Two generator 4-Engel groups

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Using known results on 4-Engel groups one can see that a 4-Engel group is locally nilpotent if and only if all its 3-generator subgroups are nilpotent. As a step towards settling the question whether all 4-Engel groups are locally nilpotent we show that all 2-generator 4-Engel groups are nilpotent.

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1 Introduction

In this paper we continue our study on 4-Engel groups. Before we discuss the contents of the paper we give a short overview on what is known on 4-Engel groups. The main open question concerning their structure is whether they need to be locally nilpotent. This was proved to be the case for 3-Engel groups by Heineken in 1961 [5]. For 4-Engel groups, some partial results have been obtained. We know that in a 4-Engel group the torsion elements form a subgroup which is modulo its centre a direct product of p-groups [12]. Furthermore we have that in any 4-Engel p-group G, G/R is of exponent divisible by p where R is the Hirsch-Plotkin radical of G. These results reduce the local nilpotence question for 4-Engel groups to groups that are either torsion-free or of prime exponent. All groups of exponent 2 and 3 are locally finite and Vaughan-Lee [15] has shown that 4-Engel groups of exponent 5 are also locally finite (see also [9] for further structure results). It follows from this and the structure results mentioned above that every 4-Engel $\{2, 3, 5\}$-group is locally nilpotent. A crucial fact for these results is the fact that any two conjugates in a 4-Engel group generate a nilpotent subgroup of class at most 4 [13]. That result also implies that 4-Engel groups satisfy a semigroup identity (see also [8]).

Whereas the local nilpotency problem still remains to be fully solved, we now have quite a good understanding of the structure of locally nilpotent 4-Engel groups. Every 4-Engel group that is locally nilpotent and without elements of order 2, 3 or 5 is nilpotent of class at most 7 [12]. If only the primes 2 and 5 are excluded the groups are not in general nilpotent but they are still soluble [1]. On the other hand there are examples of locally nilpotent 4-Engel 2-groups and 5-groups that are not soluble [2,11]. In [6] L. C. Kappe and W. K. Kappe proved that a group is 3-Engel if and only if the normal closure of any element is nilpotent of class at most 2. N. D. Gupta and F. Levin [4] have on the other hand constructed examples that show that the analogue for 4-Engel groups does not hold and that there is a locally nilpotent 4-Engel group with an element whose normal closure has class larger than 3. They also showed that a locally nilpotent n-Engel group does not need to be a Fitting group if $n \geq 5$. In [14] we proved however that all locally nilpotent 4-Engel groups are Fitting groups and furthermore that the normal closure of any element is nilpotent of class at most 4. It follows in particular that any nilpotent r-generated 4-Engel group is nilpotent of class at most $4r$.

So our picture of the structure of locally nilpotent 4-Engel groups is getting quite clear. Coming back to the local nilpotence question one can easily see using known results that a 4-Engel group is locally nilpotent if and only if all its 3-generator subgroups are nilpotent. Let us see why this is the case. We recall that a group H is said to be restrained [7] if

$$\langle a \rangle^{\langle b \rangle}$$

is finitely generated for all $a, b \in H$. It is not difficult to show that in a finitely generated restrained group all the terms of the derived series are finitely generated (see [7, Corollary 4]). Notice that in every 4-Engel group

$\langle a\rangle^{\langle b\rangle}$ is generated by a, a^b, a^{b^2} and a^{b^3}, so every 4-Engel group is restrained. We first prove an elementary lemma that we will also need in the next section.

Lemma 1.1 *Let G be a 4-Engel group and let R be the Hirsch-Plotkin radical of G. Then the Hirsch-Plotkin radical of G/R is trivial.*

Proof Let S/R be the Hirsch-Plotkin radical of G/R. It suffices to show that S is locally nilpotent. Let H be a finitely generated subgroup of S. Then $H/(H \cap R)$ is nilpotent and thus soluble of derived length, say n. As H is restrained we have that $H^{(n)}$ is a finitely generated subgroup of $H \cap R$ and thus nilpotent. Therefore H is soluble and then nilpotent by a well known theorem of Gruenberg [3]. $\Box$

In fact the same proof works for n-Engel groups where n is an arbitrary positive integer.

