Power groups of free abelian groups of finite rank

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Introduction

Let G be a group with identity element e. The set $\mathcal{P}(G)$ of subsets of G is usually called the power set of G. There is an induced semigroup structure on $\mathcal{P}(G)$. Let $H \in \mathcal{P}(G)$ be an idempotent, so $HH = H$. We are interested in subsemigroups of $\mathcal{P}(G)$ which are actually groups and have H as multiplicative identity. We will refer to these as *power groups* of G.

To take an obvious example, let H, N be subgroups of G with $H \trianglelefteq N$, then the set of cosets N/H is a power group of G. So every section of G is a power group of G. More generally for any idempotent H, let N be a subgroup of the normaliser $N_G(H)$ of the set H. Then $\mathcal{G} = \{xH : x \in N\}$ is a power group. Such a construction is called a *quasiquotient group*. Clearly every quasiquotient group of G is isomorphic to a section. In [3] it is shown that a power group with identity H is always a quasiquotient group provided $e \in H$.

However, this is not true in general if $e \notin H$. Let $G = \mathbb{Q}$, the additive group of rationals, and let $\mathbb{R}$ be the additive group of real numbers. Let

$$\mathcal{G} = \{I_\alpha : \alpha \in \mathbb{R}\}$$

where $I_\alpha = \{x \in \mathbb{Q} : x < \alpha\}$. Then $\mathcal{G}$ is a power group of G that is isomorphic to $\mathbb{R}$. What we have here is of course the well known *Dedekind cuts* construction. It is clear that $\mathcal{G}$ is not a quasiquotient group of G in fact it is not isomorphic to a section of G as it is not countable.

In this example, if we replace each I_α by the subset $J_\alpha = \{x \in \mathbb{R} : x < \alpha\}$ of $\mathbb{R}$, then $\mathcal{G}$ is isomorphic to the group $\{J_\alpha : \alpha \in \mathbb{R}\}$ which is a quasiquotient group of $\mathbb{R}$. Such types of extensions were dealt with in [4], where the author proves some interesting results about criteria for the existence of such extensions.

In view of our remarks, one class of groups of particular interest consists of those groups for which every non-empty idempotent contains the group identity e. In this event all power groups are isomorphic to sections of G. These groups are called *monoidal* groups and are dealt with in [2]. The authors prove that for a large class of groups these groups coincide with the class of groups that are torsion-by-cyclic-by-finite or torsion. It is not difficult to see that such a group is torsion, torsion-by-cyclic or torsion-by-(infinite dihedral). Although this classification holds for a very large class of groups there are however monoidal groups that are not of this type [1].

In this paper we will mostly be interested in the power groups of free abelian groups of finite rank. We will see that even here the situation is quite complicated and we will give a rich family of examples to illustrate this.

1 General observations.

We are interested in the subsets of $\mathcal{P}(G)$ that form groups. Let $\mathbb{C}_G$ be the class of these groups.

Proposition 1 *Let $H \subseteq G$ be such that $HH = H$. There is a unique largest subset $\mathcal{G}_H$ of $\mathcal{P}(G)$ that is a group with H as the identity.*

Proof Let $\mathcal{M}_H = \{HAH : A \subseteq G\}$. Now $\mathcal{M}_H$ is the unique largest monoid which is a subset of $\mathcal{P}(G)$ and has H as an identity. Let

$$\mathcal{G}_H = \{A \in \mathcal{M}_H : AB = BA = H \text{ for some } B \in \mathcal{M}_H\}.$$

Now $\mathcal{G}_H$ is clearly a group and thus the unique largest subsemigroup of $\mathcal{P}(G)$ that is a group with H as the identity. $\Box$

Notation. In the notation of Proposition 1, a group of the form $\mathcal{G}_H$ is called the maximal power group with respect to H.

If $A \in \mathcal{P}(G)$, then we will denote by A^{-1} the set $\{a^{-1} : a \in A\}$. If $A \in \mathcal{G}_H$, we will denote by $\bar{A}$ the inverse of A in the group $\mathcal{G}_H$. A subset H of G such that $HH = H$ will be called an idempotent.

