Semigroup identities in 4-Engel groups

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1 Introduction

We say that a group G or a variety of groups V satisfies a semigroup law, if it satisfies a nontrivial law of the form $u(x_1, \ldots, x_n) = v(x_1, \ldots x_n)$, where u and v are words in the free semigroup freely generated by $x_1, \ldots, x_n$. It follows from a result of J. Lewin and T. Lewin [2] that a variety V of groups which satisfies a semigroup law can be characterised by its semigroups laws. Furthermore, we have then a sufficient and necessary condition for a semigroup to be embeddable in some group in V. A semigroup S is embeddable in some group in V if and only if it is cancellative and it satisfies all the semigroup laws that hold in V. In other words we have that S is embeddable in some group in V if and only if S is a cancellative semigroup in the corresponding semigroup variety. In [4] B. H. Neumann and T. Taylor show that nilpotent groups satisfy semigroup laws. We will be using their work later on so we will now describe it in more details. Let F be a free group that is freely generated by the variables $x, y, z_1, z_2, \ldots$. We define a sequence of words $q_1, q_2, \ldots$ in the variables $x, y, z_1, z_2, \ldots$ by induction as follows.

$$q_1(x,y) = xy, \;\; q_{i+1}(x,y,z_1,\ldots,z_i) = q_i(x,y,z_1,\ldots,z_{i-1})z_i q_i(y,x,z_1,\ldots,z_{i-1}).$$

They show that a group is nilpotent of class at most c if and only if it satisfies the semigroup law $q_c(x,y,z_1,\ldots,z_{c-1}) = q_c(y,x,z_1,\ldots,z_{c-1})$. It now follows easily that every group that is an extension of a nilpotent group by a group of finite exponent satisfies a semigroup law. For some classes of groups the converse is true. In the paper by J. Lewin and T. Lewin mentioned above this is shown to be true for finitely generated soluble groups and A. Shalev[5] has shown that every finitely generated residually finite group that satisfies a semigroup law must also be nilpotent by finite.

In this paper we will be considering the corresponding question for Engel groups. We remind the reader that a group G is an n-Engel group if it satisfies the law $[\ldots[[x,\underbrace{y],y],\ldots y}_{n}] = 1$. A. I. Shirshov [6] has shown that the variety of n-Engel groups can be characterised with semigroup laws when n is either 2 or 3. In the first case only one law is needed, $xy^2x = yx^2y$, but to describe 3-Engel groups we need two semigroup laws

$$xy^2xyx^2y = yx^2yxy^2x \quad \text{and} \quad xy^2xyxyx^2y = yx^2y^2x^2y^2x.$$

Whether every n-Engel groups can be described in terms of semigroup laws is an open question and stated as problem 2.82 in the Kourovka notebook [8]. In this article we will extend Shirshov's result to 4-Engel groups and find two semigroup identities that describe the variety of 4-Engel groups. In August 1997, P. Longobardi and M. Maj [3] gave a lecture at the conference Groups St Andrews 97 in Bath in which they presented a partial solution. They proved that there is a semigroup law that every torsion-free 4-Engel group satisfies. Our generalisation is based on the following result.

Proposition 1 *Let G be a* 4*-Engel group and let $a, b \in G$. The subgroup $\langle a, a^b\rangle$ is metabelian and nilpotent of class at most* 4*.*

In the special case when G is a torsion group this was proved by the author a few years ago [7] and in the case when G is torsion-free this is also known to be true [3]. M. Vaughan-Lee [9] has recently proved the proposition in the general form using computer methods. We will give a different hand proof in next section. We will use the remaining part of this section to see how we can apply this proposition to show that the 4-Engel condition can be described in terms of semigroup laws.

Let G be a 4-Engel group. For every $a, b \in G$ we have that ab and ba are conjugate and the proposition therefore implies that the subgroup $\langle ab, ba\rangle$ is nilpotent of class at most 4. But from the result of B. H. Neumann and T. Taylor we have that every group that is nilpotent of class at most 4 satisfies the identity $q_4(x, y, 1, 1, 1) = q_4(y, x, 1, 1, 1)$. Therefore G satisfies the nontrivial identity

$$q_4(xy, yx, 1, 1, 1) = q_4(yx, xy, 1, 1, 1).$$

So 4-Engel groups satisfy a semigroup law. We mentioned earlier in the introduction the result of J. Lewin and T. Taylor that a variety which satisfies a semigroup law can in fact be characterised by its semigroup laws. In fact, given a nontrivial semigroup law $xh(x, y) = yk(x, y)$, there is a simple procedure that replaces each law in the variety by a semigroup law that is equivalent modulo the law $xh = yk$. The idea is very simple. One uses $x^{-1}y = hk^{-1}$ repeatedly to change an arbitrary word to a word of the form uv^{-1} where u and v are semigroup words.

