Milnor groups and (virtual) nilpotence

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1 Introduction

We begin by considering three well-known results. The first two give equivalent ways of stating the positive answer to the restricted Burnside problem [24],[25], in the general case and for nilpotent groups.

Theorem A. *Let n be a positive integer.*

(1) For each positive integer r there exists a largest finite r-generator group of exponent n.
(2) Every finitely generated residually finite group of exponent n is finite.
(3) The class of locally finite groups of exponent n is a variety.

Theorem B. *Let n be a positive integer.*

(1) For each positive integer r there exists a largest r-generator nilpotent group of exponent n.
(2) Every finitely generated residually nilpotent group of exponent n is nilpotent.
(3) The class of locally nilpotent groups of exponent n is a variety.

The third result on the other hand is a consequence from [23] of Zel'manov's global nilpotence theorem for n-Engel Lie algebras over a field of characteristic zero [22].

Theorem C. *Every torsion-free locally nilpotent n-Engel group is nilpo-*

tent.

Other authors have proved similar results for other varieties, also using the solution to the restricted Burnside problem. A theorem of Wilson [21] shows that Theorem B remains true if the Burnside variety is replaced by the variety of n-Engel groups. A short proof of this result can be found in [3]. Shalev [19] has also shown that an analog of Theorem A is true if one replaces the Burnside variety by any variety satisfying a positive law; here 'finite' is replaced by 'nilpotent-by-finite'. The question of characterizing the varieties for which analogs of Theorems A, B and C hold has been studied in [2] ,[4], [6]-[9]. In [4], [7] one has a characterization for varieties of type A, in [7], [8] for varieties of type B and in [6] varieties of type C.

The aim of this paper is to develop a theory unifying and generalizing the results discussed above. We will work in a more general setting than that of varieties. For each variety $\mathcal{M}$ of metabelian groups then let $\mathcal{C}_{\mathcal{M}}$ be the class of groups all of whose metabelian sections belong to $\mathcal{M}$. The main results of this paper, Theorems 3.23, 4.5 and 4.9, characterize those classes $\mathcal{C}_{\mathcal{M}}$ that are of type A, B or C. Replacing $\mathcal{C}_{\mathcal{M}}$ by a variety $\mathcal{V}$ whose metabelian groups form the variety $\mathcal{M}$, we obtain a characterization for varieties. These results, Theorem 3.24, 4.6 and 4.10, are different from those found in [4], [6], [7], [8] and stronger in that all bounds depend only on the metabelian groups in $\mathcal{V}$.

In order to prove our theorems we will introduce a class of generalized Engel groups. For each polynomial f over the integers we will introduce a class of f-Milnor groups. Our choice of name comes from a paper of F. Point [16], who introduced the notion of Milnor identities. She singled out from a paper of Milnor [15] a property for a group G, namely that the normal closure of a in $\langle a, b\rangle$ is finitely generated for all $a, b \in G$. Groups with this property have been called restrained by other authors.

In Section 2 we give a general introduction to Milnor groups. We will prove the main results for classes of type A in Section 3 and for classes of type B and C in Section 4.

In our generalized setting we are often able to use methods and arguments of other authors and this will be clearly indicated when it happens.

2 Milnor groups

Let G be any group. For $a, t \in G$ we let

$$A(a,t) := \langle a\rangle^{\langle t\rangle}/(\langle a\rangle^{\langle t\rangle})'$$

where $\langle a\rangle^{\langle t\rangle}$ is the normal closure of $\langle a\rangle$ in $\langle a, t\rangle$. Then $A(a,t)$ is an abelian section of G. Let $E(a,t)$ be the ring of all endomorphisms of $A(a,t)$. Notice that t induces an endomorphism on $A(a,t)$ by conjugation.

Definition 2.1 *Let $f(x) \in \mathbb{Z}[x]$. We say that a group G is f-Milnor if $a^{f(t)} = 1$ in $A(a,t)$ for all $a, t \in G$.*

More generally let $\mathcal{F}$ be any set of non-zero polynomials of $\mathbb{Z}[x]$. We say that a group G is $\mathcal{F}$-Milnor if for all $a, t \in G$ there exists an $f \in \mathcal{F}$ such that $a^{f(t)} = 1$ in $A(a,t)$.

Notice that if $\mathcal{F}$ is finite, say $\mathcal{F} = \{f_1, \ldots, f_n\}$ then every $\mathcal{F}$-Milnor group is a $(f_1 \cdots f_n)$-Milnor group.

Example. Let G be an n-Engel group. Calculating in $A(a,t)$, we get

$$1 = [a,_n t] = a^{(-1+t)^n},$$

so G is an e-Milnor group for $e = (x-1)^n$. Notice also that every Engel group is an $\mathcal{E}$-Milnor group where $\mathcal{E} = \{(x-1)^n : n \geq 0\}$.

Let $\mathcal{M}_f$ be the class of all f-Milnor groups. Notice that $\mathcal{M}_f$ is closed under taking subgroups and quotients.

Some classical results on Engel groups generalize easily to $\mathcal{E}$-Milnor groups. In particular, every $\mathcal{E}$-Milnor group that is either finite or finitely generated soluble is nilpotent. These statements generalize well-known theorems of Gruenberg [10] and Zorn [26] and we omit the easy proofs. However, whereas all Engel groups satisfying the maximal condition are nilpotent, this is not true for $\mathcal{E}$-Milnor groups, since any Tarsky monster is a $(x-1)$-Milnor group that satisfies the maximal condition.

3 Virtual nilpotence

Example. Let C be the infinite cyclic group and C_p be the cyclic group of order p for some prime p. The standard wreath product $C_p \operatorname{wr} C$ is an

$f(x-1)$-Milnor group for all polynomials f divisible by p. Notice that $C_p \,\mathrm{wr}\, C$ is residually a finite p-group but it is not nilpotent-by-(finite exponent).

We are interested in criteria for an f-Milnor group to be nilpotent-by-(finite exponent). In view of the example above we will assume in the rest of this section that the coefficients of f have no common prime divisor. Our aim is to show that there exist positive integers $e(f)$ and $c(f)$ such that every finite f-Milnor group G satisfies

$$\gamma_{c(f)+1}(G^{e(f)}) = 1.$$

We will in fact show that the same conclusion holds for all virtually soluble f-Milnor groups. We deal with finite soluble groups in Sections 3.1 and 3.2. In Section 3.3, 3.4 we extend the result to all virtually soluble f-Milnor groups. In Section 3.5 we will use this result to obtain the main characterization result, Theorem 3.23.

Our approach in Sections 3.1-3.3 follows in outline Shalev [19]. For the reduction from finite groups to finite soluble groups in Section 3.3 we cannot, like Shalev, use directly a result of Jones [12] that any proper subvariety of the variety of all groups, contains only finitely many finite non-abelian simple groups. Instead we describe a variant argument appropriate for our situation.

