Powerful 2-Engel groups II

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We conclude our classification of powerful 2-Engel groups of class three that are minimal in the sense that every proper powerful section is nilpotent of class at most two. In the predecessor to this paper we obtained three families of minimal groups. Here we get a fourth family of minimal examples that is described in terms of irreducible polynomials over the field of three elements. We also get one isolated minimal example of rank 5 and exponent 27. The last one has a related algebraic structure that we call a "symplectic alternating algebra". To each symplectic alternating algebra over the field of three elements there corresponds a unique 2-Engel group of exponent 27.

1 Introduction

Every finite n-Engel group is nilpotent [9]. However if $n \geq 3$, the class is not n-bounded. In contrast we know that the class is n-bounded if one adds the further condition that the group is powerful [1]. (Recall that a finite p-group, p odd, is said to be powerful if $[G, G] < G^p$. We refer to [6] for further information and description of their many abelian like properties). The proof of this result relies on deep results on Lie algebras. It does not give any precise information how the property of being powerful affects the structure of the group and in particular we have no good bounds for the nilpotence class, not even for small values of n. In this paper we conclude our study of powerful 2-Engel groups that we started in [8].

We recall that a group is said to be 2-Engel if it satisfies the commutator law $[[y,x],x]=1$ or equivalently the law $[x^y,x]=1$, i.e. any two conjugates commute. These groups have their origin in Burnside's papers [2,3] and were subsequently studied by Hopkins and Levi [4,5]. We refer to the introduction of [8] for further background. To summarize, we have the following transparent description of the variety of 2-Engel groups as the variety of groups satisfying the following identities:

$$[x,y,z] = [y,z,x] \qquad (1)$$
$$[x,y,z]^3 = 1 \qquad (2)$$
$$[x,y,z,t] = 1. \qquad (3)$$

Of course this settles the study of 2-Engel groups no more than knowing that the variety of abelian groups is characterized by the law $[x,y]=1$ settles the study of abelian groups. For example, the following well known problems raised by Caranti [7] still remain unsolved.

Problem. (a) Let G be a group of which every element commutes with all its endomorphic images. Is G nilpotent of class at most 2?

(b) Does there exist a finite 2-Engel 3-group of class three such that $\mathrm{Aut}\,G = \mathrm{Aut}_c G \cdot \mathrm{Inn}\,G$ where $\mathrm{Aut}_c G$ is the group of central automorphisms of G.

The class of powerful p-groups is quite a special class of p-groups. In some sense the groups are very abelian like but at the same time they are quite typical p-groups as the class of powerful p-groups generates the class of all groups. For this reason one is often able to reduce problems on p-groups to the class of powerful p-groups where one can make use of all the abelian like properties. Our belief is therefore that understanding the structure of powerful 2-Engel groups should be helpful in tackling various problems on 2-Engel groups like the problems mentioned above.

Turning back to our study of powerful p-groups, in [8] we proved that any powerful 2-Engel group generated by three elements is nilpotent of class at most 2. Surprisingly this result does not hold in general when the number of generators is higher. In [8] we started our classification of the minimal examples, where by minimal we mean that the group is powerful of class three but all proper powerful sections have class at most 2. We found it useful to divide the minimal examples into two subclasses.

(I) The minimal examples G where $\gamma_3(G) < [G,G]^3$.
(II) The minimal examples G where $\gamma_3(G) = [G,G]^3$.

In [8] we gave a concrete classification of all minimal examples of type I by listing them as three infinite families, one with groups of rank 5 and two with groups of rank 4. In this paper we will turn to the minimal examples of type II. It turns out that these are very different from the groups of type I. We will get one infinite family of minimal examples, with groups of any even rank greater than or equal to 4, and one isolated example of rank 5. In the former case the classification will be in terms of irreducible polynomials over the field of three elements. The isolated example of rank 5 and exponent 27 has an associated algebraic structure, symplectic alternating algebra (that has nothing to do with the classical Lie algebras). We will see that these symplectic alternating algebras correspond to a special class of 2-Engel groups of odd rank and exponent 27. It is curious that whereas there are minimal examples of any even rank greater than or equal to 4, the only odd rank that occurs is 5.

It had been our hope that within this rich class of minimal examples we would find some counter examples to the problems above raised by Caranti. This turns out not to be the case. It seems to suggest that such counter examples do not exists and perhaps there is a way of reducing these problems to powerful 2-Engel groups.

2 Minimal examples of type II

Let G be a group of type II. As $\gamma_3(G) = [G,G]^3$, it follows that $[G^9, G] = [G,G]^9 = \gamma_3(G)^3 = 1$ and G^9 is contained in the center and therefore cyclic by the minimality of G. Consider the homomorphism

$$\Phi : G \to G^9,\ a \mapsto a^9.$$

Let H be the kernel. Then G/H is cyclic and $H^9 = 1$. Notice that $H \neq G$, since otherwise $\gamma_3(G) = [G,G]^3 \leq G^9 = \{1\}$ as G is powerful. This contradicts the assumption that G has class 3. We divide the groups of type II into two subclasses.

(A) The groups of type II where $\gamma_3(H) = 1$.
(B) The groups of type II where $\gamma_3(H) \neq 1$.

2.1 Groups of type A

As $\gamma_3(G) = [G,G]^3$ and $Z(G)$ is cyclic, it follows that $[G,G]^3$ is cyclic of order three and we can identify it with the field F of three elements. As G is powerful G/G^3 is abelian of exponent three and can be thought of as a vector space over F. Choose a generator z for $[G,G]^3$. We thus get an alternating form

$$\Psi : G/G^3 \times G/G^3 \to [G,G]^3$$

that maps $(\bar{a}, \bar{b})$ to $[a,b]^3$. Choose first a standard basis for HG^3/G^3 with respect to Ψ, say $\bar{x_1}, \bar{x_2}, \ldots, \bar{x_{2r}}, \bar{x_{2r+1}}, \ldots, x_{2r+s}$ such that

$$\begin{aligned} [x_2, x_1]^3 &= z \\ &\vdots \\ [x_{2r}, x_{2r-1}]^3 &= z \\ [x_u, x_v]^3 &= 1 \quad \text{otherwise.} \end{aligned}$$

Choose $x \in G$ such that $G = \langle x, H \rangle$. By replacing x by some $x x_1^{\alpha_1} \cdots x_{2r}^{\alpha_{2r}}$, we can assume that

$$[x, x_1]^3 = \ldots = [x, x_{2r}]^3 = 1.$$

Now x^3 can't commute with all of $x_{2r+1}, \ldots, x_{2r+s}$, since otherwise $x \in C_G(G^3) \geq C_G([G,G])$ and since $\gamma_3(H) = \{1\}$ it follows then that we get the contradiction that $\gamma_3(G) = \{1\}$. We can thus suppose we have generators x and $y, x_1, x_2, \ldots, x_{2r}, x_{2r+1}, \ldots, x_{2r+s-1} \in H$ such that

$$\begin{aligned} [y, x]^3 &= z \\ [x_2, x_1]^3 &= z \\ &\vdots \\ [x_{2r}, x_{2r-1}]^3 &= z \\ [x, x_u]^3 &= 1 \text{ for all } 1 \leq u \leq 2r+s-1 \end{aligned}$$

$$\begin{aligned}[y, x_u]^3 &= 1 \text{ for all } 1 \leq u \leq 2r+s-1 \\ [x_u, x_v]^3 &= 1 \text{ otherwise.}\end{aligned}$$