Example. The word $x^{-1}yt^{-1}z$ is equivalent to $h(x, y)k(x, y)^{-1}t^{-1}z$ which is equivalent to $h(x, y)h(tk(x, y), z)k(tk(x, y), z)^{-1}$. So, modulo the law $xh(x, y) = yk(x, y)$, we have that the law $x^{-1}yt^{-1}z = 1$ is equivalent to the semigroup law $h(x, y)h(tk(x, y), z) = k(tk(x, y), z)$.

We can thus use the semigroup law we have obtained to replace the 4-Engel law by a semigroup law that is equivalent modulo the first semigroup law.

2 Subgroups generated by two conjugate elements in 4-Engel groups

In this section we give a new proof of Vaughan-Lee's recent result that every subgroup in a 4-Engel group, that is generated by two conjugate elements, is metabelian and nilpotent of class at most 4. As we mentioned in the introduction, this has already been proved in the case when the group is either a torsion group or a torsion-free group. Before we get into the proof we make a short useful remark about 4-Engel groups. When we wrote up the 4-Engel identity in the introduction, we used bracketing from the left. But

$$[[[[y, \underbrace{x], x] \cdots], x]}_{m} = [\underbrace{x^{-1}, [\cdots [x^{-1}, [x^{-1}}_{m}, y]]]]^{x^m}.$$

So it does not matter whether we use bracketing from the right or from the left in the definition. We will use both in the following calculations. The following two lemmas where proved in [7] but we include the proofs here for the convenience of the reader, since they are not long. For the rest of this section we assume that G is a 4-Engel group and that a and a^b are some fixed conjugate elements in G.

Lemma 1 *We have that* $[a, a^b]$ *and* $[a, a^b]^{aa^b}$ *commute with* $[a, a^b]^a$ *and* $[a, a^b]^{a^b}$.

Proof We have

$$\begin{aligned}
1 &= [a,[a,[a,[a,b]]]] \\
&= [a,([a,[a,b]]^{-a}\cdot[a,[a,b]])] \\
&= [a,[a,b]]^{-a}\cdot[a,[a,b]]^{a^2}\cdot[a,[a,b]]^{-a}\cdot[a,[a,b]] \\
&= [a,b]^{-a}[a,b]^{a^2}[a,b]^{-a^3}[a,b]^{a^2}[a,b]^{-a}[a,b]^{a^2}[a,b]^{-a}[a,b]
\end{aligned}$$

which implies that

$$[a,[a,b]]^{a^2} = [a,[a,b]]^a\cdot[a,[a,b]]^{-1}\cdot[a,[a,b]]^a \tag{1}$$

$$[a,b]^{a^3} = [a,b]^{a^2}[a,b]^{-a}[a,b]^{a^2}[a,b]^{-a}[a,b][a,b]^{-a}[a,b]^{a^2}. \tag{2}$$

Also

$$\begin{aligned}
1 &= [[[[b,a],a],a],a] \\
&= [[a,b][a,b]^{-a},a,a] \\
&= [[a,b]^a[a,b]^{-1}[a,b]^a[a,b]^{-a^2},a] \\
&= [a,b]^{a^2}[a,b]^{-a}[a,b][a,b]^{-a}[a,b]^{a^2}[a,b]^{-a}[a,b]^{a^2}[a,b]^{-a^3}
\end{aligned}$$

and therefore

$$[a,b]^{a^3} = [a,b]^{a^2}[a,b]^{-a}[a,b][a,b]^{-a}[a,b]^{a^2}[a,b]^{-a}[a,b]^{a^2}. \tag{3}$$

