Locally nilpotent 4-Engel groups are Fitting groups

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Abstract

Let G be a locally nilpotent 4-Engel group. We show that the normal closure of any element from G is nilpotent of class at most 4. When G has no element of order 2 or 5, the normal closure has class at most 3. These bounds are sharp.

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1 Introduction

We define $[y,_n x]$ inductively by $[y,_1 x] = [y, x] = y^{-1}x^{-1}yx$ and $[y,_{n+1} x] = [[y,_n x], x]$. A group G is an n-Engel group if it satisfies the law $[y,_n x] = 1$. In this paper we will be looking at 4-Engel groups. Whereas 2-Engel groups and 3-Engel groups are quite well understood (see for example [2,4,6,8,11,13]), relatively little is known about the structure of 4-Engel groups. In particular it is still an open question whether all 4-Engel groups are locally nilpotent. In [18] this problem was reduced to the case where the group is either of

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prime exponent or torsion-free. In [20] it was shown that 4-Engel groups of exponent 5 are locally finite. It follows that all 4-Engel $\{2,3,5\}$-groups are locally nilpotent. To this one could add that 4-Engel groups satisfy a semigroup identity [19]. Whether all n-Engel groups satisfy a semigroup identity is an open question.

We want to investigate further the structure of 4-Engel groups that are locally nilpotent. We will see that they satisfy a strong generalised nilpotence property. Before we describe our results, we list some known properties. If G is a locally nilpotent 4-Engel group without elements of order 2, 3 or 5 it is nilpotent of class at most 7 [18] (see also [3,17]). If only the primes 2 and 5 are excluded the group is not in general nilpotent but it is still soluble [1]. On the other hand there are examples of 4-Engel 2-groups and 5-groups that are not soluble [2,16].

The starting point for our investigation is a result of L. C. Kappe and W. P. Kappe [11]. They proved that a group is 3-Engel if and only if the normal closure of every element is nilpotent of class at most 2. It is easily seen that the analogous result holds for 2-Engel group. Thus a group is 2-Engel if and only if the normal closure of every element is abelian. N. D. Gupta and F. Levin [5] have on the other hand constructed an example of a 4-Engel 5-group with an element x such that the normal closure of x has class 4 and thus greater than 3. Thus the analogous result does not hold for 4-Engel groups. We will however see that something close to this is true.

We recall that a group is a Fitting group if it is the product of its normal nilpotent subgroups. Equivalently a group is a Fitting group if the normal closure of any element is nilpotent. If there is a bound on the nilpotency classes of the normal closures and n is the maximal value then we say that the group has Fitting degree n. Thus 2-Engel groups and 3-Engel groups have Fitting degrees 1 and 2 respectively. N. D. Gupta and F. Levin [5] have constructed examples that show that an n-Engel group does not have to be Fitting if $n \geq 5$ and they asked whether 4-Engel groups are Fitting groups. In this paper we will show that every locally nilpotent 4-Engel group is a Fitting group of Fitting degree at most 4. More precisely

Theorem. *Let G be a locally nilpotent* 4*-Engel group. Then G is a Fitting group of degree at most* 4*. If G has no element of order* 2 *or* 5 *then G*

has Fitting degree at most 3*.*

As there are examples of a 4-Engel 2-group and 5-group with Fitting degree 4 [5, 15], the bounds in the theorem are sharp.

We mention one corollary to this. First we introduce another generalised nilpotence property that is weaker then the Fitting property. A group is said to be a Baer group if every cyclic subgroup is subnormal. If there is a bound on the subnormal defect and n is the maximal value of the defects, then we say that the group is an n-Baer group. We also say that a group is non-torsion if it has an element of infinite order. According to a result of H. Heineken [9], a non-torsion group is a 2-Baer group if and only if it is 2-Engel. In [12] it is shown that the analogous result holds for the integer 3. Thus a non-torsion group is 3-Engel if and only if it is a 3-Baer group. In that paper it is also shown that in general every non-torsion n-Baer group is an n-Engel group. It is easy to see that as a corollary of the theorem above, we have that the converse holds for $n = 4$ if the group is locally nilpotent.

Corollary . *A locally nilpotent non-torsion group G is 4-Engel if and only if it is 4-Baer.*

Proof Suppose that G is a locally nilpotent 4-Engel group. Let $x \in G$. We want to show that $\langle x\rangle$ is subnormal of defect at most 4. This is true if and only if $[G, \langle x\rangle, \langle x\rangle, \langle x\rangle, \langle x\rangle]$ is contained in $\langle x\rangle$. But as the normal closure of x has class at most 4, we have

$$\begin{aligned} [G, \langle x\rangle, \langle x\rangle, \langle x\rangle, \langle x\rangle] &= \langle [g, x, x, x, x] : g \in G\rangle \\ &= 1. \end{aligned}$$

Hence the result. $\Box$

2 A general reduction to Lie algebras

Let F be a free group of countably infinite rank, freely generated by $\{f_0, f_1, f_2, \ldots\}$, and let L likewise be a free Lie algebra of countably infinite rank, freely generated by $\{x_0, x_1, x_2, \ldots\}$. Consider the set $\mathbb{N}_0^\infty$ of all sequences of natural numbers with almost all entries zero. We give this set a partial ordering as follows: $(d_i) \leq (e_i)$ if $d_i \leq e_i$ for all i. We will sometimes use the short hand notation $(r, 1^s)$ for the element $(r, d_1, \ldots, d_s, 0, \ldots)$ where $d_1 = \ldots = d_s = 1$. Take some fixed element $D = (d_0, d_1, \ldots, d_r, 0, \ldots)$ of $\mathbb{N}_0^\infty$. For each left normed commutator

$$w = [f_{i_1}, f_{i_2}, \ldots, f_{i_n}]$$

of multiweight D we associate the left normed Lie product

$$l(w) = x_{i_1} x_{i_2} \ldots x_{in}.$$

Let $\mathcal{V}$ be some variety of groups and consider any multiweight D. We denote by R^D the normal closure in F of the set of all commutators of multiweight higher than D. Then let $H_{\mathcal{V}}^D$ be the subgroup of F that is generated by R^D and all $w \in F$ that are laws in $\mathcal{V}$ of the form

$$w = w_1 \cdots w_m u$$

with $w_1, \ldots, w_m$ commutators of multiweight D and $u \in R^D$. To this we naturally associate the submodule $M_{\mathcal{V}}^D$ of L consisting of all

$$l(w) = l(w_1) + \cdots + l(w_m)$$

where for some $u \in R^D$, $w = w_1 \cdots w_m u \in H_{\mathcal{V}}^D$ with $w_1, \ldots, w_m$ commutators of multiweight D. An element l that lies in $M_{\mathcal{V}}^D$ for some multiweight $D \in \mathbb{N}_0^\infty$ will be called a strong Lie relator for $\mathcal{V}$. Notice that any strong Lie relator is a multihomogenous element of L.