Then $x_{2r+1}^3, \ldots, x_{2r+s-1}^3$ are elements of order 3 in the center of G and as the center is cyclic we must have that these elements are powers of x^9. It follows that the subgroup $\langle x, y, x_1, \ldots, x_{2r}\rangle$ is powerful. As $x_{2r+1}, \ldots, x_{2r+s-1} \in C_G(G^3)$ they commute with everything in $[G, G]$ and are therefore in $Z^2(G)$. Hence $\gamma_3(G) = \gamma_3(\langle x, y, x_1, \ldots, x_{2r}\rangle)$ and $G = \langle x, y, x_1, \ldots, x_{2r}\rangle$ by minimality of G. Suppose that $o(x) = 3^{s+2}$, $s \geq 1$. Since $\gamma_3(H) = [H, H]^9 = \{1\}$ and since G is powerful, it follows that

$$[H, H] = \langle y^3, x^{3^s}\rangle.$$

The reason that y^3 must be in $[H, H]$ is that $\gamma_3(G) \neq \{1\}$ and $x^{3^s} \in [H, H]$ since $[H, H]^3 \neq \{1\}$. Notice that s must be at least 2 since $[H, H, \langle y\rangle] \leq \gamma_3(H) = \{1\}$. Let $K = \langle x_1, \ldots, x_{2r}\rangle$ and consider the vector space $V = KG^3/G^3$. This is a symplectic space with respect to the alternating form $(\bar{u}, \bar{v}) = [u, v]^3$. We know from above that $(V, (\ ))$ is non-degenerate. Consider the map

$$\phi : V \to V,\ \bar{u} \mapsto \bar{v} \quad \text{if} \ \ [u, x] = v^3.$$

Let us see why this is well defined. Suppose

$$[u, x] = x_1^{3\alpha_1} \cdots x_{2r}^{3\alpha_{2r}} x^{3^s \alpha_x} y^{3\alpha_y}$$

and as $[u, x, x] = [u, x]^3 = 1$ it follows that 3 divides α_x and α_y. So $[u, x] = (ac)^3$ with

$$a = x_1^{\alpha_1} \cdots x_{2r}^{\alpha_{2r}} \in K$$

and $c \in G^3$. If also $[u, x] = (bd)^3$ with $d \in G^3$ and

$$b = x_1^{\beta_1} \cdots x_{2r}^{\beta_{2r}} \in K,$$

then we must have $\alpha_i = \beta_i$ since the alternating form is non-degenerate.

Notice that as $[u, x] \in \langle x_1^3, \ldots, x_{2r}^3, x^{3^{s+1}}\rangle$, we have $[u, x, y] = 1$. As $\gamma_3(H) = \{1\}$ and $[y, x, x] = [y, x, y] = 1$ it follows that y is in $Z^2(G)$.

Lemma 2.1 *The map $\phi : V \to V$ is self adjoint. Furthermore*

$$(\phi^i(\bar{u}), \phi^j(\bar{u})) = 0$$

for all $\bar{u} \in V$ and $i, j \geq 0$.

Proof As
$$(\phi(\bar{u}), \bar{v}) = [u, x, v] = [u, [v, x]] = (\bar{u}, \phi(\bar{v})),$$
ϕ is self adjoint. By the 2-Engel identity
$$(\phi(\bar{u}), \bar{u}) = [u, x, u] = 1, \tag{4}$$
and since ϕ is self adjoint every alternating product of the form $(\phi^i(\bar{u}), \phi^j(\bar{u}))$ is equal to one where j is either i or $i + 1$. The first is of course trivial and by (4) the second is as well. □

Lemma 2.2 *Let $u \in K$. Then $\bar{u} \in ker\phi$ if and only if $u \in K \cap Z^2(G)$.*

Proof As $[u, x, x] = 1$, $y \in Z^2(G)$ and $\gamma_3(K) = \{1\}$, we have that $u \in Z^2(G)$ if and only if $[u, x, v] = 1$ for all $v \in K$. In other words $u \in Z^2(G)$ if and only if $(\phi(\bar{u}), \bar{v}) = 0$ for all $v \in K$. As the alternating form is non-degenerate this happens if and only if $\phi(\bar{u}) = 0$. □

Lemma 2.3 *We have $im\,\phi = ker\,\phi^{\perp}$. Furthermore $ker\,\phi \subseteq im\,\phi$ and the dimension of $ker\,\phi$ is at most 2.*

Proof Let $\bar{w} = \phi(\bar{u}) \in \operatorname{im}\phi$ and $\bar{v} \in \ker\phi$. Then
$$(\bar{w}, \bar{v}) = [u, x, v] = [u, [v, x]] = (\bar{u}, \phi(\bar{v})) = (\bar{u}, 0) = 0.$$
This shows that $\operatorname{im}\phi \subseteq \ker\phi^{\perp}$. As $\dim(\operatorname{im}\phi) = \dim(\ker\phi^{\perp})$, it follows that $\operatorname{im}\phi = \ker\phi^{\perp}$. Let
$$R = \{u \in K : \bar{u} \in \operatorname{im}\phi\}.$$
Then $[K, x] = R^3$ and
$$[G, G] = \langle [H, H], [y, x], R^3 \rangle = \langle y^3, x^{3^s}, R^3 \rangle.$$
It follows that every subgroup containing $\langle x, y, R \rangle$ is powerful. By minimality of G, any proper subgroup of G containing $\langle x, y, R \rangle$ is nilpotent of class at most 2. Pick $u_1, \ldots, u_l \in R$ such that $\bar{u_1}, \ldots, \bar{u_l}$ is a basis for $\operatorname{im}\phi$. Then pick $v_1, \ldots, v_m, v_{m+1}, \ldots v_n \in G$ such that $\bar{u_1}, \ldots \bar{u_l}, \bar{v_1}, \ldots, \bar{v_m}$ is a basis for $\operatorname{im}\phi + \ker\phi$ and $\bar{u_1}, \ldots, \bar{u_l}, \bar{v_1}, \ldots, \bar{v_n}$ is a basis for V. Notice that $l = \dim(\operatorname{im}\phi)$ and $n = \dim(\ker\phi)$. We next show that $\ker\ \phi \subseteq \operatorname{im}\phi$ by showing that $m = 0$. We argue by contradiction at suppose this is not the case. Then the subgroup
$$S = \langle x, y, u_1, \ldots, u_l, v_2, \ldots, v_n \rangle$$

contains $\langle x, y, R\rangle$ and is therefore powerful. But as $v_1 \in Z^2(G)$, S is of class three. This contradicts the minimality of G. Hence $m = 0$. It remains to show that $n \leq 2$. For every $1 \leq i \leq n$, the subgroup

$$S_i = \langle x, y, u_1, \ldots, u_l, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\rangle$$

is powerful and proper and must therefore have class at most 2 by the minimality of G. Because of this and the fact that $\gamma_3(H) = \{1\}$, it follows that a non-trivial commutator of weight 3 in $x, y, u_1, \ldots, u_l, v_1, \ldots, v_n$ must include all of $x, v_1, \ldots, v_n$ but this is only possible if $n \leq 2$. $\Box$

We next show that $\dim(\ker\phi)$ cannot be 1. We argue by contradiction and suppose that we have a minimal example of this type. Let F be the field of three elements. We can think of V as a $F[x]$-module with $x \cdot v = \phi(v)$. As a module over a principal ideal domain, V can be written as a direct sum of cyclic submodules say

$$V = F[x]v_1 \oplus F[x]v_2 \oplus \cdots \oplus F[x]v_r.$$

where the "order", i.e. the characteristic polynomial of the restriction of ϕ, of each summand is a power of an irreducible polynomial. As $\dim(\ker\phi) = 1$ exactly one of the summands, say the first one, has order x^s for some positive integer s. But this implies that any $F[x]v_1 \oplus F[x]v_i$, $2 \leq i \leq r$ is cyclic and therefore by Lemma 2.1 we have that $F[x]v_1 \subseteq V^{\perp} = \{0\}$. By this contradiction it is clear that no such example can exist.

It remains to examine two cases.

CASE 1. $\dim(\ker\phi) = 2$.

We first show that in this case $\operatorname{im}\phi = \ker\phi$. Let $l = \dim(\operatorname{im}\phi)$ and $u_1, \ldots, u_l, v_1, v_2 \in K$ such that $\bar{u_1}, \ldots, \bar{u_l}$ is a basis for $\operatorname{im}\phi$ and $\bar{u_1}, \ldots, \bar{u_l}, \bar{v_1}, \bar{v_2}$ a basis for V. We have seen that a non-trivial commutator of weight 3 in $x, y, u_1, \ldots, u_l, v_1, v_2$ must contain all of x, v_1, v_2. It follows that $u_1, \ldots, u_l$ are all in $Z^2(G)$ and hence $\operatorname{im}\phi \subseteq \ker\phi$. This proves the claim. In particular it follows that $l = 2$ and therefore $2r = \dim(V) = 4$. So G must be of rank 6. We now analyze the structure further.