Proposition 2 *Let H be an idempotent in $\mathcal{P}(G)$. Then H^{-1} is also an idempotent and $\mathcal{G}_{H^{-1}}$ is isomorphic to $\mathcal{G}_H$.*

Proof Notice that if $A \in \mathcal{M}_H$ then A^{-1} is clearly in $\mathcal{M}_{H^{-1}}$. Also if $AB = BA = H$ then $B^{-1}A^{-1} = A^{-1}B^{-1} = H^{-1}$. We thus have two maps $\Phi : \mathcal{G}_H \to \mathcal{G}_{H^{-1}}$ and $\Psi : \mathcal{G}_{H^{-1}} \to \mathcal{G}_H$ given by $\Phi(A) = (\bar{A})^{-1}$ and $\Psi(A) = (\bar{A})^{-1}$. Then

$$\Psi(AB) = (\overline{AB})^{-1} = (\bar{B} \cdot \bar{A})^{-1} = \Psi(A) \cdot \Psi(B).$$

So Ψ is a homomorphism and by symmetry Φ is also a homomorphism. Now $A\bar{A} = H$ so $\bar{A}^{-1}A^{-1} = H^{-1}$. It follows that $(\bar{A})^{-1} = \overline{A^{-1}}$ by uniqueness of inverses. So

$$\Psi(\Phi(A)) = \Psi(\overline{A}^{-1}) = \overline{(\bar{A}^{-1})^{-1}} = A.$$

Similarly $\Phi(\Psi(A)) = A$ and thus Φ is an isomorphism. $\Box$

Let $\mathcal{G}$ be a powergroup of G with identity H. When the group identity e is in H the situation is well understood [3]. So we will only be interested in the case when $e \notin H$. In this case there is a natural partial order on G. This is defined by

$$x < y \text{ if and only if } x^{-1}y \in H$$

Then we have that $<$ is a left-invariant dense order on G. Conversely let $<$ be any left-invariant dense order on G and let $H = \{g \in G : g > e\}$, then H is an idempotent that does not contain the identity. So there is a one-to-one correspondence between the set of all idempotents H of G not containing the identity, and the set of all left-invariant dense orders on G. Notice that we could instead have defined the order by $x < y$ iff $yx^{-1} \in H$. This would have given us a right invariant dense order instead and we have an analogous one-to-one correspondence between idempotents and right-invariant orderings. If H is a normal set in G the two orders are the same.

Let H be any idempotent that is normalised by G. Then $K = HH^{-1}$ is the smallest subgroup in G containing H and clearly we have that K is normal in G. The next result tells us how the powerset group for H in K and the powerset group for H in G are related.

Proposition 3 *Let H be an idempotent of G such that $N_G(H) = G$ and let $K = HH^{-1}$, then*

$$\mathcal{G}_H/\mathcal{K}_H \cong G/K.$$

Proof Consider the map

$$\phi : \mathcal{G}_H \to \mathcal{P}(G),\ A \mapsto AK.$$

Then $\mathrm{Im}\,\phi$ clearly contains $\phi(aH) = aHK = aK$ for all $a \in G$. So $G/K \subseteq \mathrm{Im}\,\phi$.

Also $\phi(AB) = ABK = AKB = AKKB = AKBK = \phi(A)\phi(B)$ and ϕ preserves the multiplication. It follows that $\mathrm{Im}\,\phi$ is a group with identity K. So $\mathrm{Im}\,\phi \leq \mathcal{G}_K = G/K$ and ϕ is a surjective homomorphism. Now $A \in \mathrm{Ker}\,\phi$ if and only if $AK = K$, and this happens if and only if $A \subseteq K$. Hence $\mathrm{Ker}\,\phi = \mathcal{K}_H$. $\Box$

The Dedekind cuts in $\mathbb{Q}$ form a linearly ordered set with respect to inclusion. Notice that this power group is torsion-free. For elements of finite order the situation is very different.

Proposition 4 *Let A, B be distinct elements of finite order in a power group $\mathcal{G}$ of G. Then neither A nor B is contained in the other.*

Proof For contradiction suppose that A is contained in B. Let H be the identity element of $\mathcal{G}$. Let n be a positive integer such that $A^n = B^n = H$. Then

$$B = BA^n \subseteq B^n A = A$$

and $B = A$, a contradiction. $\square$

2 Free abelian groups of finite rank

In this section we will be looking for power groups of the free abelian groups of finite rank. In the following we mostly concentrate on the groups with rank 2. But most of the results generalise easily to higher ranks. We start with the much simpler task of finding all the idempotents of the integers.

