't, it follows that $S_1$ and $S_2$ are not isomorphic.

**Definition 5.5** . *For $x = ax_1 + bx_2 + cx_3 + dy_1 + ey_2 + fy_3$, we define the conjugate of $x$ to be*

$$\bar{x} = -dx_1 - ex_2 - fx_3 + ay_1 + by_2 + cy_3$$

*and the norm of $x$ to be*

$$|x| = (x, \bar{x}) = a^2 + b^2 + c^2 + d^2 + e^2 + f^2.$$

One readily sees that if $\bar{x} = \pm x$ then $x$ must be 0. So if $x$ is non-trivial then $x$ and $\bar{x}$ are linearly independent. Clearly

$$\bar{\bar{x}} = -x \quad \text{and} \quad |\bar{x}| = |x|.$$

**Lemma 5.6** *Let $x, y \in S_2$. Then*

*(a)* $(\bar{x}, \bar{y}) = (x, y)$
*(b)* $\bar{x} \cdot \bar{y} = -x \cdot y$
*(c)* $x\bar{x} = 0$
*(d)* $\overline{x \cdot y} = -x \cdot \bar{y} = -\bar{x} \cdot y$
*(e)* $u\bar{x}\bar{y} = uxy \quad \forall u \in S_2$
*(f)* $ux\bar{x} + u\bar{x}x = 0 \quad \forall u \in S_2$
*(g)* $u\bar{x}^2 = ux^2 \quad \forall u \in S_2$.

**Proof** As the conjugation is a linear map, it suffices to prove (a) and (b) in the case when $x$ and $y$ are basis vectors. Now

$$\begin{aligned}
&(\bar{x}_i, \bar{x}_j) = (y_i, y_j) = 0 = (x_i, x_j)\\
&(\bar{y}_i, \bar{y}_j) = (-x_i, -x_j) = 0 = (y_i, y_j)\\
&(\bar{x}_i, \bar{y}_j) = (y_i, -x_j) = \delta_{ij} = (x_i, y_j)\\
&\bar{x}_i \cdot \bar{x}_j = y_i \cdot y_j = -x_i \cdot x_j\\
&\bar{y}_i \cdot \bar{y}_j = (-x_i) \cdot (-x_j) = -y_i \cdot y_j\\
&\bar{x}_i \cdot \bar{y}_j = y_i \cdot (-x_j) = x_j \cdot y_i = -x_i \cdot y_j.
\end{aligned}$$

We next turn to (c). Using (b) we have

$$x \cdot \bar{x} = -\bar{x} \cdot \bar{\bar{x}} = (-\bar{x}) \cdot (-x) = \bar{x} \cdot x = -x \cdot \bar{x}.$$

Hence, $x \cdot \bar{x} = 0$. Then (d) follows from

$$(\overline{xy}, z) = (\overline{\overline{xy}}, \bar{z}) = -(xy, \bar{z}) = -(x, \bar{z}y) = (x, \bar{\bar{z}}\bar{y}) = -(x, z\bar{y}) = (-x\bar{y}, z).$$

The second part follows from the first, since $\overline{xy} = -\overline{yx} = y\bar{x} = -\bar{x}y$. To prove (e) we use (b) and (d). We have

$$u\bar{x}\bar{y} = -\overline{u\bar{x}}\bar{\bar{y}} = \overline{u\bar{x}}y = -u\bar{\bar{x}}y = uxy.$$

Then (g) is a special case of (e) and it remains only to show (f). But using (e) we have

$$ux\bar{x} = -u\bar{\bar{x}}\bar{x} = -u\bar{x}x. \quad \square$$

**Lemma 5.7** *Let $x \in S_2$ such that $|x| \neq 0$. If*

$$(x, y) = |x| \quad \textit{and} \quad ux^2 = uy^2 \quad \forall u \in S_2$$

*then $y = \bar{x}$.*

**Proof** Suppose

$$\begin{aligned}
x &= a_1x_1 + b_1x_2 + c_1x_3 + d_1y_1 + e_1y_2 + f_1y_3\\
y &= a_2x_1 + b_2x_2 + c_2x_3 + d_2y_1 + e_2y_2 + f_2y_3.
\end{aligned}$$