Pick $x_1, x_3 \in K$ such that $\bar{x_1}, \bar{x_3}$ is a basis for $\ker\phi$. As $\ker\phi^{\perp} = \ker\phi$ we have

$$[x_3, x_1]^3 = 1.$$

As the alternating form is non degenerate we can find $x_2 \in K$ such that

$$[x_2, x_1]^3 = z.$$

By replacing x_3 by some $x_3 x_1^{\alpha}$ if necessary we can assume that

$$[x_2, x_3]^3 = 1.$$

Now again as the alternating form is non degenerate, there is an $x_4 \in K$ such that $\bar{x_1}, \bar{x_2}, \bar{x_3}, \bar{x_4}$ is a basis for V and such that

$$[x_4, x_1]^3 = 1$$

and

$$[x_4, x_3]^3 = z.$$

By replacing x_4 by some $x_4 x_1^{\alpha}$ we can assume that

$$[x_4, x_2]^3 = 1.$$

Extend $\bar{x_1}, \bar{x_2}, \bar{x_3}, \bar{x_4}$ to a standard basis on G/G^3 by finding $y \in H \setminus K$ and $x \in G \setminus H$ such that

$$\begin{aligned} [x, y]^3 &= z \\ [x_i, y]^3 &= 1 \\ [x_i, x]^3 &= 1. \end{aligned}$$

As $[x, y], [x_2, x_1], [x_4, x_3] \in Z(G) = \langle x^9 \rangle$ and of order 9 we have (after replacing x_1, x_3, y by suitable powers if necessary) that

$$[x, y] = x^{3^s} \tag{5}$$
$$[x_2, x_1] = x^{3^s} \tag{6}$$
$$[x_4, x_3] = x^{3^s}. \tag{7}$$

We have seen previously that $[H, H] = \langle y^3, x^{3^s} \rangle$. We know that $[x_4, x_2, x] \neq 1$ but all other commutators $[x_j, x_i, x]$, $1 \leq i < j \leq 4$, are trivial. We also know that all the commutators $[x_j, x_i]$, $1 \leq i < j \leq 4$, apart from those in (6) and (7) have order dividing 3. It follows that we have the following equations (swapping the roles of x_2, x_4 if needed):

$$\begin{aligned} [x_4, x_2] &= y^3 x^{3^{s+1}\alpha(4,3)} && (8) \\ [x_j, x_i] &= x^{3^{s+1}\alpha(j,i)} \quad i < j \text{ and } \{i,j\} \neq \{1,2\},\{3,4\},\{2,4\}. && (9) \end{aligned}$$

The commutators $[x_i, y], [x_1, x], [x_3, x]$ are all in $Z(G)$ and of order dividing 3. Also $[x_2, x]^3 = [x_4, x]^3 = 1$ and

$$[x_2, x, x_4] = [x_4, x_2, x] = [y, x]^3 = x^{-3^{s+1}} = [x_3^3, x_4]$$

$$[x_4, x, x_2] = x^{3^{s+1}} = [x_1^{-3}, x_2].$$

Therefore we get

$$\begin{aligned} [x_i, y] &= x^{3^{s+1}\alpha_i} && (10) \\ [x_1, x] &= x^{3^{s+1}\beta_1} && (11) \\ [x_3, x] &= x^{3^{s+1}\beta_3} && (12) \\ [x_2, x] &= x_3^3 x^{3^{s+1}\beta_2} && (13) \\ [x_4, x] &= x_1^{-3} x^{3^{s+1}\beta_4}. && (14) \end{aligned}$$

Replacing x_1, x_2, x_3, x_4 by

$$\begin{aligned} \tilde{x_1} &= x_1 x^{-3\alpha_1} y^{3\beta_1} x_3^{-3\alpha(4,1)} \\ \tilde{x_2} &= x_2 x^{-3\alpha_2} y^{3\beta_2} x_4^{3\alpha(3,2)} \\ \tilde{x_3} &= x_3 x^{-3\alpha_3} y^{3\beta_3} x_2^{-3\alpha(3,1)} \\ \tilde{x_4} &= x_4 x^{-3\alpha_4} y^{3\beta_4} x_1^{3\alpha(4,3)} \end{aligned}$$

we get the cleaner presentation

$$\begin{aligned} [y, x] &= x^{3^s} \\ [x_2, x_1] &= x^{3^s} \\ [x_4, x_3] &= x^{3^s} \\ [x_4, x_2] &= y^3 \\ [x_4, x_1] &= 1 \end{aligned}$$

$$D(s):\quad \begin{array}{rcl} [x_3,x_2] &=& 1\\ [x_3,x_1] &=& 1\\ [x_i,y] &=& 1\\ [x_1,x] &=& 1\\ [x_3,x] &=& 1\\ [x_2,x] &=& x_3^3\\ [x_4,x] &=& x_1^{-3}\\ x^{3^{s+2}} &=& 1\\ y^9 &=& 1\\ x_i^9 &=& 1 \end{array}$$

Theorem 2.4 *The groups of type A with* $|Z^2(G)/G^3| = 27$ *form the one parameter family* $D(s)$ *of groups of rank* 6*. The group* $D(s)$ *is nilpotent of class three, has exponent* 3^{s+2} *and order* 3^{s+12}.

Proof. It follows from the relations that $D(s)$ is powerful and therefore every element in $D(s)$ can be written of the form

$$x_2^{n_2}x_4^{n_4}x_1^{n_1}x_3^{n_3}y^{n_y}x^{n_x}$$

with $0 \le n_1, n_2, n_3, n_4, n_y < 9$ and $0 \le n_x < 3^{s+2}$. It follows that $D(s)$ has at most 3^{s+12} elements. We first show that this is the exact order of $D(s)$. In order to show this we consider the set of all formal expressions

$$a_2^{n_2}a_4^{n_4}a_1^{n_1}a_3^{n_3}b^{n_b}a^{n_a}$$

where $0 \le n_1, n_2, n_3, n_4, n_y < 9$ and $0 \le n_x < 3^{s+2}$. We define a product on these formal expressions (a formula derived from the relations of $D(s)$) by setting

$$\begin{array}{rcl} a_2^{n_2}a_4^{n_4}a_1^{n_1}a_3^{n_3}b^{n_b}a^{n_a} * a_2^{m_2}a_4^{m_4}a_1^{m_1}a_3^{m_3}b^{m_b}a^{m_a} &=& a_2^{n_2+m_2}a_4^{n_4+m_4}a_1^{n_1+m_1+3n_xm_4}\\ && a_3^{n_3+m_3-3n_xm_2}b^{n_b+m_b+3n_4m_2}\\ && a^{n_a+m_a+3^sn_am_b-3^sn_1m_2-3^sn_3m_4}\\ && a^{3^{s+1}n_xm_2m_4} \end{array}$$

and where the exponents of a_1, a_2, a_3, a_4, b are calculated modulo 9 and the exponent of a is calculated modulo 3^{s+2}. Straightforward calculations show that we get a group which satisfies all the relations of $D(s)$. Therefore $D(s)$ has exactly 3^{s+12} elements. In particular x has order 3^{s+2}. As $G^{3^{s+1}} = \langle x^{3^{s+1}} \rangle$ and $[x_2, x_4, x] = x^{3^{s+1}}$, it follows that the class is 3 and the exponent is $e(G) = 3^{s+2}$. Clearly the groups in the list are pairwise non-isomorphic.

