On 3-Baer groups

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1 Introduction

A group is called a Baer group if every cyclic subgroup is subnormal. If every cyclic subgroup in G is subnormal of defect at most n then we say that G is an n-Baer group or more shortly a B_n group. It is not difficult to see that every B_n group is an $(n+1)$-Engel group. In this paper we will study 3-Baer groups. Our main results can be summarised into two theorems.

Theorem 1 *Let G be a 3-Baer group. Then G is an extension of a nilpotent group of class at most 2 by a 3-Engel group. Furthermore if G has no element of order 2 then G is abelian by 3-Engel.*

If G is a non-torsion group, that is a group that contains an element of infinite order, then we have the stronger result that G is a 3-Engel group [5]. Theorem 1 is thus essentially a theorem about 3-Baer p-groups.

In [1], Garrison and Kappe give a detailed analysis of metabelian 3-Baer groups. One of their main results is the following: if G is a 3-Baer group without elements of order 2 or 3 then $G/Z(G)$ is 3-Engel and nilpotent of class at most 4. They also give examples of 3-Baer 2 groups and 3-groups which are not centre by 3-Engel and which are nilpotent of class 5. There are non-nilpotent metabelian 3-Engel groups of exponent 4. However, we have that metabelian 3-Baer groups without involutions are nilpotent and by next result the best upper bound for the nilpotency class in that case is 5.

Theorem 2 *Let G be a metabelian 3-Baer 3-group, then G is nilpotent of class at most 5.*

2 Connection with 3-Engel groups

By a theorem of L.-C. Kappe and W. Kappe [4], we have that if G is a 3-Engel group then the normal closure of an element is always nilpotent of class at most 2. So it follows that if $x \in G$ then every subgroup of x^G is subnormal of defect at most 2. It follows in particular that $\langle x \rangle$ is subnormal of defect at most 3 in G. Hence every 3-Engel group is a 3-Baer group. In this section we will show that every 3-Baer group is an extension of a nilpotent group of class at most 2 by a 3-Engel group.

Lemma 1 *Let H be a 3-Baer group and suppose $x \in H$ is an element in H satisfying $x^{p^{2i}} = 1$ for some integer $i \geq 0$. Then if $(x^{p^i})^{x^H}$ is abelian it follows that $(x^{p^i})^H$ is abelian.*

Proof Let $b \in H$. We then have that $x^b \in x^H$ and since H is a 3-Baer group we have

$$[x^{p^i}, x^b, x^b] \in \langle x^b \rangle^{(H,3)} = \langle x^b \rangle$$

which implies that $[x^{p^i}, x^b, x^b, x^b] = 1$. Using the fact that $(x^{p^i})^{x^H}$ is abelian, we get

$$\begin{aligned} [x^{p^i}, b^{-1}x^{p^i}b] &= [x^{p^i}, x^b]^{p^i}[x^{p^i}, x^b, x^b]^{\binom{p^i}{2}} \\ &= [x^{p^i\binom{p^i}{2}}, x^b, x^b]. \end{aligned}$$

Suppose that $[x^{p^i}, x^{p^i b}] \neq 1$. From the equality above we have $[x^{p^i}, x^b, x^b]^{\binom{p^i}{2}} \neq 1$. Therefore we must have that $p = 2$, that x has order (exactly) 2^{2i} and that $[x^{2^i}, x^b, x^b]$ has order 2^i. But since $[x^{2^i}, x^b, x^b] \in \langle x^b \rangle$ and since this element has order 2^i and x^b has order 2^{2i}, we have

$$[x^{2^i}, x^b, x^b] = (x^b)^{2^i r},$$

where r is odd. It follows that $(x^b)^{2^i} = [x^{2^i}, x^b, x^b]^s$ for some s. But then

$$[x^{2^i}, b^{-1}x^{2^i}b] = [x^{2^i}, [x^{2^i}, x^b, x^b]^s] = 1,$$

since $[x^{2^i}, x^b, x^b]^s \in (x^{2^i})^{x^H}$, and we have a contradiction. Hence $[x^{p^i}, x^{p^i b}] = 1$. □

Corollary 1 *Let G be a 3-Baer group and $x \in G$ be an element satisfying $x^{p^{2i}} = 1$ for some integer $i \geq 0$. Then $(x^{p^i})^G$ is abelian.*

Proof Since G is a 3-Baer group we have that $x^{(G,3)} = \langle x \rangle$. We now apply Lemma 1 three times, letting H be first $x^{(G,2)}$ then $x^{(G,1)}$ and $x^{(G,0)}$. $\Box$

Before proceeding further we make a remark which is going to be useful in later. Suppose G is a nilpotent group and that g is an element in G satisfying $g \in [\langle g \rangle, G]$. It then follows by induction that $\langle g \rangle \leq [\langle g \rangle, \underbrace{G, \ldots, G}_{r}]$ for all positive integers. Since G is nilpotent it then follows that $g = 1$.