We know from previous results that the conditions are necessary. Let us see that they are sufficient. As $X^2 = Y^2$, the formula for these matrices gives us that

$$\begin{aligned}
&a_i^2 + d_i^2,\ b_i^2 + e_i^2,\ c_i^2 + f_i^2,\ (a_i, d_i) \circ (b_i, e_i),\ (b_i, e_i) \circ (c_i, f_i)\\
&(c_i, f_i) \circ (a_i, d_i),\ (a_i, d_i) \circ (-e_i, b_i),\ (b_i, e_i) \circ (-f_i, c_i),\ (c_i, f_i) \circ (-d_i, a_i)
\end{aligned}$$

are same for $i = 1$ and $i = 2$. In other words, any inner product of any pair from

$$(a_i, d_i),\ (b_i, e_i),\ (c_i, f_i),\ (-d_i, a_i),\ (-e_i, b_i),\ (-f_i, c_i)$$

is independent of $i$. As $x \neq 0$ we have that one of $(a_1, d_1)$, $(b_1, e_1)$, $(c_1, f_1)$ is non-zero. Suppose this is $(a_1, d_1)$ (the other cases can be dealt with similarly). Suppose that

$$\begin{aligned}
(b_i, e_i) &= \alpha(a_i, d_i) + \beta(-d_i, a_i)\\
(c_i, f_i) &= \gamma(a_i, d_i) + \delta(-d_i, a_i)\\
(a_2, d_2) &= r(a_1, d_1) + s(-d_1, a_1).
\end{aligned}$$

Notice that
$$a_2^2 + d_2^2 = (r^2 + s^2)(a_1^2 + d_1^2)$$
implies that $r^2 + s^2 = 1$. Then

$$\begin{aligned}
|x| &= (x, y) \\
&= -(a_1, d_1) \circ (-d_2, a_2) - (b_1, e_1) \circ (-e_2, b_2) - (c_1, f_1) \circ (-f_2, c_2) \\
&= -(a_1, d_1) \circ (-d_2, a_2) - (\alpha(a_1, d_1) + \beta(-d_1, a_1)) \circ (-\beta(a_2, d_2) + \alpha(-d_2, a_2)) \\
&\quad -(\gamma(a_1, d_1) + \delta(-d_1, a_1)) \circ (-\delta(a_2, d_2) + \gamma(-d_2, a_2)) \\
&= s(a_1^2 + d_1^2) + s(\alpha^2 + \beta^2)(a_1^2 + d_1^2) + s(\delta^2 + \gamma^2)(a_1^2 + d_1^2) \\
&= s(1 + \alpha^2 + \beta^2 + \delta^2 + \gamma^2)(a_1^2 + d_1^2) \\
&= s(a_1^2 + d_1^2 + b_1^2 + e_1^2 + c_1^2 + f_1^2) \\
&= s \cdot |x|.
\end{aligned}$$

It follows that $s = 1$ since $|x| \neq 0$. Hence $r = 0$ and

$$\begin{aligned}
(a_2, d_2) &= (-d_1, a_1) \\
(b_2, e_2) &= \alpha(a_2, d_2) + \beta(-d_2, a_2) \\
&= -\beta(a_1, d_1) + \alpha(-d_1, a_1) \\
&= (-e_1, b_1) \\
(c_2, f_2) &= \gamma(a_2, d_2) + \delta(-d_2, a_2) \\
&= -\delta(a_1, d_1) + \gamma(-d_1, a_1) \\
&= (-f_1, c_1)
\end{aligned}$$

and
$$y = -d_1 x_1 - e_1 x_2 - f_1 x_3 + a_1 y_1 + b_1 y_2 + c_1 y_3 = \bar{x} \quad \square$$

**Lemma 5.8** *Let $x = ax_1 + bx_2 + cx_3 + dy_1 + ey_2 + fy_3$ be a non-trivial element in $S_2$. Then the minimal polynomial of $X$ is*

$$t^3 + |x|t$$

*Furthermore, if $|x| \neq 0$ then $ux^2 = -|x|u$ for all $u \in (Fx \oplus F\bar{x})^\perp$.*

**Proof** We have already obtained a formula for $X$ and $X^2$. This formula shows that $X \neq 0$ and $X^2 + |x|I \neq 0$. Now straightforward calculations show that $X^3 + |x|X = 0$. This proves that the minimal polynomial is $t^3 + |x|t$.

We know from previous work that $\bar{x}x = xx = 0$. Hence the kernel of $X$ is of dimension at least two. Since $\overline{ux} = -u\bar{x}$, we have that $ux = 0$ if and only if $u\bar{x} = 0$. And as $\overline{ux} = -\bar{u}x$ we have that the kernel of $X$ is closed under taking conjugates. We claim that $\ker(X) = Fx + F\bar{x}$. We argue by contradiction and suppose that there is some non-zero $y \in (Fx + F\bar{x})^\perp$ such that $yx = 0$. As $(\bar{u}, \bar{v}) = (u, v)$ for all $u, v \in S_2$ we have that $\bar{y}$ is also in $(Fx + F\bar{x})^\perp$. Furthermore, we have seen previously that $y$ and $\bar{y}$ are linearly independent. So we have that $\ker(X)$ is at least 4 dimensional. We consider two cases. First suppose that $(y, \bar{y}) \neq 0$. Extend the basis $x, \bar{x}, y, \bar{y}$ to a basis for all of $S_2$ by adjoining $z, \bar{z}$ where $z \in (Fx + F\bar{x} + Fy + F\bar{y})^\perp$. It follows that for all $u \in Fx + F\bar{x} + Fy + F\bar{y}$, we have