3.1 Finite soluble groups in $\mathcal{M}_f$

Our first aim is to prove that there exists constants $c(f)$ and $e_4(f)$ such that any finite soluble f-Milnor group is an extension of a nilpotent group of class at most $c(f)$ by a group of exponent dividing $e_4(f)$. In this section we reduce this problem to the class of nilpotent f-Milnor groups. Our approach follows closely that of [19]. We let f be a fixed polynomial in $\mathbb{Z}[x]$ of degree m whose coefficients have greatest common divisor 1. Let

$$I = \mathbb{Z}[x]f(x) + \mathbb{Z}[x]f(x^2) + \ldots + \mathbb{Z}[x]f(x^{m+1}).$$

and let I_p be the natural image of I in $\mathrm{GF}(p)[x]$. Notice that $I_p \neq 0$ as the coefficients of f have no common prime divisor.

Lemma 3.1 $[x(x^{m!}-1)]^m \in I_p$.

Proof Let $\bar{f}(x), \ldots, \bar{f}(x^{m+1})$ be the images of $f(x), \ldots, f(x^{m+1})$ in I_p. The greatest common divisor g of $\bar{f}(x), \ldots, \bar{f}(x^{m+1})$ has degree at most m and $I_p = \mathrm{GF}(p)[x]g$. Suppose that g has prime factorization $g = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ for distinct prime polynomials $p_1, \ldots, p_r$. Let $J_i = \mathrm{GF}(p)[x]p_i$ then $J_1, \ldots, J_r$ are maximal ideals of $\mathrm{GF}(p)[x]$ with intersection $J = \mathrm{GF}(p)[x]p_1 \cdots p_r$. Clearly $J^m \leq I_p$. It thus suffices to show that $(x^{m!} - 1)x \in J_i$ for all i. So we work in the field $F_i = \mathrm{GF}(p)[x]/J_i$. Let u be the image of x in F_i. Then

$$f(u) = f(u^2) = \ldots = f(u^{m+1}) = 0.$$

But as f has degree m, two of these powers of u must be equal. If $u^s = u^t$ with $s < t$ then either $u = 0$ or $u^{t-s} = 1$. Hence $(u^{m!} - 1)u = 0$. □

Corollary 3.2 *Suppose $G = \langle V, t\rangle$ where V is an elementary abelian normal p-subgroup of G. If G is an f-Milnor group then $(t^{m!} - 1)^m = 0$ in End(V).*

Proof As $G \in \mathcal{M}_f$, we have that $f(t) = 0, f(t^2) = 0, \ldots, f(t^{m+1}) = 0$ in $\mathrm{End}(V)$. By Lemma 3.1 we have then that $(t^{m!} - 1)^m t^m = 0$. But as t is invertible we conclude that $(t^{m!} - 1)^m = 0$. □

Lemma 3.3 *Let G be a finite soluble f-Milnor group and let N be a minimal normal subgroup of G. Then there exist an integer $e_1 = e_1(m)$ such that $t^{e_1} = 1$ in End(N) for all $t \in G$.*

Proof Since N is an elementary abelian p-group for some prime p we can apply Corollary 3.2 to deduce that $(t^{m!} - 1)^m = 0$ in $\mathrm{End}(N)$. Suppose that t has order s. If s is coprime to p then, as $\langle N, t^{m!}\rangle$ is nilpotent, $t^{m!}$ must centralize N and thus $t^{m!} = 1$ in $\mathrm{End}(N)$.

Next consider the case when $s = p^r$. Then $H = G/C_G(N)$ is a soluble linear group over the finite field of order p. Let $R/C_G(N)$ be a normal p-subgroup of H. Then $R = PC_G(N)$ where P is the Sylow p-subgroup of R. Now $[N, R] = [N, NP]$ is a proper subgroup of N because NP is nilpotent. As N is a minimal normal subgroup in G it follows that $[N, R] = 1$ and $R \leq C_G(N)$. Thus H has no non-trivial normal p-subgroups. It now follows from Theorem B of Hall and Higman [11] that the minimal polynomial of t in $\mathrm{End}(V)$ has degree at least $p^{r-1}(p-1)$. Hence $m \cdot m! \geq (p-1)p^{r-1} \geq s/2$ and thus $s \leq 2m \cdot m!$. This shows that in this case $t^{(2m!m)!} = 1$ in $\mathrm{End}(N)$.

In the general case $s = ap^r$ for some integer a coprime to p. By the discussion above, both t^{p^r} and t^a have order dividing $e_1(m) = (2m \cdot m!)!$ and thus so has t. □

Proposition 3.4 *Let G be a finite soluble f-Milnor group. Then $G^{e_1(m)}$ is nilpotent where $e_1(m) = (2m! \cdot m)!$.*

Proof By Lemma 3.3 the subgroup $G^{e_1(m)}$ centralizes all chief factors of G and so lies in the Fitting subgroup. $\square$

Remark Notice that $e_1(m)$ only depended on the degree of f.

3.2 Finite nilpotent groups in $\mathcal{M}_f$

For a prime p we let $r = r(p, m)$ be the integer satisfying

$$p^{r-1} < m \leq p^r.$$

Recall that m is the degree of f.

Lemma 3.5 *Let G be a finite f-Milnor p-group. Then $G^{p^{r(p,m)}}$ centralizes every normal elementary abelian section $V = H/K$ with H, K normal in G.*

Proof Suppose that $f(x) = e_n(x-1)^n + \cdots + e_m(x-1)^m$ with $e_n \neq 0$. Let s be the smallest integer such that p does not divide e_s. Then $(t-1)^s = 0$ in $\mathrm{End}(V)$ for all $t \in G$ as G is nilpotent. Hence, as $s \leq p^{r(p,m)}$,

$$t^{p^{r(p,m)}} - 1 = (t-1)^{p^{r(p,m)}} = 0$$

in $\mathrm{End}(V)$. $\square$

Corollary 3.6 *Let G be a finite f-Milnor p-group. Then $G^{p^{r(p,m)}}$ is powerful if p is odd and $G^{p^{r(p,m)+1}}$ is powerful if $p = 2$.*

Proof This follows directly from Lemma 3.5 and a lemma of [18]. $\square$

We refer to [14] for the definition and properties of powerful p-groups. For our next lemma we will need a proposition of Shalev ([19], Proposition D) that is a consequence of Zel'manov's work on the restricted Burnside problem.

Proposition 3.7 *(Shalev) Let G be a d-generated nilpotent group, and let $H \trianglelefteq G$. Suppose that*

$$[h,_k g] = 1$$

for all $h \in H$ and $g \in G$. Then

$$[H,_b G] = 1$$

for some $b = b(k, d)$ depending only on k and d.

Lemma 3.8 *Suppose that G is a d-generator nilpotent $(x-1)^k$-Milnor group. Then G is nilpotent of (k,d)-bounded class.*

Proof Suppose that G has class c. Then the $[(c+2)/2]$-th term A of the lower central series is abelian. As G is a $(x-1)^k$-Milnor group, $[a,_k g]=1$ for all $a \in A$ and all $g \in G$. By Proposition 3.7 we can conclude that G has class at most $(c+2)/2+b(k,d)$. Hence $c \leq (c+2)/2+b(k,d)$ which gives that $c \leq 2+2b(k,d)$. $\square$

It follows from this that any finitely generated residually nilpotent $(x-1)^k$-Milnor group is nilpotent. This generalizes Wilson's result on k-Engel groups. Next we need the following immediate corollary of Theorem C.