Lemma 2 *Let G be a B_3 group of exponent 4. Then G is a 3-Engel group.*

Proof Let $B(3,4)$ be the relatively free group with 3 generators and of exponent 4. One can see from a power-commutator presentation of $B(3,4)$, (see p. 144 in [8] for example) that all groups of exponent 4 satisfy

$$[a, x^2, x] = [a, x, x, x].$$

We want to show that $[a, x, x, x] = 1$ for all $a, x \in G$. Since G is a 3-Baer group we have that $[a, x, x, x] \in \langle x \rangle$. If $[a, x, x, x]$ is equal to either x or x^{-1} then it follows from the remark made before the statement of the lemma, that $x = 1$. We can thus assume that $[a, x, x, x] = x^2$. But then $x^2 = [a, x^2, x] \in [\langle x^2 \rangle, G]$ and we have $x^2 = 1$ by the same remark.$\Box$

The following lemma which we state without a proof will also be useful. (see [7]).

Lemma 3 *If G is a 4-Engel group and $x \in G$ is of finite order then $\langle x, x^b \rangle$ is nilpotent of class at most 4 for all $b \in G$.*

We said in the introduction that every n-Baer group is a $(n+1)$-Engel group. That is every element is a left $(n+1)$-Engel element. If an element is either of infinite order or of prime order more can be said. Let $[a,_m x] = [\cdots [[a, \underbrace{x],], \cdots, x}_{m}]$.

Lemma 4 *Let G be a n-Baer group. If $x \in G$ is an element which is of infinite order then x is a left n-Engel element. If x is a p-element for some prime p then $[a,_n x] \in \langle x^p \rangle$. In particular if x has order p we have that x is a left n-Engel element.*

Proof Suppose first that $a, x \in G$ where x is of infinite order. Since G is a n-Baer group, we have that $[a,_n x] = x^s$ for some integer s. Being a Baer group we have that G is locally nilpotent and thus we have

$$x^{s^{r+1}} = [\underbrace{[a,_{n-1} x], \cdots, [[a,_{n-1} x]}_{r}, [a,_n x]] \cdots] = 1$$

for some integer $r \geq 1$. But since x is of infinite order we must have $s = 0$. Hence $[a,_n x] = 1$. Now suppose that x is a p-element for some prime p. Since G is a n-Baer group we have that $[a,_n x] = x^m$ for some integer m. We want to show that $x^m \in \langle x^p \rangle$. If this is not the case we must have $x = [a,_n x]^s$ for some integer s. Thus $\langle x \rangle \leq [\langle x \rangle, H]$ and by the remark before Lemma 2 we have that $x = 1$ which is a contradiction.□

Theorem 1 *Let G be a 3-Baer group. Then G is an extension of a nilpotent group of class at most 2 by a 3-Engel group. Furthermore if G does not contain an involution then G is abelian by 3-Engel.*

Proof It follows from Lemma 4 that $[a, x, x, x] = 1$ when x is of infinite order. For each prime p let $H_p = \langle [a, x, x, x] : a, x \in G \text{ and } x \text{ is a } p-\text{element} \rangle$. Let $H = \langle [a, x, x, x] : a, x \in G \rangle$. G is locally nilpotent and thus we have that the torsion elements in G form a subgroup which is a direct product of p-groups. Since $[a, x, x, x] = 1$ when x is of infinite order we have that H is a torsion group and that $H = \prod_p H_p$. It is now clearly sufficient to show that H_p is abelian when $p \neq 2$ and that H_2 is nilpotent of class at most 2. Suppose first that $p \neq 2$. Let $a, b, x, y \in G$ such that x and y are p-elements. Suppose that $[a, x, x, x] = x^m$ and that $[b, y, y, y] = y^n$ where $m = p^i r$, $n = p^j s$ and $(r, p) = (s, p) = 1$. By Lemma 4 we have that i and j are greater than 0. By Lemma 3 we have that $\langle x, x^a \rangle$ is nilpotent of class at most 4. Hence