Corollary 1.2 *A 4-Engel group G is locally nilpotent if and only if all its 3-generator subgroups are nilpotent.*

Proof Let $a, b, c \in G$. As $H = \langle a, b, c\rangle$ is nilpotent, it follows from [14] that the normal closure of b in H is nilpotent of class at most 4 (this can also be read from a polycyclic presentation of the free nilpotent 3-generator 4-Engel group). Hence $[[a, b, b, b], [a, b, b, b]^c] = 1$. This holds for all $a, b, c \in G$ and it follows that $\langle [a, b, b, b]\rangle^G$ is abelian for all $a, b \in G$. Hence G/R is 3-Engel where R is the Hirsch-Plotkin radical. But 3-Engel groups are locally nilpotent [5] and thus G/R is locally nilpotent. But by Lemma 1.1 we then have that G/R is trivial. Hence $G = R$ and thus locally nilpotent. $\Box$

As a step towards solving the local nilpotence problem we prove in the next section.

Theorem 1.3 *All 2-generator 4-Engel groups are nilpotent.*

Using this result one also gets some sharpening of known structure results. The first is an immediate corollary.

Corollary 1.4 *Let G be a 4-Engel group then the torsion elements form a subgroup which is a direct product of p-groups.*

Most of this was proved in [12] but then we could only conclude that the torsion subgroup is a product of p-groups modulo its centre.

Corollary 1.5 *Any two conjugates in a* 4*-Engel group generate a nilpotent subgroup that is nilpotent of class at most* 3*.*

Using Theorem 1.3 this can be read from a polycyclic presentation of the free nilpotent 4-Engel group on two generators [10]. This sharpens the bound 4 for the nilpotency class that was obtained in [13].

2 Two generator 4-Engel groups

Let $G = \langle x, y \rangle$ be a 2-generator 4-Engel group. We want to show that G is nilpotent. If R is the Hirsch-Plotkin radical of G this is the same as proving that G/R is trivial. By Lemma 1.1 we know that the Hirsh-Plotkin radical of G/R is trivial. By replacing G by G/R we can thus without loss of generality assume that G has a trivial Hirsch-Plotkin radical. It follows by Gruenbergs Theorem that there are no normal locally-soluble subgroups. We prove in a few steps that G must then be trivial.

From [12] we know that the torsion elements of G form a subgroup that is a direct product of p-groups modulo the centre. As the locally nilpotent radical of G is trivial we conclude that this centre is trivial and thus the torsion group is a direct product of p-groups. As we mentioned in the introduction, all $\{2, 3, 5\}$-subgroups of G are locally nilpotent. As a result, we can assume that G has no element of order 2, 3 or 5. In fact we will only need to assume that G has no element of order 2. We make use of the fact that in every 4-Engel group, a subgroup generated by two conjugates is nilpotent [13]. In particular the subgroup $\langle xy^{-1}, y^{-1}x \rangle$ of G is nilpotent. The aim is to show that this subgroup is trivial. It suffices to show that the centre is trivial. Let a be an arbitrary element of the centre. Then $a^x = a^y$ and $a^{x^{-1}} = a^{y^{-1}}$. If we let $c = a^{x^{-1}}$ then $c^x = c^y = a$ and $c^{x^2} = c^{y^2}$.