From (2) and (3) we have

$$[a,b]^{a^2}[a,b]^{-a}[a,b] = [a,b][a,b]^{-a}[a,b]^{a^2}. \tag{4}$$

But it is easy to see that this is equivalent to

$$[a,[a,b]]^a\cdot[a,[a,b]] = [a,[a,b]]\cdot[a,[a,b]]^a$$

which can also be written

$$[a, a^b]^a \cdot [a, a^b] = [a, a^b] \cdot [a, a^b]^a. \tag{5}$$

By symmetry we also have that $[a, a^b]$ commutes with $[a, a^b]^{a^b}$. It follows from this that $[a, a^b]^{a^b}$ commutes with $[a, a^b]^{aa^b}$ and that $[a, a^b]^a$ commutes with $[a, a^b]^{a^b a[a,a^b]} = [a, a^b]^{aa^b}$. □

Lemma 2 *$\langle a, a^b\rangle'$ is generated by $[a, a^b]$, $[a, a^b]^a$, $[a, a^b]^{a^b}$ and $[a, a^b]^{aa^b}$.*

Proof Since $\langle a, a^b\rangle'$ is the normal closure of $[a, a^b]$ in $\langle a, a^b\rangle$, it is sufficient to show that the group generated by $[a, a^b]$, $[a, a^b]^a$, $[a, a^b]^{a^b}$ and $[a, a^b]^{aa^b}$ is normal in $\langle a, a^b\rangle$. From (1) and (5) we have $[a, a^b]^{a^2} = [a, a^b]^{2a}[a, a^b]^{-1}$ and then also $[a, a^b]^{a^{-1}} = [a, a^b]^{-a+2}$. By symmetry have as well $[a, a^b]^{a^{2b}} = [a, a^b]^{2a^b}$ and $[a, a^b]^{a^{-b}} = [a, a^b]^{-a^b+2}$. We also have the following relations.

$$\begin{aligned}
[a, a^b]^{aa^{-b}} &= [a, a^{-b}]^{-1}[a, a^b]^{a^{-b}a}[a, a^{-b}] \\
&= [a, a^b]^{a^{-b}}[a, a^b]^{-a^b a}[a, a^b]^{2a}[a, a^b]^{-a^{-b}} \\
&= [a, a^b]^{a^{-b}}[a, a^b][a, a^b]^{-aa^b}[a, a^b]^{-1}[a, a^b]^{2a}[a, a^b]^{-a^{-b}},
\end{aligned}$$

$$[a, a^b]^{a^b a} = [a, a^b][a, a^b]^{aa^b}[a, a^b]^{-1},$$

$$\begin{aligned}
[a, a^b]^{a^b a^{-1}} &= [a^{-1}, a^b][a, a^b]^{a^{-1}a^b}[a^{-1}, a^b]^{-1} \\
&= [a, a^b]^{-a^{-1}}[a, a^b]^{-aa^b}[a, a^b]^{2a^b}[a, a^b]^{a^{-1}},
\end{aligned}$$

$$\begin{aligned}
[a, a^b]^{aa^b a} &= [a, a^b][a, a^b]^{a^2 a^b}[a, a^b]^{-1} \\
&= [a, a^b][a, a^b]^{2aa^b}[a, a^b]^{-a^b}[a, a^b]^{-1},
\end{aligned}$$

$$\begin{aligned}
[a, a^b]^{aa^b a^{-1}} &= [a^{-1}, a^b][a, a^b]^{a^b}[a^{-1}, a^b]^{-1} \\
&= [a, a^b]^{-a^{-1}}[a, a^b]^{a^b}[a, a^b]^{a^{-1}},
\end{aligned}$$

$$\begin{aligned}
[a, a^b]^{aa^b a^b} &= [a, a^{2b}]^{-1}[a, a^b]^{a^{2b}a}[a, a^{2b}] \\
&= [a, a^b]^{-a^b-1}[a, a^b]^{2a^b a}[a, a^b]^{-a}[a, a^b]^{1+a^b} \\
&= [a, a^b]^{-a^b}[a, a^b]^{2aa^b}[a, a^b]^{-1}[a, a^b]^{-a}[a, a^b]^{1+a^b}.
\end{aligned}$$

From these equalities it is clear that $\langle [a,a^b],[a,a^b]^a,[a,a^b]^{a^b},[a,a^b]^{aa^b}\rangle$ is normal in $\langle a,a^b\rangle$. $\square$

Let $x=[a,a^b]$ and $u=[x^a,x^{a^b}]$.

Lemma 3 *We have that* $[x^a,x^{a^b}]=[x^{aa^b},x]$. *The group* $\langle a,a^b\rangle''$ *is cyclic generated by* u.

