# On Torsion-by-Nilpotent Groups

**G. Endimioni**

*C.M.I. - Université de Provence, UMR-CNRS 6632,*
*39, rue F. Joliot-Curie, 13453 Marseille Cedex 13, France*
E-mail: `endimion@gyptis.univ-mrs.fr`

and

**G. Traustason**

*Department of Mathematical Sciences, University of Bath,*
*Bath BA2 7AY, United Kingdom*
E-mail: `masgt@maths.bath.ac.uk`

**Abstract.** Let $\mathcal{C}$ be a class of groups, closed under taking subgroups and quotients. We prove that if all metabelian groups of $\mathcal{C}$ are torsion-by-nilpotent, then all soluble groups of $\mathcal{C}$ are torsion-by-nilpotent. From that, we deduce the following consequence, similar to a well-known result of P. Hall: if $H$ is a normal subgroup of a group $G$ such that $H$ and $G/H'$ are (locally finite)-by-nilpotent, then $G$ is (locally finite)-by-nilpotent. We give an example showing that this last statement is false when "(locally finite)-by-nilpotent" is replaced by "torsion-by-nilpotent".

## 1. INTRODUCTION AND MAIN RESULTS

The class of nilpotent groups is not closed under forming extensions.

However, we have the following well-known result, due to P. Hall [2]:

Theorem A. *Let $H$ be a normal subgroup of a group $G$. If $G/H'$ and $H$ are nilpotent, then $G$ is nilpotent.*

This result is often very useful to prove that a group is nilpotent. In particular, by an induction on the derived length, it is easy to obtain the following consequence:

Theorem B. *Let $\mathcal{C}$ be a class of groups which is closed under taking subgroups and quotients. Suppose that all metabelian groups of $\mathcal{C}$ are nilpotent. Then all soluble groups of $\mathcal{C}$ are nilpotent.*

Since the first result of Hall, various results of a similar nature have been given (see for instance [3, Part 1, p. 57]). The aim of this paper is to see whether it is possible to obtain analogous results, when "nilpotent" is replaced by "torsion-by-nilpotent". At first, we shall prove an analogue to Theorem B:

Theorem 1.1. *Let $\mathcal{C}$ be a class of groups which is closed under taking subgroups and quotients. Suppose that all metabelian groups of $\mathcal{C}$ are torsion-by-nilpotent. Then all soluble groups of $\mathcal{C}$ are torsion-by-nilpotent.*

On the other hand, Theorem A fails to be true when "nilpotent" is replaced by "torsion-by-nilpotent". A counterexample will be given at the end of this paper. However, we shall deduce from Theorem 1.1 the following:

Theorem 1.2. *Let $H$ be a normal subgroup of a group $G$. If $G/H'$ and $H$ are (locally finite)-by-nilpotent, then $G$ is (locally finite)-by-nilpotent.*

In particular, since a locally soluble torsion group is locally finite, we obtain:

Corollary 1.3. *Let $H$ be a normal subgroup of a locally soluble group $G$. If $G/H'$ and $H$ are torsion-by-nilpotent, then $G$ is torsion-by-nilpotent.*

As we said above, Corollary 1.3 is false if "locally soluble" is omitted. Also notice that contrary to the case "nilpotent" where Theorem B is a consequence of Theorem A, we shall use Theorem 1.1 to prove Theorem 1.2.

## 2. A PRELIMINARY LEMMA

Let $x_1, \ldots, x_n$ be elements of a group $G$. As usual, we define the left-normed commutator $[x_1, \ldots, x_n]$ of weight $n$ inductively by

$$[x_1, \ldots, x_n] = [x_1, \ldots, x_{n-1}]^{-1} x_n^{-1} [x_1, \ldots, x_{n-1}] x_n.$$

If $H$ and $K$ are subgroups of $G$, we shall write $[H, K]$ for the subgroup generated by the elements of the form $[y, z]$, with $y \in H, z \in K$. For $n \geq 1$, we shall denote by $\gamma_n(G)$ the $n$th term of the descending central series of $G$. This subgroup is generated by the set of all left-normed commutators of weight $n$ in $G$.