Lemma 2.1 *The subgroups* $\langle x, x^c \rangle$ *and* $\langle y, y^c \rangle$ *are nilpotent of class at most* 3 *and furthermore* $[[c, x, x], [c, x]] = [[c, y, y], [c, y]] = 1$*.*

Proof From [13] we know that the subgroups $\langle x, x^c \rangle$ and $\langle y, y^c \rangle$ are nilpotent of class at most 4. We also know that any commutator in x and x^c with three occurrences of either x or x^c must be trivial. From this and the choice of the element c we have that $[[c,x,x],[c,x],[c,x]] = [[c,y,y],[c,y],[c,y]]$ and that this element commutes with x and y and is thus in the centre of G. But then it must be the identity. Expanding we get

$$\begin{aligned} 1 &= [x^{-c}x, x, x^{-c}x, x^{-c}x] \\ &= [x^{-c}, x, x^{-c}, x][x^{-c}, x, x, x^{-c}] \\ &= [x^c, x, x^c, x]^2. \end{aligned}$$

As G has no element of order 2 it follows that $\langle x, x^c\rangle$ and likewise $\langle y, y^c\rangle$ are nilpotent of class at most 3. Next consider the element $[[c,x,x],[c,x]] = [[c,y,y],[c,y]]$. From what we have just proved this element commutes with both x and y and is thus in the centre of G and must therefore be the identity. $\square$

Lemma 2.2 *The subgroups $\langle cx, xc\rangle$ and $\langle cy, yc\rangle$ are nilpotent of class at most 3. Furthermore $[x,c]^{x^{-1}}$ commutes with all elements in $\langle [x,c]\rangle^{\langle c\rangle}$.*

Proof Let $d = [xc, cx, (cx)^{-1}xc, (cx)^{-1}xc]$. As the conjugates xc, cx generate a subgroup that is nilpotent of class at most 4 we know that this element commutes with xc and cx. By Lemma 2.1 we know that $[x,c,x]$ commutes with $[x,c]$. Thus

$$\begin{aligned} d &= [cx[x,c], cx, [x,c], [x,c]] \\ &= [[x,c,cx,[x,c],[x,c]] \\ &= [[x,c,x][x,c,c]^x, [x,c], [x,c]] \\ &= [[x,c,c]^x, [x,c], [x,c]]. \end{aligned}$$

But this element is in $\langle c, c^x, c^{x^2}\rangle$ and by our choice of c it follows that

$$d = [[y,c,c]^y, [y,c], [y,c]] = [yc, cy, (cy)^{-1}yc, (cy)^{-1}yc]$$

and d commutes with xc, cx, yc and cy. Thus

$$d^x = d^y = d^{c^{-1}} \text{ and } d^{x^{-1}} = d^{y^{-1}} = d^c.$$

Then also

$$d^{x^2} = d^{c^{-1}x} = d^{xc^{-x}} = d^{yc^{-y}} = d^{c^{-1}y} = d^{y^2}.$$

As $\langle d\rangle^{\langle x\rangle}$ and $\langle d\rangle^{\langle y\rangle}$ are generated by $\langle d^{x^{-1}}, d, d^x, d^{x^2}\rangle$ and $\langle d^{y^{-1}}, d, d^y, d^{y^2}\rangle$ respectively, it follows that

$$d^{x^i} = d^{y^i}$$

for all $i \in Z$. In particular $[d, z_1, z_2, z_3, z_4] = [d, x, x, x, x] = 1$ for all $z_1, z_2, z_3, z_4 \in \langle x, y\rangle$ and d is in the 4-th center of G. Hence $d = 1$. As in the proof of Lemma 2.1, we use the fact that every commutator in the conjugates xc and cx with 3 occurrences of either xc or cx is trivial. It follows that

$$1 = d = [xc, cx, (cx)^{-1}, xc][xc, cx, xc, (cx)^{-1}] = [xc, cx, cx, xc]^{-2},$$

and as there are no elements of order 2 we conclude that $\langle xc, cx\rangle$ and likewise $\langle yc, cy\rangle$ are nilpotent of class at most 3. Next consider the element

$$\begin{aligned} e &= [xc, cx, (cx)^{-1}xc] \\ &= [x, c, cx, [c, x]] \\ &= [[x, c, x][x, c, c]^x, [c, x]] \\ &= [[x, c, c]^x, [c, x]] \\ &= [[y, c, c]^y, [c, y]] \\ &= [yc, cy, (cy)^{-1}yc]. \end{aligned}$$

