On 3-Baer groups

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1 Introduction

A group is called a Baer group if every cyclic subgroup is subnormal. If every cyclic subgroup in \( G \) is subnormal of defect at most \( n \) then we say that \( G \) is an \( n \)-Baer group or more shortly a \( B_n \) group. It is not difficult to see that every \( B_n \) group is an \((n + 1)\)-Engel group. In this paper we will study 3-Baer groups. Our main results can be summarised into two theorems.

Theorem 1 Let \( G \) be a 3-Baer group. Then \( G \) is an extension of a nilpotent group of class at most 2 by a 3-Engel group. Furthermore if \( G \) has no element of order 2 then \( G \) is abelian by 3-Engel.

If \( G \) is a non-torsion group, that is a group that contains an element of infinite order, then we have the stronger result that \( G \) is a 3-Engel group [5]. Theorem 1 is thus essentially a theorem about 3-Baer \( p \)-groups.

In [1], Garrison and Kappe give a detailed analysis of metabelian 3-Baer groups. One of their main results is the following: if \( G \) is a 3-Baer group without elements of order 2 or 3 then \( G/Z(G) \) is 3-Engel and nilpotent of class at most 4. They also give examples of 3-Baer 2 groups and 3-groups which are not centre by 3-Engel and which are nilpotent of class 5. There are non-nilpotent metabelian 3-Engel groups of exponent 4. However, we have that metabelian 3-Baer groups without involutions are nilpotent and by next result the best upper bound for the nilpotency class in that case is 5.
Theorem 2 Let $G$ be a metabelian 3-Baer 3-group, then $G$ is nilpotent of class at most 5.

2 Connection with 3-Engel groups

By a theorem of L.-C. Kappe and W. Kappe [4], we have that if $G$ is a 3-Engel group then the normal closure of an element is always nilpotent of class at most 2. So it follows that if $x \in G$ then every subgroup of $x^G$ is subnormal of defect at most 2. It follows in particular that $\langle x \rangle$ is subnormal of defect at most 3 in $G$. Hence every 3-Engel group is a 3-Baer group. In this section we will show that every 3-Baer group is an extension of a nilpotent group of class at most 2 by a 3-Engel group.

Lemma 1 Let $H$ be a 3-Baer group and suppose $x \in H$ is an element in $H$ satisfying $x^{p^2i} = 1$ for some integer $i \geq 0$. Then if $(x^{p^i})^H$ is abelian it follows that $(x^{p^i})^H$ is abelian.

Proof Let $b \in H$. We then have that $x^b \in x^H$ and since $H$ is a 3-Baer group we have $$[x^{p^i}, x^b, x^b] \in \langle x^b \rangle^{(H,3)} = \langle x^b \rangle$$ which implies that $[x^{p^i}, x^b, x^b] = 1$. Using the fact that $(x^{p^i})^H$ is abelian, we get $$[x^{p^i}, b^{-1}x^{p^i}b] = [x^{p^i}, x^b]^{p^i}[x^{p^i}, x^b, x^b]^{(p^i)}$$ $$= [x^{p^i(p^i)}, x^b, x^b].$$ Suppose that $[x^{p^i}, x^{p^ib}] \neq 1$. From the equality above we have $[x^{p^i}, x^b, x^b]^{(p^i)} \neq 1$. Therefore we must have that $p = 2$, that $x$ has order (exactly) $2^{2i}$ and that $[x^{2^i}, x^b, x^b]$ has order $2^i$. But since $[x^{2^i}, x^b, x^b] \in \langle x^b \rangle$ and since this element has order $2^i$ and $x^b$ has order $2^{2i}$, we have $$[x^{2^i}, x^b, x^b] = (x^b)^{2^ri},$$ where $r$ is odd. It follows that $(x^b)^{2^i} = [x^{2^i}, x^b, x^b]^s$ for some $s$. But then $$[x^{2^i}, b^{-1}x^{2^i}b] = [x^{2^i}, x^b, x^b]^s = 1,$$ since $[x^{2^i}, x^b, x^b]^s \in (x^{2^i})^H$, and we have a contradiction. Hence $[x^{p^i}, x^{p^ib}] = 1$. □
Corollary 1 Let $G$ be a 3-Baer group and $x \in G$ be an element satisfying $x^{3^i} = 1$ for some integer $i \geq 0$. Then $(x^p)^G$ is abelian.