Lemma 3.9 *(Zel'manov) For each integer r there exist a constant $l_1(r)$ such that any finite r-Engel p-group is nilpotent of r-bounded class if $p > l_1(r)$.*

The following proposition holds for any non-zero polynomial $f \in \mathbb{Z}[x]$; we do not need the assumption that the coefficients have greatest common divisor 1.

Proposition 3.10 *There exists an integer $l(f)$ such that any finite f-Milnor p-group with $p > l(f)$ is nilpotent of m-bounded class.*

Proof Recall that $f = e_n(x-1)^n + \cdots + e_m(x-1)^m$ where $e_n \neq 0$. Choose a prime p that does not divide e_n. Let G be a finite p-group in $\mathcal{M}_f$. Then G is a $(x-1)^n$-Milnor group. By Lemma 3.8 we have that G is b-Engel where b is n-bounded. By Lemma 3.9, G is nilpotent of n-bounded class if p is greater than $l(f) := \max\{e_n, l_1(b(n))\}$. $\square$

Now let $e_2(f)$ be the product of $2^{r(2,m)+1}$ and all prime powers $p^{r(p,m)}$ with $3 \leq p \leq l(f)$. It follows from Corollary 3.6 and Proposition 3.10 that for any finite nilpotent f-Milnor group G the subgroup $G^{e_2(f)}$ is a direct product of p-groups where the factors with $p > l(f)$ are nilpotent of m-bounded class and the rest of the factors are powerful. We can therefore concentrate on powerful p-groups. We first need a lemma similar to lemma 3.1. Its statement and proof closely resemble those of [17, Lemma 3.3].

Lemma 3.11 *Let I be the ideal in $\mathbb{Z}[x]$ generated by $f(x), f(x^2), \ldots, f(x^{m+1})$. Then $qx^m(x^{m!}-1)^m \in I$ for some positive integer $q = q(f)$.*

Proof Let g be the greatest common divisor of $f(x), \dots, f(x^{m+1})$ in $\mathbb{Q}[x]$. Then g has degree at most m and $\mathbb{Q}I = \mathbb{Q}[x]g$. As in the proof of Lemma 3.1 we see that $(x^{m!}-1)^m x^m$ is in $\mathbb{Q}I$ and thus $q(x^{m!}-1)^m x^m$ is in I for some positive integer q. $\square$

Lemma 3.12 *Let G be a powerful f-Milnor p-group. Then $G^{m!}$ is nilpotent of f-bounded class.*

Proof Let p^s be the largest power of p that divides q from Lemma 3.11. Then G is a $p^s(x^{m!}-1)^m$-Milnor group. Let $H = \langle A, t\rangle$ be a section of G with A an abelian normal subgroup of H. Consider the quotients

$$A/pA,\ pA/p^2A, \dots, p^{s-1}A/p^sA.$$

By Lemma 3.5 the polynomial $(t-1)^{p^{r(p,m)}}$ maps all these quotients to zero. Hence $(t-1)^{sp^{r(p,m)}}$ maps A/p^sA to 0. But $(t^{m!}-1)^m$ maps p^sA to zero, and so G is a k-Milnor group where $k = (x-1)^{sp^{r(p,m)}}(x^{m!}-1)^m$.

We use this to show that $N = G^{m!}$ is nilpotent of f-bounded class. As G is powerful we have $N = \{g^{m!} : g \in G\}$. From the previous paragraph, N is a $(x-1)^{sp^{r(p,m)}+m}$-Milnor group. It follows from Lemma 3.8 that N is an $b(f,p)$-Engel group for some integer $b(f,p)$. Now N is powerful and so nilpotent of (f,p)-bounded class by [1, Proposition 2.2]. But by Proposition 3.10 we only need to consider finitely many primes p and thus N is nilpotent of f-bounded class, as required. $\square$

As we remarked before almost all Sylow p-subgroups of $G^{e_2(f)}$ are nilpotent of f-bounded class and the remaining ones are powerful. Let $e_3(m) = m!$. It follows from Lemma 3.12 that $G^{e_3(m)e_2(f)}$ is nilpotent of f-bounded class for each finite nilpotent f-Milnor group. But by Proposition 3.4 the subgroup $G^{e_1(m)}$ is nilpotent for each finite soluble f-Milnor group G. Hence writing $e_4(f) = e_1(m)e_2(f)e_3(m)$ where $m = \deg f$, we obtain

Proposition 3.13 *There exists a positive integer $c(f)$ such that*

$$\gamma_{c(f)+1}(G^{e_4(f)}) = 1$$

for all finite soluble f-Milnor groups.

3.3 Finite f-Milnor groups

We shall extend the result of the previous section to include all finite f-Milnor groups. We first show that there are only finitely many non-abelian finite simple f-Milnor groups. Our approach is based on G. Jones [12].

Lemma 3.14 *There are finitely many alternating groups in $\mathcal{M}_f$.*

Proof Otherwise $\mathcal{M}_f$ contains all alternating groups, as $\mathcal{M}_f$ is closed under taking subgroups. But as any finite group can be embedded into an alternating group, $\mathcal{M}_f$ then contains all finite groups. However, if p is any prime not dividing f and $n > \deg f$ then C_p wr C_n is not in $\mathcal{M}_f$. $\square$

Lemma 3.15 *$PSL(2,q) \in \mathcal{M}_f$ for only finitely many prime powers q.*

Proof Let A be the subgroup of $\mathrm{SL}(2,q)$ consisting of the matrices

$$a(\alpha) = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix},\ \alpha \in \mathrm{GF}(q)$$

and let $t = \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}$, where x is a primitive $(q-1)$-th root of unity in $\mathrm{GF}(q)$. Let $G = \langle A, t\rangle$. Now

$$[a(\alpha), t^r] = a(\alpha x^{-2r} - \alpha),$$

and so $[a(\alpha), t^r] \in Z(\mathrm{SL}(2,q))$ if and only if $x^{2r} = 1$. Let $\bar{G}, \bar{A}$ and $\langle \bar{t}\rangle$ be the images of G, A and $\langle t\rangle$ in $\mathrm{PSL}(2,q)$. We have $|\bar{A}| = q$ and $|\bar{t}|$ divides $q-1$, and so $|\bar{A}|$,$|\bar{t}|$ are coprime. It follows that $\langle \bar{A}, \bar{t}^r\rangle$ is nilpotent if and only if $\bar{t}^r$ centralizes $\bar{A}$ which happens if and only if $x^{2r} = 1$. This shows that $\bar{G}/\mathrm{Fit}(\bar{G})$ has exponent $(q-1)/2$ if q is odd and $q-1$ if q is even. But if $\bar{G} \in \mathcal{M}_f$ then, as $\bar{G}$ is soluble, the exponent of $\bar{G}/\mathrm{Fit}(\bar{G})$ must divide $e_4(f)$. Hence $\mathrm{PSL}(2,q) \in \mathcal{M}_f$ for finitely many q. $\square$