It remains to establish the minimality of $D(s)$. Let H/K be a section of $G = D(s)$ that is powerful and of class 3. Notice that $Z^2(G) = \langle x_1, x_3, y \rangle G^3$ and thus if $\gamma_3(H) \neq \{1\}$, we must have elements $g_1, g_2, g_3 \in H$ of the following form

$$\begin{aligned} g_1 &= x x_1^{r_1} x_3^{s_1} y^{t_1} \\ g_2 &= x_2 x_1^{r_2} x_3^{s_2} y^{t_2} \\ g_3 &= x_4 x_1^{r_3} x_3^{s_3} y^{t_3}. \end{aligned}$$

Now for H/K to remain of class 3, we can't have $x^{3^{s+1}} \in K$. Let us see what restrictions this makes on K. Suppose

$$g = x_2^{n_2} x_4^{n_4} x_1^{n_1} x_3^{n_3} y^{n_y} x^{n_x}$$

is in K. Firstly as $[g, g_1, g_2] = x^{3^{s+1} n_4}$, $[g, g_3, g_1] = x^{3^{s+1}} n_2$ and $g^9 = x^{9 n_x}$ are in K, it follows that 3 divides n_2, n_4 and 3^s divides n_x. Then $[g_2, g]^3 = x^{3^{s+1} n_1}$, $[g_2, g]^3 = x^{3^{s+1} n_3}$ and $[g_1, g]^3 = x^{3^{s+1} n_y}$ are in K and 3 must divide n_1, n_3, n_y. Thus any $g \in K$ must be of the form

$$g = x_1^{3m_1} x_2^{3m_2} x_3^{3m_3} x_4^{3m_4} y^{3m_y} x^{3^s m_x}$$

and $K \subseteq G^3$. Hence HG^3/G^3 must be powerful, being a quotient of H/K. As

$$\begin{aligned} [g_2, g_1] G^9 &= x_3^3 G^9 \\ [g_3, g_1] G^9 &= x_1^{-3} G^9 \\ [g_3, g_2] G^9 &= y^3 G^9, \end{aligned}$$

and HG^3/G^3 is powerful, we must have that H contains elements from $x_3 G^3$, $x_1 G^3$ and $y G^3$. By this and the fact that $g_1, g_2, g_3 \in H$, it follows that $H = G$. In particular $[g, x], [g, x_i]$ are in K which, as $x^{3^{s+1}} \notin K$, implies that 3 divides m_i, m_y and $g = x^{3^s m_x}$. Again to avoid K containing $x^{3^{s+1}}$ this forces g to be trivial. Hence $K = \{1\}$ and $H/K = G$. □

CASE 2. ϕ is bijective.

Lemma 2.5 . *If U is a proper ϕ-invariant subspace of V, then U is an isotropic subspace of V, that is $(u, v) = 0$ for all $u, v \in U$.*

Proof Suppose that the basis of U is $\bar{u_1}, \ldots, \bar{u_m}$ with $u_1, \ldots, u_m \in K$. As $[H, H] \leq \langle x^{3^s}, y^3 \rangle$ we have that the U being ϕ-invariant implies that

$$\langle x, y, u_1, \ldots, u_m \rangle$$

is powerful. By minimality of G we must have that the class is at most 2. In particular $[u_i, x, u_j] = 1$ for all $1 \leq i, j \leq m$. In other words $(\phi(u), v) = 0$ for all $u, v \in U$. But as ϕ is bijective it follows that $(u, v) = 0$ for all $u, v \in U$. □

Write V as a sum of cyclic $F[x]$-modules, say

$$V = F[x]v_1 \oplus F[x]v_2 \oplus \cdots \oplus F[x]v_r$$

where the ith summand has order $p_i^{m_i}$ for some irreducible polynomial $p_i \in F[x]$. By Lemma 2.1 we must have $r \geq 2$, since otherwise $V^{\perp} = V$. Now if $r \geq 3$ then each $F[x]v_i \oplus F[x]v_j$ is a proper submodule and thus isotropic by Lemma 2.5 but his would imply again that $V^{\perp} = V$. Hence $r = 2$. Notice that $p_1 v_1$ is contained in the proper submodules $F[x]p_1v_1 \oplus F[x]v_2$ and $F[x]v_1$ and is therefore, by Lemma 2.5, in $V^{\perp} = \{0\}$. Similarly $p_2 v_2 = 0$. So we have that $m_1 = m_2 = 1$. Finally as V can't be cyclic we must have $p_1 = p_2$.

Let $p = p_1 = p_2$ be the polynomial above. Suppose

$$p = a_0 + a_1 x + \cdots + a_{r-1}x^{r-1} + x^r.$$

Suppose the cyclic summands of V are U and W. So $V = U \oplus W$ and U and W are ϕ-invariant of dimension r. Pick a non-zero vector u in U and let $u_i = \phi^{i-1}(u)$, $1 \leq i \leq r$. Then $u_1, u_2, \ldots, u_r$ is a basis for U. From Lemma 2.1 we no that U and W are isotropic subspaces of the symplectic space V. Using standard linear algebra methods, we can pick a basis $w_1, \ldots, w_r$ for W such that V is a direct sum of pairwise orthogonal subspaces

$$V = \langle u_1, w_1 \rangle \oplus \cdots \oplus \langle u_r, w_r \rangle.$$

such that $(u_i, w_j) = \delta_{ij}$. As ϕ is self adjoint we have that the matrix for ϕ with respect to $u_1, u_2, \ldots, u_r, w_1, \ldots, w_r$ is

$$\begin{bmatrix} A & 0 \\ 0 & A^t \end{bmatrix}$$

where

$$A = \begin{bmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & 0 & & \vdots \\ & & \ddots & & \vdots \\ & & & 1 & -a_{r-1} \end{bmatrix}.$$

From this data we will now decipher a clean presentation for the group G. First chose $x_i, y_i \in K,\ 1 \leq i \leq r$ such that $u_i = x_i G^3$ and $w_i = y_i G^3$. It is not difficult to see that the structure of G/G^3 as a symplectic space and the action of ϕ on $V = KG^3/G^3$ determine all commutator relations modulo $\langle x^{3^{s+1}} \rangle$. Let us look at this in more details. Firstly the action of ϕ on V gives us the following relations

$$\begin{aligned} [x_1, x] &= x_2^3 \cdot x^{3^{s+1}\alpha_1} \\ &\vdots \\ [x_{r-1}, x] &= x_r^3 \cdot x^{3^{s+1}\alpha_{r-1}} \\ [x_r, x] &= x_1^{-3a_0} x_2^{-3a_1} \cdots x_r^{-3a_{r-1}} \cdot x^{3^{s+1}\alpha_r} \\ [y_1, x] &= y_r^{-3a_0} \cdot x^{3^{s+1}\beta_1} \\ [y_2, x] &= y_1^3 y_r^{-3a_1} \cdot x^{3^{s+1}\beta_2} \\ &\vdots \\ [y_r, x] &= y_{r-1}^3 y_r^{-3a_{r-1}} \cdot x^{3^{s+1}\beta_r}. \end{aligned} \tag{15}$$

We turn next to the orthogonality relations on G/G^3 with respect to the symplectic form. Recall that $[H, H] \leq \langle y^3, x^{3^s} \rangle$. The precise exponents of y^3 and x^{3^s} modulo $\langle x^{3^{s+1}} \rangle$ are determined by (15) (using $[a, b, x] = [b, x, a]$) and the symplectic form. Furthermore $y \in Z^2(G)$ and $[y, x]^3 = x^{3^{s+1}}$ implies that $[y, x] = x^{3^s}$ modulo $\langle x^{3^{s+1}} \rangle$. The reader can check that this leads to the following relations:

$$[y, x] = x^{3^s} \cdot x^{3^{s+1}\alpha}$$

$$\begin{array}{rcl}
[x_i, x_j] & = & x^{3^{s+1}\gamma(i,j)} \quad 1 \leq j < i \leq r \\
[y_i, y_j] & = & x^{3^{s+1}\delta(i,j)} \quad 1 \leq j < i \leq r \\
[x_i, y] & = & x^{3^{s+1}\epsilon_i} \quad 1 \leq i \leq r \\
[y_i, y] & = & x^{3^{s+1}\tau_i} \quad 1 \leq i \leq r \\
[x_i, y_i] & = & y^{3(\phi(w_i),u_i)} x^{3^s} x^{3^{s+1}\sigma(i,i)}, \quad 1 \leq i \leq r \\
[x_i, y_j] & = & y^{3(\phi(w_j),u_i)} \cdot x^{3^{s+1}\sigma(i,j)}, \quad i \neq j
\end{array} \tag{16}$$

We now make some small manipulations in order to get a cleaner presentation. Firstly, by replacing y by

$$yx_1^{3\tau_1} \cdots x_r^{3\tau_r} y_1^{-3\epsilon_1} \cdots y_r^{-3\epsilon_r}$$

we can assume that all the τ_i, ϵ_i in (16) are zero. Then replacing y with $y^{1-3\alpha}$ we can assume that α in (16) is zero as well. We next deal with the rest of the relations of (16). First let

$$s(i,j) = \left(\begin{array}{ll} \gamma(i,j) & \text{if } \ i > j \\ -\gamma(i,j) & \text{if } \ i < j \end{array} \right.$$

and

$$t(i,j) = \left(\begin{array}{ll} -\gamma(i,j) & \text{if } \ i > j \\ \gamma(i,j) & \text{if } \ i < j \end{array} \right.$$

If one replaces x_i, y_i by

$$x_i x_1^{-3\sigma(i,1)} \cdots x_r^{-3\sigma(i,r)} \cdot y_1^{3s(i,1)} \cdots y_r^{3s(i,r)}$$

and

$$y_i x_1^{3t(i,1)} \cdots x_r^{3t(i,r)} \cdot y_1^{-3\sigma(1,i)} \cdots y_r^{-3\sigma(r,i)},$$

one arrives at the presentation as (16) but with all of $\gamma(i,j), \delta(i,j), \sigma(i,j)$ zero.