Let H be any non-empty idempotent of $\mathbb{Z}$. Suppose $\mathbb{Z}H = d\mathbb{Z}$ where $d \geq 0$ (here $\mathbb{Z}H$ denotes the set of products of an integer and an element of H; this amounts to $H - H$). If $d = 0$, then $H = \{0\}$ so let us suppose that $d \geq 1$. As H/d is also an idempotent it suffices to assume that $\mathbb{Z}H = \mathbb{Z}$. If H has both positive and negative numbers. Then it is easy to see that $H = h\mathbb{Z}$ where h is the smallest positive number in H. So we can further suppose that H contains either only positive numbers or only negative numbers. Without loss of generality we may suppose that all the numbers are positive.

Let a be the smallest number in H. Since $\mathbb{Z}H = \mathbb{Z}$ there are some $h, k \in H$ such that $h - k = 1$. Let $0 \leq i \leq a$ then

$$ak + i = i(k+1) + (a-i)k = ih + (a-i)k \in H.$$

(Note that this is also true when $i = 0$ or $i = a$). As $a \in H$ we then have that $ak + ra + i \in H$ for all $r \geq 0$ and all $0 \leq i \leq a$. In other words, all numbers greater than or equal to ak are in H. This shows

Lemma 1 *Let H be an idempotent of $\mathbb{Z}$ and suppose that $\mathbb{Z}H = d\mathbb{Z}$. If H is neither empty nor a subgroup, then $(1/d)H$ is a cofinite subset of the positive numbers or the negative numbers (including 0).*

We now turn to the problem of determining the idempotents H of the free abelian group of rank 2 where $0 \notin H$. We will use some (elementary) geometric arguments so we embed $\mathbb{Z}^2$ into the standard Euclidean plane $\mathbb{R}^2$.

Proposition 5 *For each non-zero $(x, y) \in \mathbb{Z}^2$ we have that the line $\mathbb{R}(x, y)$ contains an element from H. Furthermore the elements of H on the line are all either in $\mathbb{R}^+(x, y)$ or $\mathbb{R}^-(x, y)$.*

Proof We can assume that x and y are coprime. Now choose $(u, v) \in \mathbb{Z}^2$ such that (x, y) and (u, v) generate the group $\mathbb{Z}^2$. Let

$$K = \{a \in \mathbb{Z} : a(u, v) + t(x, y) \in H \text{ for some } t \in \mathbb{Z}\}.$$

Now K is clearly an idempotent of $\mathbb{Z}$. By Lemma 1, every idempotent of $\mathbb{Z}$ contains 0. Therefore $0 \in K$ and we have proved the first statement. Also the latter statement has to be true since otherwise we would be forced to have $(0,0)$ in H. $\square$

Remark. Notice that Proposition 5 does not hold in general for higher ranks than 2. For example, take any idempotent H of $\mathbb{Z}^2 \times 0 \subseteq \mathbb{R}^3$ that does not contain the identity. Then all one dimensional subspaces, that are not contained in $\mathbb{R}^2 \times 0$, do not contain any elements from H. However for same reasons as before we can never have that $\mathbb{R}^+v$ and $\mathbb{R}^-v$ both contain elements from H when v is any non-trivial element of $\mathbb{R}^n$ and H is an idempotent of $\mathbb{Z}^n$ without the identity.

Proposition 6 *There is a line $y = wx$ where w is an irrational number and all elements of H are either above the line or under the line.*

Proof We know that for each line going through the origin and another point of $\mathbb{Z}^2$, one (and only one) of the half-lines contains elements from H. We let $K = \mathbb{Q}^+H$. Then K contains exactly one of the (strict) half-lines from each line in the rational plane. We next show that K is also an idempotent.