Proof It follows from lemmas 1 and 2 that every element in $\langle a,a^b\rangle'$ can be written in the form tz with $t\in\langle x^{aa^b},x\rangle$ and $z\in\langle x^a,x^{a^b}\rangle$, since it follows from Lemma 1 that the elements in $\langle x^{aa^b},x\rangle$ commute with the elements in $\langle x^a,x^{a^b}\rangle$. This fact will sometimes be used in the following calculations without mention.

Let $y=[a,[a,bx]]$. From Lemma 1, we have that y commutes with y^a. It then follows from the 4-Engel law that

$$1=[a,[a,y]]=y^{a^2}y^{-a}yy^{-a}.$$

We next expand y.

$$\begin{aligned}
y &= [a,([a,x][a,b][[a,b],x])]\\
&= [a,(x^{-a+1}[a,b]x^{-a^{-1}a^b+1})]\\
&= x^{-a}x^{a^{-1}a^ba}[a,b]^{-a}x^{-a+a^2}x^{-a+1}[a,b]x^{-a^{-1}a^b+1}\\
&= x^{-a}x^{a^b[a,a^b]^{-1}}[a,[a,b]]x^{(a-2)a^b+1}\\
&= x^{-a+a^b+1+aa^b-2a^b+1}\\
&= x^{-a+1+aa^b-a^b+1}.
\end{aligned}$$

We then have

$$\begin{aligned}
y^{a^2} &= x^{-a^3+a^2+aa^ba^2-a^ba^2+a^2}\\
&= x^{-a^3+a^2+(a-1)a^2a^b(a^2\circ a^b)^{-1}+a^2}\\
&= x^{-a^3+a^2+(a^3-a^2)a^b(a\circ a^b)^{-(a+1)}+a^2}\\
&= x^{2+(a-1)a^b+a-2},
\end{aligned}$$

and

$$\begin{aligned}
y^{-a} &= x^{-a+a^ba-aa^ba-a+a^2}\\
&= x^{-a+1+aa^b-a^2a^b-1-a+a^2}\\
&= x^{-a+1+(-a+1)a^b+a-2}.
\end{aligned}$$

Therefore

$$\begin{aligned}1 &= y^{a^2-a+1-a}\\ &= x^{2+aa^b-a^b+a-2}x^{-a+1-aa^b+a^b+a-2}\\ &\quad x^{-a+1+aa^b-a^b+1}x^{-a+1-aa^b+a^b+a-2}\\ &= x^{2+aa^b-2+1-aa^b-2+1+aa^b+1+1-aa^b-2}\\ &\quad x^{-a^b+a-a+a^b+a-a-a^b-a+a^b+a}\\ &= x^{2+aa^b-1}x^{-aa^b-1+aa^b+1}x^{-a^b-a+a^b+a}x^{1-aa^b-2}.\end{aligned}$$

Conjugation with x^{2+aa^b-1} gives

$$1 = [x^{aa^b}, x][x^{a^b}, x^a].$$

Therefore $[x^a, x^{a^b}] = [x^{aa^b}, x]$ and by Lemma 1 and Lemma 2 we have that $\langle a, a^b\rangle''$ is the normal closure of u in $\langle a, a^b\rangle'$. Since $u = [x^{aa^b}, x] = [x^{a^b}, x^a]$. We have by Lemma 1 that u commutes with all elements in $\langle a, a^b\rangle'$ and thus $\langle a, a^b\rangle'' = \langle u\rangle$. $\Box$

By Lemma 3 we have that $\langle a, a^b\rangle$ is soluble. By a theorem of Gruenberg [1], every finitely generated soluble Engel group is nilpotent. It thus follows that $\langle a, a^b\rangle$ is nilpotent. It is in fact easy to see that the nilpotency class is at most 5. We can see this as follows. By Lemma 1 we have that $[a, a^b]$ commutes with $[a, [a, a^b]]$. Since $[[a, [a, a^b]], a] = 1$ by the 4-Engel identity, it follows that $\langle [a, a^b], a^b\rangle$ is nilpotent of class at most 2. Thus $[[a, a^b], a, [a, a^b]] = 1$. Modulo $\gamma_6(\langle a, a^b\rangle)$ this gives that

$$1 = [[a, a^b], a, [a, a^b]] = [a^b, a, a, a^b, a].$$

From $1 = [[a, a^b], [a, a^b], a]$ we then also have $[a^b, a, a^b, a, a]$. Hence, every commutator of weight 5 with two occurances of a is trivial. By symmetry this is also true for commutators of weight 5 with two occurances of a^b (and thus three occurances of a). Because of this and the 4-Engel law, it is now clear that every commutator of weight 5 in a and a^b is trivial modulo $\gamma_6(\langle a, a^b\rangle)$. Therefore $\langle a, a^b\rangle$ is nilpotent of class at most 4 and then also metabelian. This finishes our proof of Proposition 1.