Finally, replacing x by

$$xx_1^{3\beta_1} \cdots x_r^{3\beta_r} y_1^{-3\alpha_1} \cdots y_r^{-3\alpha_r}$$

we can assume furthermore that all the α's and β's in (15) are zero. We thus get the following presentation

$$E(p,s): \quad \begin{array}{rcl} [x_1, x] &=& x_2^3 \\ &\vdots& \\ [x_{r-1}, x] &=& x_r^3 \\ [x_r, x] &=& x_1^{-3a_0} x_2^{-3a_1} \cdots x_r^{-3a_{r-1}} \\ [y_1, x] &=& y_r^{-3a_0} \\ [y_2, x] &=& y_1^3 y_r^{-3a_1} \\ &\vdots& \\ [y_r, x] &=& y_{r-1}^3 y_r^{-3a_{r-1}} \\ [y, x] &=& x^{3^s} \\ [x_i, x_j] &=& 1, \quad 1 \leq j < i \leq r \\ [y_i, y_j] &=& 1, \quad 1 \leq j < i \leq r \\ [x_i, y] &=& 1, \quad 1 \leq i \leq r \\ [y_i, y] &=& 1, \quad 1 \leq i \leq r \\ [x_i, y_i] &=& y^{3(\phi(w_i), u_i)} x^{3^s}, \quad 1 \leq i \leq r \\ [x_i, y_j] &=& y^{3(\phi(w_j), u_i)}, \quad i \neq j \end{array}$$

Remark. We have that $H = \langle x_1, \cdots, x_r, y_1, \ldots, y_r, y, x^{3^s} \rangle$ and $Z(H) = \langle y, x^{3^s} \rangle$. Now x acts naturally on $H/Z(H)$ by conjugation and

$$\begin{array}{rcl} [x_1 Z(H), x] &=& x_2^3 Z(H) \\ &\vdots& \\ [x_{r-1} Z(H), x] &=& x_{r-1}^3 Z(H) \\ [x_r Z(H), x] &=& x_1^{-3\alpha_0} \cdots x_r^{-3\alpha_{r-1}} Z(H). \end{array}$$

As $[H, HG^3] \leq Z(H)$ the action remains the same if we replace x by any xu with $u \in HG^3$ and therefore induces the same self-adjoint map ϕ on V. All the other elements in G that generate G together with H are in $x^{-1}HG^3$. These give a different action that induce the map $-\phi$ on V. If p is the (monic) minimal polynomial of ϕ then $q = (-1)^{\deg(p)} p(-x)$ is the minimal polynomial of $-\phi$. In particular in the list above we have that $E(p,s)$ and $E(q,s)$ are isomorphic.

Theorem 2.6 *Let $\mathcal{P}$ be the set of all monic irreducible polynomials over the field of three elements excluding the polynomial t. The groups of type A with $|Z^2(G)/G^3| = 3$ form the two parameter family $D(p,s)$, $p \in \mathcal{P}, s \geq 2$. Suppose that p is of degree r. Then $E(p,s)$ has rank $2r+2$, class 3, exponent 3^{s+2} and order 3^{s+4r+4}. Furthermore $E(p_1,s_1)$ and $E(p_2,s_2)$ are isomorphic if and only if $s_1 = s_2$ and $p_2 \in \{p_1, (-1)^{deg\,(p)} p_1(-t)\}$.*

Proof. It follows from the relations that $E(p,s)$ is powerful and therefore every element can be written of the form

$$x^a y^b \prod x_i^{a_i} \prod y_i^{b_i}$$

with a calculated modulo 3^{s+2} and all the other exponents modulo 9. It follows that $E(p,s)$ has at most $3^{s+2(2r+2)}$ elements. We first show that this is the exact order of $E(p,s)$. In order to show this we consider the set of all formal expressions

$$X^a Y^b \prod X_i^{a_i} \prod Y_i^{b_i}$$

where a ranges between 0 and $3^{s+2}-1$ and the other exponents between 0 and 8. We define a product on these formal expressions (a formula derived from the relations of $E(p,s)$) by setting

$$X^a Y^b \prod X_i^{s_i} \prod Y_i^{b_i} * X^c Y^d \prod X_i^{c_i} \prod Y_i^{d_i} = X^u X^v \prod X_i^{u_i} \prod X_i^{v_i}$$

with

$$\begin{aligned}
u &= a + c + 3^s bc - 3^s \sum b_i c_i - 3^{s+1}(b_2 c c_1 + \cdots b_r c c_{r-1}) \\
&\quad +3^{s+1}(\alpha_0 b_1 c c_r + \cdots + \alpha_{r-1} b_r c c_r) \\
v &= b + d + 3(b_2 c_1 + \cdots + b_r c_{r-1}) - 3(\alpha_0 b_1 c_r + \cdots + \alpha_{r-1} b_r c_r) \\
u_1 &= a_1 + c_1 - 3\alpha_0 a_r c \\
u_2 &= a_2 + c_2 + 3a_1 c - 3\alpha_1 a_r c \\
&\vdots \\
u_r &= a_r + c_r + 3a_{r-1} c - 3\alpha_r c \\
v_1 &= b_1 + d_1 + 3b_2 c \\
v_{r-1} &= b_{r-1} + d_{r-1} + 3b_r c \\
v_r &= b_r + d_r - 3(\alpha_0 b_1 c + \cdots + 3\alpha_{r-1} b_r c).
\end{aligned}$$

and where the exponent of X is calculated modulo 3^{s+2} and the other exponents modulo 9.

Messy but straightforward calculations show that we get a group which satisfies all the relations of $E(p,s)$. Therefore $E(p,s)$ has exactly 3^{s+4r+4} elements. In particular x has order 3^{s+2}. As $G^{3^{s+1}} = \langle x^{3^{s+1}} \rangle$ and $[x_1, x, y_2] = x^{3^{s+1}}$, it follows that the class is 3 and the exponent is $e(G) = 3^{s+2}$. Now suppose $E(p_1, s_1)$ and $E(p_2, s_2)$ are isomorphic. As the exponent is the same we clearly must have $s_1 = s_2$ furthermore $p_2 \in \{p_1, (-1)^{\deg(p)} p_1(-t)\}$ by the remark made before the statement of the theorem.

It remains to establish the minimality of $E(p,s)$. Let R/K be a section of $G = E(p,s)$ that is powerful and of class 3. We can suppose that this is minimal. Clearly R/K must be of type A as well. And is therefore either one of the groups $D(l)$ or one of the groups $E(q,l)$. As $\gamma_3(H) = \{1\}$, $\gamma_3(G)^3 = 1$ and R has class 3, R must contain an element of the form xu with $u \in HG^3$. Suppose

$$p = \alpha_0 + \alpha_1 t + \cdots + \alpha_{r-1} t^{r-1} + t^r.$$

Let $S = H \cap R$ and pick any element in a from $S \setminus Z(H)H^3$. Pick a sequence $a_0, \ldots, a_r$ from H such that a_0 and $a_i^3 = [a_i, xu]$. As p is the minimal polynomial of ϕ we have

$$a_0^{\alpha_0} a_1^{\alpha_1} \cdots a_{r-1}^{\alpha_{r-1}} a_r \in Z(H)H^3$$

It follows that this same expression is in $Z(H)H^3K$. Hence if R/K is isomorphic to $E(q,l)$ then q must divide p. But p is irreducible and hence $q = p$. Notice also that R/K can't be isomorphic to $D(l)$ since the minimal polynomial of ϕ in that case is x^2. Hence R/K is isomorphic to some $E(p,l)$. In fact this must be $E(p,s)$ since the class of R is three. As R is nilpotent of class 3 there are some elements $h_1, h_2 \in R$ such that $[h_1, x, h_2] \neq 1$. It follows that $RZ(H)H^3/KZ(H)H^3$ as a $F(t)$ module is not cyclic (otherwise it would be isotropic). Hence $RZ(H)/KZ(H)$ has rank $2r$ and as G is powerful, R must contain elements of the form $h = xy^n, h_1 = x_1 y^{n_1}, k_1 = y_1 y^{m_1}, \cdots, h_r = x_r y^{n_r}, k_r = y_r y^{m_r}$. Let us next deduce from this that K must be trivial. Let