Lemma 3.16 *There are only finitely many Suzuki groups $Sz(q)$ in $\mathcal{M}_f$.*

Proof Let $q = 2^{2n+1}$ and let θ be the field automorphism of $\mathrm{GF}(q)$ that maps x to $x^{2^{n+1}}$. According to [20] the group $Sz(q)$ can be described as the subgroup of $\mathrm{GL}(4,q)$ generated by the matrices

$$a(\alpha,\beta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ \alpha^{1+\theta} + \beta & \alpha^{\theta} & 1 & 0 \\ \alpha^{2+\theta} + \alpha\beta + \beta^{\theta} & \beta & \alpha & 1 \end{bmatrix},$$

$$t(\gamma) = \begin{bmatrix} \gamma^{1+\theta} & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma^{-1} & 0 \\ 0 & 0 & 0 & \gamma^{-1-\theta} \end{bmatrix}$$

and

$$x = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

with $\alpha, \beta, \gamma \in \mathrm{GF}(q)$ where $\gamma \neq 0$. The set $A = \{a(\alpha, \beta) |\ \alpha, \beta \in \mathrm{GF}(q)\}$ is a subgroup. Let $t = t(\gamma)$ where γ is a primitive $(q-1)$-th root of unity and let $G = \langle A, t\rangle$. This is a semidirect product of A by $\langle t\rangle$. As $t^{-r}a(\alpha, \beta)t^r = a(\alpha\gamma^r, \beta\gamma^{r(1+\theta)})$, the elements t^r, $a(\alpha, \beta)$ commute if and only if $\gamma^r = 1$. As $|A| = q^2$ and $|t| = q-1$ are coprime, $\langle A, t^r\rangle$ is nilpotent if and only if $t^r = 1$. Hence the exponent of $G/\mathrm{Fit}(G)$ is $q-1$. But if $G \in \mathcal{M}_f$, the exponent of $G/\mathrm{Fit}(G)$ must divide $e_4(f)$. Hence there are only finitely many groups $Sz(q)$ in $\mathcal{M}_f$. □

Proposition 3.17 *There are finitely many finite non-abelian simple groups in* $\mathcal{M}_f$*.*

Proof According to the classification of the finite simple groups there are, apart from the alternating groups and the 26 sporadic simple groups, 16 infinite families of simple groups of Lie type. We have already dealt with the alternating groups and the Suzuki groups. The remaining 15 families are the Chevalley groups

$$A_l(q), B_l(q), C_l(q), D_l(q), G_2(q), F_4(q), E_6(q), E_7(q), E_8(q)$$

and six families of twisted simple groups of Lie type. The Steinberg groups ${}^2A_l(q),{}^2D_l(q),{}^2E_6(q),{}^3D_4(q)$ and the Ree groups ${}^2F_4(2^{2m+1}),{}^2G_2(3^{2m+1})$. We refer to [5] for details. To deal with these 15 remaining families we apply the work of Jones [12]. From [12, Lemma 2], each group $X_l(q)$ contains an alternating group $A_{l'}$ as a section where $l' \to \infty$ as $l \to \infty$. By Lemma 3.14 the rank l is therefore be bounded for groups in $\mathcal{M}_f$. But [12, Lemma 4] demonstrates that for each X_l the group $X_l(q)$ has $\mathrm{PSL}(2, q')$ as a section, where $q' \to \infty$ as $q \to \infty$. It follows from Lemma 3.15 that $X_l(q)$ is in $\mathcal{M}_f$ for only finitely many q. This finishes the proof. □

Proposition 3.18 *Let G be a finite f-Milnor group. There exists a positive integer $e_5(f)$ such that $G^{e_5(f)}$ is soluble.*

Proof Suppose $f = c_n X^n + \cdots + c_m X^m$. Let $b(f)$ be the least common multiple of the exponents of the automorphism groups of all finite non-abelian simple groups in $\mathcal{M}_f$ and let $e_5(f) = b(f) \cdot m!$. We want to show that $G^{e_5(f)}$ is soluble. We argue by contradiction and assume that M is a non-abelian G-invariant section of $G^{e_5(f)}$ that is a chief factor of G. Then M is a direct product of isomorphic simple groups $S_1, \ldots, S_k$. Fix $x \in G$. The action of x on M permutes the direct factors S_i. Consider some orbit $S_i, S_i^x, \ldots, S_i^{x^{N-1}}$ with $S_i^{x^N} = S_i$. We show that $x^{e_5(f)}$ centralises S_i; it then follows that $G^{e_5(f)}$ centralizes M and we get the contradiction that M is abelian.

Let $H = \langle S_i, x \rangle$. Let P be some Sylow subgroup of S_i. By Sylow's Theorem $P^{x^N} = P^{s^{-1}}$ for some $s \in S_i$. Then

$$(xs)^N = x^N s^{x^{N-1}} \cdots s^x s = x^N s(s^{x^{N-1}} \cdots s^x)$$

normalises P. Let $a \in P$ and let $Q = \langle a \rangle^{\langle (xs)^N \rangle}$. We have

$$\begin{aligned}\langle a \rangle^{\langle xs \rangle} &= Q \times Q^{xs} \times \cdots \times Q^{(xs)^{N-1}} \\ &= Q \times Q^x \times \cdots \times Q^{x^{N-1}}\end{aligned}$$

Then

$$A(a, xs) = Q/Q' \times Q^x/(Q^x)' \times \cdots \times Q^{x^{N-1}}/(Q^{X^{k-1}})'.$$

Now as $\gcd(c_n, c_{n+1}, \ldots, c_m) = 1$, we must have that $N \leq m$. Otherwise $a^{f(x)} = a^{c_n x^n} a^{c_{n+1} x^{n+1}} \cdots a^{c_m x^m} \neq 0$ in $A(a, xs)$. So N divides $m!$ and $x^{m!}$ normalises S_i. Then $x^{e_5(f)} = x^{m! b(f)}$ centralises S_i as we wanted to show. (Notice that $Q/Q' \neq 1$ as Q is nilpotent.) $\square$

Taking $e(f) = e_4(f) e_5(f)$ and applying Propositions 3.13 and 3.18 we get our first main result on f-Milnor groups.

Theorem 3.19 *There exist positive integers $e(f)$ and $c(f)$ such that*

$$\gamma_{c(f)+1}(G^{e(f)}) = 1$$

for all finite f-Milnor groups.

3.4 Finitely generated virtually soluble f-Milnor groups

In this section we extend the result from the previous section to all finitely generated virtually soluble f-Milnor groups, using an elegant idea from [3].