Suppose that we want to show that any locally nilpotent group G in $\mathcal{V}$ is Fitting of Fitting degree d. As any commutator involves only a finite number of elements from G, we can without loss of generality assume that G is finitely generated and thus nilpotent. But then it is well known that G is residually a finite p-group. We can thus assume that G is a finite p-group. As G is nilpotent it suffices to show that for any non-negative integer e and arbitrary elements $g_0, g_1, \ldots, g_e$, any commutator of multiweight $(d+1, 1^e)$ in $g_0, g_1, \ldots, g_e$ is in the normal closure of commutators in $g_0, g_1, \ldots, g_e$ that are of higher multiweight. Let $D = (d+1, 1^e)$. From what we have said above, it follows that in order to show that any locally nilpotent group in $\mathcal{V}$ has Fitting degree d, it suffices to show that any commutator of weight D in F lies in $H_{\mathcal{V}}^D$. This is equivalent to saying that any Lie product in L of multiweight D is in $M_{\mathcal{V}}^{D}$.

Now take the ideal $I_{\mathcal{V}}$ in L generated by all $\omega(x_0, l_1, \ldots, l_e)$, where $l_1, \ldots, l_e \in L$, $w(x_0, x_1, \ldots, x_e) \in M_{\mathcal{V}}^{D}$ and where D runs over all multiweights of the form $(r, 1^s)$ with r, s non-negative integers. (Notice that x_0 is kept fixed in ω). We make two simple observations. Firstly if l is a strong Lie relator then lx_i is also a strong Lie relator for $i = 0, 1, \ldots$. Secondly, if $\omega(x_0, x_1, \ldots, x_n)$ is in $M_{\mathcal{V}}^{D}$ for some $D = (r, 1^n)$ and if $l_1, \ldots, l_n \in L$ then $\omega(x_0, l_1, \ldots, l_n)$ is a linear combination of elements of the form $\omega(x_0, c_1, \ldots, c_n)$ where $c_1, \ldots, c_n$ are Lie products of the generators $x_0, x_1, \ldots$, and furthermore the elements $\omega(x_0, c_1, \ldots, c_n)$ are all strong Lie relators. From these two observations, it follows easily that the elements of $I_{\mathcal{V}}$ are linear combinations of elements $\omega(x_0, x_{i_1}, \ldots, x_{i_e})$ where $i_1, \ldots, i_e \geq 1$ and $\omega(x_0, x_1, \ldots, x_e) \in M_{\mathcal{V}}^{(r,1^e)}$ for some non-negative integers r, e. We see from this that $I_{\mathcal{V}}$ is multigraded and as L is also multigrated it follows that a Lie product in L of weight D_0 is in $I_{\mathcal{V}}$ if and only if it lies in $M_{\mathcal{V}}^{D}$. We thus obtain:

Proposition 2.1 *If the ideal in $L/I_{\mathcal{V}}$ generated by $x_0 + I_{\mathcal{V}}$ is nilpotent of class at most d then any locally nilpotent group in $\mathcal{V}$ has Fitting degree d.*

Our problem has thus been reduced to a problem on Lie algebras. We want to reduce the problem further. It will be convenient to deal separately with finite p-groups in $\mathcal{V}$ for each prime p. So for the rest of this section we work with a fixed prime p and finite p-groups of $\mathcal{V}$.

Consider the subvariety $\mathcal{V}(p^h)$ of all groups in $\mathcal{V}$ that are of exponent p^h. As $f_1^{p^h} \in H_{\mathcal{V}(p^h)}^{(1)}$ we have that $p^h x_1 \in M_{\mathcal{V}(p^h)}^{(1)}$ and thus $p^h L \leq I_{\mathcal{V}(p^h)}$. It thus follows that $L/I_{\mathcal{V}(p^h)}$ is a quotient of $L/(p^h L + I_{\mathcal{V}})$. To prove that any finite group in $\mathcal{V}$ of exponent p^h has Fitting degree d it suffices then to show that the ideal in $L/(p^h L + I_{\mathcal{V}})$ generated by $x_0 + p^h L + I_{\mathcal{V}}$ is nilpotent of class at most d. We will next show that p^h can be replaced by p.

Lemma 2.2 *Suppose the ideal generated by $x_0 + pL + I_{\mathcal{V}}$ of $L/(pL + I_{\mathcal{V}})$ is nilpotent of class d then the same is true of the ideal generated by $x_0 + p^h L + I_{\mathcal{V}}$ of $L/(p^h L + I_{\mathcal{V}})$.*

Proof Let $Q = L/(p^h L + I_{\mathcal{V}})$ and let $y_i = x_i + p^h L + I_{\mathcal{V}}$. Let $u = y_{i_1} \cdots y_{i_n}$ be any left normed Lie product in Q with $d + 1$ occurrences of y_0. By our assumption, we have that

$$u = pu_1$$

for some $u_1 \in Q$. As Q is multigraded it follows that we can assume that u_1 has the same multidegree as u in $y_0, y_1, \ldots$. Similarly we get elements $u_2, \ldots, u_h$ of same multidegree as u such that $u_i = pu_{i+1}$ for $i = 1, \ldots, h-1$. Thus $u = pu_1 = p^2u_2 = \ldots = p^hu_h = 0$ in Q. □

The final reduction uses an argument of G. Higman [10] (see also [7,14]). It allows us to add the assumption that the elements $x_1, x_2, \ldots$ commute. Let I_{ab} be the ideal of L generated by $x_ix_j,\ i, j \geq 1$. Let $L^p_{\mathcal{V}} = L/I$ where $I = I_{ab} + pL + I_{\mathcal{V}}$. For the next lemma we will need the following property of $I_{\mathcal{V}}$, which follows directly from the definition of this ideal. If $w(x_0, x_1, x_2, \ldots, x_n) \in I_{\mathcal{V}}$ then we have that $w(x_0, l_1, \ldots, l_n) \in I_{\mathcal{V}}$ for all $l_1, \ldots, l_n \in L$.

Lemma 2.3 *If in $L^p_{\mathcal{V}}$ we have that the ideal generated by $x_0 + I$ is nilpotent of class at most d then the same holds for the ideal generated by $x_0 + pL + I_{\mathcal{V}}$ in $L/(pL + I_{\mathcal{V}})$.*

Proof Let M be the subalgebra of L generated by $\{x_1, x_2, \ldots\}$. Let $e_1, \ldots, e_t$ be any elements of M. We prove by induction on t that all products of weight $(d+1, 1^t)$ in $x_0, e_1, \ldots, e_t$ are in $pL + I_{\mathcal{V}}$. We first deal with the case $r = 1$. The product $x_1x_0^{d+1}$ is in I by the assumption. As $M_{\mathcal{V}}^{(d+1,1)} \cap I_{ab} = \{0\}$, we have that $M_{\mathcal{V}}^{(d+1,1)} \cap I \subseteq pL + I_{\mathcal{V}}$. As $I/(pL + I_{\mathcal{V}})$ is multigraded it then follows that $x_1x_0^{d+1} \in pL + I_{\mathcal{V}}$. But then $lx_0^{d+1} \in pL + I_{\mathcal{V}}$ for all $l \in L$ by the remark made just before the statement of the lemma.