Proof Since $G$ is a 3-Baer group we have that $x^{(G,3)} = \langle x \rangle$. We now apply Lemma 1 three times, letting $H$ be first $x^{(G,2)}$ then $x^{(G,1)}$ and $x^{(G,0)}$. □

Before proceeding further we make a remark which is going to be useful in later. Suppose $G$ is a nilpotent group and that $g$ is an element in $G$ satisfying $g \in \langle \langle g \rangle, G \rangle$. It then follows by induction that $\langle g \rangle \leq \langle \langle g \rangle, G, \ldots, G \rangle$ for all positive integers. Since $G$ is nilpotent it then follows that $g = 1$.

Lemma 2 Let $G$ be a $B_3$ group of exponent 4. Then $G$ is a 3-Engel group.

Proof Let $B(3,4)$ be the relatively free group with 3 generators and of exponent 4. One can see from a power-commutator presentation of $B(3,4)$, (see p. 144 in [8] for example) that all groups of exponent 4 satisfy

$$[a, x^2, x] = [a, x, x, x].$$

We want to show that $[a, x, x, x] = 1$ for all $a, x \in G$. Since $G$ is a 3-Baer group we have that $[a, x, x, x] \in \langle x \rangle$. If $[a, x, x, x]$ is equal to either $x$ or $x^{-1}$ then it follows from the remark made before the statement of the lemma, that $x = 1$. We can thus assume that $[a, x, x, x] = x^2$. But then $x^2 = [a, x^2, x] \in \langle x^2, G \rangle$ and we have $x^2 = 1$ by the same remark. □

The following lemma which we state without a proof will also be useful. (see [7]).

Lemma 3 If $G$ is a 4-Engel group and $x \in G$ is of finite order then $\langle x, x^b \rangle$ is nilpotent of class at most 4 for all $b \in G$.

We said in the introduction that every $n$-Baer group is a $(n + 1)$-Engel group. That is every element is a left $(n + 1)$-Engel element. If an element is either of infinite order or of prime order more can be said. Let $[a, m] = [\cdots [a, x], \cdots, x]$.

Lemma 4 Let $G$ be a $n$-Baer group. If $x \in G$ is an element which is of infinite order then $x$ is a left $n$-Engel element. If $x$ is a $p$-element for some prime $p$ then $[a, n] x] \in \langle x^p \rangle$. In particular if $x$ has order $p$ we have that $x$ is a left $n$-Engel element.
Proof Suppose first that $a, x \in G$ where $x$ is of infinite order. Since $G$ is a $n$-Baer group, we have that $[a, n x] = x^s$ for some integer $s$. Being a Baer group we have that $G$ is locally nilpotent and thus we have

$$x^{s+1} = [[a, n x], \ldots, [[a, n x], [a, n x]] \ldots] = 1$$

for some integer $r \geq 1$. But since $x$ is of infinite order we must have $s = 0$. Hence $[a, n x] = 1$. Now suppose that $x$ is a $p$-element for some prime $p$. Since $G$ is a $n$-Baer group we have that $[a, n x] = x^m$ for some integer $m$. We want to show that $x^m \in \langle x^p \rangle$. If this is not the case we must have $x = [a, n x]^s$ for some integer $s$. Thus $\langle x \rangle \leq [\langle x \rangle, H]$ and by the remark before Lemma 2 we have that $x = 1$ which is a contradiction. \hfill $\square$

Theorem 1 Let $G$ be a 3-Baer group. Then $G$ is an extension of a nilpotent group of class at most 2 by a 3-Engel group. Furthermore if $G$ does not contain an involution then $G$ is abelian by 3-Engel.

Proof It follows from Lemma 4 that $[a, x, x, x] = 1$ when $x$ is of infinite order. For each prime $p$ let $H_p = \langle [a, x, x, x] : a, x \in G \text{ and } x \text{ is a } p-\text{element} \rangle$. Let $H = \langle [a, x, x, x] : a, x \in G \rangle$. $G$ is locally nilpotent and thus we have that the torsion elements in $G$ form a subgroup which is a direct product of $p$-groups. Since $[a, x, x, x] = 1$ when $x$ is of infinite order we have that $H$ is a torsion group and that $H = \prod_p H_p$. It is now clearly sufficient to show that $H_p$ is abelian when $p \neq 2$ and that $H_2$ is nilpotent of class at most 2. Suppose first that $p \neq 2$. Let $a, b, x, y \in G$ such that $x$ and $y$ are $p$-elements. Suppose that $[a, x, x, x] = x^m$ and that $[b, y, y, y] = y^n$ where $m = p^r r$, $n = p^s s$ and $(r, p) = (s, p) = 1$. By Lemma 4 we have that $i$ and $j$ are greater than 0. By Lemma 3 we have that $\langle x, x^a \rangle$ is nilpotent of class at most 4. Hence