$$(zx, u) = (xu, z) = (0, z) = 0$$

and as furthermore $(zx, \bar{z}) = (\bar{z}z, x) = (0, x) = 0$ and $(zx, z) = 0$, it follows that $zx \in S_2^{\perp} = \{0\}$ and then also $\bar{z}x = 0$. Hence $x$ is in the center which contradicts the simplicity of $S_2$.

We are left with the case when $(y, \bar{y}) = 0$. Extend to a basis for whole of $S_2$

$$x_1, \bar{x_1}, y, z, \bar{y}, w$$

where $(y, z) = (\bar{y}, w) = 1$ but so that $z$ is orthogonal to $x_1, \bar{x_1}, \bar{y}, w$ and $w$ is orthogonal to $x_1, \bar{x_1}, y, z$. It follows that $zx$ is orthogonal to everything except possibly $w$ and that $wx$ is orthogonal to everything except possibly $z$. This gives

$$zx = a\bar{y}, \qquad wx = -ay$$

for some $a \in F$. But then $zx^2 = wx^2 = 0$, and as the minimal polynomial of $X$ is $(t^2+|x|)t$, this can only happen if $zx = wx = 0$. Thus we again get the contradiction that $x$ is in the center.

So we have shown that $\ker X = Fx + F\bar{x}$. By the Primary Decomposition Theorem for linear maps we have that $\ker(X^2 + |x|I) = \mathrm{im}(X)$. It follows that $\ker(X^2 + |x|I)$ is 4-dimensional and as every $ux$ is orthogonal to both $x$ and $\bar{x}$ it follows that $\ker(X^2 + |x|I) = (Fx + F\bar{x})^{\perp}$. □

**Lemma 5.9** *Let $\phi \in Aut(S_2)$ and $x \in S_2$. Then $\phi(\bar{x}) = \overline{\phi(x)}$ and $|\phi(x)| = |x|$.*

**Proof** The minimal polynomial of $x$ is the same as for $\phi(x)$. Lemma 5.8 therefore implies that $|\phi(x)| = |x|$. We turn to the proof of $\phi(\bar{x}) = \overline{\phi(x)}$. Since both the conjugation and $\phi$ are linear, it suffices to prove this in the case when $x$ is a basis vector. In that case $|x| \neq 0$ and we can use Lemma 5.7. Now

$$(\phi(x), \phi(\bar{x})) = (x, \bar{x}) = |x| = |\phi(x)|$$

and since $ux^2 = u\bar{x}^2$ for all $u \in S_2$, it follows that

$$u\phi(x)^2 = u\phi(\bar{x})^2$$

for all $u \in S_2$. By Lemma 5.7 we then have $\phi(\bar{x}) = \overline{\phi(x)}$. □

**Proposition 5.10** *Let $x, y \in S_2$. The linear map induced from*

$$\begin{array}{ll} \phi(x_1) = -\bar{x} & \phi(y_1) = x \\ \phi(x_2) = -\bar{y} & \phi(y_2) = y \\ \phi(x_3) = \overline{xy} & \phi(y_3) = -xy \end{array}$$

*is an automorphism if and only if $|y| = |x| = 1$ and $(y, x) = (y, \bar{x}) = 0$. Furthermore every automorphism is of this form.*

**Proof** First let us see that all automorphisms are of this form. Let $L = S_2$ and let $\phi \in \mathrm{Aut}(L)$. Then let $x = \phi(y_1)$ and $y = \phi(y_2)$. As $\phi$ respects the alternating form and the norm it is clear that $|x| = |y| = 1$ and that $(y, x) = 0$. Then using previous established properties

$$\begin{aligned} \phi(x_1) &= \phi(-\bar{y_1})) = -\phi(\bar{y_1}) = -\overline{\phi(y_1)} = -\bar{x} \\ \phi(x_2) &= \phi(-\bar{y_2}) = -\phi(\bar{y_2}) = -\overline{\phi(y_2)} = -\bar{y} \\ \phi(y_3) &= \phi(-y_1y_2) = -\phi(y_1)\phi(y_2) = -xy \\ \phi(x_3) &= \phi(-\bar{y_3}) = -\phi(\bar{y_3}) = -\overline{\phi(y_3)} = \overline{xy}. \end{aligned}$$