Now suppose we know that our hypothesis is true for some t. Let $e_1, \ldots, e_{t+1}$ be any elements of M and take any product u of weight $(d+1, 1^{t+1})$ in $x_0, e_1, \ldots, e_{t+1}$. By our assumption we know that u is in I and as I is multigraded we furthermore have that u is modulo $pL + I_{\mathcal{V}}$ in the linear span of products in I of the form

$$e_ie_ju_1 \cdots u_{t-1}$$

where the sum of the degrees of $e_i, e_j, u_1, \ldots, u_{t-1}$ is $(d+1, 1^{t+1})$ in $x_0, e_1, \ldots, e_{t+1}$. But such a product is a product of weight $(d+1, 1^t)$ in $x_0, f_1, \ldots, f_t$ where $f_1 = e_ie_j, f_2, \ldots, f_t$ are elements of M. By the induction hypothesis, u must be in $pL + I_{\mathcal{V}}$. □

It is time to summarise. Let $x = x_0 + I$ and $a_i = x_i + I$ for $i \geq 1$. The Lie algebra $L^p_{\mathcal{V}}$ is a Lie algebra over the field $GF(p)$ in which the elements a_i,

$i \geq 1$, commute. For any given word w of weight $D = (r, 1^s)$ in $M_{\mathcal{V}}^{D}$ we have that $L_{\mathcal{V}}^{p}$ satisfies the identity

$$w(x, l_1, \cdots, l_s) = 0$$

for all $l_1, \ldots, l_s \in L_{\mathcal{V}}^{p}$. We have also proved the following.

Theorem 2.4 *If the ideal generated by x in $L_{\mathcal{V}}^{p}$ is nilpotent of class at most d then every finite p-group in $\mathcal{V}$ has Fitting degree at most d.*

For a given variety $\mathcal{V}$ we are thus faced with two problems. The first is to calculate (partly or fully) the multigraded Lie algebra $L_{\mathcal{V}}^{p}$ for each prime and the second is to show that the ideal generated by x in $L_{\mathcal{V}}^{p}$ is nilpotent and to determine its class. For the first problem we can apply the general approach of G. Wall [21] to generate the multilinear identities of $I_{\mathcal{V}}$. This we shall do in the next section for the variety of 4-Engel groups.

3 Multilinear identities for 4-Engel groups

Let $\mathcal{E}$ be the variety of 4-Engel groups. In this section we investigate the ideal $I_{\mathcal{E}}$ in L. In particular we will calculate explicitly some of these identities that we will need later for the main result of this paper.

We next give some notation. Let r be an arbitrary positive integer and let $F_1, F_2, \ldots, F_l$ be a list of all the non-empty subsets of $\{f_0, \ldots, f_r\}$ ordered in such a way that $\|F_i| \leq \|F_j\|$ whenever $i \leq j$, where $\|F_i\|$ is the size of the set F_i. In particular we have that $F_l = \{f_0, \ldots, f_r\}$. The following two elementary lemmas will be useful.

Lemma 3.1 *Suppose that the group G satisfies a law*

$$w_1 \cdots w_l = u_1 \cdots u_l,$$

where w_i and u_i are products of simple commutators involving all the elements of F_i and only these. Then G satisfies the laws

$$w_i = u_i$$

for $i = 1, \ldots l$.

Proof Putting all the variables in $F_l \setminus F_1$ equal to 1 we obtain $w_1 = u_1$. Then we have that G satisfies $w_2 \cdots w_l = u_2 \cdots u_l$. Continuing in this manner we get the result. $\square$

Lemma 3.2 *Let $w_1, \ldots, w_l$ be as in Lemma 3.1 with the extra hypothesis that w_l is a product of simple commutators of weight $(d, 1^r)$. Suppose that G satisfies a law*

$$w_1 \cdots w_l = u$$

where u is a product of simple commutators with at least $d+1$ occurrences of f_0. Then G satisfies a law of the form

$$w_l = u_l$$

where u_l is a product of simple commutators of weight at least $(d+1, 1^r)$.

Proof Using Hall's collection process we can write

$$u = u_1 \cdots u_l.$$

Where $u_1, \ldots, u_l$ are like in Lemma 3.1 with the further property that all the simple commutators that occur as factors in u_i have at least $d+1$ occurrences of f_0. By Lemma 3.1 we have that G satisfies the law $w_l = u_l$. But all the factors of u_l involve all the elements $f_1, \ldots, f_r$ so they are of multidegree at least $(d+1, 1^r)$. $\square$

We need some more notations. Let $b, a_1, \ldots, a_n$ be arbitrary group elements. For a subset $S = \{s_1, \ldots, s_m\} \subseteq \{1, \ldots, n\}$ with $s_1 < s_2 < \cdots s_m$, we let

$$[b, a_S] = [b, a_{s_1}, \ldots, a_{s_m}].$$

The following identity is well known and easily proved by induction.

$$[b, a_1 \cdots a_m] = \prod_{\emptyset \neq S \subseteq \{1,\ldots,r\}} [b, a_S], \tag{1}$$

where the factors on the right hand side come in a certain order. From the next lemma we will deduce the identities in $I_{\mathcal{E}}$ that we will need.

Lemma 3.3 *For each positive integer r we have that the variety $\mathcal{E}$ satisfies a law of the form*

$$[f_1, f_0, f_2, \ldots, f_r, f_0, f_0, f_0] = u,$$

where u is a product of commutators of multiweight at least $(5, 1^r)$.

Proof Using the identity $[f_1, f_0, f_0, f_0, f_0] = 1$ and calculating modulo $R^{(5)}$ we have:

$$\begin{aligned} 1 &= [f_1 \cdots f_r, f_0, f_0, f_0, f_0] \\ &= [[f_2 \cdots f_r, f_0][f_1, f_0][f_1, f_0, f_2 \cdots f_r], f_0, f_0, f_0] \\ &= [f_2 \cdots f_r, f_0, f_0, f_0, f_0][f_1, f_0, f_0, f_0, f_0][f_1, f_0, f_2 \cdots f_r, f_0, f_0, f_0] \\ &= [f_1, f_0, f_2 \cdots f_r, f_0, f_0, f_0]. \end{aligned}$$

Using identity (1) this gives

$$\begin{aligned} 1 &= \prod_{\emptyset \neq S \subseteq \{2,\ldots,r\}} [f_1, f_0, f_S, f_0, f_0, f_0] \\ &= w_1 \cdots w_{l-1} [f_1, f_0, f_2, \ldots, f_r, f_0, f_0, f_0]. \end{aligned}$$

Where $w_1, \ldots, w_{l-1}$ are like in Lemma 3.1. But then it follows from Lemma 3.2 that $\mathcal{E}$ satisfies a law of the form

$$[f_1, f_0, f_2, \ldots, f_r, f_0, f_0, f_0] = u,$$

with u a product of simple commutators of weight at least $(5, 1^r)$. $\Box$

From this we immediately derive the following law in $L^p_{\mathcal{E}}$

$$l_1 x l_2 \cdots l_r x x x = 0, \tag{2}$$

for all $r \geq 1$ and $l_1, \ldots, l_r \in L^p_{\mathcal{E}}$.