We let G be a fixed group with these properties. Suppose that G^m is soluble of derived length l. It suffices to show that G is residually finite. By a well known result of P. Hall, every finitely generated abelian-by-nilpotent group is residually finite. Let $e = e(f)$ be as in Theorem 3.19 and consider the chain

$$N_0 = G^m, \quad , N_{i+1} = N_i^e.$$

Then each N_i is finitely generated (by the solution to the restricted Burnside problem). As

$$N_i/[\gamma_{c(f)+1}(N_i), \gamma_{c(f)+1}(N_i)]$$

is abelian-by-nilpotent, it is residually finite by Hall's result and so from Theorem 3.19 the image $\overline{N_{i+1}}$ of N_{i+1} in this quotient is nilpotent of class at most $c(f)$. So we get

$$\gamma_{c(f)+1}(N_{i+1}) \leq [\gamma_{c(f)+1}(N_i), \gamma_{c(f)+1}(N_i)].$$

It follows by induction that $\gamma_{c(f)+1}(N_l)$ is in the lth term of the derived series of G^m and thus trivial. As G/N_l is of finite exponent it is finite. So G is nilpotent-by- finite and thus residually finite. This together with Theorem 3.19 gives the result we wanted.

Theorem 3.20 *Let f be a polynomial in $\mathbb{Z}[x]$ whose coefficients have no common prime factor. Then there exists a constant $e(f)$ such that for any finitely generated virtually soluble group G in $\mathcal{M}_f$ the subgroup $G^{e(f)}$ is nilpotent of f-bounded class.*

Theorem 3.19 has also the following generalization.

Theorem 3.21 *Let f be any polynomial in $\mathbb{Z}[x]$ and let π be the set of all primes that divide f. There exists a constant $e(f)$ such that for any finite π'-group G in $\mathcal{M}_f$ the subgroup $G^{e(f)}$ is nilpotent of f-bounded class.*

Proof Suppose that $f = mq$ where m is the greatest common divisor of the coefficients of f. As all sections of G are of m-prime order we see that G is q-Milnor. The result now follows from Theorem 3.19. □

3.5 Criteria for virtual nilpotence

Lemma 3.22 *The variety $\mathcal{A}_p\mathcal{A}$ is generated by $C_p \, wr \, C$.*

Proof Suppose that the law $w = 1$ is satisfied by $C_p \,\mathrm{wr}\; C$. We want to show that it is then satisfied by all groups in $\mathcal{A}_p\mathcal{A}$. Let $\mathcal{V}$ be the variety defined by the law $w = 1$. We need to show that $\mathcal{V}$ contains $\mathcal{A}_p\mathcal{A}$. But if this were not the case then [9] would yield that every soluble group in $\mathcal{V}$ is nilpotent-by-(finite exponent). But this is absurd as $C_p \,\mathrm{wr}\; C$ is a metabelian group in $\mathcal{V}$ that is not nilpotent-by-(finite exponent). □

Before stating the main result of this section we need some more notation. Let $\mathcal{M}$ be a variety of metabelian groups. Denote by $\mathcal{C}_{\mathcal{M}}$ the class of all groups G all of whose metabelian sections belong to $\mathcal{M}$. It is clear that $\mathcal{C}_{\mathcal{M}}$ is closed under taking subgroups and quotients. The connection with f-Milnor groups is as follows. Every metabelian f-Milnor group satisfies the law $[y, x]^{f(x)} = 1$. Let $\mathcal{M}$ the variety of all metabelian groups satisfying this law. Then $\mathcal{M}_f \leq \mathcal{C}_{\mathcal{M}} \leq \mathcal{M}_{(x-1)f}$.

Theorem 3.23 *Let $\mathcal{M}$ be a variety of metabelian groups and suppose that C is a class that is closed under taking subgroups and quotients such that $\mathcal{M} \leq \mathcal{C} \leq \mathcal{C}_{\mathcal{M}}$. The following are equivalent:*

(1) the groups in $\mathcal{C}$ are f-Milnor for some polynomial $f \in \mathbb{Z}[x]$ whose coefficients have greatest common divisor 1*;*
(2) there exist constants c and e, depending only on $\mathcal{M}$, such that

$$\gamma_{c+1}(G^e) = 1$$

for all finitely generated virtually soluble groups G in $\mathcal{C}$;
(3) every finitely generated residually finite group in $\mathcal{C}$ is nilpotent-by-finite;
(4) C_p wr $C \notin \mathcal{M}$ for each prime p;
(5) $\mathcal{M}$ does not contain $\mathcal{A}_p\mathcal{A}$ for any prime p.

Proof (1)⇒(2): As $\mathcal{C}$ is f-Milnor if and only if the subclass of the metabelian groups is f-Milnor, this follows immediately from Theorem 3.20.

(2)⇒(3): It follows from the solution to the restricted Burnside problem that the locally finite groups of exponent e form a variety $\mathcal{B}_e^*$ (as the class is residually closed and closed under taking quotients). Now take any finitely generated residually finite group G in $\mathcal{C}$. It follows from (2) that G is residually (nilpotent of class at most c)-by-$\mathcal{B}_e^*$ and thus G itself is (nilpotent of class at most c)-by-$\mathcal{B}_e^*$. As G is finitely generated is therefore nilpotent-by-finite.

(3)$\Rightarrow$(4): This is clear as $C_p \,\mathrm{wr}\, C$ is residually finite but not nilpotent-by-finite.

(4)$\Leftrightarrow$(5): This is lemma 3.22.

(5)$\Rightarrow$(1): As $\mathcal{M}$ is a soluble variety, the main result of [9] tells us that $\mathcal{M}$ is nilpotent-by-(finite exponent). Thus $\mathcal{M}$ satisfies a law of the form $[y,_a x^b] = 1$ for some integers $a, b > 0$ and the groups in $\mathcal{C}$ are $(x^b - 1)^a$-Milnor groups. $\Box$

In particular we can ofcourse take $\mathcal{C} = \mathcal{C}_{\mathcal{M}}$. Another interesting special case is if $\mathcal{C} = \mathcal{V}$ is a variety whose subvariety of metabelian groups is $\mathcal{M}$. Let $L_v\mathcal{V}$ be the subclass of all groups in $\mathcal{V}$ that are locally nilpotent-by-finite. For varieties, the result can be rewritten as follows; cf. [7, Theorem 2].

Theorem 3.24 *Let $\mathcal{V}$ be a variety. The following are equivalent:*

(1) every finitely generated residually finite group in $\mathcal{V}$ is nilpotent-by-finite;
(2) C_p wr $C \notin \mathcal{V}$ for any prime p;
(3) $\mathcal{V}$ does not contain $\mathcal{A}_p\mathcal{A}$ for any prime p
(4) there exist constants c and e that depend only on the metabelian groups of $\mathcal{V}$ such that

$$\gamma_{c+1}(G^e) = 1$$

for all finitely generated virtually soluble groups in $\mathcal{V}$.
(5) $L_v\mathcal{V}$ is a subvariety.

Proof Statements (1) to (4) are just restatements of (2)-(5) from Theorem 3.23. It remains to show that (1) and (5) are equivalent.