In particular $(y, \bar{x}) = (y_2, x_1) = 0$. It remains to show that these conditions are also sufficient. Notice that $(y, x) = (y, \bar{x}) = (x, \bar{y}) = 0$. Now from Lemma 5.8, it follows that

$$xy^2 = -x \quad \text{and} \quad yx^2 = -y.$$

We will make use of this. First we show that $\phi$ preserves the alternating form. It suffices to check these for the basis elements. We have $(\phi(x_1), \phi(y_1)) = (-\bar{x}, x) = (x, \bar{x}) = |x| = 1 = (x_1, y_1)$ and similarly $(\phi(x_2), \phi(y_2)) = (x_2, y_2)$. Then

$$(\phi(x_3), \phi(y_3)) = (\overline{xy}, -xy) = (-\bar{x}y, -xy) = (\bar{x}, xy^2) = (\bar{x}, -x) = (x, \bar{x}) = 1 = (x_3, y_3).$$

By the assumptions $\phi(x_1)$ is orthogonal to $\phi(x_2), \phi(y_2)$ and $\phi(x_2)$ is orthogonal to $\phi(y_1)$. We are also assuming that $\phi(y_1)$ and $\phi(y_2)$ are orthogonal. It remains to show that $\phi(x_3)$ and $\phi(y_3)$ are orthogonal to $\phi(x_1)$, $\phi(x_2)$, $\phi(y_1)$ and $\phi(y_2)$. As $(\bar{u}, \bar{v}) = (u, v)$, it suffices to prove this for $\phi(y_3)$. But this follows from

$$(xy, \bar{x}) = (\bar{x}x, y) = 0 \quad \text{and} \quad (xy, \bar{y}) = (x, \bar{y}y) = 0.$$

As $xy$ is orthogonal to both $x$ and $y$, this finishes the proof that $\phi$ respects the alternating form.

We finish the proof by showing that $\phi$ preserves the multiplication. Again it suffices to work with the basis elements. We have

$$\begin{aligned}
&\phi(x_1x_2) = \phi(y_3) = -xy = (-\bar{x}) \cdot (-\bar{y}) = \phi(x_1) \cdot \phi(x_2) \\
&\phi(x_2x_3) = \phi(y_1) = x = -xyy = y(xy) = (-\bar{y}) \cdot \overline{xy} = \phi(x_2) \cdot \phi(x_3) \\
&\phi(x_3x_1) = \phi(y_2) = y = -yxx = xy \cdot x = \overline{xy} \cdot (-\bar{x}) = \phi(x_3) \cdot \phi(x_1) \\
&\phi(y_1y_2) = \phi(-y_3) = xy = \phi(y_1) \cdot \phi(y_2) \\
&\phi(y_2y_3) = \phi(-y_1) = -x = y \cdot (-xy) = \phi(y_2) \cdot \phi(y_3) \\
&\phi(y_3y_1) = \phi(-y_2) = -y = (-xy) \cdot x = \phi(y_3) \cdot \phi(y_1).
\end{aligned}$$

As the product is determined by these, this finishes the proof. □

From the last proposition it is easy to determine the automorphism group. The matrix with respect to the given basis is the matrix with column vectors $-\bar{x}, -\bar{y}, \overline{xy}, x, y$ and $-xy$ where $x$ and $y$ are as described in the proposition. To determine the order of this group we need to count the number of pairs $(x, y)$ that satisfy the criteria. First we choose

$$x = a_1x_1 + a_2x_2 + a_3x_3 + b_1y_1 + b_2y_2 + b_3y_3.$$

The only requirement is that $|x| = 1$ and there are two ways for which this can happen. Either exactly one of the coefficients is non-zero or exactly four are non-zero. The number of solutions of the former type is $6 \cdot 2$ and the number of latter is $\binom{6}{4} \cdot 2^4 = 15 \cdot 2^4$. So in total we have

$$3^2 \cdot 2^2 \cdot 7 = 3^2(3^3 + 1).$$

It remains to choose $y$. Take any automorphism $\phi$ that maps $y_1$ to $x$. Let $u_2 = \phi(x_2)$, $u_3 = \phi(x_3)$, $v_2 = \phi(y_2)$ and $v_3 = \phi(y_3)$. Then

$$y = c_2u_2 + c_3u_3 + d_2v_2 + d_3v_3$$

subject to $1 = |y| = c_2^2 + c_3^2 + d_2^2 + d_3^2$. Again either exactly one of the coefficients is non-zero or all four of them. The number of solutions for $y$ is therefore

$$4 \cdot 2 + 2^4 = 24 = 3 \cdot (3^2 - 1).$$

We conclude that

$$|\mathrm{Aut}(L)| = 3^3 \cdot (3^2 - 1) \cdot (3^3 + 1)$$

and therefore

$$|S_2^G| = \frac{(3^6-1)(3^4-1)(3^2-1)3^9}{(3^3+1)(3^2-1)3^3} = (3^4-1)(3^3-1)3^6.$$