We will next calculate identities in $M_{\mathcal{E}}^{(0,1^m)}$ for $m \geq 1$. We follow the general approach of G. Wall [21] and adopt some of his notations. The general idea is simple. For each pair of integers (r, s) with $r \geq 1$ and $s \geq 4$ we expand the identity

$$1 = [f_1 \cdots f_r, f_{r+1} \cdots f_{r+s}, f_{r+1} \cdots f_{r+s}, f_{r+1} \cdots f_{r+s}, f_{r+1} \cdots f_{r+s}],$$

and use the Hall's collection process to rewrite this as a law of the form

$$w_1 \cdots w_l = 1.$$

Where $w_1, \ldots, w_l$ are like in Lemma 3.1. From this we deduce the law

$$w_l = 1.$$

But we can write $w_l = wu$ where w is a product of simple commutators of weight $(0, 1^{r+s})$ and u is a product of simple commutators of higher multi-weight. From this we obtain the associated multilinear Lie identity

$$t^{(r,s)}(x_1, \ldots, x_r; x_{r+1}, \ldots, x_{r+s}) = 0, \tag{3}$$

where $t^{(r,s)} = l(w)$. Now for the details: first we deal with the case $r = 1$. In this case we start with

$$1 = [f_1, f_2 \cdots f_{1+s}, f_2 \cdots f_{1+s}, f_2 \cdots f_{1+s}, f_2 \cdots f_{1+s}]$$

and using the identity (1) it is not difficult to see that we obtain

$$t^{(1,s)}(x_1; x_2, \ldots, x_{1+s}) = \sum_{(S(1),S(2),S(3),S(4))} x_1 x_{S(1)} x_{S(2)} x_{S(3)} x_{S(4)}, \tag{4}$$

where the sum is taken over all partitions of $\{2, \ldots, 1 + s\}$ into non-empty pairwise disjoint sets. In particular we have that

$$t^{(1,4)}(x_1; x_2, \ldots, x_5) = \sum_{\sigma \in \mathrm{Sym}\{2,3,4,5\}} x_1 x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} x_{\sigma(5)},$$

and the identity $t^{(1,4)} = 0$ is the linearised 4-Engel identity.

Now we deal with the case when $r \geq 2$. We can here make use of Lemma 3.3. Replacing f_0 by $f_{r+1} \cdots f_{r+s}$ in Lemma 3.3 leads to an identity of the form

$$\sum_{(S(1),S(2),S(3),S(4))} x_1 x_{S(1)} x_2 \cdots x_r x_{S(2)} x_{S(3)} x_{S(4)} = y.$$

Where the sum is taken over all partitions of $\{r+1, \ldots r+s\}$ into non-empty pairwise disjoint sets $S(1), \ldots, S(4)$ and y is the Lie word associated to the linearisation of u from Lemma 3.3. In case $s = 4$ we must have $y = 0$ as u is a product of simple commutators with a least 5 occurrences of f_0. This namely implies that after replacing f_0 by $f_{r+1}f_{r+2}f_{r+3}f_{r+4}$ and expanding no commutator will have multiweight $(0, 1^{r+4})$. So for $s = 4$ we obtain

$$t^{(r,4)} = \sum_{\sigma \in Sym\{r+1,r+2,r+3,r+4\}} x_1 x_{\sigma(r+1)} x_2 \cdots x_r x_{\sigma(r+2)} x_{\sigma(r+3)} x_{\sigma(r+4)}. \tag{5}$$

In fact we will only use identities (2), (4) and (5).

4 The algebra $L_{\mathcal{E}}^p$ for $p \neq 2, 3$

Let p be a fixed prime and $L = L_{\mathcal{E}}^p$. In the following we will also be working within $A = A_{\mathcal{E}}^p$ where $A_{\mathcal{E}}^p$ is the associative subalgebra of End (L), the algebra of linear maps from L to L, generated by $\{\mathrm{ad}\,(u) : u \in L\}$. To distinguish between elements from L and A we will use small letters for elements in L and capital letters for elements in A.

Let

$$A^* = \bigcup_{i=4}^{\infty} \underbrace{A \times \cdots \times A}_{i}.$$

We let $E : A^* \to A$ be the function given by

$$E(Y_1, Y_2, \ldots, Y_m) = \sum_{(S(1),\ldots,S(4))} Y_{S(1)} Y_{S(2)} Y_{S(3)} Y_{S(4)}$$

where the sum is taken over all partitions $(S(1), \ldots, S(4))$ of $\{1, 2, \ldots, m\}$ into non-empty subsets. It follows from section 3 that $L_{\mathcal{E}}^p$ satisfies

$$uE(Y_1, Y_2, \ldots, Y_m) = 0$$

for all $u \in L$ and $Y_1, \ldots, Y_m \in \mathrm{ad}\,(L)$ when $m \geq 4$.

We want to show that the ideal generated by x in $L_{\mathcal{E}}^p$ is nilpotent. For the primes $p \geq 5$ this can be dealt with quickly using known results. We have seen that $L_{\mathcal{V}}^p$ satisfies the linearised 4-Engel identity which is equivalent to the 4-Engel identity when $p \geq 5$. But in [17] it is shown that all principal ideals in a 4-Engel Lie algebra are nilpotent of class at most 3 if the underlying field has characteristic greater than 5. It follows that any locally nilpotent 4-Engel group without elements of order $2, 3$ or 5 is Fitting with Fitting degree at most 3 which is clearly the best possible bound.

Dealing with $L_{\mathcal{V}}^5$ requires more work. In [14, Lemma 6] the authors prove that this Lie algebra is nilpotent of class 6 and that the ideal generated by x is nilpotent of class 4 (although the statement there is given in the context of 4-Engel groups of exponent 5, the only Lie identities used come from $I_{\mathcal{E}}$). The result was obtained with the aid of a computer. The calculations can also be done by hand using a similar approach as for the case $p = 3$ in Section 6. It follows that 4-Engel 5-groups have Fitting degree at most 4. By the example of Gupta and Levin that we mentioned in the introduction, this bound is sharp.

This leaves the primes 2 and 3 to be dealt with. These two cases differ from the others in that the Lie algebras $L^3_{\mathcal{E}}$ and $L^2_{\mathcal{E}}$ are not nilpotent. The approach will be quite different for the two primes, the case $p = 2$ being considerably harder. We will give the proofs in sections 5 and 6.