It is convenient to introduce a map $\delta_G$ on the set of normal subgroups of $G$, defined by $\delta_G(H) = [H, G]$. Note that $\delta_G(HK) = \delta_G(H)\delta_G(K)$ for any normal subgroups $H, K$ of $G$. By the Three Subgroups Lemma (see for instance [3, Lemma 2.13]), we have

$$\delta_G([H, K]) \leq [\delta_G(H), K][H, \delta_G(K)].$$

It follows by induction that we have the Leibniz formula:

$$\delta_G^n([H, K]) \leq \prod_{i=0}^{n} \left[\delta_G^i(H), \delta_G^{n-i}(K)\right].$$

Lemma 2.1. *Let $H, K$ be normal subgroups of a group $G$. Suppose that for some integer $c > 0$, we have $\delta_G^c(H) \leq K$. Then, for any integer $t > 0$, we have*

$$\delta_G^{t(c-1)+1}\left(\gamma_t(H)\right) \leq \delta_H^{t-1}\left(K\right).$$

*Proof.* The proof is by induction on $t$. The case $t = 1$ is covered by the hypothesis. So consider an integer $t > 1$ and suppose that the result is true for $t-1$. Since $\delta_G^{t(c-1)+1}(\gamma_t(H)) = \delta_G^{t(c-1)+1}([\gamma_{t-1}(H), H])$, the Leibniz formula gives

$$\delta_G^{t(c-1)+1}(\gamma_t(H)) \leq \prod_{i=0}^{t(c-1)+1} \left[\delta_G^i(\gamma_{t-1}(H)), \delta_G^{t(c-1)+1-i}(H)\right].$$

It follows from the inductive hypothesis that for $i \geq (t-1)(c-1)+1$, we have

$$\left[\delta_G^i(\gamma_{t-1}(H)), \delta_G^{t(c-1)+1-i}(H)\right] \leq \left[\delta_H^{t-2}(K), H\right] = \delta_H^{t-1}(K).$$

If $i < (t-1)(c-1)+1$, then $t(c-1)+1-i \geq c$ and so we can write

$$\left[\delta_G^i(\gamma_{t-1}(H)), \delta_G^{t(c-1)+1-i}(H)\right] \leq \left[\gamma_{t-1}(H), K\right].$$

Using the Three Subgroups Lemma and an induction, it is easy to show that the inclusion $[\gamma_{t-1}(H), K] \leq \delta_H^{t-1}(K)$ holds. Thus, as in the preceding case, we obtain again

$$\left[\delta_G^i(\gamma_{t-1}(H)), \delta_G^{t(c-1)+1-i}(H)\right] \leq \delta_H^{t-1}(K).$$

Therefore, we have $\delta_G^{t(c-1)+1}(\gamma_t(H)) \leq \delta_H^{t-1}(K)$, as required. ■

## 3. PROOF OF THEOREM 1.1

If $H$ is a subgroup of a group $G$, we shall write $\sqrt{H}$ for the isolator of $H$ in $G$. Recall that $\sqrt{H}$ is the set of elements $x \in G$ such that, for some integer $e > 0$, we have $x^e \in H$. It is well-known that if $G$ is nilpotent, then $\sqrt{H}$ is a subgroup.

Now consider a class of groups $\mathcal{C}$, closed under taking subgroups and quotients. We assume that for some integer $d > 2$, all soluble groups in $\mathcal{C}$ of derived length at most $d-1$ are torsion-by-nilpotent. Under these conditions, from Lemma 3.1 to Lemma 3.6, we suppose that $G$ is a soluble group in $\mathcal{C}$ of derived length $\leq d$.

Lemma 3.1. *The set of torsion elements of $G$ is a subgroup.*

*Proof.* Let $a, b$ be elements of $G$ of finite order. We want to show that $H = \langle a, b \rangle$ is a torsion group. The derived length of $H'$ is at most $d-1$ and this subgroup is therefore torsion-by-nilpotent. It follows that the torsion elements of $H'$ form a subgroup $T$. Let $K = H/T$. The quotient $K/K'' \in \mathcal{C}$ is metabelian, and therefore torsion-by-nilpotent. Since $K/K''$ is generated by the images of $a, b$, this quotient is then a torsion group. We have thus in particular that $K'/K''$ is a torsion group. As $K' = H'/T$ is nilpotent it follows then that $K'$ is a torsion group. We have shown that $K/K'$ and $K'$ are torsion groups. Hence, $K = H/T$ is a torsion group and this implies that $H$ is a torsion group. ■

The next lemma is an immediate consequence of Lemma 3.1:

Lemma 3.2. *If $H$ is a normal subgroup of $G$, then $\sqrt{H}$ is a subgroup.*

Lemma 3.3. *Let $a, b$ be elements of $G$ such that $[a^r, b^s] = 1$ for some integers $r, s > 0$. If $G$ is torsion-free, then $a$ and $b$ commute.*

*Proof.* Let $H = \langle a, b \rangle$. We need to show that $H$ is abelian. The derived length of $H'$ is at most $d-1$ and this subgroup is therefore torsion-by-nilpotent. But as $G$ is torsion-free, $H'$ is torsion-free and nilpotent. Since the quotient $H/\sqrt{H''} \in \mathcal{C}$ is metabelian, it is also torsion-free and nilpotent. Since this quotient is generated by the images of $a, b$, it is abelian. It follows that $\sqrt{H''}$ contains $H'$ and so $H'/H''$ is a torsion group. It follows that $H'$ is a torsion group as $H'$ is nilpotent. But then $H'$ is both torsion-free and a torsion group. It follows that $H'$ is trivial. ■

Lemma 3.4. *If $H$ and $K$ are normal subgroups of $G$, then:*
(i) $[\sqrt{H}, \sqrt{K}] \leq \sqrt{[H,K]}$;
(ii) $\delta_G(\sqrt{H}) \leq \sqrt{\delta_G(H)}$.

*Proof.* (i). Let $a \in \sqrt{H}$ and $b \in \sqrt{K}$; then $a^r \in H$ and $b^s \in K$ for some integers $r, s > 0$. Put $L = G/\sqrt{[H,K]}$. The images $\overline{a}, \overline{b}$ of $a, b$ in $L$ satisfy the relation $[\overline{a}^r, \overline{b}^s] = 1$. Since $L \in \mathcal{C}$ is a torsion-free group of derived length at most $d$, Lemma 3.3 implies the relation $[\overline{a}, \overline{b}] = 1$. In other words, $[a, b]$ belongs to $\sqrt{[H,K]}$, and the first part of the lemma follows.
(ii). We have $\delta_G(\sqrt{H}) = \left[\sqrt{H}, G\right] = \left[\sqrt{H}, \sqrt{G}\right]$ and $\sqrt{\delta_G(H)} = \sqrt{[H,G]}$; hence the result follows from (i). ■

Lemma 3.5. *Let $H$ be a normal subgroup of $G$. Suppose that for some integer $c > 0$, we have $\delta_G^c(H) \leq \sqrt{H'}$. Then, for any integer $t > 0$, we have:*
(i) $\delta_G^{t(c-1)+1}(\gamma_t(H)) \leq \sqrt{\gamma_{t+1}(H)}$;
(ii) $\delta_G^{t(c-1)+1}\left(\sqrt{\gamma_t(H)}\right) \leq \sqrt{\gamma_{t+1}(H)}$.

*Proof.* (i). We can apply Lemma 2.1, with $K = \sqrt{H'}$; so we obtain

$$\delta_G^{t(c-1)+1}(\gamma_t(H)) \leq \delta_H^{t-1}\left(\sqrt{H'}\right).$$

By Lemma 3.4, we have $\delta_H^{t-1}\left(\sqrt{H'}\right) \leq \sqrt{\delta_H^{t-1}(H')}$. It remains to notice that $\delta_H^{t-1}(H') = \gamma_{t+1}(H)$ and we have proved (i).
(ii). By Lemma 3.4, $\delta_G^{t(c-1)+1}\left(\sqrt{\gamma_t(H)}\right) \leq \sqrt{\delta_G^{t(c-1)+1}(\gamma_t(H))}$ and so, by using (i), $\delta_G^{t(c-1)+1}\left(\sqrt{\gamma_t(H)}\right) \leq \sqrt{\sqrt{\gamma_{t+1}(H)}}$. Since $\sqrt{\sqrt{\gamma_{t+1}(H)}} = \sqrt{\gamma_{t+1}(H)}$, the proof is complete. ■

Lemma 3.6. *Let $H$ be a normal subgroup of $G$. Suppose that for some integer $c > 0$, we have $\delta_G^c(G) \leq \sqrt{H'}$. Then, for any integer $t > 0$, we have*

$$\delta_G^{f(t)}(G) \leq \sqrt{\gamma_{t+1}(H)}, \text{ with } f(t) = \frac{t(t+1)(c-1)}{2} + t.$$