Let $a, b \in H$ and r, s be some positive rationals. Choose an integer n such that nr, ns are integers. Then $nra + nsb \in H$ and thus $ra + rb \in K$. This shows that $K + K \subseteq K$. Conversely suppose $a \in H$ and r some positive rational. Chose $b, c \in H$ such that $b + c = a$. Then $ra = rb + rc$. Hence $K = K + K$ and K is an idempotent.

Note that if we take two rational half-lines L_1, L_2 contained in K then the whole cone $L_1 + L_2$ that these two half-lines generate is also contained in K (as $K + K \subseteq K$). Moving to the real plane. We say that a cone (R_1, R_2) is *good* if all the rational lines between R_1, R_2 are contained in K. (We don't exclude the possibility that R_1 and R_2 are opposite lines. In that case between means either above the line or under the line, so the rational half lines contained in K are then all either above or below the line so there is no ambiguity). It is clear that a good cone exists. Now if R_1 and R_2 are not opposite then we can take some rational line through the origin such that R_1, R_2 are one the same side of this line. Now one of the half lines of this new line is contained in K. Taking the corresponding real half line and then the sum of it and the good cone, we obtain a larger good cone. By Zorn's lemma there is a maximal good cone. And the preceding argument shows that its half-lines must be opposite. Now take the corresponding real line. This line cannot have rational slope. Because otherwise, one of the half-lines would be contained in K. Take the element of H on this line that is closest to the origin and notice that it cannot be in $H + H$. $\square$

Proposition 6 characterises all the maximal idempotents of $\mathbb{Z}^2$. This can be generalised to a characterisation for $\mathbb{Z}^n$, $n \geq 2$. We describe this now briefly, leaving the proofs to the reader.

Let $\bar{a} = (a_1, a_2, \dots, a_n)$ be an arbitrary non-zero element of $\mathbb{R}^n$ and let U be the normal complement. We think of $\mathbb{R}$ as a vectorspace over $\mathbb{Q}$ and consider the linear map

$$\phi : \mathbb{Q}^n \to \mathbb{R};\ (x_1, \dots, x_n) \mapsto x_1 a_1 + \cdots + x_n a_n.$$

Then $U \cap \mathbb{Q}^n = \ker \phi$ and the rank of the abelian group $U \cap \mathbb{Z}^n$ is the same as the rank of $U \cap \mathbb{Q}^n = n - \text{rank } (\mathbb{Q}a_1 + \cdots \mathbb{Q}a_n)$. Let H_0 be a maximal idempotent of $U \cap \mathbb{Z}^n$ without the identity (notice that this always exists by Zorn's Lemma and when the rank is less than 2 then this is the empty set) then

$$H = H_0 \cup [(U + \mathbb{R}^+ a) \cap \mathbb{Z}^n]$$

is a maximal idempotent of $\mathbb{Z}^n$ without the identity. This recursively gives all the maximal idempotents of $\mathbb{Z}^n$ that do not contain the identity.

Although this characterisation is simple, it does not seem to be a simple problem to find the subidempotents. The next example illustrates this by giving a rich variety of subidempotents of $\mathbb{Z}^2$.

Example. Let w be an irrational number and embed the free abelian group of rank 2 as $G = \mathbb{Z} + \mathbb{Z}w$ in the additive group of real numbers. Let T be any set of positive real numbers such that $\inf T = 0$. Let S be the set of all subgroups of rank 2 in G (and thus all subgroups of G that are dense in $\mathbb{R}$). Take $L : T \to S$ to be an increasing function where the order in S is the inclusion order. Let

$$H_t = \{z \in L(t) : z > t\}$$

and

$$H = \bigcup_{t \in T} H_t.$$

Lemma 2 *H is an idempotent of G.*

Proof Firstly if $t_1 \leq t_2$ then $L(t_1) \leq L(t_2)$ and thus

$$\begin{aligned} H_{t_1} + H_{t_2} &= \{h \in L(t_1) : h > t_1\} + \{h \in L(t_2) : h > t_2\} \\ &\subseteq \{h \in L(t_2) : h > t_1 + t_2\} \\ &\subseteq H_{t_2}. \end{aligned}$$