5 The algebra $L^2_{\mathcal{E}}$

In this section we deal with the case when $p = 2$. We let $L = L^2_{\mathcal{E}}$ and $A = A^2_{\mathcal{E}}$. We will work with the function $E : A^* \to A$ that was described in section 4. The aim is show that $\mathrm{Id}\,(x)^5 = 0$. The following consequence of the Engel identities is very special for the prime 2 case and we will make much use of it later.

Lemma 5.1 *Let $U_1, \ldots, U_r \in A$ and $1 \leq i \leq r-1$. If for all $\sigma \in Sym(r)$, the product $U_{\sigma(1)} \cdots U_{\sigma(r)}$ is symmetric in U_i, U_{i+1}, then*

$$E(U_1, \ldots, U_i, U_{i+1}, \ldots U_r) = E(U_1, \ldots, U_{i-1}, U_iU_{i+1}, U_{i+2}, \ldots U_r).$$

Proof We have $E(U_1, \ldots, U_i, U_{i+1}, \ldots, U_r)$ is a sum

$$V_1 + V_2 + \cdots + V_s,$$

where $V_1, \ldots, V_s$ are all the products of the form $U_{S(1)}U_{S(2)}U_{S(3)}U_{S(4)}$, where $(S(1), S(2), S(3), S(4))$ is a partition of $\{1, 2, \ldots, r\}$ into non-empty sets. Suppose that for each product V_k, $i \in S(\alpha(k))$ and $i+1 \in S(\beta(k))$. Let $\sum_1$ be the sum of all V_k where $\alpha(k) < \beta(k)$. Let $\sum_2$ be the sum of all V_k where $\alpha(k) > \beta(k)$. Finally let $\sum_3$ be the sum of all V_k where $\alpha(k) = \beta(k)$. As every product is symmetric in U_i, U_{i+1} by assumption, we have that $\sum_1 = \sum_2$. Hence

$$\begin{aligned} E(U_1, \ldots, U_i, U_{i+1}, \ldots U_s) &= \sum_1 + \sum_2 + \sum_3 \\ &= \sum_3 \\ &= E(U_1, \ldots, U_{i-1}, U_iU_{i+1}, U_{i+2}, \ldots, U_s). \quad \square \end{aligned}$$

As we have seen, it follows from the Engel identities that L satisfies the linearised 4-Engel identity. When the characteristic is 2, more is true. The next lemma shows that modulo the centre we have that L satisfies the linearised 3-Engel identity.

Lemma 5.2 *Let $w, u_1, u_2, u_3 \in L$ and let $W = ad(w)$ and $V_i = ad(u_i)$ then*

$$\sum_{\sigma \in \mathit{Sym(3)}} U_{\sigma(1)} U_{\sigma(2)} U_{\sigma(3)} W = 0.$$

Proof From the linearised 4-Engel identity we have.

$$\begin{aligned}
0 &= \sum_{\sigma\in\mathrm{sym}(4)} w u_{\sigma(1)} u_{\sigma(2)} u_{\sigma(3)} u_{\sigma(4)} \\
&= \sum_{\sigma\in\mathrm{sym}(3)} w u_4 u_{\sigma(1)} u_{\sigma(2)} u_{\sigma(3)} + \sum_{\sigma\in\mathrm{sym}(4)} w u_{\sigma(1)} u_4 u_{\sigma(2)} u_{\sigma(3)} + \\
&\quad \sum_{\sigma\in\mathrm{sym}(4)} w u_{\sigma(1)} u_{\sigma(2)} u_4 u_{\sigma(3)} + \sum_{\sigma\in\mathrm{sym}(4)} w u_{\sigma(1)} u_{\sigma(2)} u_{\sigma(3)} u_4 \\
&= \sum_{\sigma\in\mathrm{sym}(3)} u_4 w u_{\sigma(1)} u_{\sigma(2)} u_{\sigma(3)} + \sum_{\sigma\in\mathrm{sym}(4)} u_4 (w u_{\sigma(1)}) u_{\sigma(2)} u_{\sigma(3)} + \\
&\quad \sum_{\sigma\in\mathrm{sym}(4)} u_4 (w u_{\sigma(1)} u_{\sigma(2)}) u_{\sigma(3)} + \sum_{\sigma\in\mathrm{sym}(4)} u_4 (w u_{\sigma(1)} u_{\sigma(2)} u_{\sigma(3)}) \\
&= 4 \sum_{\sigma\in\mathrm{sym}(4)} u_4 w u_{\sigma(1)} u_{\sigma(2)} u_{\sigma(3)} + 6 \sum_{\sigma\in\mathrm{sym}(4)} u_4 u_{\sigma(1)} w u_{\sigma(2)} u_{\sigma(3)} + \\
&\quad 4 \sum_{\sigma\in\mathrm{sym}(4)} u_4 u_{\sigma(1)} u_{\sigma(2)} w u_{\sigma(3)} + \sum_{\sigma\in\mathrm{sym}(4)} u_4 u_{\sigma(1)} u_{\sigma(2)} u_{\sigma(3)} w \\
&= \sum_{\sigma\in\mathrm{sym}(4)} u_4 u_{\sigma(1)} u_{\sigma(2)} u_{\sigma(3)} w \quad \Box
\end{aligned}$$

We will next see that the previous lemma still holds if V_1 is replaced by X^2.

Lemma 5.3 *Let $V_1 = X^2$ and $V_2 = ad(v_2)$, $V_3 = ad(v_3)$, $W = ad(w) \in$*

ad(L). *Then*

$$0 = \sum_{\sigma \in Sym(3)} V_{\sigma(1)} V_{\sigma(2)} V_{\sigma(3)} W.$$

Proof Let $V_4 = \mathrm{ad}(v_4) \in \mathrm{ad}(L)$. Using Lemma 5.1, we have

$$\begin{aligned} 0 &= wE(X, X, V_2, V_3, V_4) \\ &= wE(X^2, V_2, V_3, V_4) \\ &= \sum_{\sigma \in \mathrm{Sym}(4)} wV_{\sigma(1)} V_{\sigma(2)} V_{\sigma(3)} V_{\sigma(4)}. \end{aligned}$$

Now $u(vX^2) = u(vxx) = uvx^2 + ux^2v = uvX^2 + uX^2v$ as we are working over a field of characteristic 2. We can thus finish the proof just as in the proof of Lemma 5.2. □

The following lemma is a useful application of the previous lemmas.

Lemma 5.4 *Let* $W \in ad(L)$. *Then*

$$\begin{aligned} A_i X A_j W &= A_j X A_i W \\ A_i X^2 A_j W &= A_j X^2 A_i W \\ X A_i X^2 W &= X^2 A_i X W. \end{aligned}$$

Proof From Lemma 5.2 we have

$$\begin{aligned} 0 &= A_i A_j X W + A_j A_i X W + X A_i A_j W + \\ &\quad X A_j A_i W + A_i X A_j W + A_j X A_i W \\ &= A_i X A_j W + A_j X A_i W. \end{aligned}$$

This proves the first part. The other parts are proved in a similar way using Lemma 5.3. □

Our aim is to prove that all Lie products with 5 occurrences of x are trivial. The next two propositions are steps towards this. The first one is an immediate corollary from the first two parts of the lemma above.