*Proof.* The proof is by induction on $t$, the case $t = 1$ being covered by the hypothesis. Suppose that $\delta_G^{f(t-1)}(G) \leq \sqrt{\gamma_t(H)}$. It follows that $\delta_G^{f(t)}(G) \leq \delta_G^{t(c-1)+1}\left(\sqrt{\gamma_t(H)}\right)$. The hypothesis of our lemma implies that $\delta_G^c(H) \leq \sqrt{H'}$ and so we can apply Lemma 3.5. We obtain $\delta_G^{t(c-1)+1}\left(\sqrt{\gamma_t(H)}\right) \leq \sqrt{\gamma_{t+1}(H)}$; hence $\delta_G^{f(t)}(G) \leq \sqrt{\gamma_{t+1}(H)}$, as required. ■

*Proof of Theorem 1.1.* We argue by induction on the derived length $d$, the case $d \leq 2$ being clear. Suppose that for some integer $d > 2$, all soluble groups in $\mathcal{C}$ of derived length at most $d-1$ are torsion-by-nilpotent. Let $G$ be a soluble group in $\mathcal{C}$ of derived length $d$. By Lemma 3.1, the set of torsion elements of $G$ forms a subgroup. Hence we can assume that $G$ is torsion-free without loss of generality. We must prove that $G$ is nilpotent. With that in mind we let $H = G'$. Then, by the inductive hypothesis, $G/\sqrt{H'}$ and $H$ are nilpotent (and torsion-free). Let $c, k$ be positive integers such that $\gamma_{c+1}(G) \leq \sqrt{H'}$ and $\gamma_{k+1}(H) = \{1\}$. Since $\gamma_{c+1}(G) = \delta_G^c(G)$, we can apply Lemma 3.6. It follows that $\delta_G^{f(t)}(G) \leq \sqrt{\gamma_{t+1}(H)}$ for any positive integer $t$. By taking $t = k$, we obtain $\delta_G^{f(k)}(G) \leq \sqrt{\{1\}}$. But $G$ is torsion-free and hence $\sqrt{\{1\}} = \{1\}$. We conclude that $\delta_G^{f(k)}(G) = \gamma_{f(k)+1}(G)$ is trivial and the result follows. ■

## 4. PROOF OF THEOREM 1.2

We write $\zeta(G)$ for the centre of a group $G$.

LEMMA 4.1. *Let $G$ be a metabelian group such that $G/\zeta(G)$ is torsion-by-nilpotent. Then $G$ is torsion-by-nilpotent.*

*Proof.* By assumption, there exists an integer $k$ such that $\gamma_k(G/\zeta(G))$ is a torsion group; we can assume that $k \geq 2$. Hence, for any $x_1, \ldots, x_{k+1} \in G$, there exists an integer $e > 0$ such that $[[x_1, \ldots, x_k]^e, x_{k+1}] = 1$. But in a metabelian group, the relation $[a^e, b] = [a, b]^e$ holds for any element $a$ in the derived subgroup. Therefore, we have the equality $[x_1, \ldots, x_k, x_{k+1}]^e = 1$. Also note that $\gamma_{k+1}(G) \leq G'$ is abelian. It follows that $\gamma_{k+1}(G)$ is a torsion group, and the lemma follows. ■

LEMMA 4.2. *Let $G$ be a soluble group such that $G/\zeta(G)$ is torsion-by-nilpotent. Then $G$ is torsion-by-nilpotent.*

*Proof.* Let $\mathcal{C}$ be the class of soluble groups $G$ such that $G/\zeta(G)$ is torsion-by-nilpotent. It is easy to see that $\mathcal{C}$ is closed under taking subgroups and quotients. Thus the result follows from Theorem 1.1 and Lemma 4.1. ■

Recall without proof the following extension of a well-known result due to Schur (see for instance [3, Part 1, p. 102]):

LEMMA 4.3. *Let $G$ be a group such that $G/\zeta(G)$ is locally finite. Then $G'$ is locally finite.*

LEMMA 4.4. *Let $G$ be a group such that $G/\zeta(G)$ is (locally finite)-by-nilpotent. Then $G$ is (locally finite)-by-nilpotent.*

*Proof.* Denote by $\varphi(G)$ the locally finite radical of $G$, namely the product of all the normal locally finite subgroups of $G$. Since the class of locally finite groups is closed under forming extensions, $\varphi(G)$ is locally finite and $\varphi(G/\varphi(G))$ is trivial. Therefore, by replacing $G$ by $G/\varphi(G)$, we can assume that $G$ has no non-trivial normal locally finite subgroup. Then we must prove that $G$ is nilpotent. Let $L$ be the normal subgroup of $G$ containing