$$1 = [[a, x, x], [a, x, x, x]] = x^{m^2}.$$

By corollary to Lemma 1, we have that $(x^m)^G$ is abelian. Similarly $y^{n^2} = 1$ and $(y^n)^G$ is abelian. We can without loss of generality assume that $i \leq j$. We now have

$$\begin{aligned} [x^m, y^n] &= [x^m, y]^n [x^m, y, y]^{\binom{n}{2}} [x^m, y, y, y]^{\binom{n}{3}} \\ &= [x^m, y, y, y]^{\binom{n}{3}}. \end{aligned} \tag{1}$$

If $[x^m, y^n] \neq 1$ we thus must have $p = 3$ and $i = j$. We also must have that x has order 3^{2i} and $[x^m, y, y, y]$ has order 3^i. Since G is a 3-Baer group, $[x^m, y, y, y] \in \langle y \rangle$. We thus must have

$$[x^m, y, y, y] = y^{3^i t}$$

for some t which is coprime to 3, and thus

$$y^n = [x^m, y, y, y]^l$$

for some l. But then $[x^m, y^n] = [x^m, [x^m, y, y, y]^l] = 1$, since $(x^m)^G$ is abelian, which is a contradiction. We have thus proved that H_p is abelian when $p \neq 2$. We now show that H_2 is nilpotent of class at most 2. Let $a, b, c, x, y, z \in G$ such that x, y and z are 2-elements. Let $u = [[a, x, x, x], [b, y, y, y]]$. We want to show that u commutes with $[c, z, z, z]$. From equation like equation (2) we see that u has order either 1 or 2. We consider two cases. Suppose first that

$$[c, z, z, z] = z^{4m}$$

for some integer m. We know that u^{H_2} is abelian. Therefore

$$[u, [c, z, z, z]] = [u, z]^{4m} [u, z, z]^{\binom{4m}{2}} [u, z, z, z]^{\binom{4m}{3}} = 1.$$

By Lemma 4 we have that $[c, z, z, z] \in \langle z^2 \rangle$, we can thus assume that

$$[c, z, z, z] = z^{2m}$$

where m is odd. By similar argument as before we have that $z^4 = 1$ and $(z^2)^{H_2}$ is abelian. By Lemma 2 we have that $[a, x, x, x] \in G^4$. Therefore $u \in G^4$. Suppose that $u = u_1^4 \cdots u_l^4$. But

$$[z^{2m}, u_i^4] = [z^{2m}, u_i]^4 [z^{2m}, u_i, u_i]^6 [z^{2m}, u_i, u_i, u_i]^4 = 1.$$

So again $[u, [c, z, z, z]] = 1$. □

From our knowledge on 3-Engel groups we get the following corollary.

Corollary 2 *Let G be a 3-Baer group which is 5-torsion free. Then G is soluble of derived length at most 5. Furthermore if G does not have an involution then G has derived length at most 4.*

Remarks (1) What are the best upper bounds in Corollary 2? We know it must be $3, 4$ or 5 in the general case and 3 or 4 when G is also 2-torsion free. If G is a torsion 4-Engel group that is $\{2, 3, 5\}-$torsion free it is known [6,7] that G is nilpotent of class at most 7 so in that case the derived length is at most 3.

(2) Is it possible to strengthen Theorem 1 so that every 3-Baer group is abelian by 3-Engel?

3 Metabelian 3-Baer groups

It is well known that there are metabelian 3-Engel groups of exponent 4 that are non-nilpotent. Take for example the standard wreath product of a cyclic group of order 2 with a countably infinite elementary abelian 2-group. So metabelian 3-Baer groups need not be nilpotent. However if a metabelian 3-Baer group is without an involution then it is nilpotent. Garrison and Kappe [1] have shown the best upper bound for the nilpotency class is 4 if one furthermore assumes that there are no elements of order 3. They also give an example of a metabelian 3-Baer 3-group which is nilpotent of class 5. In this section we will prove that 5 is in fact the best upper bound for the nilpotency class of 3-Baer 3-groups. We will first need the following elementary lemma.