By what we have just seen, this element commutes with xc, cx, yc, cy and the same argument as we used previously shows that $e = 1$. So $[x, c]$ commutes with $[x, c, c]^x$, that is to say $[x, c]^{-x}[x, c]^{cx}$ and by Lemma 2.1, $[x, c]$ commutes then with $[x, c]^{cx}$ or equivalently $[x, c]^c$ commutes with $[x, c]^{x^{-1}}$. Replacing c by c^{-1} it follows that $[x, c^{-1}]^{c^{-1}}$ commutes with $[x, c^{-1}]^{x^{-1}}$ or equivalently $[c, x]^{c^{-2}}$ commutes with $[c, x]^{c^{-1}x^{-1}}$. Conjugating by cx we get that $[c, x]^{c^{-1}x}$ commutes with $[c, x]^{[c,x]} = [c, x]$ and $[x, c]^{x^{-1}}$ commutes with $[x, c]^{c^{-1}}$. From this and Lemma 2.1 we now know that $[x, c]^{x^{-1}}$ commutes with $[x, c]$, $[x, c]^c$ and $[x, c]^{c^{-1}}$. Let $u = [c, x]^{c^{-1}}$ and $v = [c, x]^{x^{-1}}$. Then v commutes with u, $[u, c]$ and $[u, c, c]$ and thus

$$\begin{aligned} 1 &= [u, vc, vc, vc, vc] \\ &= [u, c, vc, vc, vc] \\ &= [u, c, c, vc, vc] \\ &= [u, c, c, c, vc] \\ &= [u, c, c, c, v]^c. \end{aligned}$$

As $\langle [x,c]\rangle^{\langle c\rangle}$ is generated by u, u^c, u^{c^2} and u^{c^3}, this concludes the proof of the Lemma. $\Box$

Lemma 2.3 *If* $u \in Z(\langle c, c^x, c^{x^2}\rangle)$ *then* $u = 1$.

First notice that

$$\begin{aligned} 1 &= [c, ux, ux, ux, ux] = [c, x, ux, ux, ux] = [c, x, x, ux, ux] \\ &= [c, x, x, x, ux] = [c, x, x, x, u]^x \end{aligned}$$

and u commutes with c^{x^3} and as $c^{\langle x\rangle}$ is generated by c, c^x, c^{x^2} and c^{x^3}, it follows that u commutes with c^{x^i} for all $i \in \mathbb{Z}$. As $c^x = c^y$ and $c^{x^2} = c^{y^2}$ it follows from the symmetry of the role of x and y that u commutes with c^{y^i} for all $i \in \mathbb{Z}$. We next show by induction on r that $u^{z_1 \cdots z_n}$ commutes with all elements in $\langle c\rangle^{\langle x\rangle}$ and $\langle c\rangle^{\langle y\rangle}$, where $z_1, \ldots, z_n \in \{x, y, x^{-1}, y^{-1}\}$. We have already dealt with the case $n = 0$. Suppose now that $n \geq 1$ and that the result holds for all smaller integers. Let $v = [u, z_1, \ldots, z_{n-1}]$. Then using the induction hypothesis we have

$$\begin{aligned} [v, x]^c &= [v, x[x, c]] \\ &= [v, x]^{[x,c]} \\ &= [v, x[x, [x, c]]] \\ &= [v, x]^{[x,[x,c]]} \\ &\vdots \\ &= [v, x]^{[x,[x,[x,[x,c]]]]} \\ &= [v, x]. \end{aligned}$$

So $[v, x]$ commutes in particular with c, c^x, c^{x^2} which, as we have seen, implies that $[v, x]$ commutes with all elements in $\langle c\rangle^{\langle x\rangle}$ and $\langle c\rangle^{\langle y\rangle}$. As $[v, x] = u^{z_1 \cdots z_{n-1} x}$ modulo conjugates of lower weights, it follows that our hypothesis holds for $z_n = x$. The proof for $z_n = x^{-1}$, y or y^{-1} is similar and the induction hypothesis holds.