$\zeta(G)$ such that $L/\zeta(G) = \varphi\left(G/\zeta(G)\right)$. Then $L/\zeta(L)$ is locally finite, since it is a quotient of $L/\zeta(G)$. It follows from Lemma 4.3. that $L'$ is locally finite. But $G$ contains no non-trivial normal locally finite subgroup, so we must have $L' = \{1\}$. Since $G/L$ is nilpotent, it follows that $G$ is soluble and by Lemma 4.2 we see that $G$ is torsion-by-nilpotent. As $G$ is soluble it is therefore (locally finite)-by-nilpotent. Finally, as $\varphi(G)$ is trivial, we have proved that $G$ is nilpotent, as required. ■

*Proof of Theorem 1.2.* Suppose that $H$ is (locally finite)-by-(nilpotent of class $k$). We prove the theorem by induction on $k$, the result being obvious when $k \leq 1$. By replacing $G$ by $G/\varphi(H)$, we can assume that $H$ is torsion-free and nilpotent of class $k > 1$. It follows from the inductive hypothesis that $G/\gamma_k(H)$ is (locally finite)-by-nilpotent. Thus there exists an integer $c$ such that $\gamma_{c+1}(G)\gamma_k(H)/\gamma_k(H)$ is locally finite. In particular, we have $\gamma_{c+1}(G) \leq \sqrt{\gamma_k(H)}$. It is clear that this implies that $\delta_G^c(H) \leq K$, where $K = H \cap \sqrt{\gamma_k(H)}$. Now we can apply Lemma 2.1. By taking $t = k$, we have

$$\delta_G^{k(c-1)+1}\left(\gamma_k(H)\right) \leq \delta_H^{k-1}\left(K\right).$$

The group $\delta_H^{k-1}\left(K\right)$ is generated by the elements of the form $[z, y_1, \ldots, y_{k-1}]$, with $z \in K$ and $y_1, \ldots, y_{k-1} \in H$. Consider such a generator; let $e$ be a positive integer such that $z^e \in \gamma_k(H)$. Since $H$ is nilpotent of class $k$, we may write

$$[z, y_1, \ldots, y_{k-1}]^e = [z^e, y_1, \ldots, y_{k-1}] = 1.$$

But as $H$ is torsion-free, it follows that $[z, y_1, \ldots, y_{k-1}] = 1$. This proves that $\delta_H^{k-1}\left(K\right)$ is trivial which implies that $\delta_G^{k(c-1)+1}\left(\gamma_k(H)\right)$ is trivial. If we denote by $\left(\zeta_n(G)\right)_{n\geq 0}$ the upper central series of $G$, this means that $\gamma_k(H)$ is included in $\zeta_{k(c-1)+1}(G)$. Since $G/\gamma_k(H)$ is (locally finite)-by-nilpotent, then so is $G/\zeta_{k(c-1)+1}(G)$. By iterated application of Lemma 4.4, we conclude that $G$ is (locally finite)-by-nilpotent. ■

## 5. EXAMPLE

In this last part, we show that Theorem 1.2 is false if one substitutes "torsion-by-nilpotent" for "(locally finite)-by-nilpotent".

With this aim, consider an odd integer $e \geq 665$ and an integer $m \geq 2$. In [1, Chap. VII], Adian gives an example of a non soluble torsion-free group $A(m,e)$ such that $\zeta(A(m,e))$ is cyclic (non trivial) and $A(m,e)/\zeta(A(m,e))$ is $m$-generated of exponent $e$. For convenience, put $A = A(m,e)$. Let $B$ be a torsion-free nilpotent group of class 2 whose centre is cyclic and coincides with $B'$ (for example the group of $3 \times 3$ unitriangular matrices with entries in the ring of integers).

Suppose $\zeta(A) = \langle a \rangle$ and $\zeta(B) = \langle b \rangle$. Let $G = (A \times B)/C$, where $C = \langle (a,b) \rangle$, and let $f : B \to G$ is the homomorphism defined by $f(z) = (1,z)C$. For $H = f(B)$ one can easily check that:

- $H$ is nilpotent of class 2;
- $G/H'$ is (exponent $e$)-by-abelian;
- $G$ is torsion-free and is not nilpotent.

Therefore, $G/H'$ and $H$ are torsion-by-nilpotent whereas $G$ is not torsion-by-nilpotent.

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