Lemma 5 *Let $G = \langle x, y\rangle$ be a metabelian 3-Baer 3-group. Let $H \lhd G$ then*

$$[H, G^{3^i}] \leq [H, G]^{3^i}[H, G, G]^{3^{i-1}}.$$

Furthermore, if $H \leq \gamma_2(G)$ then

$$[H, G^{3^i}] \leq [H, G]^{3^i}.$$

Proof Let $h \in H$ and $g \in G$. Since G is a metabelian 4-Engel group, we have

$$[h, g^{3^i}] = [h, g]^{3^i}[h, g, g]^{\binom{3^i}{2}}[h, g, g, g]^{\binom{3^i}{3}} \qquad (2)$$

which is in $[H, G]^{3^i}[H, G, G, G]^{3^{i-1}}$. (We will not need this slightly stronger version). This proves the first part of the lemma. Now suppose furthermore that $h \in \gamma_2(G)$. If $g \in \gamma_2(G)$ then $[h,g]=1$. So let $g \in G \setminus \gamma_2(G)$; we may assume that $G = \langle g, f\rangle$ for some $f \in G$. Then $h \in \langle g\rangle^G$ and since G is a 3-Baer group we have $[h,g,g] \in \langle g\rangle$ so $[h,g,g,g]=1$. It follows now from (3) that $[h, g^{3^i}] \leq [H,g]^{3^i}$. $\Box$

We are now ready to prove the main result of this section.

Proposition 1 *Let G be a metabelian 3-Baer group and suppose $t \in G$ is a 3-element. Then $[x,t,t,t,y,z]=1$ for all $x,y,z \in G$.*

Proof From [2] we have that a metabelian n-Engel group satisfies $[x,y,z,_{n-1}u] = 1$ (this follows from $[z,_n u[x,y]]=1$). Since G is a metabelian 4-Engel group, it follows that

$$1 = [t,[x,y],t,t,z] = [t,x,y,t,t,z][t,y,x,t,t,z]^{-1}$$

and thus

$$[x,t,t,t,y,z] = [y,t,t,t,x,z] \quad \text{for all } x,y,z \in G. \tag{3}$$

Suppose for some given $x,y,z,t \in G$ we have $[x,t,t,t,y,z] \neq 1$. Since G is a metabelian B_3 group we have that $[x,t,t,t]$, $[x,t,t,t,y] = [x,t,y,t,t]$ and $[x,t,t,t,z]$ are in $\langle t\rangle$. Furthermore suppose

$$\langle [x,t,t,t]\rangle = \langle t^{3^i}\rangle, \quad \langle [t^{3^i}, y]\rangle = \langle t^{3^{i+u}}\rangle, \quad \langle [t^{3^i}, z]\rangle = \langle t^{3^{i+e}}\rangle.$$

Assume that x is chosen so that i is minimal and then that y and z are chosen so that u and e are minimal. We then have $u=e$ and we can assume that $y=z$. We also have $\langle [x,t,t,t,y,y]\rangle = \langle t^{3^{i+2u}}\rangle$ and thus $t^{3^{i+2u}} \neq 1$. Now suppose further that

$$\langle [y,t,t,t]\rangle = \langle t^{3^j}\rangle \text{ and } \langle [t^{3^i}, x]\rangle = \langle t^{3^{i+h}}\rangle.$$

Since x was chosen such that i is minimal we must have $i \leq j$. It follows now from identity (4) that $j+h = i+u$ and therefore $h \leq u$. We thus have $t^{3^{i+2h}} \neq 1$ and therefore $[x,t,t,t,x,x] \neq 1$. We will now show that this leads to a contradiction.