## 5.3 The remaining symmetric algebras

We will see that the remaining symmetric algebras are the 5 non-simple ones that we have found earlier. These are

$$O \oplus O,\ O \oplus A_1, O \oplus C_1, O \oplus B_5, O \oplus B_6.$$

For later use and also for calculating the size of their orbits, we now calculate their automorphism groups. We start with

$$O \oplus C_1 : \begin{array}{ll} x_2x_3 = 0 & y_2y_3 = y_1 \\ x_3x_1 = 0 & y_3y_1 = 0 \\ x_1x_2 = 0 & y_1y_2 = 0 \end{array}$$

The center is

$$O \cap (C_1^2)^\perp + Z(C_1) = Fx_2 + Fx_3 + Fy_1$$

which is preserved by any automorphism. Replacing the basis above with $-y_1, x_2, x_3, x_1, y_2, y_3$ we get the following presentation for $R_1 = O \oplus C_1$

$$R_1 : \begin{array}{ll} x_2x_3 = 0 & y_2y_3 = -x_1 \\ x_3x_1 = 0 & y_3y_1 = -x_2 \\ x_1x_2 = 0 & y_1y_2 = -x_3 \end{array}$$

Every automorphism must map $Z(R_1)$ to itself. This means that its matrix with respect to the above basis is of the form (since it is a symplectic map)

$$\begin{bmatrix} (A^{-1})^t & B \\ 0 & A \end{bmatrix} = \begin{bmatrix} I & BA^{-1} \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} (A^{-1})^t & 0 \\ 0 & A \end{bmatrix}$$

where $BA^{-1}$ is symmetric. Furthermore the reader can check that the condition $(y_1^g y_2^g, y_3^g) = (y_1y_2, y_3) = -1$ implies that $\det(A) = 1$ and that this condition is sufficient for $g$ to be a automorphism. Hence

$$\mathrm{Aut}(R_1) = \{\begin{bmatrix} (A^{-1})^t & B \\ 0 & A \end{bmatrix} : \ \det(A) = 1 \text{ and } BA^{-1} \text{ is symmetric}\}.$$

The number of these is

$$|\mathrm{Aut}(R_1)| = \frac{(3^3-1)\cdot(3^3-3)\cdot(3^3-3^2)}{2} \cdot 3^6$$

which implies that

$$|(O \oplus C_1)^G| = \frac{(3^6-1)(3^4-1)(3^2-1)3^9 \cdot 2}{(3^3-1)(3^2-1)(3-1)3^9} = (3^3+1)\cdot(3^4-1).$$

We next move to

$$O \oplus A_1 : \begin{array}{ll} x_2x_3 = 0 & y_2y_3 = y_1 \\ x_3x_1 = 0 & y_3y_1 = y_2 \\ x_1x_2 = 0 & y_1y_2 = y_3 \end{array}$$

Recall that there $O = Fx_1 + Fx_2 + Fx_3$ is the unique proper non-trivial ideal of $L = O \oplus A_1$ and we have previously calculate the automorphism group of $L/O$ which consisted of orthogonal linear maps with determinant 1. As $I$ is invariant the matrix of a automorphism with respect to the basis above is of the form

$$\left[\begin{array}{cc} A & 0 \\ 0 & (A^{-1})^t \end{array}\right] \cdot \left[\begin{array}{cc} I & B \\ 0 & I \end{array}\right]$$

where $B$ is symmetric and $A$ is orthogonal with determinant 1. Furthermore the condition $y_1^g \cdot y_2^g = y_3^g$, $y_2^g \cdot y_3^g = y_1^g$ and $y_3^g y_1^g = y_2^g$ implies that that the trace of $B$ must be zero. These conditions can also be checked to be sufficient. Hence for $R_2 = O \oplus A_1$

$$\mathrm{Aut}(R_2) = \{ \left[\begin{array}{cc} (A^{-1})^t & B \\ 0 & A \end{array}\right] : \begin{array}{c} A \text{ is orthogonal with } \det(A) = 1 \\ \text{and } BA^{-1} \text{ is symmetric with trace } 0. \end{array} \}$$

The number of these is $|\mathrm{Aut}(R_2)| = 2^3 \cdot 3 \cdot 3^5 = (3^2-1)3^6$ and hence

$$|(O \oplus A_1)^G| = \frac{(3^6-1)(3^4-1)(3^2-1)3^9}{(3^2-1)3^6} = (3^6-1)(3^4-1)3^3.$$