Hence $H + H \subseteq H$. Now suppose $h \in H_t$. Then $h \in L(t)$ and $h > t$. As $L(t)$ is a dense subgroup of $\mathbb{R}$ there is $h_1 \in L(t)$ such that $h > h_1 + t > t$. As $h_1 > 0$ and $\inf T = 0$ there exists some $s \in T$ such that $h_1 > s$ and $t \geq s$. Since $L(s)$ is dense in $\mathbb{R}$ there is $h_2 \in L(s)$ such that $s < h_2 < h_1$, then $h > h_1 + t > h_2 + t$ As $L(s) \leq L(t)$, we have that $h - h_2 \in L(t)$ and as $h - h_2 > t$, it follows that $h - h_2 \in H_t$. Clearly $h_2 \in H_s$. Hence $h \in H_t + H_s \subseteq H + H$. $\square$

We now turn to the problem of finding the elements of $\mathcal{G}_H$. First notice that $A \in \mathcal{M}_H$ if and only if it is the union of sets of the form $a+H$, $a \in G$. If such an element has an inverse B in $\mathcal{M}_H$ then $A+B=H$ is bounded below by 0 and thus A must be bounded below. So we can restrict our attention to those elements of $\mathcal{M}_H$ that are bounded below.

Let $L(0)=\bigcap_{t\in T} L(t)$ and $\overline{G}=G/L(0)$. Let X be a transversal of $L(0)$ in G such that $0 \in X$. For an element $g \in G$ we will as usual denote the image in $\overline{G}$ by $\overline{g}$. Let $A \in \mathcal{M}_H$, then

$$A=\bigcup_{x\in X} A(x)$$

where

$$A(x)=\bigcup\{z+H \subseteq A : \overline{z}=\overline{x}\}.$$

Suppose that $a<b$ and $\overline{a}=\overline{b}$. Let $t \in T$. As $a+L(t)=b+L(t)$ we have

$$\begin{aligned} b+H_t &= \{z \in L(t)+a : z>t+b\} \\ &\subseteq \{z \in L(t)+a : z>t+a\} \\ &= a+H_t. \end{aligned}$$

Hence $b+H \subseteq a+H$. Now assume that A is bounded below and let $a(x)=\inf A(x)$. By the remark just made we have that for $x \in X$

$$A(x)=\bigcup_{t\in T} A_t(x)$$

where

$$A_t(x)=\{z \in L(t)+x : z>a(x)+t\}.$$

We now describe a more general type of elements. We will later see that every element in $\mathcal{G}_H$ can be written in this way. Let r be any real number and $x: T \to G$ be a function such that

$$x(s)+L(t)=x(t)+L(t) \quad \text{when} \quad s \le t.$$

we let

$$G(x,r)=\bigcup_{t\in T} G_t(x,r)$$

Where

$$G_t(x,r)=\{z \in L(t)+x(t) : z>r+t\}.$$

Taking x as a constant function and $r=a(x)$ we have that $G(x,r)$ is like $A(x)$ above.

Lemma 3 *Let $x,y: T \to G$ be some functions of the above mentioned type and $r,s \in \mathbb{R}$, then*

$$G(x,r)+G(y,s)=G(x+y,r+s)$$

Proof Firstly if $t_1 \le t_2$ then $L(t_1) \le L(t_2)$ and

$$\begin{aligned} G_{t_1}(x,r) + G_{t_2}(y,s) &= \{z \in L(t_1) + x(t_1) : z > r + t_1\} + \\ &\quad \{z \in L(t_2) + y(t_2) : z > s + t_2\} \\ &\subseteq \{z \in L(t_2) + x(t_2) + y(t_2) : z > r + s + t_1 + t_2\} \\ &\subseteq G_{t_2}(x+y, r+s). \end{aligned}$$

Conversely suppose that $z \in G(x+y, r+s)$ so $z \in G_t(x+y, r+s)$ for some $t \in T$. Then $z \in L(t) + x(t) + y(t)$ and $z > r + s + t$. As $L(t) + y(t)$ is a dense subset in $\mathbb{R}$, there exists $z_1 \in L(t) + y(t)$ such that

$$z > r + t + z_1 > r + s + t.$$

Since $z_1 > s$ and $\inf T = 0$, there exists some $u \in T$ such that $z_1 > s + u$ and $t \ge u$. Since $L(u) + y(u)$ is dense in $\mathbb{R}$ there exists $z_2 \in L(u) + y(u)$ such that $s + u < z_2 < z_1$, then

$$z > r + t + z_1 > r + t + z_2 > r + s + t.$$

As $L(u) \le L(t)$, we have $z - z_2 \in L(t) + x(t) + y(t) - y(u) = L(t) + x(t)$. As we also have $z - z_2 > r + t$ we have that $z - z_2 \in G_t(x,r)$. Clearly $z_2 \in G_u(y,s)$ and thus $z \in G_t(x,r) + G_u(y,s) \subseteq G(x,r) + G(y,s)$. □