As we said in the introduction, it follows that 4-Engel groups can be described in terms of two semigroup identities. We end by deriving an explicit description of two such semigroup identities. We have seen that 4-Engel groups satisfy the identity $q_4(xy, yx, 1, 1, 1) = q_4(yx, xy, 1, 1, 1)$ which when expanded becomes

$$xy^2xyx^2y^2x^2yxy^2xyx^2yxy^2x^2y^2xyx^2y = yx^2yxy^2x^2y^2xyx^2yxy^2xyx^2y^2x^2yxy^2x. \tag{6}$$

We get the second identity by replacing the 4-Engel identity by a semigroup identity which is equivalent modulo the identity (6). We have

$$[y, [y, [y, [y, x]]]] = [y, [y, [y, y^x]]],$$

and therefore the 4-Engel identity is equivalent to

$$[y, [y, [y, y^x]]] = 1. \tag{7}$$

But (7) is equivalent to the identity

$$[xy, [xy, [xy, yx]]] = 1. \tag{8}$$

The identity (8) follows from (7) since the words xy and yx are conjugate and to see that (7) follows from (8) replace y with $x^{-1}y$ in (8). We want to find a semigroup identity which is equivalent to (8) modulo the identity (6). Let $u = xy$ and $v = yx$. Expansion of $[u, [u, [u, v]]]$ gives (8) the form

$$u^{-1}v^{-1}u^{-1}vu^{-1}v^{-1}uvuv^{-1}u^{-1}vuv^{-1}uv = 1.$$

We now introduce some more notation. We let $a(x, y)$, $b(x, y)$ and $c(x, y)$ be the semigroup words in x and y such that the identity (6) can be written in the following three ways:

$$\begin{aligned} xa(x, y) &= ya(y, x) \\ b(x, y)x &= b(y, x)y \\ xyc(x, y) &= yxc(y, x). \end{aligned}$$

Identity (6) then gives $v(vu)^{-1} = b(v, vu)^{-1}b(vu, v)$ and $v^{-1}u^{-1}vu = c(u, v)c(v, u)^{-1}$. This implies that modulo (6) the identity (8) is equivalent to

$$1 = u^{-1}v^{-1}u^{-1}b(v, vu)^{-1}b(vu, v)uvuc(u, v)c(v, u)^{-1}v^{-1}uv.$$

Using (6) again we have $(vc(v,u))^{-1}uv = a(vc(v,u),uv)a(uv,vc(v,u))^{-1}$. Using this together with the last identity we reach the semigroup identity

$$b(v,vu)uvua(uv,vc(v,u)) = b(vu,v)uvuc(u,v)a(vc(v,u),uv). \tag{9}$$

A group is therefore a 4-Engel group if and only if it satisfies the semigroup identities (6) and (9). The identity (9) is too long to be written down as word in x and y. Both sides have weight 576 in both x and y. It is likely however that one can find a description with simpler semigroup identities.

References

[1] K. W. Gruenberg. Two theorems on Engel groups. *Math. Proc. Camb. Phil. Soc.* **49** (1953), 377-380.

[2] J. Lewin and T. Lewin. Semigroup laws in varieties of solvable groups. *Proc. Camb. Phil. Soc.* **65** (1969), 1-9.

[3] P. Longobardi and M. Maj. Semigroup identities and Engel groups. *Preprint.*

[4] B. H. Neumann and T. Taylor. Subsemigroups of nilpotent groups. *Proc. Roy. Soc. Ser. A* **274** (1963), 1-4.

[5] A. Shalev. Combinatorial conditions in residually finite groups, II. *Journal of Algebra* **157** (1993), 51-62.

[6] A. I. Shirshov. Certain almost Engel groups. *Algebra i Logika, Seminar* **2**, No. 5 (1963), 5-18.

[7] G. Traustason. On 4-Engel groups. *Journal of Algebra* **178** (1995), 414-429.

[8] *Unsolved Problems in Group Theory, The Kourovka notebook* (Novosibirsk 1995).

[9] M. R. Vaughan-Lee, A note on 4-Engel groups, *Preprint.*