Suppose first that x is of infinite order. Let $u = [x,t,t,t,x]$. Since G is a 3-Baer group we have that $[x,u] = x^m$ for some integer m. Since G is locally nilpotent the argument that was used in the proof of Lemma 4 gives that $m = 0$ and thus $[x, t, t, t, x, x] = 1$. We can thus assume that x is of finite order. Since G is locally nilpotent we have that x commutes with t if x has order coprime to 3. We can thus assume that x is also a 3-element. From [2] we have that a metabelian n-Engel group G satisfies the identities $\gamma_{n+1}(G)^{(n+1)\cdot(n-1)!\cdot 1!\cdot 2!\cdots(n-1)!} = 1$ and $\gamma_{n+2}(G)^{1!\cdots(n-1)!} = 1$. Since G is a metabelian 4-Engel group we have then that $\gamma_5(G)^{9\cdot 40} = \gamma_6(G)^{3\cdot 4} = 1$. Thus $[x, t, t, t, x]^9 = [x, t, t, t, x, x]^3 = 1$. We therefore have that $t^{3^{i+h+2}} = 1$ and $t^{3^{i+2h}} \neq 1$ which implies that we must have $h = 1$. Since $i \geq 1$ we have that $|t| \geq 3^4$. Similarly we have $|x| \geq 3^4$. Now $\langle[x, t, t, t]\rangle = \langle t^{3^i}\rangle$. Suppose further that $\langle[t, x, x, x]\rangle = \langle x^{3^j}\rangle$. If $\langle[x^{3^j}, t]\rangle = \langle x^{3^{j+k}}\rangle$ then by the same argument as before we have $k = 1$. From what we have already shown we have

$$
\begin{aligned}
&t^{3^{i+3}} = x^{3^{j+3}} = [t^{3^{i+2}}, x] = [x^{3^{j+2}}, t] = \\
&[t^{3^{i+1}}, x, x] = [t^{3^i}, x, t] = [x^{3^j}, t, x] = [x^{3^{j+1}}, t, t] = 1. \qquad (5)
\end{aligned}
$$

Let $H = \langle x, t\rangle$ and suppose further that t is chosen in $H \setminus \Phi(G)$ such that it has minimal order. Notice that we still have $[x, t, t, t, x, x] \neq 1$ for some other generator x for H. Since t has minimal order we have that $i \leq j$. We now have $\langle[x, t, t, t, x, x]\rangle = \langle t^{3^{i+2}}\rangle = \langle x^{3^{j+2}}\rangle$. Suppose that

$$t^{3^{i+2}} = x^{-3^{j+2}r}$$

for some r not divisible by 3. Let $u = tx^{3^{j-i}r}$. We get a contradiction by showing that $u^{3^{i+2}} = 1$ (since then t is not of minimal order in $H \setminus \Phi(G)$). Let $y = x^{3^{j-i}r}$ and $n = 3^{i+2}$. We will show that $(ty)^n = t^n y^n$. Since $t^n = y^{-n}$ it follows then that $u^n = 1$.

We let $H = \langle t, y\rangle$ and for each integer $m \geq 1$ we let $H_m = \gamma_2(H)^{3^m}\gamma_3(H)^{3^{m-1}}$. We show by induction on m that

$$(ty)^{3^m} = t^{3^m}y^{3^m} \pmod{H_m}. \qquad (6)$$

Modulo $\gamma_3(H)$ we have $(yt)^3 = y^3t^3[t, y]^3$ and (6) is therefore true for $m = 1$. Suppose now that (6) is true for some $m \geq 1$. Then $(yt)^{3^m} = y^{3^m}t^{3^m}h$ for some $h \in H_m$. It follows from Lemma 5 that y^{3^m}, t^{3^m} and h commute modulo H_{m+1}. Also $h^3 \in H_{m+1}$. Therefore we have modulo H_{m+1} that $(yt)^{3^{m+1}} = y^{3^{m+1}}t^{3^{m+1}}$. This finishes the proof of our inductive hypothesis.

From (5) we have that $H_{i+2} = \{1\}$ and thus (6) gives us $(ty)^{3^{i+2}} = t^{3^{i+2}} y^{3^{i+2}}$ as we wanted. $\Box$

It is well known [3] that metabelian 3-Engel groups without involutions are nilpotent of class at most 3. Therefore we have as a corollary the following result.

Theorem 2 *Let G be a metabelian 3-Baer 3-group. Then G is nilpotent of class at most 5.*

Remarks. (1) As we mentioned in the introduction, there are examples of metabelian 3-Baer groups which have class 5 so this is the best upper bound. (2) The following result of P. Hall is well known: If G is a group with normal subgroup H such that $G/[H, H]$ and H are nilpotent then G is nilpotent. One also gets a bound for the nilpotency class of G in terms of the nilpotency classes of the quotient and the subgroup. Using this result, Corollary 2 and what we know about metabelian groups one can show that 3-Baer 3-groups are nilpotent of class at most 965. When G is $\{2, 3, 5\}$-torsion free we however have a much better upper bound since, as we mentioned in section 2, 4-Engel $\{2, 3, 5\}$-torsion free groups are nilpotent of class at most 7.

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