$$g = x^{3^s n} y^m x_1^{n_1} \cdots x_r^{n_1} y_1^{m_1} \cdots y_r^{m_r}$$

be an element of K. As R/K is of class 3, K can't contain $x^{3^{s+1}}$. Using this fact we take commutators of g with h_i^3, k_i^3 and deduce that all of n_i, m_i must be divisible by 3. Taking then commutator with h we see that 9 must divide m. Finally taking commutators with h_i, k_i we see that all of n_i, m_i must be divisible by 9 and $g = x^{3^s n}$. But as K can't contain $x^{3^{s+1}}$ this forces g to be trivial. Hence K is trivial and R is a powerful subgroup of G containing $h, h_1, k_1, \ldots, h_r, k_r$. Then $[h_1, k_2] = y^{-3}$ and as R is powerful it must contain some yv with $v \in G^3$. Hence RG^3/G^3 has rank $2r+2$ and $R = G$. This finishes the proof. $\Box$

Examples. Let us say that two polynomials q and p are equivalent if $q = p$ or $q = (-1)^{\deg(p)} p(-x)$.

1) The irreducible monic polynomials of degree 1 (apart from t) are $t-1$ and $t+1$ that are however equivalent. So in the list above, the minimal polynomials of rank 4 are

$$E(t-1,s):\quad \begin{aligned} [x_1, x] &= x_1^3 \\ [y_1, x] &= y_1^3 \\ [y, x] &= x^{3^s} \\ [x_1, y] &= 1 \\ [y_1, y] &= 1 \\ [x_1, y_1] &= y^{-3} x^{3^s} \end{aligned}$$

2) The irreducible polynomials of degree 2 belong to the following two equivalence classes $\{ t^2+1 \}$, $\{ t^2+t-1, t^2-t-1 \}$. This leads to two families of groups of rank 6 of which the first one is:

$$E(t^2+1,s):\quad \begin{aligned} [x_1, x] &= x_2^3 \\ [x_2, x] &= x_1^{-3} \\ [y_1, x] &= y_2^{-3} \\ [y_2, x] &= y_1^3 \\ [y, x] &= x^{3^s} \\ [x_2, x_1] &= 1 \end{aligned}$$

$$\begin{aligned}
[y_2, y_1] &= 1 \\
[x_1, y_1] &= x^{3^s} \\
[x_2, y_2] &= x^{3^s} \\
[x_1, y_2] &= y^{-3} \\
[x_2, y_1] &= y^3
\end{aligned}$$

In general these groups are all of even rank and as there are irreducible polynomials of arbitrary degree we get minimal examples of any even rank.

2.2 Groups of type B

We now turn to the final class of minimal examples. We will see that the search will lead us naturally to certain algebraic structures "symplectic alternating algebras" that we will explore later on. We are looking here at all the minimal examples G satisfying $[G, G]^3 = \gamma_3(G)$ and $\gamma_3(H) \neq \{1\}$ where H is the subgroup consisting of the elements of order dividing 9.

Lemma 2.7 *Every group of type B is of exponent* 27*.*

Proof Suppose $G = \langle x, H\rangle$ where x has order 3^{s+2} where as we have seen $s \geq 1$. Since G is powerful we have $[G, G] \leq \langle x^3, H^3\rangle$. But $[G, G]^9 = \gamma_3(G)^3 = 1$ and thus

$$[G, G] \leq \langle x^{3^s}, H^3\rangle.$$

If $s \geq 2$ then $\langle x^3, H\rangle$ would be powerful and nilpotent of class 3, since $\gamma_3(H) \neq \{1\}$. Hence by minimality $s = 1$ and x has order 27. $\Box$

Notice that in particular $\gamma_3(G) = [G, G]^3 = G^9 = \langle x^9\rangle$. As before we consider the alternating form

$$\Psi : HG^3/G^3 \times HG^3/G^3 \to [G, G]^3$$

that maps $(\bar{a}, \bar{b})$ to $[a, b]^3$. This is clearly well defined as $[G, G]^9 = \{1\}$. We choose a standard basis $\bar{x_1}, \bar{x_2}, \cdots, \bar{x_{2r+m}}$ with respect to this form such that

$x_i \in H$ and

$$\begin{aligned} [x_2, x_1]^3 &= x^9 \\ &\vdots \\ [x_{2r}, x_{2r-1}]^3 &= x^9 \\ [x_u, x_v]^3 &= 1 \quad \text{otherwise} \end{aligned}$$

By replacing x by a suitable $xx_1^{\alpha_1} \cdots x_{2r}^{\alpha_{2r}}$, we can assume that

$$[x, x_1]^3 = \cdots [x, x_{2r}]^3 = 1.$$

We claim that $x^3 \in Z(G)$. We argue by contradiction and suppose that $[x^3, x_{2r+1}] \neq 1$. By replacing $x_{2r+2}, \cdots, x_{2r+m}$ by a suitable $x_i x_{2r+1}^{\alpha_i}$, we get orthogonality relations on G/G^3 as follows

$$\begin{aligned} [x_2, x_1]^3 &= x^9 \\ &\vdots \\ [x_{2r}, x_{2r-1}]^3 &= x^9 \\ [x, x_{2r+1}]^3 &= x^9 \\ [x, x_u]^3 = [x_u, x_v]^3 &= 1 \quad \text{otherwise.} \end{aligned}$$

As G is powerful we get an equation of the form

$$[x, x_{2r+1}] = x^{3\alpha} x_1^{3\alpha_1} \cdots x_{2r+m}^{3\alpha_{2r+m}}.$$

Since $[x, x_{2r+1}, x_{2r+1}] = 1$, it is clear that $3|\alpha$. But then we get the contradiction that $x^9 = [x, x_{2r+1}]^3 = 1$. Thus $x^3 \in Z(G)$.

Notice that $x_{2r+1}^3, \ldots, x_{2r+m}^3 \in Z(G)$ and thus in $\langle x^9 \rangle$. It follows that the group

$$L = \langle x, x_1, \ldots, x_{2r} \rangle$$

is powerful and $\gamma_3(L) = \gamma_3(G) \neq \{1\}$, since $x_{2r+1}, \ldots, x_{2r+m} \in Z^2(G)$. By minimality $G = L$. In particular the rank of G is $2r + 1$.

As $x^3 \in Z(G)$ we have that $x \in Z^2(G)$ ($[u, v] = c^3$ implies that $[x, u, v] = [u, v, x] = [c, x]^3 = [c, x^3] = 1$). Suppose

$$\begin{aligned} [x, x_{2i}] &= [x_{2i-1}, x_{2i}]^{3\alpha_i} \\ [x, x_{2i-1}] &= [x_{2i}, x_{2i-1}]^{3\beta_i}. \end{aligned}$$

Replacing x by $xx_1^{-3\alpha_1}x_2^{-3\beta_1}\cdots x_{2r-1}^{-3\alpha_{2r-1}}x_{2r}^{-3\beta_{2r}}$ we can assume that $x \in Z(G)$. Let us summarize.

Lemma 2.8 *Let G be of type B. Then HG^3/G^3 is of even rank, $2r \geq 4$. Also $G = \langle x, H\rangle$ with $x \in Z(G)$ of order 27. Furthermore we can choose generators $x_1, \ldots, x_{2r}$ for H such that the following relations hold:*

$$\begin{aligned} [x_2, x_1]^3 &= x^9 \\ &\vdots \\ [x_{2r}, x_{2r-1}]^3 &= x^9 \\ [x_u, x_v]^3 &= 1 \quad \textit{otherwise} \end{aligned}$$

To clarify the structure further we introduce an associated linear structure that captures the powerfulness of G. Let $V = HG^3/G^3$. Then, as we have seen, V is a symplectic vector space with non-degenerate alternating form

$$(\ ,\ ) : V \times V \to [G, G]^3$$

defined by $(\bar{a}, \bar{b}) = [a, b]^3$. We give V a product as follows

$$\bar{a} \cdot \bar{b} = \bar{c} \quad \text{if} \quad [a, b]Z(G) = c^3 Z(G).$$

If $c_1^3 Z(G) = c_2^3 Z(G)$ then $(c_1 c_2^{-1}[c_1, c_2])^3 Z(G) = Z(G)$ which implies that $c_1 c_2^{-1}[c_1, c_2] \in G^3$ and thus $c_1 c_2^{-1} \in G^3$. This shows that the product is well defined. The following lemma lists some of its properties.