It follows in particular that the elements of this type form a group $\mathcal{G}$ with identity $G(0,0) = H$. What is this group?

Consider the inverse system $(G/L(t))_{t\in T}$ of quotients of $G/L(0)$ with the natural homomorphisms $G/L(t) \to G/L(s)$ as map when $t \le s$. The group of all functions $\hat{x} : T \to G/L(0)$, defined by $\hat{x}(t) = x(t) + L(0)$, is the inverse limit of this system. Denote this by $G(L)$. There is a natural homomorphism from $G(L) \oplus \mathbb{R}$ to $\mathcal{G}$, taking $(\hat{x}, r)$ to $G(x,r)$. It should be clear that this is well defined. To show that we have an isomorphism we need to show that if $G(x,r) = G(0,0)$, then $\hat{x} = \hat{0}$ and $r = 0$. As r is the greatest lower bound of $G(x,r)$ the latter is clear. The next lemma settles the other one.

Lemma 4 *If $G(x,0) \subseteq H$, then $\hat{x} = \hat{0}$.*

Proof We argue by contradiction. Suppose not. As $\hat{x} \ne \hat{0}$ there exists a real number r such that

$$x(t) + L(t) \ne L(t) \quad \text{when} \quad t < r. \tag{1}$$

Now let $h < r$. As h is in $G(x,0)$ and H, there exits $t_1, t_2 \le h$ such that $h \in L(t_1) + x(t_1)$ and $h \in L(t_2)$. Take t be the larger of the two. Then h is in both the cosets $L(t)$ and $L(t) + x(t)$. Hence these cosets must be equal. But this contradicts (1). □

Now suppose B is the inverse of A. If $G/L(0)$ is finite then $a(x)+b(y)$ must be zero for some x, y with $A(x), B(y)$ non-empty. Hence by Lemma 3 and Lemma 4, $A(x)+B(y)=H$. But then

$$H = A(x) + B(y) \subseteq A + B(y) \subseteq A + B = H$$

and $A + B(y) = H$. As the inverse of $B(y)$ is uniquely determined we must have $A = A(x)$. Hence for the case $G/L(0)$ finite we have that all the elements of $\mathcal{G}_H$ are of the form $G(x, r)$ for some constant function x. We have also seen that the group of these elements is isomorphic to $G(L) \oplus \mathbb{R}$. But in this case we have $G(L) = G/L(0)$ and hence:

Proposition 7 *If $G/L(0)$ is finite then $\mathcal{G}_H$ is isomorphic to $G/L(0) \oplus \mathbb{R}$.*

In particular, if H is one of the maximal idempotents then $L(0) = G$ and $\mathcal{G}_H$ is isomorphic to $\mathbb{R}$.

We now deal with the case when $G/L(0)$ is infinite we want to show that every element of $\mathcal{G}_H$ is of the form $G(x, r)$ where $\hat{x}$ is in $G(L)$. The following simple lemma is going to be crucial.

Lemma 5 *Let ε be an arbitrary small real number and let H_ε be the set of all elements in H that are smaller than ε. Then*

$$H_\varepsilon + H = H.$$

Proof One inclusion is obvious. Now let $h \in H$ as $H = H + H$ we have that $h = h_0 + h_1$ for some $h_0, h_1 \in H$ then one of them must be at most half of h. Suppose this is h_1 then we write $h_1 = h_2 + h_3$ and now one of h_2 and h_3 is at most one fourth of h. Iterate this process n times such that $(1/2)^n < \varepsilon$. Then h is a sum of elements one of which is less than ε. Let k_1 be this element and k_2 be the sum of the others. Then $h = k_1 + k_2$ is in $H_\varepsilon + H$. $\square$