There still remain two symmetric algebras to consider. First we deal with

$$O \oplus B_5 : \begin{array}{ll} x_2x_3 = 0 & y_2y_3 = y_1 \\ x_3x_1 = 0 & y_3y_1 = y_2 \\ x_1x_2 = 0 & y_1y_2 = 0 \end{array}$$

Here

$$Z(O \oplus B_5) = O \cap (B_5^2)^\perp + Z(B_5) = Fx_3$$

and $(O \oplus B_5)^2 = Z(O \oplus B_5)^\perp = Fx_1 + Fx_2 + Fx_3 + Fy_1 + Fy_2$ are invariant. Working on the automorphism conditions in more detail one can check that a symplectic map is an automorphism on $(O \oplus B_5)$ if and only if its matrix with respect to the above basis is of the form

$$\left[\begin{array}{cccccc} a_1 & -\epsilon a_2 & 0 & b_1 & \epsilon b_2 & e_1 \\ a_2 & \epsilon a_1 & 0 & b_2 & -\epsilon b_1 & e_2 \\ c_1 & c_2 & \epsilon & c_3 & c_4 & e \\ a_3 & \epsilon a_4 & 0 & b_3 & -\epsilon b_4 & e_3 \\ a_4 & -\epsilon a_3 & 0 & b_4 & \epsilon b_3 & e_4 \\ 0 & 0 & 0 & 0 & 0 & \epsilon \end{array}\right]$$

subject to the conditions that

$$\begin{array}{rcl} \epsilon & = & \pm 1 \\ c_1 & = & \epsilon \left| \begin{array}{cc} -a_2 & e_3 \\ a_1 & e_4 \end{array} \right| - \epsilon \left| \begin{array}{cc} e_1 & a_4 \\ e_2 & -a_3 \end{array} \right| \\ c_2 & = & - \left| \begin{array}{cc} a_1 & e_3 \\ a_2 & e_4 \end{array} \right| + \left| \begin{array}{cc} e_1 & a_3 \\ e_2 & a_4 \end{array} \right| \\ c_3 & = & -\epsilon \left| \begin{array}{cc} b_2 & e_3 \\ -b_1 & e_4 \end{array} \right| + \epsilon \left| \begin{array}{cc} e_1 & -b_4 \\ e_2 & b_3 \end{array} \right| \\ c_4 & = & \left| \begin{array}{cc} b_1 & e_3 \\ b_2 & e_4 \end{array} \right| - \left| \begin{array}{cc} e_1 & b_3 \\ e_2 & b_4 \end{array} \right| \end{array}$$

and

$$\begin{bmatrix} a_4 & -a_3 & a_2 & -a_1 \\ -a_3 & -a_4 & a_1 & a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The reader can verify that the this implies that for $R_3 = O \oplus B_5$, $|\mathrm{Aut}(R_3)| = (3-1)(3^4-1)3^7$ and hence

$$|(O \oplus B_5)^G| = \frac{(3^6-1)(3^4-1)(3^2-1)3^9}{(3^4-1)(3-1)3^7} = (3^6-1)(3+1)3^2.$$

Similar analysis of

$$O \oplus B_6 : \begin{array}{ll} x_2x_3 = 0 & y_2y_3 = y_1 \\ x_3x_1 = 0 & y_3y_1 = -y_2 \\ x_1x_2 = 0 & y_1y_2 = 0 \end{array}$$

shows that the automorphism group consists of matrices of the form

$$\begin{bmatrix} a_1 & \epsilon a_2 & 0 & b_1 & \epsilon b_2 & e_1 \\ a_2 & \epsilon a_1 & 0 & b_2 & \epsilon b_1 & e_2 \\ c_1 & c_2 & \epsilon & c_3 & c_4 & e \\ a_3 & \epsilon a_4 & 0 & b_3 & \epsilon b_4 & e_3 \\ a_4 & \epsilon a_3 & 0 & b_4 & \epsilon b_3 & e_4 \\ 0 & 0 & 0 & 0 & 0 & \epsilon \end{bmatrix}$$

subject to the conditions that

$$\begin{aligned} \epsilon &= \pm 1 \\ c_1 &= -\epsilon a_2e_4 - \epsilon a_1e_3 + \epsilon e_1a_3 + \epsilon e_2a_4 \\ c_2 &= -a_1e_4 - a_2e_3 + e_1a_4 + e_2a_3 \\ c_3 &= -\epsilon b_2e_4 - \epsilon b_1e_3 + \epsilon e_1b_3 + \epsilon e_2b_4 \\ c_4 &= -b_1e_4 - b_2e_3 + e_1b_4 + e_2b_3 \end{aligned}$$

and

$$\begin{bmatrix} a_4 & a_3 & -a_2 & -a_1 \\ -a_3 & -a_4 & a_1 & a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The reader can verify that the this implies that for $R_4 = O \oplus B_6$, $|\mathrm{Aut}(R_4)| = (3^2-1)^2(3-1)3^7$ and hence

$$|(O \oplus B_6)^G| = \frac{(3^6-1)(3^4-1)(3^2-1)3^9}{(3^2-1)^2(3-1)3^7} = (3^4+3^2+1)(3^2+1)(3+1)3^2.$$