(1)$\Rightarrow$(5): To show that $L_v\mathcal{V}$ is a variety it suffices to show that this class is closed under taking quotients and that it is residually closed. It is clear that the first condition holds. Suppose that G is residually in $L_v\mathcal{V}$. As $\mathcal{V}$ is a variety we have $G \in \mathcal{V}$. Furthermore G is residually a group that is locally nilpotent-by-finite. Let H be a finitely generated subgroup of G. Since $H \in \mathcal{V}$ and every finitely generated nilpotent-by-finite group is residually finite, H is residually finite, and hence nilpotent-by-finite by (1). Thus G is locally nilpotent-by-finite and so in $L_v\mathcal{V}$.

(5)⇒(1): Suppose that (5) holds. Then every r-generator nilpotent-by-finite group in $\mathcal{V}$ is the image of the relatively free one. Thus all finite r-generator groups satisfy the same law of the form

$$[x_1^e, \ldots, x_{c+1}^e] = 1.$$

Then the same is true for any residually finite r-generator group. By the solution to the restricted Burnside problem, such a group is an extension of a group that is nilpotent of class c by a finite group. □

Let M be a metabelian variety and let $\mathcal{C}$ be a class that is closed under taking subgroups and quotients such that $\mathcal{M} \leq \mathcal{C} \leq \mathcal{C}_{\mathcal{M}}$. We will say that such a class is a weak Milnor class if every finitely generated residually finite group in $\mathcal{C}$ is nilpotent-by-finite. We also say that an identity $w = 1$ is a weak Milnor identity if the corresponding variety is a weak Milnor variety. Theorem 3.23 gives a simple criterion for deciding whether $\mathcal{C}$ is a weak Milnor class: we only need to check that no group $C_p \operatorname{wr} \ C$ satisfies the laws of $\mathcal{M}$. This is quite straightforward if we know the laws of $\mathcal{M}$. For example, as no group $C_p \operatorname{wr} \ C$ is periodic or Engel it follows that for any $n > 0$ the Burnside variety of groups of exponent n and the variety of n-Engel groups are weak Milnor varieties. Thus Theorem 3.24 includes Wilson's theorem on Engel groups and the solution to the restricted Burnside problem. (Of course the latter was used to prove Theorem 3.24).

4 Nilpotence

For virtual nilpotence we needed to exclude the wreath products $C_p \operatorname{wr} \ C$. Here we need to exclude some more groups.

Example Consider $C \operatorname{wr} C_p$, the standard wreath product of the cyclic groups C and C_p. This group is an $f(x-1)$-Milnor group for all polynomials f divisible by $x^p - 1$. Notice that $C_p \operatorname{wr} \ C$ is residually a finite p-group but that it is not nilpotent.

We are interested in criteria for a finitely generated residually nilpotent f-Milnor group to be nilpotent. For this reason and in view of the example above we will assume in the rest of the section that the polynomial f is neither divisible by p nor $x^p - 1$ for any prime p. Our aim is to show that for each r there exists a a positive integer $c(r, f)$ such that any r-generator nilpotent f-Milnor group has class at most $c(r,f)$. This problem will be dealt with in section 4.1. In that section, we will also prove an analogous result for global nilpotence of locally nilpotent groups that are torsion free. In section 4.2 we will then use these results to obtain our main characterisation results.

4.1 Milnor groups and nilpotence

Let f be a polynomial that is not divisible by p and $x^p - 1$ for all primes p. We want to show first that there is a constant $c(r,f)$ such that any r-generated f-Milnor group that is residually nilpotent must be nilpotent of class at most c. In the first lemma our setting is slightly more general.

Lemma 4.1 *Let g and h be polynomials with the property that g is not divisible by p and h is not divisible by $x^p - 1$ for any prime p. There exist positive integers m and l such that every metabelian group that is both g-Milnor and h-Milnor satisfies the law $[y,_m x]^l = 1$.*

Proof Let $\mathcal{M}$ be the variety of all metabelian groups satisfying the law $[y,x]^{g(x)} = 1$ and $[y,x]^{h(x)} = 1$. It is clear that $\mathcal{M}$ contains all metabelian groups that are both g-Milnor and h-Milnor. We want to prove that there exists integers m and l so that all groups in $\mathcal{M}$ satisfy the law $[y,_m x]^l = 1$. Let F be the free group in $\mathcal{M}$ on two generators x, y and let T be the torsion subgroup of F'. It suffices to show that $(x-1)^m$ acts trivially on F'/T for some positive integer m. We argue by contradiction and assume that this is not the case. We obtain a contradiction by showing that the polynomial $(X-1)h$ is divisible by $(X-1)(X^p-1)$ for some prime p or in other words that is divisible by $(X-1)^2$ and $X^p - 1$.

Consider the vector space $V = \mathbb{C} \otimes_{\mathbb{C}} \frac{F'}{T}$ over the complex numbers. As F is metabelian and g-Milnor, it follows from Theorem 3.20 that it is nilpotent by finite and thus satisfies max. Hence F'/T is finitely generated and V is a finite dimensional vector space. The conjugation by x on F'/T extends to a linear automorphism Φ on V. Let α be the minimal polynomial of Φ. If he only eigenvalue is 1 then the minimal polynomial is a power of $(X-1)$ which contradicts our assumptions. So we can assume that there is an eigenvalue $\lambda \neq 1$ for α. Then λ is a root of $(X-1)h(X)$. As we can replace x in the law $[y,x]^{h(x)}$ by any power of x, we have that λ^r is a root of $(X-1)h(X)$ for all $r \geq 0$. But as there are only finitely many roots it follows that λ must be an n-th root of unity for some $n \geq 2$ and $X^n - 1$

divides $(X-1)h(X)$.

It remains to show that $(X-1)^2$ divides $(X-1)h(X)$. But if that was not the case then $(X-1)h(X)$ is of the form

$$l_1(X-1)+\cdots+l_n(X-1)^n.$$

Replacing x, y by $[y, x]$ and x respectively in the law $[y, x]^{h(x)}=1$ we get the law $[y, x, x]^{l_1}=1$ which implies that $\mathcal{M}$ satisfies the 2-Engel law. Again this gives a contradiction to our assumptions. $\square$

Theorem 4.2 *Let g and h be polynomials as above. A nilpotent r-generator group that is both g-Milnor and h-Milnor is nilpotent of (r, g, h)-bounded class.*

Proof Let l and m be the integers from Lemma 4.1 and suppose that $l=p_1^{k_1}\cdots p_s^{k_s}$ for some distinct primes $p_1, \ldots, p_s$. By Lemma 4.1 we can w.l.o.g. assume that

$$h=l(X-1)^m.$$

Suppose

$$g=l_{m'}(X-1)^{m'}+\cdots+l_n(X-1)^n$$

where $l_{m'}, \ldots, l_n$ are coprime integers. By multiplying one of g and h by a power of $(X-1)$ if necessary we can assume that $m=m'$. Let c be the largest number of $p_1^{k_1}, \ldots p_s^{k_s}$. Now any finitely generated nilpotent group is residually finite so it suffices to deal with finite p-groups. We will prove that any finite metabelian p-group that is both g-Milnor and h-Milnor is a $(cn+1)$-Engel group. Then by Lemma 3.8 we have that any r-generator nilpotent group that is both g-Milnor and h-Milnor is nilpotent of $(r, cn+1)$-bounded class and therefore (r, g, h)-bounded class. We therefore assume that G is a finite metabelian p-group that is both g-Milnor and h-Milnor. We consider two cases.