Suppose $A + B = H$ then we must have $a(x) + b(y) \geq 0$ for all $x, y \in X$ with $A(x), B(y)$ non-empty. Let $a = \inf A$ and $b = \inf B$. Replacing A, B by $A + G(0, -a), B + G(0, -b)$ we can assume that the infimum is 0 for both A and B. Now let $X(\varepsilon) = \{x \in X : a(x) \leq \varepsilon\}$ and

$$A_\varepsilon = \bigcup_{x \in X(\varepsilon)} A(x).$$

Then $A_\varepsilon + B$ contains all the elements of H that are less than ε and as $A_\varepsilon + B = A_\varepsilon + B + H$ we have by the previous lemma that $A_\varepsilon + B = H$ therefore

Lemma 6 *$A = A_\varepsilon$ for all $\varepsilon > 0$.*

To make use of this we need to introduce new idempotents that are derived from H. For each $s \in T$ we let $L_s : T \to S$ be the function defined by $L_s(t) = L(t)$ if $t > s$ and $L_s(t) = L(s)$ if $t \leq s$. Let $H(s)$ be the idempotent that is defined from L_s just as H was defined from L. One can check that $H \subset H(s)$ and $H + H(s) = H(s)$. Now there is a group homomorphism from $\mathcal{G}_H$ to $\mathcal{G}_{H(s)}$ that takes A to $A + H(s)$. The elements in $A + H(s) = A_\varepsilon + H(s)$ are the same as in $A = A_\varepsilon + H$ when we are above $s+\varepsilon$. This is true for all $\varepsilon > 0$, so the elements of $A + H(s)$ are the same as in A when we are above s. As $G/L(s)$ is finite we know from the previous discussion that $A + H(s) = A(x(s)) + H(s)$ for some $x(s)$ in G. In particular we have that $a(x) = 0$ and A looks like $\{z \in L(s) + x(s) : z > s\}$ when we are above s. If we let x be the corresponding function then we have that $A = G(x, 0)$. Hence

Theorem 1 *The group $\mathcal{G}_H$ is isomorphic to $G(L) \oplus \mathbb{R}$.*

We have the same type of construction for any free abelian group of rank n by embedding it first in $\mathbb{R}$. Let T be any set of positive real numbers such that $\inf T = 0$. Let S be the set of all subgroups of rank n in G. As before let $L : T \to S$ be an increasing function. Let $G(L)$ be the corresponding inverse limit. Then $G(L) \oplus \mathbb{R}$ is the group $\mathcal{G}_H$ as before.

Our construction gives a number of maximal powergroups in a free abelian group of finite rank but complete classification seems to be very difficult. This would mean finding all possible idempotents H and we have seen that they can be quite complicated. The problem of calculating the corresponding powergroup then adds to the problem. The problem of finding all powergroups related to a finite abelian group of finite rank seems much simpler. Here we are looking for those groups that arise as subgroups of some $\mathcal{G}_H$. This includes all the subgroups of $\mathbb{R}$ and thus all torsion free abelian groups of order up to $2^{\aleph_0}$. So there is no restriction on torsion free groups. However in all the examples we have calculated, the torsion subgroup has rank at most 2. One might ask if this is always the case?

Remark There is a different way to find many of the groups arising from the construction applying Proposition 3. Let r be any irrational number and let $F = \mathbb{Z} + r\mathbb{Z}$. Furthermore we take G to be the subgroup $m\mathbb{Z} + nr\mathbb{Z}$. Let H be the set of all positive elements of G. One can prove without much difficulty that $\mathcal{G}_H$ is isomorphic to $\mathbb{R}$. As H generates G we then have by Proposition 3 that

$$\mathcal{F}_H/\mathcal{G}_H \cong F/G \cong \mathbb{Z}_n \oplus \mathbb{Z}_m.$$

So $\mathcal{F}_H$ is an extension of $\mathbb{R}$ by $\mathbb{Z}_n \oplus \mathbb{Z}_m$. But a divisible group is always a direct summand and hence $\mathcal{F}_H = \mathbb{R} \oplus \mathbb{Z}_n \oplus Z_m$.

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