Proposition 5.5 *The Lie product*

$$x a_1 \cdots a_r x a_{r+1} \cdots a_s x a_{s+1} \cdots a_t x a_{t+1} \cdots a_k x$$

is symmetric in $a_1, \ldots, a_k$ *for all* $0 \le r \le s \le t \le k$.

Proposition 5.6 *Let $u_1, \ldots, u_r, w_1, \ldots, w_s \in \{x, a_1, a_2, \ldots\}$ where $s \geq 1$ and let $u = u_1 \cdots u_r$. Suppose all products with the same multidegree as $ua_ia_jxa_kw_1 \cdots w_s$ are symmetric in the a_i's that occur. Then*

$$ua_ia_jxa_kw_1 \cdots w_s = ua_kxa_ia_jw_1 \cdots w_s.$$

Similarly if all products of the same multidegree as $ua_ia_jx^2a_kw_1 \cdots w_s$ are symmetric in the a_i's, then

$$ua_ia_jx^2a_kw_1 \cdots w_s = ua_kx^2a_ia_jw_1 \cdots w_s.$$

Proof Let $V_1 = A_iA_j$, $V_2 = A_k$, $V_3 = X$ and $V_4 = \mathrm{ad}(u)$. Also let $W_i = \mathrm{ad}(w_i)$ and $W = W_2 \cdots W_s$. Using Lemma 5.1, we have

$$\begin{aligned} 0 &= w_1E(A_i, A_j, A_k, X, V_4)W \qquad (6) \\ &= w_1E(A_iA_j, A_k, X, V_4)W \\ &= \sum_{\sigma \in \mathrm{Sym}(4)} w_1V_{\sigma(1)}V_{\sigma(2)}V_{\sigma(3)}V_{\sigma(4)}W. \end{aligned}$$

Before finishing the proof we make one observation. Suppose that the sum of the multidegrees of the elements $z, v, a_i, a_j, z_1, \ldots, z_l$ is the multidegree of $ua_ia_jxa_kw_1 \cdots w_s$. Then

$$\begin{aligned} z(vV_1)z_1 \cdots z_l &= z(va_ia_j)z_1 \cdots z_l \\ &= zva_ia_jz_1 \cdots z_l + za_ja_ivz_1 \cdots z_l + \\ &\quad\; za_iva_jz_1 \cdots z_l + za_jva_iz_1 \cdots z_l \\ &= zvV_1z_1 \cdots z_l + zV_1vz_1 \cdots z_l \end{aligned}$$

where for the second last identity we made use of the symmetry hypothesis. Because of this we can, just as in the proof of Lemma 5.2, deduce from (6) that

$$0 = \sum_{\sigma \in \mathrm{Sym}(3)} uV_{\sigma(1)}V_{\sigma(2)}V_{\sigma(3)}W_1W.$$

But

$$\begin{aligned} \sum_{\sigma \in \mathrm{Sym}(3)} V_{\sigma(1)}V_{\sigma(2)}V_{\sigma(3)} &= (A_iA_j)A_kX + A_k(A_iA_j)X + X(A_iA_j)A_k + \\ &\quad\; XA_k(A_iA_j) + A_kX(A_iA_j) + (A_iA_j)XA_k \end{aligned}$$

and by the symmetry hypothesis the first part of the lemma follows.

The second part is proved similarly. First we apply the Engel identities.

$$\begin{aligned} 0 &= w_1 E(A_i, A_j, A_k, X, X, U) \\ &= w_1 E(A_i, A_j, A_k, X^2, U) \\ &= w_1 E(A_i A_j, A_k, X^2, U). \end{aligned}$$

As before this gives

$$0 = \sum_{\sigma \in \mathrm{Sym}(3)} u V_{\sigma(1)} V_{\sigma(2)} V_{\sigma(3)} W_1 W$$

where $(V_1, V_2, V_3) = (A_i A_j, A_k, X^2)$. The rest of the proof follows just as for the first part. $\Box$

We are now ready for the main result of this section.

Proposition 5.7 *The ideal generated by x in $L^2_{\mathcal{E}}$ is nilpotent of class at most 4.*

Proof We use the propositions above to show that the product

$$x a_1 \cdots a_r x a_{r+1} \cdots a_s x a_{s+1} \cdots a_t x a_{t+1} \cdots a_m x$$

is zero. We use induction on m. This is clear from identity (2) when $m = 1$. Now suppose that $m \geq 2$ and that we know that all products of the type above with shorter length are 0. Then all products of the same multidegree not ending in x are 0 by induction hypothesis. By Proposition 5.5, we have that all products of this multidegree are symmetric. Now all the propositions above are available to us. Because of Proposition 5.6 and identity (2) all products are equal to either 0 or one of the following.

$$\begin{array}{l} x a_1 x a_2 x a_3 x a_4 \cdots a_m x; \\ x a_1 x a_2 x a_3 \cdots a_m x^2; \\ x a_1 x a_2 x^2 a_3 \cdots a_m x; \\ x a_1 x^2 a_2 x a_3 \cdots a_m x; \\ x a_1 x^2 a_2 \cdots a_m x^2. \end{array} \tag{7}$$

Suppose we take one of these products of the form $xa_1xa_2xu_1\cdots u_{m+1}$. By the Jacobi identity, we have

$$\begin{aligned}0 &= [(xa_1)(xa_2)x + (xa_2x)(xa_1) + x(xa_1)(xa_2)]u_1\cdots u_{m+1}\\ &= [(xa_1)(xa_2)x + (xa_2x)(xa_1) - (xa_1x)(xa_2)]u_1\cdots u_{m+1}\\ &= (xa_1)(xa_2)xu_1\cdots u_{m+1},\end{aligned}$$

by the symmetry hypothesis. It follows that

$$xa_1xa_2xu_1\cdots u_{m+1} = xa_1a_2x^2u_1\cdots u_{m+1}.$$

So we only need to consider the last two products in (7). Now using Lemma 5.4 and Proposition 5.6 we have

$$\begin{aligned}xa_1x^2a_2xa_3\cdots a_mx &= xa_1xa_2x^2a_3\cdots a_mx\\ &= a_1x^2a_2xxa_3\cdots a_mx\\ &= a_1xa_2x^2xa_3\cdots a_mx\\ &= 0\end{aligned}$$

by identity (2) and also

$$\begin{aligned}xa_1x^2a_2\cdots a_mx^2 &= xa_1\cdots a_{m-1}x^2a_mxx\\ &= xa_1\cdots a_{m-1}xa_mx^2x\\ &= 0.\end{aligned}$$

We thus see that all the products are 0. □

From this result and Theorem 2.4 we have that any locally finite 4-Engel 2-group is Fitting with Fitting degree at most 4. One can give an explicit example that shows that this bound is sharp. We have however chosen to omit this as the referee has pointed out that this can also be derived from [15]. In this paper Nickel has computed the largest nilpotent quotient of the free 3-generator 4-Engel group. From this one can see that $[z, y, y, y, x, y]$ has order 10 modulo the 7-th term of the lower central series. Hence there are finite 3-generator 2-groups and 5-groups of class 6 which have Fitting degree 4.