Let us see that we have got all the symmetric presentation matrices. We have

$$\begin{array}{rcl} |S_1^G| + |S_2^G| + |(O \oplus A_1)^G| & & (3^4-1)(3^3+1)3^6 + (3^4-1)(3^3-1)3^6 \\ +|(O \oplus C_1)^G| + |(O \oplus O)^G| & = & (3^6-1)(3^4-1)3^3 + (3^4-1)(3^3+1) + 1 \\ +|(O \oplus B_5)^G| + |(O \oplus B_6)^G| & & (3^6-1)(3+1)3^2 + (3^4+3^2+1)(3^2+1)(3+1)3^2 \\ & & \\ & & 2 \cdot 3^9(3^4-1) + 3^9(3^4-1) \\ & = & -(3^4-1)3^3 + (3^4-1)(3^3+1) + 1 \\ & & 3^2(3+1)(2 \cdot 3^6 + 2 \cdot 3^4 + 2 \cdot 3^2) \\ & & \\ & = & 3^{10}(3^4-1) + 3^4 + 3^4(2 \cdot 3^5 + 2 \cdot 3^4 + \cdots 2 \cdot 1) \\ & = & 3^{10}(3^4-1) + 3^{10} \\ & = & 3^{14}, \end{array}$$

which fits!

## 5.4 The remaining simple algebras

The strategy is now the same as for the non-simple algebras. For any of the 7 symmetric algebras $L$ above we consider the action of $\mathrm{Aut}(L)$ on the set of all anti-symmetric matrices. We will skip over the technical details. Although at times messy these are essentially straightforward.

We start with $L_1 = O \oplus O$. Here there are no new algebras that turn up. There is only one non-trivial orbit that corresponds to the unique non-trivial non-simple anti-symmetric algebra $O \oplus B_1$.

We next turn to

$$R_2 = \begin{bmatrix} 0 & -I \\ 0 & 0 \end{bmatrix}$$

which as we have seen is a presentation matrix for $O \oplus C_1$. In this case there are only two non-trivial $C_G(R_2)$-orbits

$$\begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}^G \text{ and } \begin{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}^G$$

of size $3^3 - 1$ and $3^6 - 3^3$ respectively. These correspond to the two non-simple algebras $O \oplus B_3$ and $O \oplus A_4$. So we don't get anything new here either.

Then consider $L = O \oplus B_5$. The $\mathrm{Aut}(L)$-orbits are

$$\begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{bmatrix}^G \begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \end{bmatrix}^G \text{ and } \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}^G$$

of sizes 2, $3^5 - 3$ and $3^6 - 3^5$. The first two correspond to the two non-symmetric non-simple algebras with symmetric part $O \oplus B_5$, $O \oplus B_4$ and $O \oplus A_5$. The third one must therefore be new and simple. This is

$$S_3 : \begin{array}{ll} x_2x_3 = -x_2 & y_2y_3 = y_1 \\ x_3x_1 = x_1 & y_3y_1 = y_2 \\ x_1x_2 = 0 & y_1y_2 = 0 \end{array} .$$

Next in the list is $L = O \oplus B_6$. Here there are four non-trivial $\mathrm{Aut}(L)$-orbits. These are

$$\begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}^G, \begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \end{bmatrix}^G,$$

$$\begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \end{bmatrix}^G, \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}^G$$

of sizes 2, $(3^2 - 1)^2 \cdot 3$, $(3^2 - 1) \cdot (3 - 1) \cdot 3$ and $(3 - 1)3^5$. The first three give us the non-simple algebras $O \oplus C_2$, $O \oplus A_6$ and $O \oplus A_7$, whereas the last one is new and therefore simple. This is

$$S_4 : \begin{array}{ll} x_2x_3 = -x_2 & y_2y_3 = y_1 \\ x_3x_1 = x_1 & y_3y_1 = -y_2 \\ x_1x_2 = 0 & y_1y_2 = 0 \end{array} .$$