First suppose that p does not divide l or l_m. As G is both g-Milnor and h-Milnor it then follows that G satisfies $[y,_m x]=1$ and thus $[y,_{cn+1} x]=1$.

Then suppose that p divides both l and l_m. Suppose further that $l=p^k l'$ where p does not divide l'. Let r be the smallest integer in $\{m, m+1, \ldots, n\}$ such that p does not divide l_r. Working in the polynomial ring $\mathbb{Z}[X]$ we have

$$l_m(X-1)^m+\cdots+l_n(X-1)^n=p(X-1)^m P(X)+(X-1)^r(l_r+Q(X))$$

for some polynomials $P(X)$ and $Q(X)$ where $Q(X)$ is divisible by $(X-1)$. Now raising the right hand side to the power p^k we obtain

$$p^k(X-1)^m P'(X) + (X-1)^{p^k r}(l_r^{p^k} + Q'(X))$$

for some polynomials $P'(X)$ and $Q'(X)$ where $X-1$ divides $Q'(X)$. Now as G is g-Milnor and $p^k(X-1)^m$-Milnor it follows that G is $(X-1)^{rp^k}(l_r^{p^k} + Q'(X))$-Milnor. But p does not divide l_r and thus it follows that G is $(X-1)^{rp^k}$-Milnor and thus it satisfies $[y,_{cn+1} x] = 1$. □

By Theorem C, every locally nilpotent n-Engel group that is torison-free is nilpotent of n-bounded class. We next consider this question for f-Milnor groups.

Theorem 4.3 *Let $f \neq 0$ be a polynomial in $Z[x]$. There exists a positive integer $c(f)$ such that any f-Milnor group that is residually a finite p-group for all primes p is nilpotent of class at most $c(f)$.*

Proof This follows from Proposition 3.10. □

Remark. Let f be a nonzero polynomial. It follows from the theorem above that any locally nilpotent torsion-free f-Milnor group is nilpotent of f-bounded class.

4.2 Criteria for nilpotence

Proposition 4.4 *The variety $\mathcal{A}\mathcal{A}_p$ is the variety generated by the group $C \,\mathrm{wr}\, C_p$.*

Proof Let $w = 1$ be some law satisfied by $C \,\mathrm{wr}\, C_p$ and let $\mathcal{V}$ be the variety defined by this law. If $\mathcal{V}$ does not contain $\mathcal{A}\mathcal{A}_p$, every soluble group in $\mathcal{V}$ is finite exponent by nilpotent [9]. But $C \,\mathrm{wr}\, C_p$ belongs to $\mathcal{V}$ and is not finite exponent by nilpotent. This contradiction shows that $w = 1$ must be a law in $\mathcal{A}\mathcal{A}_p$. □

We are now ready for the main result of this section.

Theorem 4.5 *Let $\mathcal{M}$ be a metabelian variety and suppose that $\mathcal{C}$ is a class that is closed under taking subgroups and quotients such that $\mathcal{M} \leq \mathcal{C} \leq \mathcal{C}_{\mathcal{M}}$. The following are equivalent.*

(1) The groups in $\mathcal{C}$ are g-Milnor and h-Milnor where the polynomials g, h are as in Lemma 4.1.
(2) For each r there exists a constant $c(r, \mathcal{M})$ depending only on the metabelian groups in $\mathcal{C}$ so that all r-generator nilpotent groups in $\mathcal{C}$ are nilpotent of class at most $c(r, \mathcal{M})$.
(3) Every finitely generated group in $\mathcal{C}$ that is residually nilpotent is nilpotent
(4) The groups $C_p\, wr\, C$ and $C\, wr\, C_p$ do not belong to $\mathcal{M}$ for any prime p.
(5) The variety $\mathcal{M}$ contains neither $\mathcal{A}\mathcal{A}_p$ nor $\mathcal{A}_p\mathcal{A}$ as a subvariety for any prime p.

Proof (1)⇒(2): As $\mathcal{C}$ is f-Milnor if and only if $\mathcal{M}$ is f-Milnor, this follows immediately from Theorem 4.2.

(2)⇒(3): Let G be a residually nilpotent group in $\mathcal{C}$ generated by r elements. It follows from (2) that G is nilpotent of class at most $c(r, \mathcal{M})$.

(3)⇒(4): This is clear as both these wreath products are residually nilpotent but neither is nilpotent.

(4)⇔(5): This follows immediately from Proposition 4.4 and Lemma 3.22.

(5)⇒(1): As $\mathcal{M}$ is a soluble variety, we can apply the main result of [9] to deduce that $\mathcal{M}$ is both nilpotent by finite exponent and finite exponent by nilpotent. Thus we have that the groups in $\mathcal{M}$ satisfy laws of the form $[y,_a x^b] = 1$ and $[y,_c x]^d = 1$. It follows that the groups in $\mathcal{C}$ are both $(x^b-1)^a$-Milnor and $d(x-1)^c$-Milnor. □

In particular we can take $\mathcal{C} = \mathcal{C}_{\mathcal{M}}$. Another interesting case is when $\mathcal{C} = \mathcal{V}$ is a variety whose subvariety of metabelian groups is $\mathcal{M}$. Let $L\mathcal{V}$ be the subclass of all groups in $\mathcal{V}$ that are locally nilpotent. For varieties, the result can be rewritten as follows. This result is stronger than the analogous result from [8] in that the bound for the nilpotency class depends only on the metabelian groups of $\mathcal{V}$.

Theorem 4.6 *Let $\mathcal{V}$ be a variety. The following are equivalent.*

(1) Every finitely generated group in $\mathcal{V}$ that is residually nilpotent is nilpotent.
(2) The groups $C_p\, wr\, C$ and $C\, wr\, C_p$ do not belong to $\mathcal{V}$ for any prime p.

(3) The variety $\mathcal{V}$ contains neither the variety $\mathcal{A}_p\mathcal{A}$ nor the variety $\mathcal{A}\mathcal{A}_p$ as subvarietes for any prime p.

(4) For each positive integer r there exists a coefficient c depending only on the metabelian groups of $\mathcal{V}$ such that every r-generator nilpotent group in $\mathcal{V}$ is nilpotent of class at most c.

(5) The subclass $L\mathcal{V}$ is a subvariety.

Proof The statements (1) to (4) are just rewritings of the statements (2) to (5) from Theorem 4.5. It remains to show that (1) and (5) are equivalent.

(1)⇒(5): To show that $L\mathcal{V}$ is a variety it suffices to show that this class is closed under taking quotients and that it is residually closed. It is clear that the first condition holds. Suppose then that G is residually in $L\mathcal{V}$. Then G is $\mathcal{V}$ and G is residually a locally nilpotent group. Let H be a finitely generated subgroup of G then H is also in $\mathcal{V}$ and is residually nilpotent. By (1) we have that H is nilpotent. This shows that G is locally nilpotent and thus in $L\mathcal{V}$.