6 The algebra $L^3_{\mathcal{E}}$

Let $L = L^3_{\mathcal{E}}$. We let V be the subspace of L generated by $\{a_1, a_2, \ldots\}$.

Lemma 6.1 *Let $a, b \in V$. Then*

$$\begin{aligned} xbxa &= -xaxb + 2xabx & (8)\\ xbx^2a &= xabx^2 & (9)\\ xax^2b &= xabx^2. & (10) \end{aligned}$$

We also have that every product of weight $(3,1,1)$ in x, a and b is in the linear span of $xaxbx$ and $xabx^2$. Every product of weight $(4,1,1)$ is trivial.

Proof The first identity follows from $xb(xa) = -xa(xb)$. The Jacobi identity then gives us

$$\begin{aligned} 0 &= xa(xb)x + xbx(xa) + x(xa)(xb)\\ &= 2xaxbx - xbxax - xabx^2 - xax^2b + xbx^2a\\ &= xbx^2a - xax^2b \end{aligned}$$

where we used (8) in the last equality. Also the Engel identities give

$$\begin{aligned} 0 &= t^{(1,4)}(x; a, b, x, x)\\ &= 2xabx^2 + xaxbx + xbxax + xax^2b + xbx^2a\\ &= xabx^2 - xax^2b \end{aligned}$$

and we have (9) and (10).

We next turn to products of weight $(4,1,1)$. As $xax^3 = 0$ by (2), we only need to consider products ending in x. As $xax^2bx = xabx^3 = 0$ it follows from (8)-(10) that all products are multiples of $xaxbx^2$ and all products not ending in xx are trivial. Now we apply the Engel identities and see that

$$\begin{aligned} 0 &= t^{(1,4)}(x; bx, a, x, x)\\ &= x(bx)ax^2 + x(bx)xax + x(bx)x^2a\\ &\quad +xa(bx)x^2 + xax(bx)x + xax^2(bx)\\ &= xbxax^2 - xaxbx^2 + xaxbx^2\\ &= xbxax^2. \end{aligned}$$

Thus all products of weight $(4,1,1)$ are trivial. $\Box$

Lemma 6.2 *Let $a, b, c \in V$ then all products of weight $(2,1,1,1)$ in x, a, b and c are multiples of $xaxbc$ and symmetric in a, b and c. Furthermore*

$$xabxc = xabcx = xaxbc \tag{11}$$

Proof The Engel identities and Lemma 6.1 give us

$$\begin{aligned} 0 &= t^{(1,4)}(b; x, x, a, a) \\ &= bx^2a^2 + bxaxa + bxaax \\ &= -xbxa^2 - xabxa - xa^2bx \\ &= xaxba - 3xabxa - xa^2bx \end{aligned}$$

and

$$xa^2bx = xaxab = xa^2xb. \tag{12}$$

Also

$$\begin{aligned} 0 &= t^{(1,4)}(x; b, x, a, a) \\ &= xbxa^2 + 2xaxab + 2xabxa + xaaxb + 3xaabx \\ &= 3xaxab + xbxa^2 + 2xabxa. \end{aligned}$$

Using this and Lemma 6.1 we have $xbxa^2 = xabxa = -xaxba - xbxaa = -xa^2xb - xbxa^2$ and thus

$$xbxa^2 = xa^2xb = xa^2bx. \tag{13}$$

Now replace a by $a + c$ in (13) and we have (11). As $xabcx$ is symmetric in a b and c the lemma follows. $\square$

Lemma 6.3 *Let $a, b, c \in V$ then any product of weight $(3,1,1,1)$ in x, a, b and c is a multiple of $xaxbxc$ and symmetric in a, b and c. Furthermore we have*

$$xabcx^2 = xaxbcx = xabxcx = xabx^2c = xax^2bc = xaxbxc \tag{14}$$

Proof The proof is on the same line as the proof of the previous lemma. First we consider products of weight $(3,2,1)$ in x, a and b. We know from the previous lemmas that all such products are in the span of xa^2xbx, xa^2x^2b,

$xaxbxa$ and $xabx^2a$. Using the Jacobi identity and Lemmas 6.1 and 6.2. We have

$$\begin{aligned} 0 &= xaa(xb)x + xbx(xaa) + x(xaa)(xb) \\ &= xa^2xbx - xa^2bx^2 + xbx^2a^2 + xbxaxa \\ &\quad +xbxa^2x - xa^2x^2b + xa^2xbx \\ &= 2xa^2xbx - xa^2x^2b + xabx^2a - xaxbxa + 2xabx^2a \end{aligned}$$

and thus

$$xa^2xbx + xa^2x^2b + xaxbxa = 0. \tag{15}$$

Also

$$\begin{aligned} 0 &= xab(xa)x + xax(xab) + x(xab)(xa) \\ &= xabxax - xabax^2 + xax^2ab - xaxaxb \\ &\quad -xaxbxa + xaxabx - xabx^2a + xabxax \\ &= 2xa^2xbx - xaxbxa - xabx^2a \end{aligned}$$

which gives together with (15)

$$xabx^2a = xa^2xbx. \tag{16}$$

We get the third identity by applying the Engel identities. Now

$$\begin{aligned} 0 &= t^{(1,4)}(xb; a, a, x, x) \\ &= xba^2x^2 + xbaxax + xbxa^2x + xbx^2a^2 + xbxaxa + xbax^2a \\ &= xabx^2a - xaxbxa + 2xabx^2a + xabx^2a \end{aligned}$$

and therefore

$$xabx^2a = xaxbxa. \tag{17}$$

It follows by these equations and the previous lemmas that all non-zero products of this multiweight and beginning in x are equal. In particular

$$xa^2xbx = xbxa^2x = xa^2xbx = xbx^2a^2 = xa^2bx^2.$$

Replacing a by $a + c$ gives then

$$xacx^2b = xbxacx = xacxbx = xcx^2ab = xabcx^2.$$

All these products are in particular symmetric in a, b and c. Notice that these are all the left normed products of this multiweight where we have somewhere in the product two adjacent elements from V. Next we apply

$$xaxaxc = xa^2cx = xcxaxa.$$

This gives

$$\begin{aligned} xaxbxc + xbxaxc &= 2xabcx^2 \\ xcxaxb + xcxbxa &= 2xabcx^2. \end{aligned}$$