$$1 = [[a, x, x], [a, x, x, x]] = x^{m^2}.$$

By corollary to Lemma 1, we have that $(x^m)^G$ is abelian. Similarly $y^{n^2} = 1$ and $(y^n)^G$ is abelian. We can without loss of generality assume that $i \leq j$. We now have

$$[x^m, y^n] = [x^m, y]^{n^2} [x^m, y, y]^\binom{n}{2} [x^m, y, y, y]^\binom{n}{3}$$

$$= [x^m, y, y, y]^\binom{n}{3}. \quad (1)$$

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If \([x^m, y^n] \neq 1\) we thus must have \(p = 3\) and \(i = j\). We also must have that \(x\) has order \(3^2i\) and \([x^m, y, y, y]\) has order \(3^i\). Since \(G\) is a 3-Baer group, \([x^m, y, y, y]\) ∈ \(\langle y \rangle\). We thus must have

\[
[x^m, y, y, y] = y^{3t}
\]

for some \(t\) which is coprime to 3, and thus

\[
y^n = [x^m, y, y, y]^l
\]

for some \(l\). But then \([x^m, y^n] = [x^m, [x^m, y, y, y]^l]\) = 1, since \((x^m)^G\) is abelian, which is a contradiction. We have thus proved that \(H_p\) is abelian when \(p \neq 2\).

We now show that \(H_2\) is nilpotent of class at most 2. Let \(a, b, c, x, y, z \in G\) such that \(x, y\) and \(z\) are 2-elements. Let \(u = [[a, x, x], [b, y, y, y]]\). We want to show that \(u\) commutes with \([c, z, z, z]\). From equation like equation (2) we see that \(u\) has order either 1 or 2. We consider two cases. Suppose first that

\[
[c, z, z, z] = z^{4m}
\]

for some integer \(m\). We know that \(u^{H_2}\) is abelian. Therefore

\[
[u, [c, z, z, z]] = [u, z]^{4m} [u, z, z]^{(4m)} [u, z, z, z]^{(4m)} = 1.
\]

By Lemma 4 we have that \([c, z, z, z] \in \langle z^2 \rangle\), we can thus assume that

\[
[c, z, z, z] = z^{2m}
\]

where \(m\) is odd. By similar argument as before we have that \(z^4 = 1\) and \((z^2)^{H_2}\) is abelian. By Lemma 2 we have that \([a, x, x, x] \in G^4\). Therefore \(u \in G^4\). Suppose that \(u = u^4_1 \cdots u^4_l\). But

\[
[z^{2m}, u^4_i] = [z^{2m}, u^4_i] [z^{2m}, u_i] [z^{2m}, u_i, u_i] [z^{2m}, u_i, u_i, u_i] = 1.
\]

So again \([u, [c, z, z, z]] = 1\). □

From our knowledge on 3-Engel groups we get the following corollary.

**Corollary 2** Let \(G\) be a 3-Baer group which is 5-torsion free. Then \(G\) is soluble of derived length at most 5. Furthermore if \(G\) does not have an involution then \(G\) has derived length at most 4.
Remarks  (1) What are the best upper bounds in Corollary 2? We know it must be 3, 4 or 5 in the general case and 3 or 4 when $G$ is also 2-torsion free. If $G$ is a torsion 4-Engel group that is \{2, 3, 5\}—torsion free it is known [6,7] that $G$ is nilpotent of class at most 7 so in that case the derived length is at most 3.

(2) Is it possible to strengthen Theorem 1 so that every 3-Baer group is abelian by 3-Engel?

3 Metabelian 3-Baer groups

It is well known that there are metabelian 3-Engel groups of exponent 4 that are non-nilpotent. Take for example the standard wreath product of a cyclic group of order 2 with a countably infinite elementary abelian 2-group. So metabelian 3-Baer groups need not be nilpotent. However if a metabelian 3-Baer group is without an involution then it is nilpotent. Garrison and Kappe [1] have shown the best upper bound for the nilpotency class is 4 if one furthermore assumes that there are no elements of order 3. They also give an example of a metabelian 3-Baer 3-group which is nilpotent of class 5. In this section we will prove that 5 is in fact the best upper bound for the nilpotency class of 3-Baer 3-groups. We will first need the following elementary lemma.