(5)⇒(1): Suppose that (5) holds. Then all r-generator nilpotent groups of $\mathcal{V}$ are the images of the freeest one. So there is a bound for the class. This implies that all r-generator residually nilpotent groups in $\mathcal{V}$ are nilpotent. □

Let $\mathcal{M}$ be a metabelian variety and let $\mathcal{C}$ be a class that is closed under taking subgroups and quotients such that $\mathcal{M} \leq \mathcal{C} \leq \mathcal{C}_{\mathcal{M}}$. We will say that such a class is a strong Milnor class if every finitely generated group in $\mathcal{C}$ that is residually nilpotent, is nilpotent. We also say that an identity $w = 1$ is a strong Milnor identity if the corresponding identity is a strong Milnor variety. Therorem 4.5 gives a nice critera for deciding whether C is a strong Milnor class. We only need to check that none of the groups $C_p \,\mathrm{wr}\, C$ or $C \,\mathrm{wr}\, C_p$ satisfies all the laws of $\mathcal{M}$. As the structure of these groups is simple this is quite straightforward if we know the laws of $\mathcal{M}$. Let us look at some examples.

Example 1. As none of these wreath products are periodic or Engel it follows that the Burnside variety of groups of exponent n and the variety of n-Engel groups are strong Milnor varieties. One can in both cases replace this by the class $\mathcal{C}_{\mathcal{M}}$ where $\mathcal{M}$ is the subvariety of the metabelian groups.

Example 2. In [7] Endimioni proves that in a variety in which every polycyclic group is nilpotent each finitely generated residually finite group is nilpotent. Our next result can be seen as a generalization of this.

Theorem 4.7 *Let $\mathcal{M}$ be a metabelian variety of which all finite groups are nilpotent. Then every finitely generated residually finite group in $\mathcal{C}_{\mathcal{M}}$ is nilpotent.*

Proof Notice first that all finite groups in $\mathcal{C}_{\mathcal{M}}$ are nilpotent. This can be proved as follows. We argue by contradiction and let G be a counterexample in $\mathcal{C}_{\mathcal{M}}$ of minimal order. Thus every proper subgroup of G is nilpotent and we know from a well known result of Schmidt that G is an extension of a p-group P by a cyclic q-group Q where p and q are some different primes. By the minimality of G we also know that its centre must be trivial. It follows that $Z(P)$ is not centralised by Q and thus $G = \langle Z(P), Q\rangle$, again by minimality of G. It follows that G is metabelian and thus in $\mathcal{M}$. But then it follows from our assumption that G is nilpotent. This contradiction shows that all finite groups in $\mathcal{C}_{\mathcal{M}}$ are nilpotent and thus all residually finite groups in $\mathcal{C}_{\mathcal{M}}$ are residually nilpotent.

We can now easily prove the theorem using Theorem 4.5. We argue by contradiction. Otherwise,by Theorem 4.5, $\mathcal{M}$ contains some wreath product $C\,\mathrm{wr}\ C_p$ or $C_p\,\mathrm{wr}\ C$. But in both cases it then follows that $\mathcal{M}$ contains some finite quotient $C_p\,\mathrm{wr}\ C_q$ for some primes $p \neq q$. But this group must then be nilpotent which is absurd. □

Remark. We could have replaced $\mathcal{C}_{\mathcal{M}}$ here by a variety $\mathcal{V}$ in which all the finite groups are nilpotent. The result we obtain in this case is also implicit in [7].

We end this section by considering the global nilpotence question for locally nilpotent torsion-free groups. In Theorem 4.3 we dealt with this question for f-Milnor groups. For the main characterisation we need the following example.

Example Consider $C\,\mathrm{wr}\ C$, the standard wreath product of the infinite cyclic group with itself. This is a torsion-free group and residually finite p-group for any prime p. But it is not nilpotent.

Lemma 4.8 *The variety $\mathcal{AA}$ of all metabelian groups is generated by $C\,\mathrm{wr}\,C$.*

Proof Otherwise there would be some law $w = 1$ that is satisfied by $C \operatorname{wr} C$ but not by all groups in $\mathcal{AA}$. Modulo the laws of $\mathcal{AA}$ it is not difficult to see that $w = 1$ would imply a law of the form $[x, y]^{P(x,y)} = 1$ where P is a non-trivial polynomial in two variables. Replacing x and y by some x^i and $x^j[y, x]$ it would follow that $C \operatorname{wr} C$ satisfies a law of the form $[x, [y, x]]^{Q(x)} = 1$ for some non-trivial polynomial Q in one variable. But $C \operatorname{wr} C$ satisfies no such law. □

Theorem 4.9 *Let $\mathcal{M}$ be a variety of metabelian groups and suppose that $\mathcal{C}$ is a class that is closed under taking subgroups and quotients such that $\mathcal{M} \leq \mathcal{C} \leq \mathcal{C}_{\mathcal{M}}$. The following are equivalent.*

(1) The groups in $\mathcal{C}$ are f-Milnor for some non-zero polynomial $f \in \mathbb{Z}[x]$.
(2) There exists a constant $c = c(\mathcal{M})$ such that every group in $\mathcal{C}$ that is residually a finite p-group for all primes p is nilpotent of class at most c.
(3) The group $C \operatorname{wr} C$ doesn't belong to $\mathcal{M}$.
(4) $\mathcal{M} \neq \mathcal{AA}$.

Proof (1)⇒(2): As $\mathcal{C}$ is f-Milnor if and only if the subclass of metabelian groups is f-Milnor, this follows from Theorem 4.3.

(2)⇒(3): This is clear as $C \operatorname{wr} C$ is residually finite p-group for all primes p but it is not nilpotent.

(3)⇔(4): This is Lemma 4.8.

(4)⇒(1): By a result of Kargapolov and Čurkin [13] we know that $\mathcal{M} \neq \mathcal{AA}$ implies that $\mathcal{M}$ is torsion-by-nilpotent-by-torsion and thus satisfies a law of the form $[y,_a x^b]^c = 1$ and (1) follows. □

In particular we can take $\mathcal{C} = \mathcal{C}_{\mathcal{M}}$ or we can take $\mathcal{C} = \mathcal{V}$ a variety whose subvariety of metabelian groups is $\mathcal{M}$. In the latter case the result can be written as follows.

Theorem 4.10 *Let $\mathcal{V}$ be a variety of groups. The following are equivalent.*

(1) There exists a constant c depending only on the metabelian groups of $\mathcal{V}$ such that every group G in $\mathcal{V}$ that is residually a finite p-group for all primes p is nilpotent of class at most c.

(2) *The variety $\mathcal{V}$ does not contain the variety of all metabelian groups as a subvariety.*
(3) *The variety $\mathcal{V}$ does not contain the group C wr C.*

A weaker version can be deduced from a result of G. Endimioni [6] (see his Theorem 1). As a consequence of Theorem 4.10 it is quite straightforward to check if the variety defined by a given law satisfies (1). We just have to check if $C\,\mathrm{wr}\ C$ satisfies this law.

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