In particular we have that u commutes with $c^{z_1 \cdots z_n}$ for all $z_1, \ldots, z_n \in \{x, y, x^{-1}, y^{-1}\}$ and thus u is in the centre of $\langle c\rangle^{\langle x,y\rangle}$. Hence the normal closure of u in G is abelian and thus $u = 1$, as G has no proper normal soluble subgroup. $\Box$

Lemma 2.4 *The subgroup $\langle c, c^x\rangle$ is nilpotent of class at most 3. Furthermore $[x, c, c, c] = 1$.*

Proof By Lemma 2.3 we have that $[x, c, c, c]^x$ commutes with $[x, c]$ and thus

$$[x, c, c, c]^{xc} = [x, c, c, c]^{cx[x,c]} = [x, c, c, c]^{x[x,c]} = [x, c, c, c]^x$$

and c commutes with $[x, c, c, c]^x = [c^{-x^2}, c^x, c^x]$. Replacing c by c^{-1} we see that c commutes also with $[c^{x^2}, c^x, c^x]$. As

$$[c^{-x^2}, c^x, c^x] = [c^{x^2}, c^x, c^x]^{-1}[c^{x^2}, c^x, c^{x^2}, c^x],$$

it follows that $[c^{x^2}, c^x, c^{x^2}, c^x]$ commutes with c. As this element commutes also with c^x and c^{x^2}, it follows from Lemma 2.3 that $[c^{x^2}, c^x, c^x, c^{x^2}] = 1$. Hence the subgroup $\langle c^x, c^{x^2}\rangle$ is nilpotent of class at most 3 and then of course the same is true for the subgroup $\langle c, c^x\rangle$. We have therefore seen that the element $[c^{x^2}, c^x, c^x]$ commutes with c and c^x and c^{x^2} and again Lemma 2.3 implies that $[c^{x^2}, c^x, c^x] = 1$ and conjugation by x^{-1} gives the second statement of the Lemma. $\square$

Lemma 2.5 *The element c is trivial.*

Proof From Lemma 2.3 and Lemma 2.4 we know that $[x, c, c]$ commutes with c and that $[x, c, c]^x$ commutes with $[x, c]$. Thus

$$[x, c, c]^{xc} = [x, c, c]^{cx[x,c]} = [x, c, c]^{x[x,c]} = [x, c, c]^x,$$

and c commutes with

$$[x, c, c]^x = [c^{-x^2}c^x, c^x] = [c^{-x^2}, c^x][c^{-x^2}, c^x, c^x] = [c^{-x^2}, c^x].$$

Replacing c by c^{-1} we see that c commutes with $[c^{x^2}, c^x]^{-1}$. As

$$[c^{-x^2}, c^x] = [c^{x^2}, c^x]^{-1}[c^{x^2}, c^x, c^{x^2}]$$

it follows that $[c^{x^2}, c^x, c^{x^2}]$ commutes with c and by Lemma 2.4 it commutes also with c^x and c^{x^2}. By Lemma 2.3 this element is then trivial. Thus $[c^{x^2}, c^x]$ commutes with c^{x^2} and c but by the second part of Lemma 2.4 it also commutes with c^x. It follows again from Lemma 2.3 that $[c^{x^2}, c^x] = 1$ and conjugation by x^{-1} gives that $[x, c, c] = 1$. From Lemma 2.1 we have also that $[c, x]$ commutes with $[c, x, x]$. So $[c, x]$ commutes with c, c^x and c^{x^2} and Lemma 2.3 implies that $[c, x] = 1$. As $[c, x] = [c, y]$ this implies that c is in the centre of G and thus trivial. □

Proof of theorem 1.3 From the previous lemmas we have seen that the centre of $\langle xy^{-1}, y^{-1}x\rangle$ is trivial. But this group is nilpotent because it is generated by two conjugates. It follows that it must also be trivial. Thus $x = y$ and G is cyclic and thus trivial as the Hirsch-Plotkin radical of G is trivial. □

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