**Lemma 5** Let $G = \langle x, y \rangle$ be a metabelian 3-Baer 3-group. Let $H \triangleleft G$ then

$$[H, G^3] \leq [H, G]^3 [H, G, G]^{3^{-1}}.$$  

Furthermore, if $H \leq \gamma_2(G)$ then

$$[H, G^3] \leq [H, G]^3.$$  

**Proof** Let $h \in H$ and $g \in G$. Since $G$ is a metabelian 4-Engel group, we have

$$[h, g^3] = [h, g][h, g][h, g, g]^{(3)}[h, g, g, g]^{(3)} (2)$$

which is in $[H, G]^3 [H, G, G]^{3^{-1}}$. (We will not need this slightly stronger version). This proves the first part of the lemma. Now suppose furthermore
that $h \in \gamma_2(G)$. If $g \in \gamma_2(G)$ then $[h, g] = 1$. So let $g \in G \setminus \gamma_2(G)$; we may assume that $G = \langle g, f \rangle$ for some $f \in G$. Then $h \in \langle g \rangle$ and since $G$ is a 3-Baer group we have $[h, g, g] \in \langle g \rangle$ so $[h, g, g, g] = 1$. It follows now from (3) that $[h, g^3] \leq [H, g]^3$. □

We are now ready to prove the main result of this section.

**Proposition 1** Let $G$ be a metabelian 3-Baer group and suppose $t \in G$ is a 3-element. Then $[x, t, t, y, z] = 1$ for all $x, y, z \in G$.

**Proof** From [2] we have that a metabelian $n$-Engel group satisfies $[x, y, z, u] = 1$ (this follows from $[z, u][x, y] = 1$). Since $G$ is a metabelian 4-Engel group, it follows that

$$1 = [t, [x, y], t, t, z] = [t, x, y, t, t, z][t, y, x, t, t, z]^{-1}$$

and thus

$$[x, t, t, y, z] = [y, t, t, t, x, z] \quad \text{for all } x, y, z \in G. \quad (3)$$

Suppose for some given $x, y, z, t \in G$ we have $[x, t, t, y, z] \neq 1$. Since $G$ is a metabelian $B_3$ group we have that $[x, t, t, t], [x, t, t, y] = [x, t, y, t, t]$ and $[x, t, t, t, z]$ are in $\langle t \rangle$. Furthermore suppose

$$\langle[x, t, t, t]\rangle = \langle t^3 \rangle, \quad \langle[t^3, y]\rangle = \langle t^{3+u} \rangle, \quad \langle[t^3, z]\rangle = \langle t^{3+e} \rangle.$$ 

Assume that $x$ is chosen so that $i$ is minimal and then that $y$ and $z$ are chosen so that $u$ and $e$ are minimal. We then have $u = e$ and we can assume that $y = z$. We also have $\langle[x, t, t, t, y] \rangle = \langle t^{3+2u} \rangle$ and thus $t^{3+2u} \neq 1$. Now suppose further that

$$\langle[y, t, t, t]\rangle = \langle t^3 \rangle \quad \text{and} \quad \langle[t^3, x]\rangle = \langle t^{3+h} \rangle.$$ 

Since $x$ was chosen such that $i$ is minimal we must have $i \leq j$. It follows now from identity (4) that $j + h = i + u$ and therefore $h \leq u$. We thus have $t^{3+2h} \neq 1$ and therefore $[x, t, t, t, x, x] \neq 1$. We will now show that this leads to a contradiction.

Suppose first that $x$ is of infinite order. Let $u = [x, t, t, t, x]$. Since $G$ is a 3-Baer group we have that $[x, u] = x^m$ for some integer $m$. Since $G$ is
locally nilpotent the argument that was used in the proof of Lemma 4 gives that \( m = 0 \) and thus \([x, t, t, t, x, x] = 1\). We can thus assume that \( x \) is of finite order. Since \( G \) is locally nilpotent we have that \( x \) commutes with \( t \) if \( x \) has order coprime to 3. We can thus assume that \( x \) is also a 3-element.