Lemma 2.9 *The product on V is bilinear and alternating (i.e. $uu = 0$ for all $u \in V$). Furthermore it satisfies*

$$(uv, w) = (vw, u)$$

for all $u, v, w \in V$.

Proof It is clear from the definition that the product is bilinear and alternating. Suppose $u = \bar{a}, v = \bar{b}, w = \bar{c}, uv = \bar{d}$ and $vw = \bar{e}$. Then

$$(uv, w) = [d, c]^3 = [d^3, c] = [a, b, c] = [b, c, a] = [e^3, a] = [e, a]^3 = (vw, u).$$

This finishes the proof. □

Remark. This "Jacobi like" property can also be written

$$(uv, w) = (vw, u) = (-wv, u) = (u, wv).$$

This means that any multiplication from the right by an element of V is a self adjoint linear map with respect to the alternating form.

We will now see how the symplectic alternating algebra encodes the structure of G. Let $x_1, x_2, \dots, x_{2r} \in H$ be as in Lemma 2.8. Let $u_i = \bar{x_i}$ be the element in V that corresponds to x_1. As G is powerful, we have commutator relations of the form $(i < j)$

$$[x_j, x_i] = x_1^{3\alpha_1^{ji}} \cdots x_{2r}^{3\alpha_{2r}^{ji}} x^{3\alpha^{ji}} x^{9\beta^{ji}}$$

with $\alpha_1^{ji}, \cdots, \alpha_{2r}^{ji}, \alpha^{ji}, \beta^{ji} \in \{-1, 0, 1\}$. By replacing each x_i by a suitable $x_i x_1^{3n(i,1)} \cdots x_{2r}^{3n(i,2r)}$ we can assume that all the β^{ji} are zero. Notice then that

$$\begin{aligned} \alpha_{2k-1}^{ji} &= (u_j u_i, -u_{2k}) \\ \alpha_{2k}^{ji} &= (u_j u_i, u_{2k-1}). \end{aligned}$$

Also $\alpha^{ji} = (u_j, u_i)$ which is 1 if i is odd and $j = i + 1$ and zero otherwise. In other words

$$[x_j, x_i] = x_1^{(u_j u_i, -u_2)} x_2^{(u_j u_i, u_1)} \cdots x_{2r-1}^{(u_j u_i, -u_{2r})} x_{2r}^{(u_j u_i, u_{2r-1})} x^{3(u_j, u_i)}.$$

Definition 2.10 *Let F be a field. A symplectic alternating algebra over F is a triple $(V, (\ ,\ ), \cdot)$ where V is a symplectic vector space over F with respect to a non-degenerate alternating form $(\ ,\ )$ and $\cdot$ is a bilinear and alternating binary operation on V such that*

$$(u \cdot v, w) = (v \cdot w, u)$$

for all $u, v, w \in V$.

Now let S be any symplectic alternating algebra over F of rank $2r$, where F is the field of two elements, with a standard basis $u_1, u_2, \dots, u_{2r}$. In other words $(u_{2k}, u_{2k-1}) = 1$ but $(u_j, u_i) = 0$ for all other pairs $(i, j),\ i < j$. We let $F(S)$ be the largest 2-Engel group on generators $x, x_1, x_2, \ldots, x_{2r}$ satisfying the relations

$$\text{F(S):}\quad \begin{aligned} x^{27} &= 1 \\ x_i^9 &= 1 \\ [x, x_i] &= 1 \\ [x_j, x_i] &= x_1^{3(u_ju_i,-u_2)} x_2^{3(u_ju_i,u_1)} \cdots x_{2r-1}^{3(u_ju_i,-u_{2r})} x_{2r}^{3(u_ju_i,u_{2r-1})} x^{3(u_j,u_i)}. \end{aligned}$$

Proposition 2.11 *The group $F(S)$ has rank $2r+1$, exponent 27, order 3^{2r+3} and class 3 except when S is abelian in which case the class is 2. Furthermore $F(S)$ is isomorphic to $F(T)$ if and only if S is isomorphic to T.*

Proof Let $a_1 = x_1, \cdots, a_{2r} = x_{2r}, a_{2r+1} = x_1^3, \ldots, a_{4r} = x_{2r}^3, a_{4r+1} = x, a_{4r+2} = x^3, a_{4r+3} = x^9$. From these we get the following power-commutator relations (where we have only written the non-trivial ones)

$$\begin{aligned} [a_j, a_i] &= a_{2r+1}^{(u_ju_i,-u_2)} a_{2r+2}^{(u_ju_i,u_1)} \cdots a_{4r-1}^{(u_ju_i,-u_{2r})} a_{4r}^{(u_ju_i,u_{2r-1})} a_{4r+2}^{(u_j,u_i)},\ 1 \le i < j \le r \\ [a_{j+2r}, a_i] &= a_{4r+3}^{(u_j,u_i)},\ 1 \le i, j \le r \\ a_i^3 &= a_{i+2r},\ 1 \le i \le r \\ a_{4r+1}^3 &= a_{4r+2} \\ a_{4r+2}^3 &= a_{4r+3}. \end{aligned}$$

There is a mechanical procedure to check whether this power-commutator presentation is consistent. We skip over the messy calculations but mention only that the consistency relies on the fact that $(uv, w) = (vw, u)$ and the fact that F has characteristic three and therefore $(uv, w) + (vw, u) + (wu, v) = 0$. It follows that the rank is $2r + 1$ and that the group has order 3^{2r+3}. As $[x_2, x_1] \neq 1$, the class is at least two. If S is abelian the relations imply that the class is exactly two. Suppose then that S is not abelian. This means that there are some $1 \le i, j \le r$ such that $u_iu_j \neq 0$. As the alternating form is non-degenerate it follows that $(u_iu_j, u_k) \neq 0$ for some $1 \le k \le r$. This means that $[x_i, x_j, x_k] \neq 1$. Hence the class is three in this case. It is now clear from the presentation that the exponent is 27. Finally it is clear from what we have said above that $F(S)$ is isomorphic to $F(T)$ if and only if S and T are isomorphic. □

Remark. Recall that a subgroup K of a group 3-group G is said to be powerfully embedded if $[K, G] \leq K^3$. Notice that under the correspondence above between groups and symplectic alternating algebras, powerful subgroups correspond to subalgebras and powerfully embedded subgroups correspond to ideals.

The problem of determining all symplectic alternating algebras of rank $2r$ up to isomorphism seems to be very difficult. The case $r = 3$ is already a surprisingly difficult problem. However it is easy to find all symplectic alternating algebras of rank 2 or 4. Let us deal with these here. First we prove a useful lemma.

Lemma 2.12 *Let L be any symplectic alternating algebra and $u, v \in L$. The elements*

$$u, uv, uv^2, \ldots$$

are pairwise orthogonal.