Hence $xaxbxc = -xbxaxc - xabcx^2 = xbxcxa$ and thus

$$xaxbxc = xbxcxa = xcxaxb. \tag{18}$$

We thus have that all products are in the linear span of $xaxbxc$ and $xbxaxc$. Now the last application of the Engel identities. We have

$$\begin{aligned} 0 &= a(xb)cx^2 + a(xb)xcx + a(xb)x^2c \\ &\quad +ax(xb)cx + ax(xb)xc + axc(xb)x \\ &\quad +axx(xb)c + ax^2c(xb) + axcx(xb) \\ &= 3(xabcx^2 + 0 + (xabx^2c - xaxbxc) + 0 \\ &\quad +(xaxbxc - xax^2bc) + (xaxcbx - xaxcxb) + 0 \\ &= xabcx^2 - xaxcxb. \end{aligned}$$

The lemma now follows. □

Lemma 6.4 *Let $a, b, c \in V$. All products of weight $(4, 1, 1, 1)$ in x, a, b and c are trivial.*

Proof If such a product ends in an element of V it is trivial by Lemma 6.1 and if it ends in x it is a multiple of $xabcx^3 = 0$ by (2) and Lemma 6.3. □

We are now ready for more general results.

Proposition 6.5 *Let $n \geq 3$. All products of multiweight $(2, 1^n)$ in $x, a_1, \ldots, a_n$ are symmetric in the a_i's and multiples of $xa_1xa_2 \cdots a_n$.*

Proof We use induction on n. By Lemma 6.2 this is true when $n = 3$. Now suppose this is true for n and let c be a product of multiweight $(2, 1^{n+1})$ in $x, a_1, \ldots, a_{n+1}$. By the induction hypothesis we have that c is symmetric and a multiple of $xa_1xa_2\cdots a_{n+1}$ if c ends in some a_i. So we only need to consider $c = xa_1 \ldots a_{n+1}x$. This is clearly symmetric and thus all products are symmetric. It remains to show that c is a multiple of $xa_1xa_2\cdots a_{n+1}$. We consider two cases. Firstly suppose n is odd. Modulo multiples of $xa_1xa_2\cdots a_{n+1}$, we have

$$\begin{aligned} xa_1\cdots a_{n+1}x &= xa_1\cdots a_{n-1}(xa_{n+1}a_n) \\ &= -xa_{n+1}a_n(xa_1\cdots a_{n-1}) \\ &= (-1)^n xa_1\cdots a_{n+1}x \\ &= -xa_1\cdots a_{n+1}x \end{aligned}$$

and thus c is a multiple of $xa_1xa_2\cdots a_{n+1}$. Now suppose that n is even. We now apply the Engel identities. Modulo multiples of $xa_1xa_2\cdots a_{n+1}$, we have

$$\begin{aligned} 0 &= 2a_{n+1}(xa_1\cdots a_{n-2})a_{n-1}a_nx + 2a_{n+1}xa_{n-1}a_n(xa_1\cdots a_{n-2}) \\ &= xa_1\cdots a_{n+1}x + (-1)^{n-2}xa_1\cdots a_{n+1}x \\ &= 2c \end{aligned}$$

and c is again a multiple of $xa_1xa_2\cdots a_{n+1}$. $\square$

Proposition 6.6 *Let $n \geq 3$. All products of weight $(3, 1^n)$ in $x, a_1, \ldots, a_n$ are symmetric in the a_i's and multiples of $xa_1xa_2xa_3\cdots a_n$.*

Proof When $n = 3$ this follows from Lemma 6.3. We now assume the lemma is true for some $n \geq 3$. By the induction hypothesis and Proposition 6.5 all products of weight $(3, 1^{n+1})$ in $x, a_1\cdots, a_{n+1}$ are symmetric in the a_i's. It remains to show that $c = xa_1xa_2\cdots a_{n+1}x$ is a multiple of $xa_1xa_2xa_3\cdots a_{n+1}$. So we calculate modulo multiples of $xa_1xa_2xa_3\cdots a_{n+1}$. We use the Jacobi and Engel identities. Now

$$\begin{aligned} 0 &= (xa_1xa_2\cdots a_{n-1})(xa_{n+1}a_n) + xa_{n+1}a_n(xa_1xa_2\cdots a_{n-1}) \\ &= xa_1xa_2\cdots a_{n+1}x + (-1)^{n-2}xa_{n+1}a_n\cdots a_2(xa_1x) \end{aligned}$$

and

$$\begin{aligned} 0 &= 2a_{n+1}(xa_1xa_2\cdots a_{n-2})a_{n-1}a_nx + 2a_{n+1}xa_na_{n-1}(xa_1xa_2\cdots a_{n-2}) \\ &= xa_1xa_2\cdots a_{n+1}x + (-1)^{n-3}xa_{n+1}a_n\cdots a_2(xa_1x). \end{aligned}$$

It follows from these two identities that c is a multiple of $xa_1xa_2xa_3\cdots a_{n+1}$.
$\square$

Now we can prove the main result of this section.

Theorem 6.7 *The ideal generated by x in $L^3_{\mathcal{E}}$ is nilpotent of class at most* 3.

Proof Let c be any product of multiweight $(4, 1^n)$ in $x, a_1, \ldots, a_n$. We prove by induction on $n \geq 1$ that c is trivial. It follows from Lemma 6.1 and 6.4 that this is true if $n \leq 3$. Now suppose the result is true for some $n \geq 3$. Let $c = xa_1xa_2xa_3\cdots a_{n+1}x$. It follows from Proposition 6.6 and the induction hypothesis that it suffices to show that c is trivial. We know that all products ending in a_i are trivial. We thus have

$$\begin{aligned} 0 &= (xa_1xa_2xa_3\cdots a_{n-1})(xa_{n+1}a_n) + xa_{n+1}a_n(xa_1xa_2xa_3\cdots a_{n-1}) \\ &= xa_1xa_2xa_3\cdots a_{n+1}x + (-1)^{n-3}xa_{n+1}a_n\cdots a_2(xa_1xa_2x) \end{aligned}$$

and

$$\begin{aligned} 0 &= 2a_{n+1}(xa_1xa_2xa_3\cdots a_{n-2})a_{n-1}a_nx + 2a_{n+1}xa_na_{n-1}(xa_1xa_2xa_3\cdots a_{n-2}) \\ &= xa_1xa_2xa_3\cdots a_{n+1}x + (-1)^{n-4}xa_{n+1}a_n\cdots a_3(xa_1xa_2x). \end{aligned}$$

It follows from these identities that c is trivial. $\square$

As a corollary to Theorem 2.4 and the work in sections 4-6, we get now the main result of this paper.

Theorem 6.8 *Let G be a locally nilpotent* 4*-Engel group. Then G is a Fitting group of degree at most* 4*. If G has no elements of order* 2 *or* 5 *then G has Fitting degree at most* 3.

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