From [2] we have that a metabelian \( n \)-Engel group \( G \) satisfies the identities \( γ_{n+1}(G)^{(n+1)-(n-1)!} = 1 \) and \( γ_{n+2}(G)^{(n-1)!} = 1 \). Since \( G \) is a metabelian 4-Engel group we have then that \( γ_5(G)^{9\cdot40} = γ_6(G)^{3\cdot4} = 1 \). Thus \([x, t, t, t, x]^3 = [x, t, t, t, x, x]^3 = 1\). We therefore have that \( t^{3j+1+2} = 1 \) and \( t^{3j+2} ≠ 1 \) which implies that we must have \( h = 1 \). Since \( i ≥ 1 \) we have that \( |t| ≥ 3^4 \). Similarly we have \([x] ≥ 3^4 \). Now \([x, t, t, t] = \langle t^3 \rangle \). Suppose further that \([x, t, t, x] = \langle x^{3^j} \rangle \). If \([x, t, t] = \langle x^{3^j+1} \rangle \) then by the same argument as before we have \( k = 1 \). From what we have already shown we have

\[
\begin{align*}
t^{3j+3} &= x^{3j+3} = [t^{3j+2}, x] = [x^{3j+2}, t] = [t^{3j+1}, x, x] = [t^3, x, t] = [x^3, t, x] = [x^{3j+1}, t, t] = 1. \\
\end{align*}
\]

(5)

Let \( H = \langle x, t \rangle \) and suppose further that \( t \) is chosen in \( H \setminus \Phi(G) \) such that it has minimal order. Notice that we still have \([x, t, t, t, x] ≠ 1 \) for some other generator \( x \) for \( H \). Since \( t \) has minimal order we have that \( i ≤ j \). We now have \([x, t, t, t, x] = (t^{3j+2}) = \langle x^{3j+2} \rangle \). Suppose that

\[
t^{3j+2} = x^{-3j+2}r
\]

for some \( r \) not divisible by 3. Let \( u = tx^{3j-r} \). We get a contradiction by showing that \( u^{3j+2} = 1 \) (since then \( t \) is not of minimal order in \( H \setminus \Phi(G) \)).

Let \( y = x^{3j-r} \) and \( n = 3j+2 \). We will show that \((ty)^n = t^ny^n \). Since \( t^n = y^{-n} \), it follows then that \( u^n = 1 \).

We let \( H = \langle t, y \rangle \) and for each integer \( m ≥ 1 \) we let \( H_m = γ_2(H)^{3^m}γ_3(H)^{3^{m-1}} \).

We show by induction on \( m \) that

\[
(ty)^{3^m} = t^{3^m}y^{3^m} \pmod{H_m}. \tag{6}
\]

Modulo \( γ_3(H) \) we have \((gt)^3 = y^3t^3[ty]^3 \) and (6) is therefore true for \( m = 1 \). Suppose now that (6) is true for some \( m ≥ 1 \). Then \((gt)^{3^m} = y^{3^m}t^{3^m}h \) for some \( h ∈ H_m \). It follows from Lemma 5 that \( y^{3^m}, t^{3^m} \) and \( h \) commute modulo \( H_{m+1} \). Also \( h^3 ∈ H_{m+1} \). Therefore we have modulo \( H_{m+1} \) that \((yt)^{3^m+1} = y^{3^m+1}t^{3^m+1} \). This finishes the proof of our inductive hypothesis.
From (5) we have that \( H_{i+2} = \{1\} \) and thus (6) gives us \((ty)^{3^{i+2}} = t^{3^{i+2}} y^{3^{i+2}}\) as we wanted. □

It is well known [3] that metabelian 3-Engel groups without involutions are nilpotent of class at most 3. Therefore we have as a corollary the following result.

**Theorem 2** Let \( G \) be a metabelian 3-Baer 3-group. Then \( G \) is nilpotent of class at most 5.

**Remarks.** (1) As we mentioned in the introduction, there are examples of metabelian 3-Baer groups which have class 5 so this is the best upper bound. (2) The following result of P. Hall is well known: If \( G \) is a group with normal subgroup \( H \) such that \( G/[H, H] \) and \( H \) are nilpotent then \( G \) is nilpotent. One also gets a bound for the nilpotency class of \( G \) in terms of the nilpotency classes of the quotient and the subgroup. Using this result, Corollary 2 and what we know about metabelian groups one can show that 3-Baer 3-groups are nilpotent of class at most 965. When \( G \) is \( \{2, 3, 5\} \)-torsion free we however have a much better upper bound since, as we mentioned in section 2, 4-Engel \( \{2, 3, 5\} \)-torsion free groups are nilpotent of class at most 7.

**References**