Proof We want to show that any (uv^i, uv^j) is zero. Using the "Jacobi" property every such expression can be transformed into an expression of the same form where either $j = i$ or $j = i + 1$. In the first case the symplectic product is clearly zero and for the latter, if we write $w = uv^i$ then $(wv, w) = (ww, v) = (0, v) = 0$. □

As $(uv, v) = (uv, u) = 0$ the only symplectic alternating algebra of rank 2 is the abelian one. We will now see that there are only two algebras of rank 4. Let L be of rank 4. We can then choose two elements x_1, y_1 whose product is non-zero. Clearly we can assume that these are not orthogonal and furthermore that $(x_1, y_1) = 1$. Then $x_2 = x_1 y_1$ is orthogonal to both x_1 and y_1 and therefore x_1, y_1, x_2 are linearly independent. Extend this to a standard basis x_1, y_1, x_2, y_2 where $(x_i, y_i) = 1$ and $(x_1, x_2) = (y_1, y_2) = (x_1, y_2) = (x_2, y_1) = 0$. By lemma 2.12 we have that the elements

$$x_2 x_1 = -y_1 x_1^2, \quad x_2 y_1 = x_1 y_1^2$$

are orthogonal to x_1, y_1 and x_2 and therefore multiples of x_2. Say $x_2x_1 = ax_2$ and $x_2y_1 = bx_2$. It follows from these equations that

$$u = -bx_1 + ay_1 + x_2$$

satisfies $ux_1 = ux_2 = uy_1 = 0$. Now if v is any of x_1, x_2, y_1 we have $(uy_2, v) = (uv, y_2) = (0, y_2) = 0$ and as $(uy_2, y_2) = 0$, uy_2 is in $L^\perp = \{0\}$. Therefore $u \in Z(L)$. This shows that $Z(L) \neq \{0\}$. But as L is non-abelian it follows that $Z(L)$ cannot have dimension larger than 1. (Since otherwise we would have a basis with at least two elements from the center and we could not have a non-trivial triple (uv, w)). So $Z(L)$ has dimension 1. We can then choose a new basis for L so that one of the basis vectors spans $Z(L)$. We can therefore assume that within our standard basis x_1, y_1, x_2, y_2, the vector x_2 is in the center. It follows that the only candidate for a non-trivial triple (uv, w) in the basis vectors must be for $\{u, v, w\} = \{x_1, y_1, y_2\}$. In particular $x_1y_1 \neq 0$. As x_1y_1 is orthogonal to both x_1 and y_1 as well as x_2, it follows that x_1y_1 is a multiple of x_2 and by replacing x_2 by this multiple we can assume that $x_1y_1 = x_2$. Hence $(x_1y_1, y_2) = 1$ and the structure of L is determined. One can check that L has the following presentation

$$L: \quad \begin{aligned} x_1x_2 &= 0 \\ y_1y_2 &= -y_1 \\ x_1y_1 &= x_2 \\ x_1y_2 &= -x_1 \\ x_2y_1 &= 0 \\ x_2y_2 &= 0 \end{aligned}$$

The corresponding group is $G(L) = \{x_1, y_1, x_2, y_2, x\}$ with the presentation:

$$G(L): \quad \begin{aligned} [x_1, x_2] &= 1 \\ [y_1, y_2] &= y_1^{-3} \\ [x_1, y_1] &= x_2^3x^3 \\ [x_1, y_2] &= x_1^{-3} \\ [x_2, y_1] &= 1 \\ [x_2, y_2] &= x^3 \end{aligned}$$

with furthermore x in the center of order 27 and x_1, y_1, x_2, y_2 all of order 9. Notice that in the case when F is the field of three elements, $G(L)$ must be a minimal group of type B. This is because otherwise there would be a proper section of exponent 27 that we can take to be minimal but from our classification the only minimal examples of exponent 27 must be of type B. Hence the group would have to be of odd rank $2s+1$ and order 3^{4s+3}, where $s \geq 2$. As $G(L)$ is of the smallest possible order there can't be any such proper section. We have thus found the unique example of type B that is of rank 5. In fact this turns out to be the only minimal example of type B.

Theorem 2.13 *Let L be the unique non-abelian symplectic alternating algebra over the field of three elements. The group $G(L)$ is the only minimal group of type B.*

Proof We have already seen that $G(L)$ is minimal. Now let S be any symplectic alternating algebra over F, the field of three elements, of rank $2r$ where $r \geq 3$. If S is abelian then $G(S)$ is nilpotent of class 2 and so can't be minimal. We can therefore assume that S is non-abelian and hence there exist some $x_1, y_1 \in S$ such that $x_2 = x_1 y_1 \neq 0$. It is not difficult to see that we can assume that $(x_1, y_1) = 1$. In that case x_2 is orthogonal to both x_1 and y_1 and in particular x_1, y_1, x_2 are linearly independent. Extend this to a standard basis $x_1, y_1, x_2, y_2, \ldots, x_r, y_r$ for S. Suppose that

$$\begin{aligned}
x_1 x_2 &= a_1 x_1 + a_2 x_2 + a_3 x_3 + b_3 y_3 + \cdots + a_r x_r + b_r y_r \\
y_1 y_2 &= d_1 y_1 + d_2 y_2 + c_3 x_3 + d_3 y_3 + \cdots + c_r x_r + d_r y_r \\
x_1 y_1 &= x_2 \\
x_1 y_2 &= e_1 x_1 + f_2 y_2 + e_3 x_3 + f_3 y_3 + \cdots + e_r x_r + f_r y_r \\
x_2 y_1 &= h_1 y_1 + g_2 x_2 + g_3 x_3 + h_3 y_3 + \cdots + g_r x_r + h_r y_r \\
x_2 y_2 &= l_2 x_2 + m_2 y_2 + \cdots + l_r x_r + m_r y_r
\end{aligned}$$

The corresponding group is $G(S) = \{x_1, y_1, x_2, y_2, x\}$ with a presentation including the following relations:

$$\begin{aligned}
[x_1, x_2] &= x_1^{3a_1} x_2^{3a_2} x_3^{3a_3} y_3^{3b_3} \cdots x_r^{3a_r} y_r^{3b_r} \\
[y_1, y_2] &= y_1^{3d_1} y_2^{3d_2} x_3^{3c_3} y_3^{3d_3} \cdots x_r^{3c_r} y_r^{3d_r} \\
[x_1, y_1] &= x_2^3 x^3 \\
[x_1, y_2] &= x_1^{3e_1} y_2^{3f_2} x_3^{3e_3} y_3^{3f_3} \cdots x_r^{3e_r} y_r^{3f_r} \\
[x_2, y_1] &= y_1^{3h_1} x_2^{3g_2} x_3^{3g_3} y_3^{3h_3} \cdots x_r^{3g_r} y_r^{3h_r} \\
[x_2, y_2] &= x_2^{3l_2} y_2^{3m_2} \cdots x_r^{3l_r} y_r^{3m_r} x^3.
\end{aligned}$$

Since $N = \langle x_3^3, y_3^3, \ldots, x_r^3, y_r^3 \rangle$ is an abelian subgroup of $G(S)$ whose elements commute with x_1, y_1, x_2, y_2, x, we have that the elements

$$x_1^{n_1} y_1^{m_1} x_2^{n_2} y_2^{m_2} x_3^{3n_3} y_3^{3m_3} \cdots x_r^{3n_r} y_r^{3m_r} x^n$$

with $0 \leq n_1, m_1, n_2, m_2 \leq 8$, $0 \leq n_3, m_3, \ldots, n_r, m_r \leq 2$ and $0 \leq n \leq 26$, form a subgroup of M of $G(S)$ with N a normal subgroup. By Proposition 2.11 (or the proof of it) we have that M/N has presentation

$$\begin{aligned}
[x_1, x_2] &= x_1^{3a_1} x_2^{3a_2} \\
[y_1, y_2] &= y_1^{3d_1} y_2^{3d_2} \\
[x_1, y_1] &= x_2^3 x^3 \\
[x_1, y_2] &= x_1^{3e_1} y_2^{3f_2} \\
[x_2, y_1] &= y_1^{3h_1} x_2^{3g_2} \\
[x_2, y_2] &= x_2^{3l_2} y_2^{3m_2} x^3
\end{aligned}$$

and $o(x) = 27$, $o(x_1) = o(x_2) = o(y_1) = o(y_2) = 9$. This is a presentation corresponding to the unique symplectic alternating algebra L of rank 4. This group is powerful and of class 3, hence $G(S)$ is not minimal. This finishes the proof. $\Box$

This finishes the classification of minimal powerful 2-Engel groups. The next corollary lists few consequences.

Corollary 2.14 *Let G be a powerful 2-Engel group and let H be the group of elements of order dividing 9. If G has one of the following properties then G is nilpotent of class at most 2.*

(a) G has rank at most 3.
(b) $G^9 = 1$
(c) $[G, G]^3 = 1$
(d) $G^{27} = 1$ and $\gamma_3(H) = \{1\}$.
(e) G has order at most 3^6.
(f) $\gamma_3(H) = \{1\}$ and $|G|$ divides 3^9.

Remark. One can check that none of our minimal examples answers the questions of Caranti mentioned in the introduction. However there is of course the possibility that some of the groups that correspond to the symplectic alternating algebras would provide us with a counterexample. In this case a necessary condition would be that the automorphism group of the corresponding symplectic alternating algebra is trivial.

Problem. Does there exist a (non-trivial) symplectic alternating algebra over the field of three elements that has a trivial automorphism group?

We know that no such algebras exist of rank up to 6.

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