We now deal with the final symmetric non-simple algebra $O \oplus A_1$. Here we get 6 non-trivial orbits

$$\begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}^G, \quad \begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \end{bmatrix}^G, \quad \begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \end{bmatrix}^G,$$

$$\begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}^G, \quad \begin{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}^G, \quad \begin{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}^G$$

with orders 6, 12, 8, $2 \cdot 3^4$, $2^2 \cdot 3^4$ and $3^5 - 3^3$. Here the first three are non-simple. These are $O \oplus A_2$, $O \oplus B_2$ and $O \oplus A_3$. The remaining ones are all new simple algebras. These are

$$S_5: \begin{array}{ll} x_2x_3 = -x_2 & y_2y_3 = y_1 \\ x_3x_1 = x_1 & y_3y_1 = y_2 \\ x_1x_2 = 0 & y_1y_2 = y_3 \end{array}, \qquad S_6: \begin{array}{ll} x_2x_3 = -x_2 - x_3 & y_2y_3 = y_1 \\ x_3x_1 = x_1 & y_3y_1 = y_2 \\ x_1x_2 = x_1 & y_1y_2 = y_3 \end{array},$$

$$S_7: \begin{array}{ll} x_2x_3 = -x_2 + x_3 & y_2y_3 = y_1 \\ x_3x_1 = x_1 - x_3 & y_3y_1 = y_2 \\ x_1x_2 = -x_1 + x_2 & y_1y_2 = y_3 \end{array}.$$

We are now only left with the two symmetric simple algebras $S_1$ and $S_2$. Here we get 5 non-trivial $\mathrm{Aut}(S_1)$-orbits. These are

$$\begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}^G, \qquad \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}^G,$$

$$\begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}^G, \qquad \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}^G,$$

$$\begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \end{bmatrix}^G,$$

with sizes $3^3 - 1$, $3^3 - 1$, $3^5 - 3^2$, $3^5 - 3^2$ and $(3^3 - 1)(3^2 - 1)$. The five new simple algebras are

$$S_8: \begin{array}{ll} x_2x_3 = y_1 & y_2y_3 = -y_2 + x_1 \\ x_3x_1 = y_2 & y_3y_1 = y_1 + x_2 \\ x_1x_2 = y_3 & y_1y_2 = x_3 \end{array}, \qquad S_9: \begin{array}{ll} x_2x_3 = -x_2 + y_1 & y_2y_3 = x_1 \\ x_3x_1 = x_1 + y_2 & y_3y_1 = x_2 \\ x_1x_2 = y_3 & y_1y_2 = x_3 \end{array},$$

$$S_{10}: \begin{array}{ll} x_2x_3 = -x_2 + y_1 & y_2y_3 = -y_2 + x_1 \\ x_3x_1 = x_1 + y_2 & y_3y_1 = y_1 + x_2 \\ x_1x_2 = y_3 & y_1y_2 = x_3 \end{array}, \qquad S_{11}: \begin{array}{ll} x_2x_3 = -x_2 + y_1 & y_2y_3 = y_2 + x_1 \\ x_3x_1 = x_1 + y_2 & y_3y_1 = -y_1 + x_2 \\ x_1x_2 = y_3 & y_1y_2 = x_3 \end{array},$$

$$S_{12}: \begin{array}{ll} x_2x_3 = -x_2 + y_1 & y_2y_3 = -y_3 + x_1 \\ x_3x_1 = x_1 + y_2 & y_3y_1 = x_2 \\ x_1x_2 = y_3 & y_1y_2 = x_3 + y_1 \end{array}.$$

We are now only left with the case where the symmetric part is $O \oplus S_2$. Here there are three $\mathrm{Aut}(S_2)$-orbits

$$\begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}^G, \begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \end{bmatrix}^G, \begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \end{bmatrix}^G,$$

of order $3^5 + 3^2$, $3^5 + 3^2$ and $(3^2 - 1)(3^3 + 1)$. This gives us the last three simple algebras:

$$S_{13}: \begin{array}{ll} x_2x_3 = y_1 & y_2y_3 = -y_1 - y_2 \\ x_3x_1 = y_2 & y_3y_1 = y_1 - y_2 \\ x_1x_2 = y_3 & y_1y_2 = -y_3 \end{array}, \qquad S_{14}: \begin{array}{ll} x_2x_3 = y_1 & y_2y_3 = -y_1 - y_2 - y_3 \\ x_3x_1 = y_2 & y_3y_1 = y_1 - y_2 \\ x_1x_2 = y_3 & y_1y_2 = y_1 - y_3 \end{array},$$

$$S_{15}: \begin{array}{ll} x_2x_3 = y_1 & y_2y_3 = -y_1 - y_2 - y_3 \\ x_3x_1 = y_2 & y_3y_1 = y_1 - y_2 - y_3 \\ x_1x_2 = y_3 & y_1y_2 = y_1 + y_2 - y_3 \end{array}.$$

So we have finished the classification. There are in total 31 algebras of which 15 are simple.

# References


[1] P. Moravec and G. Traustason, Powerful 2-Engel groups. Submitted.

[2] G. Traustason, Powerful 2-Engel groups II. Submitted.